Weinberg's Soft Theorem to Three Loops in QCD

陈文 (Wen Chen)

Zhejiang University
chenwenphy@gmail.com

In collaboration with
罗民兴 (CSRC), 杨通智 (UZH), 朱华星 (ZJU)

At USTC, Hefei
May 15, 2023

Outline

Soft Theorem

Soft theorem (QED)

The amplitude for the emission of soft photons takes the factorized form: (Weinberg, QFTt Vol I)

$$\mathcal{M}(q_1, q_2, \cdots; p_1, p_2, \dots) \simeq \prod_i \left[\sum_j \eta_j Q_j \frac{\varepsilon(q_i) \cdot p_j}{q_i \cdot p_j} \right] \mathcal{M}(p_1, p_2, \dots). \qquad (1)$$

q_i : momenta of soft photons, $\quad p_i$: momenta of hard-scattering particles

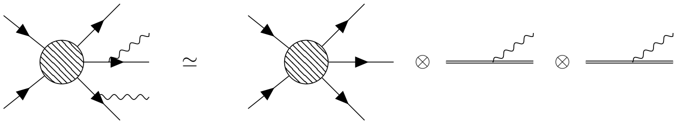


Figure: Soft factorization for the soft-photon emission.

Soft theorem holds to all orders in perturbation theory in massive QED.

Soft Factorization in QCD

$$
\begin{aligned}
&|\mathcal{M}^{\mu_1\mu_2\dots\mu_m}_{s_1 s_2\dots s_m, c_1 c_2\dots c_m}(q_1, q_2, \dots, q_m; p_1, p_2, \dots)\rangle \\
\simeq &(g_s\mu^{\epsilon})^m \underbrace{\boxed{J}}_{\boxed{\text{soft current}}}(q_1, q_2, \dots, q_m)\, |\underbrace{\boxed{\mathcal{M}}}_{\boxed{\text{hard-scattering amplitude}}}(p_1, p_2, \dots)\rangle\,.
\end{aligned}
\tag{2}
$$

The soft current $J(q)$ is process independent and can be calculated perturbatively.

One-loop soft current for a single gluon emission: Catani&Grazzini(2000)

$$
\begin{aligned}
J_a = &-\frac{(g_s\mu^{\epsilon})^3}{(4\pi)^2}\frac{1}{\epsilon^2}\frac{\Gamma^3(1-\epsilon)\Gamma^2(1+\epsilon)}{\Gamma(1-2\epsilon)} \\
&\times i f_{abc}\sum_{i\neq j} T_i^b T_j^c \left(\frac{\varepsilon(q)\cdot p_i}{p_i\cdot q} - \frac{\varepsilon(q)\cdot p_j}{p_j\cdot q}\right)\left[\frac{4\pi p_i\cdot p_j e^{-i\lambda_{ij}\pi}}{2p_i\cdot q p_j\cdot q e^{-i\lambda_{iq}\pi}e^{-i\lambda_{jq}\pi}}\right]^{\epsilon}.
\end{aligned}
\tag{3}
$$

Phenomenological Applications

Why soft currents:

Provide the subtraction terms for fixed-order calculations

Necessary ingredients for the resummation of large logarithms

Serve as the "boundary conditions" of the full amplitudes

Soft currents with two hard partons:
$e^+e^- \to$ dijet, deep inelastic scattering, Drell-Yan, etc.

State-of-art calculations for the single-gluon emission:

- One loop:
  Bern and Chalmers (1995), Bern et al. (1998), Bern et al. (1999), Catani & Grazzini (2000)
- Two loops:
  Li & Zhu (2013), Duhr & Gehrmann (2013) (double hard partons)
  Dixon et al. (2020) (triple hard partons)

Soft Currents in SCET

soft collinear effective theory (SCET):

A effective theory for the soft can collinear modes.

$$\boxed{\text{QCD}} \xrightarrow{\text{integrate out the hard mode}} \boxed{\text{SCET}}$$

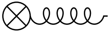
single-gluon soft current in SCET:

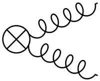
$$J(q) = \langle g(q)|Y_n^\dagger Y_{\bar{n}}|\Omega\rangle \,. \tag{4}$$

Soft Wilson line:

$$Y_n^\dagger(x) \equiv \exp\left(ig_s T^a \int_0^\infty ds\; n\cdot A^a(x+sn)\right) \tag{5}$$

Feynman rules of the Wilson line:





$\cdots$

$$-g_s n^\mu \frac{T^{a_1}}{-n\cdot q_1} \qquad g_s^2 \frac{n^{\mu_1} n^{\mu_2}}{-n\cdot(q_1+q_2)}\left(\frac{T^{a_1}T^{a_2}}{-n\cdot q_1} + \frac{T^{a_2}T^{a_1}}{-n\cdot q_2}\right) \qquad \cdots$$

The soft function can be calculated perturbatively in SCET:

$$J(q) = g_s \sum_{i=0}^{\infty} \left[\left(4\pi e^{-\gamma_E}\right)^{\epsilon} \frac{\alpha_s}{4\pi} \right]^i \varepsilon(q) \cdot J^{(i)}(q). \tag{6}$$

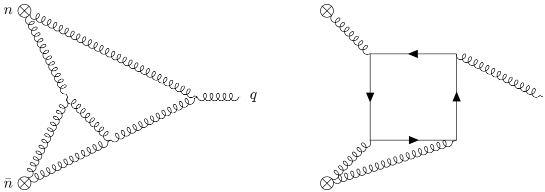


Figure: Representative diagrams for the three-loop soft current.

Reduction of Feynman Integrals

Amplitudes $\xrightarrow{\text{tensor reduction}}$ Scalar integrals $\xrightarrow{\text{IBP reduction}}$ Master integrals

Integration by parts

- Momentum space Tkachov (1981), Chetyrkin & Tkachov (1981)
- Baikov representation Baikov (1997), Larsen & Zhang (2016)
- block-triangular form Guan, Liu, Ma (2020)
- Parametric representation Lee (2014), Chen (2020a, 2020b, 2021)

Advantages of the parametric representation:

- The tensor reduction is trivial
- No auxiliary propagators
- The symmetries under permutations of indices are transparent
- All the indices are nonnegative (for normal loop integrals)

Parametric Representation

Schwinger alpha parametrization

$$\frac{1}{D_i^{\lambda_i+1}} = \frac{e^{-\frac{\lambda_i+1}{2}i\pi}}{\Gamma(\lambda_i+1)} \int_0^\infty dx_i \; e^{i x_i D_i} x_i^{\lambda_i}, \qquad \mathrm{Im}\{D_i\} > 0.$$

parametrization of a scalar integral

$$\boxed{\sum_i x_i D_i}$$

$$\int \mathrm{d}^d l_1 \mathrm{d}^d l_2 \cdots \mathrm{d}^d l_L \frac{1}{D_1^{\lambda_1+1} D_2^{\lambda_2+1} \cdots D_n^{\lambda_n+1}}$$

$$\to \int \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n \prod_{i=1}^{n} x_i^{\lambda_i} \int \mathrm{d}^d l_1 \mathrm{d}^d l_2 \cdots \mathrm{d}^d l_L \, \exp\left[i \boxed{\left(A_{ij} l_i \cdot l_j + 2B_i \cdot l_i + C\right)}\right]$$

$$\to \cdots \to \frac{\Gamma(-\lambda_0)}{\prod_{i=1}^{n+1} \Gamma(\lambda_i+1)} \int d\Pi^{(n+1)} \mathcal{F}^{\lambda_0} \prod_{i=1}^{n+1} x_i^{\lambda_i} \equiv \int d\Pi^{(n+1)} \mathcal{I}^{(-n-1)}.$$

integration measure: $\mathrm{d}\Pi^{(n+1)} \equiv \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n \delta(1 - E^{(1)})$

$E^{(n)}$: a positive definite homogeneous function of x_i of degree n.

Linear Reduction

parametric IBP

$$0 = \int d\Pi^{(n+1)} \frac{\partial}{\partial x_i} \mathcal{I}^{(-n)} + \delta_{\lambda_i 0} \int d\Pi^{(n)} \, \mathcal{I}^{(-n)} \Big|_{x_i=0} . \qquad (7)$$

A Feynman integral of a specific topology can be understood as a function of the indices.

$$\begin{aligned}
\mathcal{R}_i I(\lambda_0, \ldots, \lambda_i, \ldots, \lambda_n) =& (\lambda_i + 1) I(\lambda_0, \ldots, \lambda_i + 1, \ldots, \lambda_n), \\
\mathcal{D}_i I(\lambda_0, \ldots, \lambda_i, \ldots, \lambda_n) =& I(\lambda_0, \ldots, \lambda_i - 1, \ldots, \lambda_n), \\
\mathcal{A}_i I(\lambda_0, \ldots, \lambda_i, \ldots, \lambda_n) =& \lambda_i I(\lambda_0, \ldots, \lambda_i, \ldots, \lambda_n).
\end{aligned}$$

$$\begin{aligned}
A_{n+1} \equiv & -(L+1) A_0 - \sum_{i=1}^{n} (A_i + 1) - 1, \\
R^i_{n+1} \equiv & (A_{n+1} + 1)(A_{n+1} + 2) \cdots (A_{n+1} + i).
\end{aligned}$$

$$\mathcal{D}_0 \left(\frac{\partial \mathcal{F}}{\partial \mathcal{R}_i} - \mathcal{D}_i \right) \mathcal{R}_{n+1} I = 0. \qquad (8)$$

Reduction of Tensor Integrals

$$\int \mathsf{d}^d l_1 \mathsf{d}^d l_2 \cdots \mathsf{d}^d l_L \frac{l_{i_1}^{\mu_1} l_{i_2}^{\mu_2} \cdots l_{i_r}^{\mu_r}}{D_1^{\lambda_1+1} D_2^{\lambda^2+1} \cdots D_n^{\lambda_n+1}} \tag{9}$$
$$\rightarrow \left[P_{i_1}^{\mu_1} P_{i_2}^{\mu_2} \cdots P_{i_r}^{\mu_r} I(\lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_n) \right]_{p^\mu=0},$$

$$P_i^\mu(p) \equiv -\frac{\partial}{\partial p_{i,\mu}} - \widetilde{B}_i^\mu + \frac{1}{2} \sum_{j=1}^{L} \widetilde{A}_{ij} p_j^\mu, \tag{10}$$

$$\widetilde{A}_{ij} \equiv \mathcal{D}_0 U (A^{-1})_{ij},$$
$$\widetilde{B}_i^\mu \equiv \sum_{j=1}^{L} \widetilde{A}_{ij} B_j^\mu$$

$$f_1(\tilde{A}, \tilde{B}) + f_2(\tilde{A}, \tilde{B}) \frac{\partial \mathcal{F}}{\partial x_i} = 0.$$

Integrals with Theta Functions

$$w_\lambda(D_i) \equiv e^{-\frac{\lambda+1}{2} i\pi} \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{1}{x^{\lambda+1}} e^{ix\, D_i} \,. \tag{11}$$

$$w_0(u) = \theta(u), \qquad w_{-1}(u) = \delta(u).$$

$$\begin{aligned} &\int \mathrm{d}^d l_1 \mathrm{d}^d l_2 \cdots \mathrm{d}^d l_L \, \frac{w_{\lambda_1}(D_1) w_{\lambda_2}(D_2) \cdots w_{\lambda_m}(D_m)}{D_{m+1}^{\lambda_{m+1}+1} D_{m+2}^{\lambda_{m+2}+1} \cdots D_n^{\lambda_n+1}} \\ \rightarrow & \frac{\Gamma(-\lambda_0)}{\prod_{i=m+1}^{n+1} \Gamma(\lambda_i+1)} \int \mathrm{d}\Pi^{(n+1)} \mathcal{F}^{\lambda_0} \prod_{i=1}^{n+1} x_i^{\lambda_i}. \end{aligned} \tag{12}$$

$$\begin{aligned} \hat{x}_i &= \begin{cases} \mathcal{D}_i \quad , & i = 1,\ 2, \ldots,\ m, \\ \mathcal{R}_i \quad , & i = m+1,\ m+2, \ldots,\ n+1, \end{cases} \\ \hat{z}_i &= \begin{cases} -\mathcal{R}_i \quad , & i = 1,\ 2, \ldots,\ m, \\ \mathcal{D}_i \quad , & i = m+1,\ m+2, \ldots,\ n+1, \end{cases} \\ \hat{a}_i &= \begin{cases} -\mathcal{A}_i - 1 \quad , & i = 1,\ 2, \ldots,\ m, \\ \mathcal{A}_i \quad , & i = m+1,\ m+2, \ldots,\ n+1. \end{cases} \end{aligned}$$

$$\mathcal{D}_0 \left(\frac{\partial \mathcal{F}}{\partial \hat{x}_i} - \hat{z}_i \right) \hat{z}_{n+1} I = 0. \tag{13}$$

Representation Transformation

representation transformation Chen (2023)

$$u_i \equiv \frac{1}{\mathcal{F}} \frac{\partial \mathcal{F}}{\partial x_i} \tag{14a}$$

$$\mathcal{G} \equiv \frac{1}{\mathcal{F}}. \tag{14b}$$

duality

$$x_i = \frac{1}{\mathcal{G}} \frac{\partial \mathcal{G}}{\partial u_i} \tag{15a}$$

$$\mathcal{F} = \frac{1}{\mathcal{G}}. \tag{15b}$$

(rescaled) Baikov polynomial

$$\begin{aligned} P(v_1, v_2, \ldots, v_n) \equiv & \frac{\det(q_i \cdot q_j)}{\det(p_i \cdot p_j)} \\ = & -\mathcal{G}(v_1, v_2, \cdots, v_n, -1). \end{aligned} \tag{16}$$

Calculation of Master Integrals

- Mellin-Barnes method
- Sector decomposition
- Differential-equation method Kotikov (1991), Remiddi (1997)

  Canonical basis Henn (2013)
  Auxiliary mass flow Liu, Ma, Wang (2018), Liu, Ma (2023)
  ...

Differential Equations in the Parametric Representation

$$\frac{\partial}{\partial y} I = -\mathcal{D}_0 \frac{\partial \mathcal{F}}{\partial y} I. \tag{17}$$

For soft integrals, the scale-dependence is trivial. An auxiliary scale can be introduced by inserting a delta function.

$$\begin{aligned} I(\lambda_0, \lambda_1, \ldots, \lambda_n) &= \int \mathrm{d}y \mathrm{d}\Pi^{(n+1)} \ \delta(y - E^{(0)}(x)) \mathcal{I}^{(-n-1)} \\ &\equiv \int \mathrm{d}y I_y. \end{aligned} \tag{18}$$

Equation (13) is a consequence of the homogeneity of the integrand. Thus it also holds for the integral I_y in eq. (18).
A simple choice of $E^{(0)}$:

$$E^{(1)} = \frac{x_i}{x_j}.$$

Asymptotic Expansion

convex-hull method Pak, Smirnov (2011)

$$\mathcal{F} = \sum_{a=1}^{A} \left(C_{\mathcal{F},a} \prod_{i}^{n+1} x_i^{\Lambda_{ai}} \right), \tag{19}$$

a region r is associated with a subset S_r of $\{1,\ 2, \cdots, A\}$ and a vector $\boldsymbol{k}_r$, such that the number of elements of S_r is not less than n, and

$$\sum_{k=1}^{n+1} \Lambda_{ak} k_{r,k} = k_{r,0}, \quad a \in S_r, \tag{20a}$$

$$\sum_{k=1}^{n+1} \Lambda_{ak} k_{r,k} > k_{r,0}, \quad a \notin S_r. \tag{20b}$$

For the y-dependent integral I_y (Remember that $\mathcal{F}'$ is obtained from $\mathcal{F}$ through the replacement $x_i \to y x_j$.)

$$\mathcal{F}' = \sum_{a=1}^{A} \left(C_{\mathcal{F},a} \prod_{i}^{n+1} x_i^{\Lambda'_{ai}} \right). \tag{21}$$

$$\Lambda'_{aj} = \Lambda_{aj} + \Lambda_{ai}, \tag{22a}$$

$$\Lambda'_{ak} = \Lambda_{ak}, \qquad k \neq j. \tag{22b}$$

Regions of the y-dependent integral:

$$\sum_{k=1}^{n+1} \Lambda'_{ak} k'_{r,k} = k_{r,0}, \quad a \in S_r, \tag{23a}$$

$$\sum_{k=1}^{n+1} \Lambda'_{ak} k'_{r,k} > k_{r,0}, \quad a \notin S_r, \tag{23b}$$

$$k'_{r,i} \equiv k_{r,i} - k_{r,j}, \tag{24a}$$

$$k'_{r,k} \equiv k_{r,k}, \qquad k \neq i. \tag{24b}$$

Rules on choosing $E^{(0)}$

$$R_{ij} = \{r | k_{r,i} > k_{r,j}\}. \tag{25}$$

(1) We choose the pair $\{i,\ j\}$ such that the cardinal number of R_{ij} is minimized, where R_{ij} is defined in eq. (25).

(2) Among all the pairs satisfying the first rule, we choose the one such that $\max\{N_r | r \in R_{ij}\}$ is minimized, where N_r is the cardinal number of S_r.

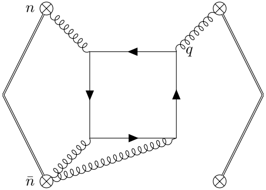
Boundary Integrals

$$
\begin{aligned}
I(\lambda_0, \lambda_1, \ldots, \lambda_n) =& \frac{\Gamma(-\lambda_0)}{\prod_{i=1}^{n+1} \Gamma(\lambda_i + 1)} \int \mathsf{d}\Pi^{(n+1)} \mathcal{F}^{\lambda_0} \prod_{i=1}^{n+1} x_i^{\lambda_i} \\
=& \frac{(L+1) \prod_{a=1}^{n+1} \left[\Gamma(\bar{\lambda}_a) C_{\mathcal{F},a}^{-\bar{\lambda}_a} \right]}{\parallel \Lambda \parallel \prod_{i=1}^{n+1} \Gamma(\lambda_i + 1)} .
\end{aligned}
\tag{26}
$$

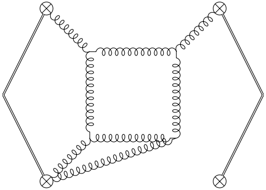
Result

QCD result

$$J^a_\mu(q) = -\,\frac{g_s}{2}\left(\frac{n_1^\mu}{n_1\cdot q}-\frac{n_2^\mu}{n_2\cdot q}\right)\Bigg[\left(\mathbf{T}_1^a-\mathbf{T}_2^a\right)\,A_{12}+2if^{abc}\left(\mathbf{T}_1^b\mathbf{T}_2^c-\mathbf{T}_2^b\mathbf{T}_1^c\right)\,B_{12}$$
$$-\left(\mathbf{T}_1^b\mathbf{T}_1^c\mathbf{T}_2^d-\mathbf{T}_2^b\mathbf{T}_2^c\mathbf{T}_1^d\right)\left(C_{12}\,d_A^{abcd}+D_{12}\,d_F^{abcd}N_f\right)\Bigg]+\mathcal{O}(\alpha_s^4)\,, \quad (27)$$



$d_F^{abcd}d_F^{abcd}$



$d_F^{abcd}d_A^{abcd}$

$$S_{12}^{(l)}(q) = \frac{1}{4N_R C_R} \mathsf{Tr}\left\{ \left[\varepsilon^{\mu} J_{\mu}^{a(l)}\right] \left[\varepsilon^{\nu} J_{\nu}^{a(0)}\right]^{*}(q) \right\}, \tag{28}$$

$$S_{\epsilon} = \left(4\pi S_{12}^{(0)} \mu^2 e^{-\gamma_E} \frac{e^{-i\lambda_{12}\pi}}{e^{-i\lambda_{1q}\pi} e^{-i\lambda_{2q}\pi}}\right)^{\epsilon},$$

$$S_{12}^{(l)}(q) = S_{12}^{(0)}(q) S_{\epsilon}^{l}\, r_{12}^{(l)},$$

$$B_{12}^{(l)} = S_{\epsilon}^{l}\, b_{12}^{(l)}, C_{12}^{(l)} = S_{\epsilon}^{l}\, c_{12}^{(l)}, D_{12}^{(l)} = S_{\epsilon}^{l}\, d_{12}^{(l)}.$$

$$
\begin{aligned}
b_{12}^{(3)} =& C_A^2 \Bigg\{ -\frac{1}{6\epsilon^6} + \frac{11}{12\epsilon^5} + \frac{1}{\epsilon^4}\left(\frac{119}{324} - \frac{3\zeta_2}{4}\right) + \frac{1}{\epsilon^3}\left(\frac{649\zeta_2}{216} + \frac{2\zeta_3}{3} - \frac{1517}{486}\right) \\
&+ \frac{1}{\epsilon^2}\left(\frac{2501\zeta_2}{648} - \frac{2101\zeta_3}{108} - \frac{1487\zeta_4}{288} - \frac{7271}{486}\right) \\
&+ \frac{1}{\epsilon}\left(\frac{11\zeta_3\zeta_2}{18} + \frac{437\zeta_2}{972} + \frac{2575\zeta_3}{36} - \frac{22583\zeta_4}{576} + \frac{98\zeta_5}{5} - \frac{446705}{8748}\right) + \dots \Bigg\} \\
&+ C_A N_f \Bigg\{ -\frac{1}{6\epsilon^5} + \frac{43}{162\epsilon^4} + \frac{1}{\epsilon^3}\left(\frac{895}{486} - \frac{59\zeta_2}{108}\right) \\
&+ \frac{1}{\epsilon^2}\left(-\frac{31\zeta_2}{324} + \frac{239\zeta_3}{54} + \frac{2603}{486}\right) \\
&+ \frac{1}{\epsilon}\left(\frac{3265\zeta_2}{972} - \frac{4945\zeta_3}{162} + \frac{2437\zeta_4}{288} + \frac{24169}{2187}\right) + \dots \Bigg\} \\
&+ C_F N_f \Bigg\{ \frac{1}{9\epsilon^3} + \frac{1}{\epsilon^2}\left(\frac{55}{54} - \frac{8\zeta_3}{9}\right) + \frac{1}{\epsilon}\left(\frac{\zeta_2}{6} - \frac{76\zeta_3}{27} - \frac{4\zeta_4}{3} + \frac{1819}{324}\right) + \dots \Bigg\} \\
&+ N_f^2 \Bigg\{ -\frac{4}{81\epsilon^4} + -\frac{40}{243\epsilon^3} + \frac{1}{\epsilon^2}\left(-\frac{2\zeta_2}{27} - \frac{8}{27}\right) \\
&+ \frac{1}{\epsilon}\left(-\frac{20\zeta_2}{81} + \frac{260\zeta_3}{81} - \frac{704}{2187}\right) + \dots \Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
c_{12}^{(3)} =&\frac{-32\zeta_2\zeta_3 - 16\zeta_5}{\epsilon} - 192\zeta_3^2 + \frac{64\zeta_3}{3} - 64\zeta_2 + \frac{1760\zeta_5}{3} - 940\zeta_6 + \dots\,,\\
d_{12}^{(3)} =&128\zeta_2 - \frac{128\zeta_3}{3} - \frac{640\zeta_5}{3} + \dots\,.
\end{aligned}
$$

$\mathcal{N} = 4$ SYM result

$$S^{(3)}_{12,\mathcal{N}=4}(q) = S^0_{12}(q) S^3_\epsilon \Bigg[C_A^3 \Bigg\{ -\frac{1}{6\epsilon^6} - \frac{3\zeta_2}{4\epsilon^4} + \frac{2\zeta_3}{3\epsilon^3} - \frac{1487\zeta_4}{288\epsilon^2} + \frac{1}{\epsilon}\left(\frac{284\zeta_5}{15} - \frac{13\zeta_2\zeta_3}{18}\right) + \frac{5\zeta_3^2}{36} + \frac{174959\zeta_6}{6912} + \dots \Bigg\} + \frac{3}{2} C_A \left\{ \frac{-32\zeta_2\zeta_3 - 16\zeta_5}{\epsilon} - 192\zeta_3^2 - 940\zeta_6 + \dots \right\} \Bigg],$$

Bern-Dixon-Smirnov (BDS) ansatz

$$S^{(l)}_{12,\mathcal{N}=4}(q) = 2^l S^0_{12}(q) S^l_\epsilon C^l_A r^{(l)}_S(\epsilon) + \text{sub-leading color contribution},$$

$$r^{(3)}_S(\epsilon) = -\frac{1}{3}\left(r^{(1)}_S(\epsilon)\right)^3 + r^{(1)}_S(\epsilon) r^{(2)}_S(\epsilon) + f^{(3)}(\epsilon) r^{(1)}_S(3\epsilon) + \mathcal{O}(\epsilon), \qquad (29)$$

$$f^{(3)}(\epsilon) = \frac{11\zeta_4}{2} + (5\zeta_2\zeta_3 + 6\zeta_5)\epsilon + a\epsilon^2 + \mathcal{O}(\epsilon^3),$$

$$a = 31\zeta_3^2 + \frac{1909\zeta_6}{48} \simeq 85.25374611,$$

Summary

We calculated the single-gluon soft current with two hard-scattering partons to three loops.

We developed a systematic method to calculate Feynman integrals recursively basing on the parametric representation.

We confirmed the prediction on the three-loop soft current in SYM based on the BDS ansatz.

Our results provide an indispensable ingredient for the $\mathrm{N^4LO}$ QCD corrections.

Thanks for your attention!

- Z. Bern and G. Chalmers, Nucl. Phys. B **447** (1995), 465-518 [arXiv:hep-ph/9503236 [hep-ph]].
- Z. Bern, V. Del Duca and C. R. Schmidt, Phys. Lett. B **445** (1998), 168-177 [arXiv:hep-ph/9810409 [hep-ph]].
- Z. Bern, V. Del Duca, W. B. Kilgore and C. R. Schmidt, Phys. Rev. D **60** (1999), 116001 [arXiv:hep-ph/9903516 [hep-ph]].
- S. Catani and M. Grazzini, Nucl. Phys. B **591** (2000), 435-454 [arXiv:hep-ph/0007142 [hep-ph]].
- Y. Li and H. X. Zhu, JHEP **11** (2013), 080 d[arXiv:1309.4391 [hep-ph]].
- C. Duhr and T. Gehrmann, Phys. Lett. B **727** (2013), 452-455 [arXiv:1309.4393 [hep-ph]].
- L. J. Dixon, E. Herrmann, K. Yan and H. X. Zhu, JHEP **05**, 135 (2020) [arXiv:1912.09370 [hep-ph]].
- Z. Bern, L. J. Dixon and V. A. Smirnov, Phys. Rev. D **72**, 085001 (2005) [arXiv:hep-th/0505205 [hep-th]].

- F. V. Tkachov, Phys. Lett. **100B**, 65 (1981).
- K. G. Chetyrkin and F. V. Tkachov, Nucl. Phys. B **192**, 159 (1981).
- P. A. Baikov, Nucl. Instrum. Meth. A **389**, 347 (1997).
- K. J. Larsen and Y. Zhang, Phys. Rev. D **93**, no.4, 041701 (2016) [arXiv:1511.01071 [hep-th]].
- X. Guan, X. Liu and Y. Q. Ma, Chin. Phys. C **44** (2020) no.9, 093106 [arXiv:1912.09294 [hep-ph]].
- R. N. Lee, [arXiv:1405.5616 [hep-ph]].
- W. Chen, JHEP **02**, 115 (2020) [arXiv:1902.10387 [hep-ph]].
- W. Chen, Eur. Phys. J. C **81**, no.3, 244 (2021) [arXiv:1912.08606 [hep-ph]].
- W. Chen, Eur. Phys. J. C **80**, no.12, 1173 (2020) [arXiv:2007.00507 [hep-ph]].
- W. Chen, [arXiv:2303.12427 [hep-ph]].

A. V. Kotikov, Phys. Lett. B **254** (1991), 158-164.

E. Remiddi, Nuovo Cim. A **110** (1997), 1435-1452 [arXiv:hep-th/9711188 [hep-th]].

J. M. Henn, Phys. Rev. Lett. **110** (2013), 251601 [arXiv:1304.1806 [hep-th]].

R. N. Lee, JHEP **04** (2015), 108 [arXiv:1411.0911 [hep-ph]].

X. Liu, Y. Q. Ma and C. Y. Wang, Phys. Lett. B **779** (2018), 353-357 doi:10.1016/j.physletb.2018.02.026 [arXiv:1711.09572 [hep-ph]].

X. Liu and Y. Q. Ma, Comput. Phys. Commun. **283** (2023), 108565 [arXiv:2201.11669 [hep-ph]].

A. Pak and A. Smirnov, Eur. Phys. J. C **71** (2011), 1626 [arXiv:1011.4863 [hep-ph]].