An Introduction to NeatIBP 1.0:

A small-size IBP system generator

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2305.08783

Nature of the problem

IBP reduction is a critical step in Feynman integral computation.

IBP reduction computation is heavy for IBP systems with:

1. multiple scales
2. large size

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Idea of solution:

1. numerical reduction & expression reconstruction
2. finding a smaller sized IBP system

NeatIBP

Features of NeatIBP

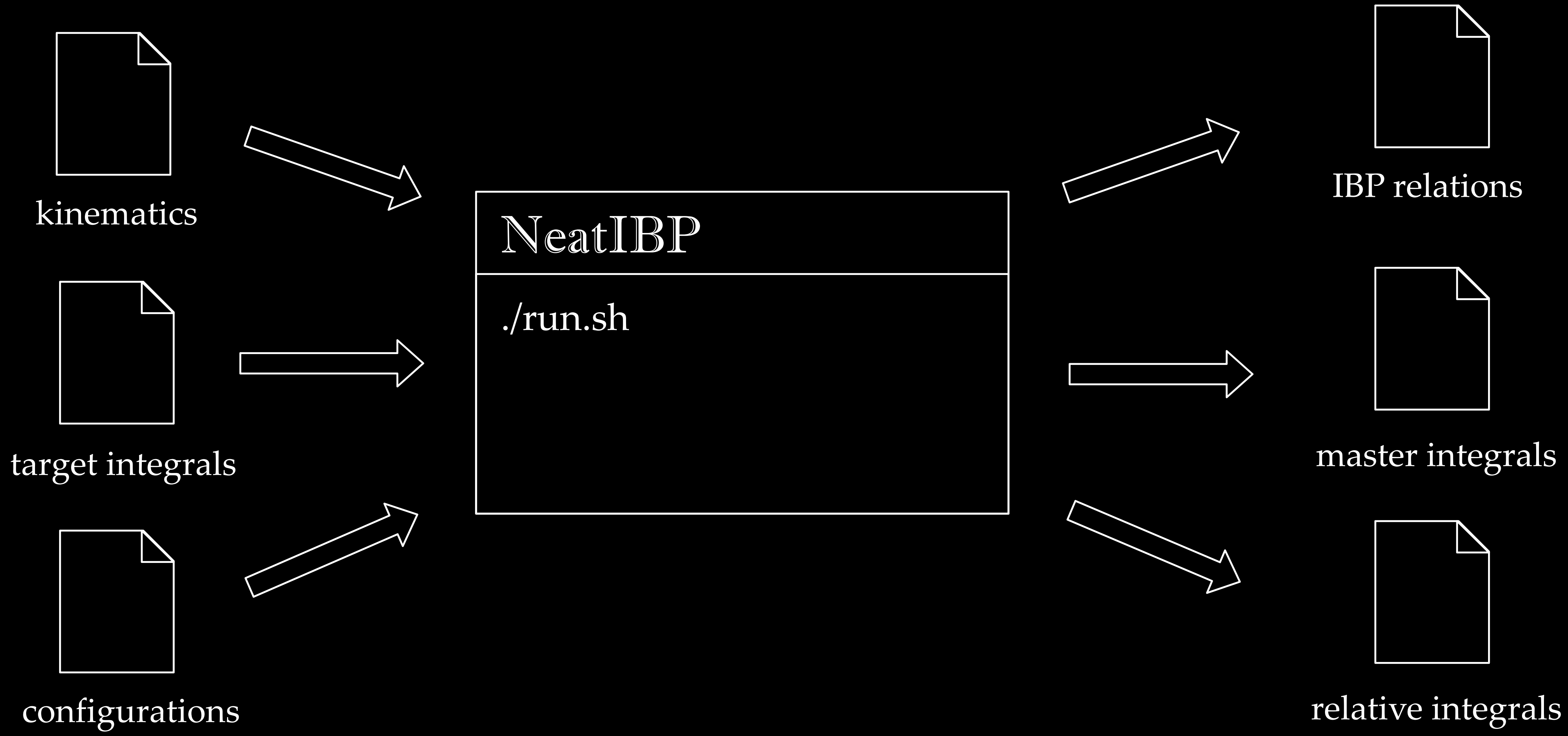
1. Controlling the size of the IBP system by avoiding integrals with denominator power increased.

2. Generating IBP relations in Baikov representation, using syzygy equations and module intersection methods.

3. Adopting row reduction methods on finite field in IBP generation and selection.

4. Employing parallization between sectors.

Inputs and outputs



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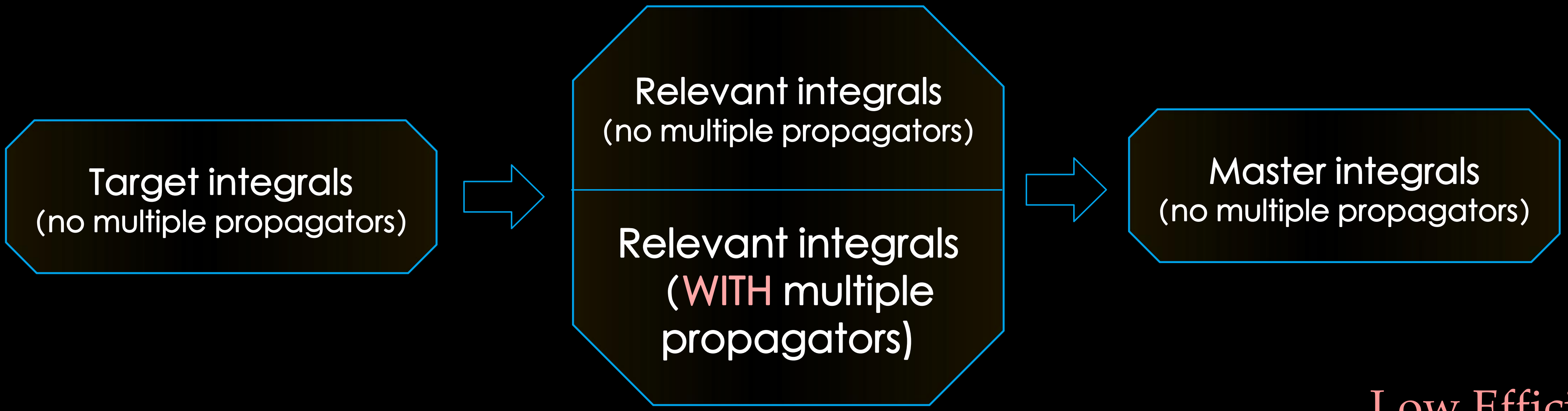
IBP relations with multiple propagators

$$0 = \int \frac{\mathrm{d}^D l_1}{i\pi^{D/2}} \cdots \frac{\mathrm{d}^D l_L}{i\pi^{D/2}} \frac{\partial}{\partial l_k^\mu} \frac{v^\mu}{D_1^{\alpha_1} \cdots D_n^{\alpha_n}} = \int \frac{\mathrm{d}^D l_1}{i\pi^{D/2}} \cdots \frac{\mathrm{d}^D l_L}{i\pi^{D/2}} \frac{\frac{\partial v^\mu}{\partial l_k^\mu} - v^\mu \sum_{i=1}^n \frac{\partial D_i}{\partial l_k^\mu} \frac{\alpha_k}{D_i}}{D_1^{\alpha_1} \cdots D_n^{\alpha_n}}$$

Introducing multiple propagators



Low Efficiency

The Baikov representation

Feynman integrals in momentum space:

$$I[\alpha_1, \cdots, \alpha_n] = \int \frac{\mathrm{d}^D l_1}{i\pi^{D/2}} \cdots \frac{\mathrm{d}^D l_L}{i\pi^{D/2}} \frac{1}{D_1^{\alpha_1} \cdots D_n^{\alpha_n}}$$

Rather than integrating over loop momenta, Baikov representation integrates directly over propagators z_i

$$I[\alpha_1, \cdots, \alpha_n] = C \int \mathrm{d}z_1 \cdots \mathrm{d}z_n P(z)^\alpha \frac{1}{z_1^{\alpha_1} \cdots z_n^{\alpha_n}}$$

IBP relations in Baikov representation

$$0 = \int \mathrm{d}z_1 \cdots \mathrm{d}z_n \sum_{i=1}^{n} \frac{\partial}{\partial z_i}\left(a_i(z) P^{\alpha} \frac{1}{z_1^{\alpha_1} \cdots z_n^{\alpha_n}}\right)$$

$$= \int \mathrm{d}z_1 \cdots \mathrm{d}z_n \left(\sum_{i=1}^{n} \frac{\partial a_i}{\partial z_i} P^{\alpha} + \sum_{i=1}^{n} \alpha a_i \boxed{\frac{\partial P}{\partial z_i} P^{\alpha-1}} - P^{\alpha} \sum_{i=1}^{n} \alpha_i \frac{a_i}{\boxed{z_i}}\right) \frac{1}{z_1^{\alpha_1} \cdots z_n^{\alpha_n}}$$

not Baikov polynomial

multiple propagators

$$\boxed{\left(\sum_{i=1}^{n} a_i(z) \frac{\partial P}{\partial z_i}\right) + b(z) P = 0}$$

$$\boxed{a_i(z) = b_i(z) z_i \quad \text{for } i \in \{j | \alpha_j > 0\}}$$

$$0 = \int \mathrm{d}z_1 \cdots \mathrm{d}z_n \left(\sum_{i=1}^{n}\left(\frac{\partial a_i}{\partial z_i} - \alpha_i b_i\right) + \alpha b\right) P^{\alpha} \frac{1}{z_1^{\alpha_1} \cdots z_n^{\alpha_n}}$$

Without multiple propagators

The syzygy equations and module intersection

$$\left(\sum_{i=1}^{n} a_i(z)\frac{\partial P}{\partial z_i}\right) + b(z)P = 0 \quad \Leftrightarrow \begin{pmatrix} a_i \\ b \end{pmatrix} \in M_1$$

$$a_i(z) = b_i(z) z_i \quad \text{for } i \in \{j|\alpha_j > 0\} \Leftrightarrow \begin{pmatrix} a_i \\ b \end{pmatrix} \in M_2$$

$$M_1 =< f_1, f_2, \cdots >$$

$$M_2 =< g_1, g_2, \cdots >$$

General expressions exist

$$\begin{pmatrix} a_i \\ b \end{pmatrix} \in M_1 \cap M_2$$

Module intersection

SINGULAR

Seeding and IBP relation generation

Formal IBP relations

$$0 = \int \mathrm{d}z_1 \cdots \mathrm{d}z_n \left(\sum_{i=1}^{n} \left(\frac{\partial a_i}{\partial z_i} - \alpha_i b_i \right) + \alpha b \right) P^{\alpha} \frac{1}{z_1^{\alpha_1} \cdots z_n^{\alpha_n}}$$

Seeding

$$\vec{\alpha} \to (1, \cdots, 1, 1, -2, -3)$$
$$\vec{\alpha} \to (1, \cdots, 1, 0, -5)$$
$$\vdots$$

IBP relations generated from Formal IBPs via seeding

$$0 = \sum_j c_{ij} I_j$$

⟸ Numeric row reduction on finite field to determine whether the system is enough

SpaSM

IBP relation selection

An enough IBP system $0 = \sum_j c_{ij} I_j$

Column reduction (numeric + finite field)

Linearly independent system $0 = \sum_j \tilde{c}_{ij} I_j$

Row reduction (numeric + finite field) $R_{ik} = L_{ij} \tilde{\mathbf{c}}_{jk}$

Remove the unneeded relations for reducing the targets

Small-size IBP system minimally needed

Remarks on numeric + finite field matrix reductions

IBP relations with symbolic variables

⇓ Enumerate at a generic numerical point $s_{12} \to \frac{1}{97}, \cdots, m_1^2 \to \frac{1}{9001}, \cdots, d \to \frac{1}{137}$

Numerical IBP relations

⇓ Row reduce modulo p

RREF form

Risk: numeric point not general enough

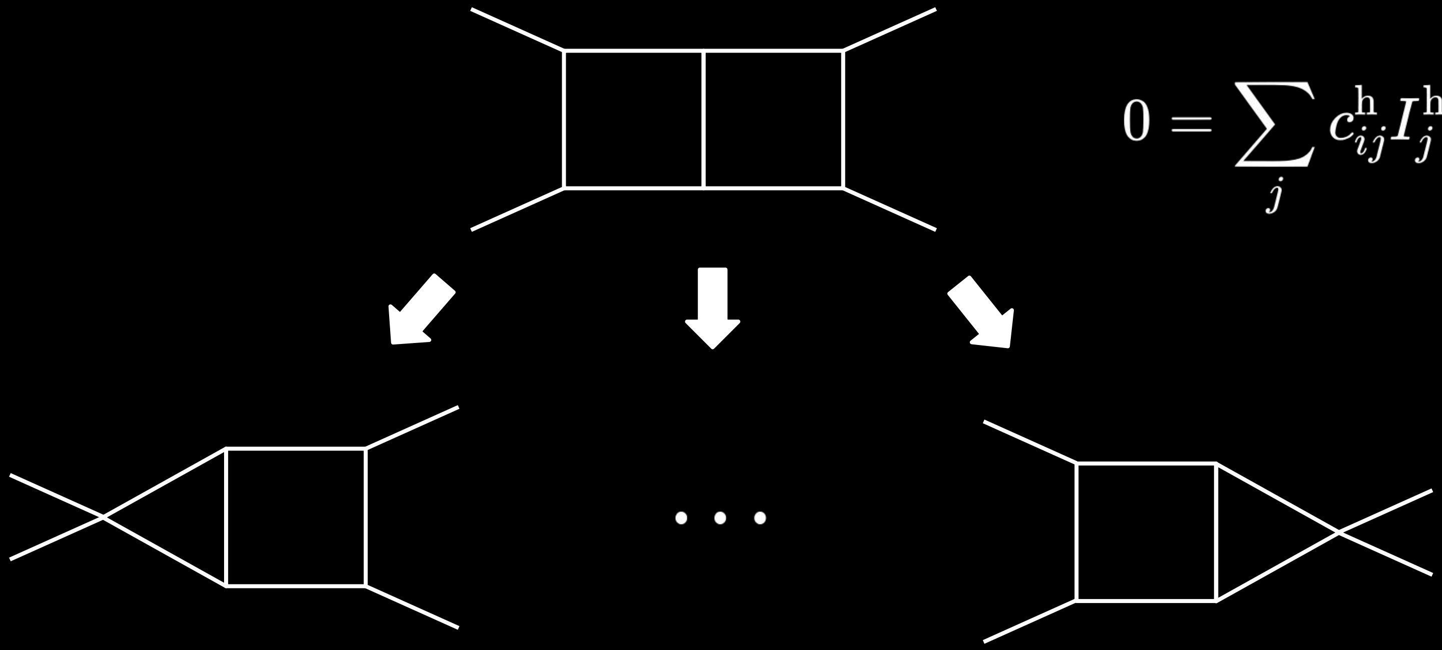
Current solution: check on other numeric points afterwards

Tail mask strategy

current sector:

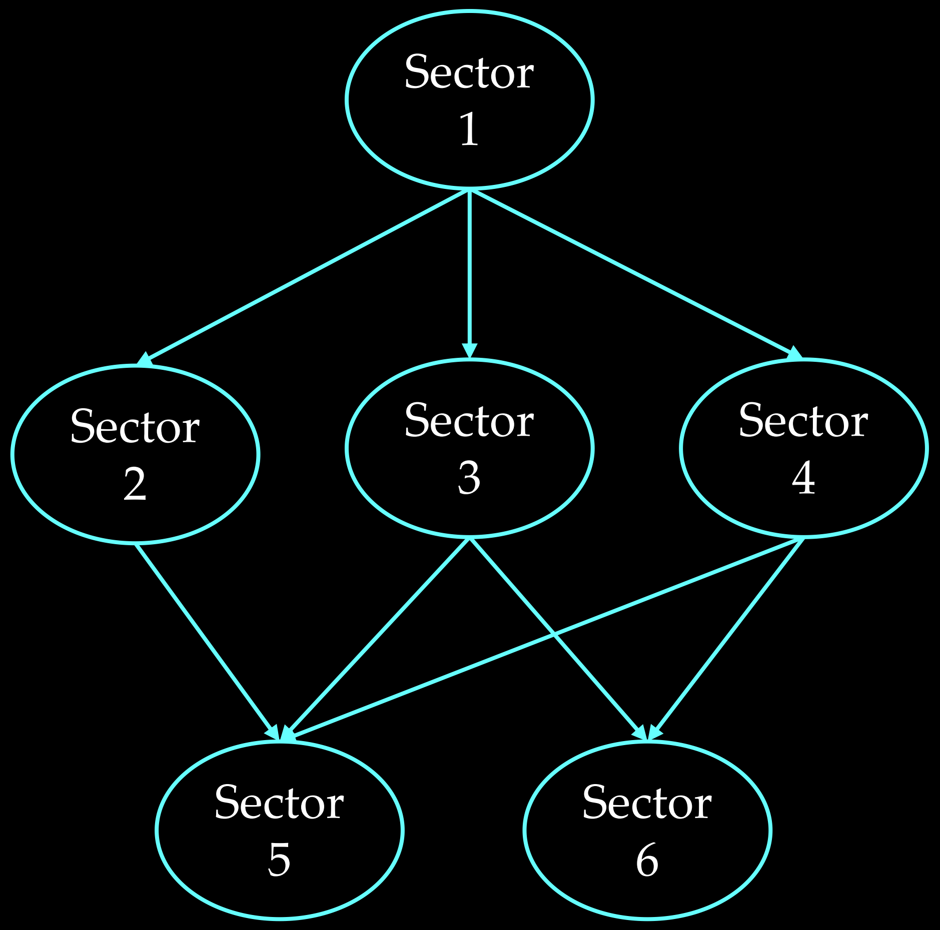
$$0 = \sum_j c^{\mathrm{h}}_{ij} I^{\mathrm{h}}_j + \sum_k c^{\mathrm{t}}_{ik} I^{\mathrm{t}}_k$$

sub sectors:

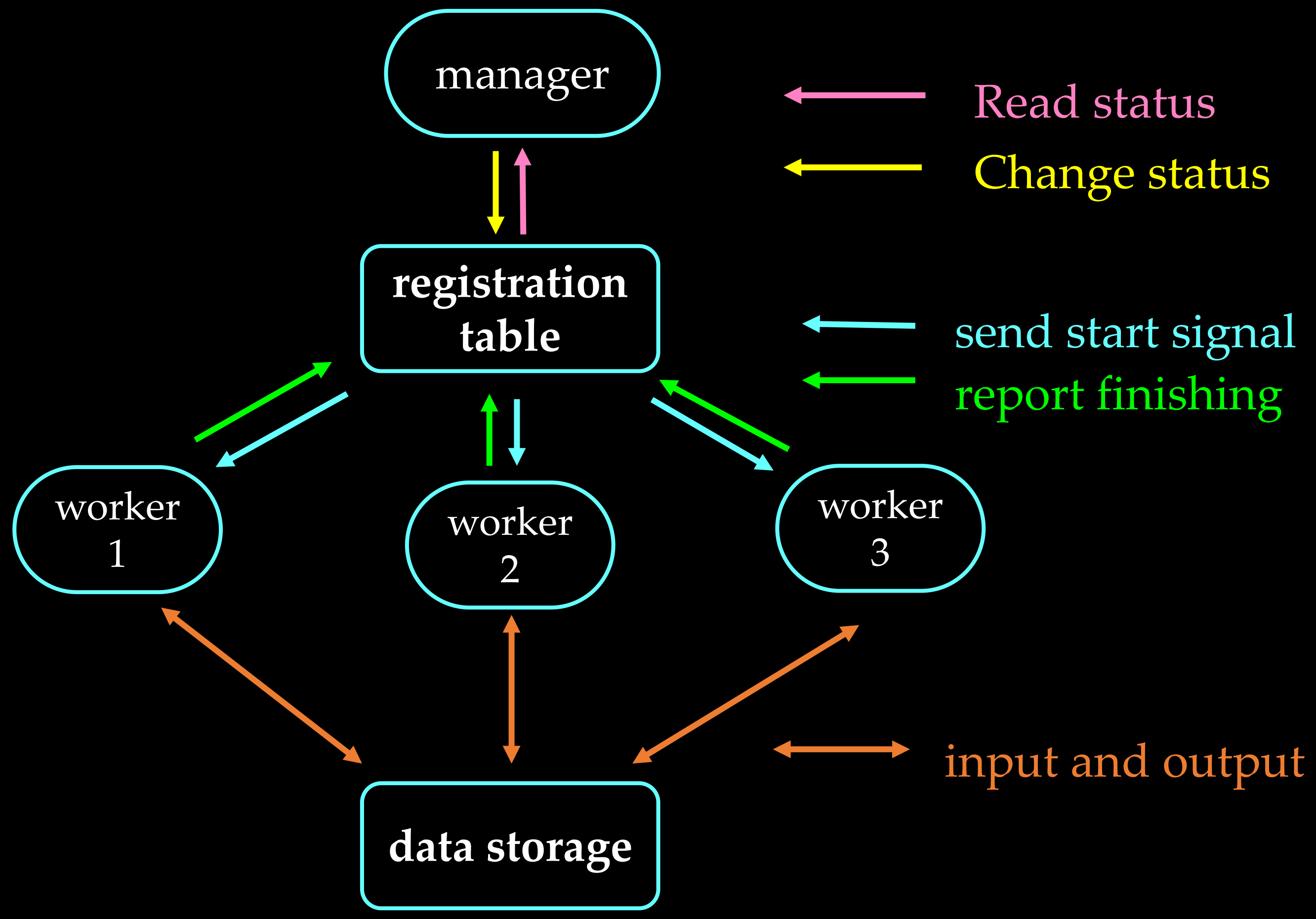


Parallelization between sectors

Web structure of the sectors



Parallelization scheme in NeatIBP 1.0



Package Installation

https://github.com/yzhphy/NeatIBP

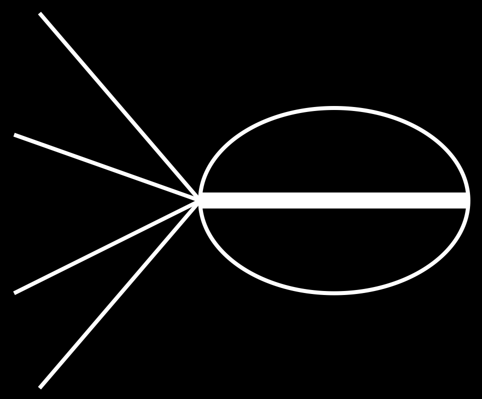
Symmetries in NeatIBP

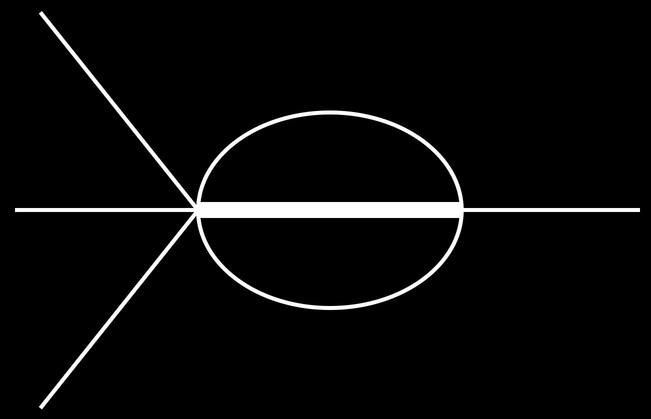
Polynomial symmetry of $G = U + F$ $\Longleftarrow$ Momentum map

$x_i \to x_{\sigma(i)}$

$l_i \to A_{ij} l_j + B_{ij} p_j$
$p_i \to C_{ij} p_j$

A counterexample





Propagator cuts in NeatIBP

Cuts in Baikov representation

$$I_{\alpha_1,\cdots,\alpha_n}\big|_{\mathcal{C}-\text{cut}} \propto \oint_0 \prod_{i\in\mathcal{C}} \mathrm{d}z_i \int \prod_{i\notin\mathcal{C}} \mathrm{d}z_i P^\alpha \frac{1}{z_1^{\alpha_1}\cdots z_n^{\alpha_n}}$$

In NeatIBP 1.0

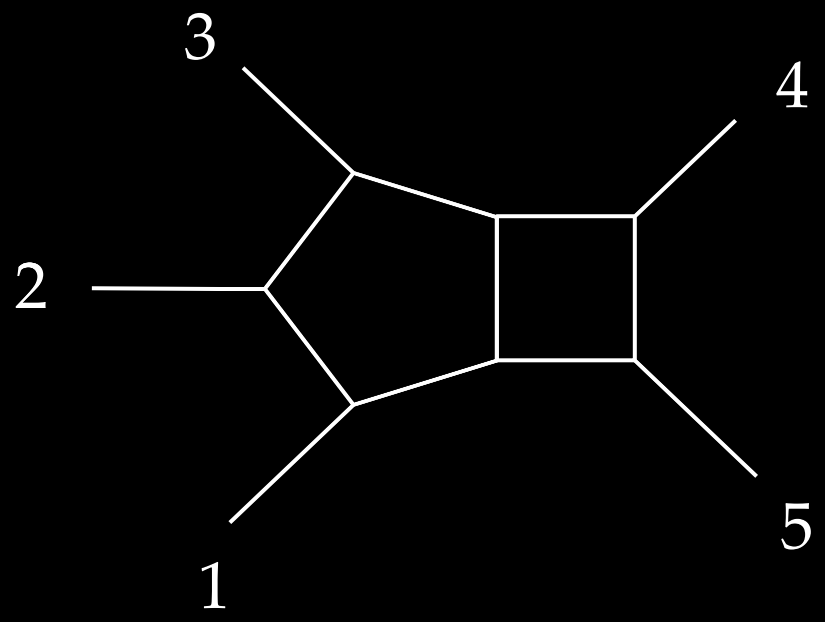
$$\alpha_i < 2, i \in \mathcal{C}$$

For sectors such that $\alpha_i = 1, i \in \mathcal{C}$

$$P \to P\big|_{z_i\to 0, i\in\mathcal{C}}$$

Example I

J. Gluza, K. Kajda, D. A. Kosower, 1009.0742



Target integrals with high-degree numerators

Quantity: 2483

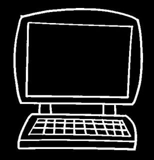
Max numerator degree: 5

Max denominator power: 1

#MI: 61

#IBP: 14120

#IBP (FIRE6): 11207942

Time used: 27m at  CPU cores: 10
RAM: 128GB

Example I

J. Gluza, K. Kajda, D. A. Kosower, 1009.0742

Target integrals for differential equations

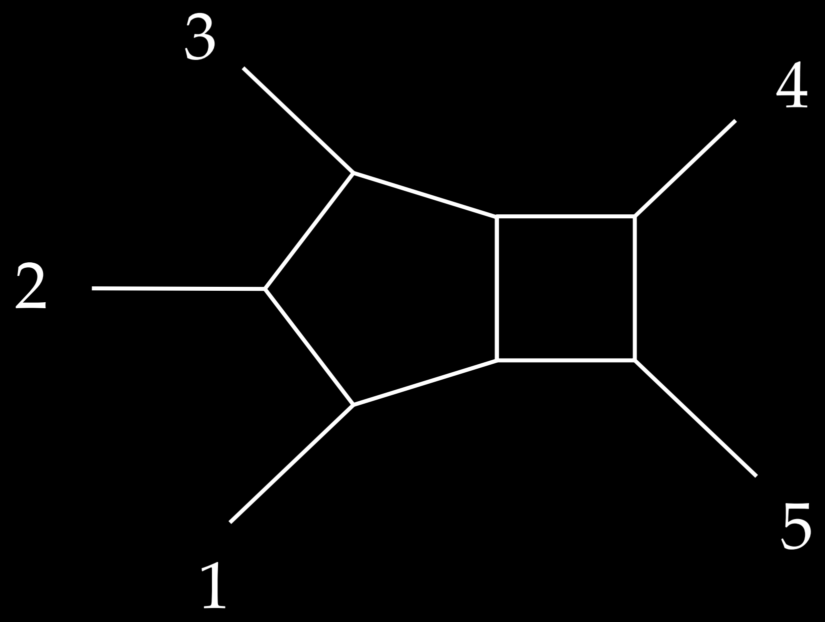
Quantity: 880

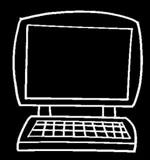
Denominator power: >1

#MI: 61

#IBP: 3313

#IBP (FIRE6): 1010236

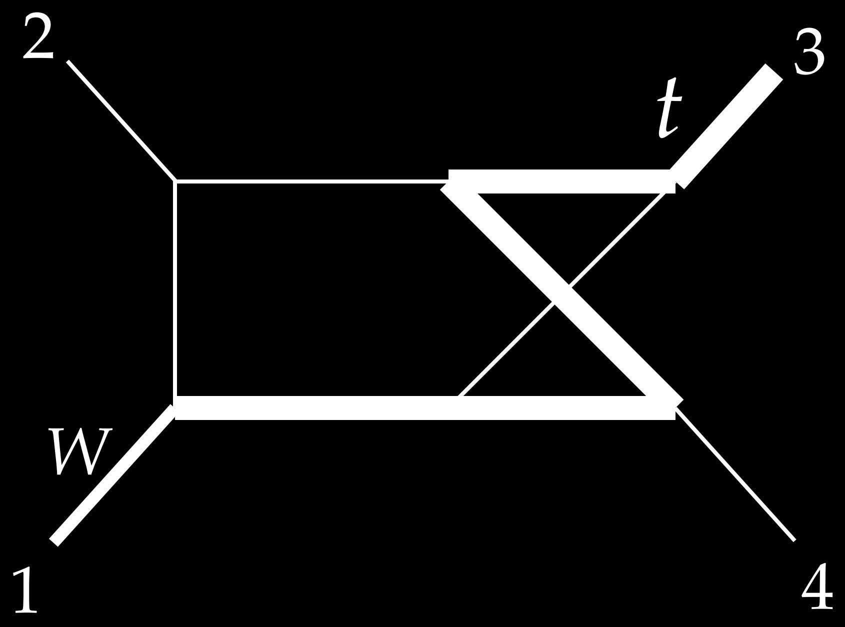


Time used: 17m at 

CPU cores: 10
RAM: 128GB

Example II

L.-B. Chen, L. Dong, H. T. Li, Z. Li, J. Wang, Y. Wang, 2212.07190



Target integrals from amplitudes

Quantity: 597

#MI: 90

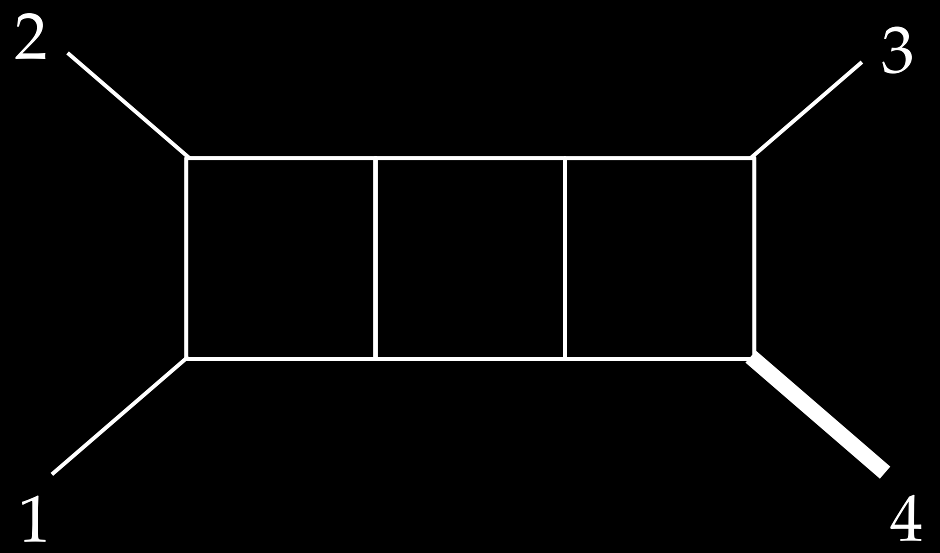
#IBP: 7169

Time used: 1h30m at

CPU cores: 10
RAM: 128GB

Example III

J. M. Henn, J. Lim, W. J. Torres Bobadilla, 2302.12776



Target integrals with high-degree numerators

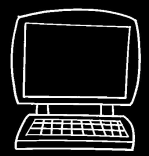
Quantity: 21185

Max numerator degree: 6

Max denominator power: 1

#MI: 83

#IBP: 200074

Time used: 6h at  CPU cores: 50
RAM: 1.5TB

Conclusions

NeatIBP is a parallelized program generating small-size IBP system.

NeatIBP generates IBP relations from Baikov representation using syzygy and module intersection.

The generated small-size IBP system could make the subsequent computations much lighter. Including:

1. Numerical reduction & analytic reconstruction.
2. Analytic reduction.
3. As an input for Blade.
4. …

Current version of NeatIBP is v1.0. Possible future upgrades:

1. Parallelization inside sectors.
2. To support cutting indices larger than 1. Auto detection of spanning cuts.
3. Code optimizations.
4. …