Feynman Integrals, Symbology and Twistor geometries

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2203.16112; 2206.04609 w. 张扬,马柔柔,吴子昊; 何颂, 李振杰;

2207.13482 w.何颂,唐一朝,刘家昊;

2212.09762 w. Matthias Wilhelm, Anne Spiering, Roger Morales and 张驰 from NBI

Feynman Integrals : Mathematical structures and calculation

- Bottom-up: direct integration, reduction, canonical differential equation⋯

- Up-bottom: physical singularities for certain integrals can be predicted before results are known

(cluster algebras, kinematics quivers, Schubert analysis⋯)

Planar, massless propagators in 4D

- → bootstrapping at symbol/function level

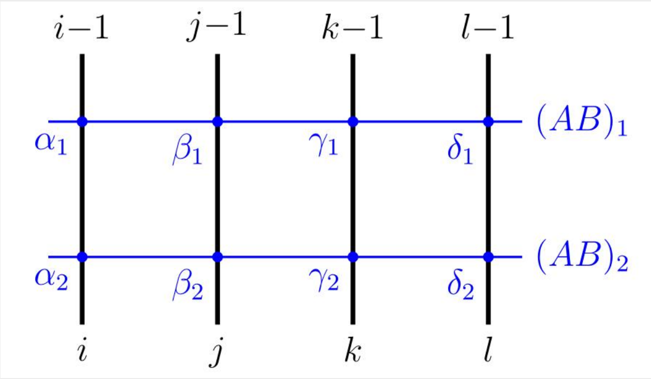
$$\mathrm{d}G^{(w)} = \sum_i G_i^{(w-1)} \mathrm{d}\log x_i \implies \mathcal{S}[G^{(w)}] = \sum_i \mathcal{S}[G_i^{(w-1)}] \otimes \log x_i$$

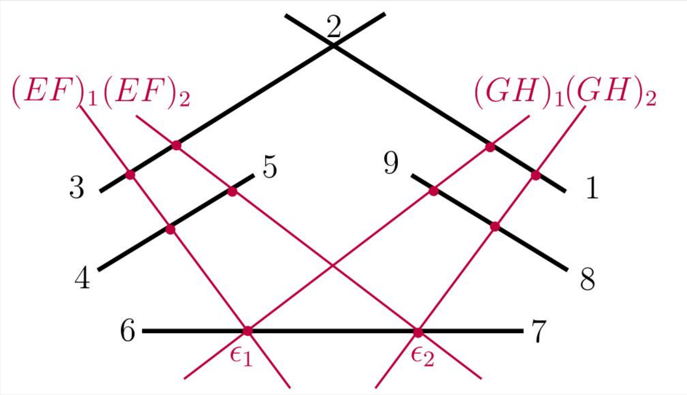
Twistor geometries associated to Feynman Integrals

- Three parts:
- 1. geometries (lines) from kinematics
- 2. geometries (lines) from Leading Singularities
- 3. Intersection points

geometrical invariants In twistor space

Covering all possible physical singularities for individual integrals!





1.Twistor geometries from kinematics

Momentum twistors:

$$\lambda_a \tilde{\lambda}_a = p_a = x_{a+1} - x_a$$

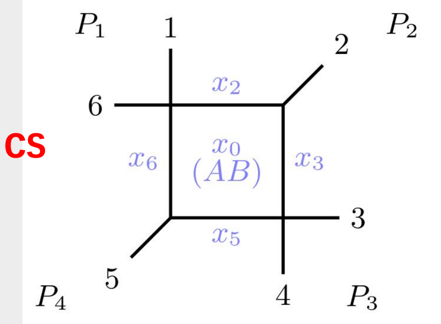
$$Z_a^I = (\lambda_a^\alpha,\ x_{a,\alpha\dot{\alpha}} \lambda_a^\alpha) \in \mathbb{P}^3, \quad x_a \sim Z_{a-1} \wedge Z_a$$

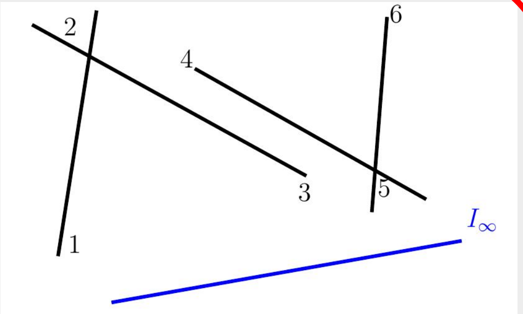
$$(x_a - x_b)^2 = \frac{\langle a{-}1ab{-}1b \rangle}{\langle a{-}1aI_\infty \rangle \langle b{-}1bI_\infty \rangle}$$

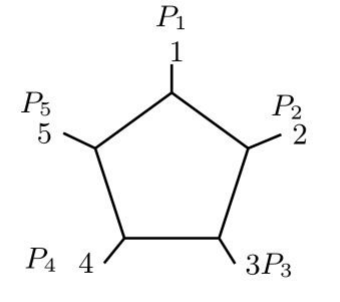
Dual momenta: lines in P^3

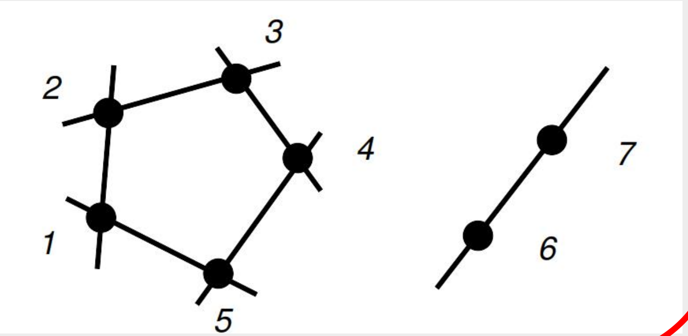
Kinematics: several intersected lines, together with reference I_inf

General kinematics

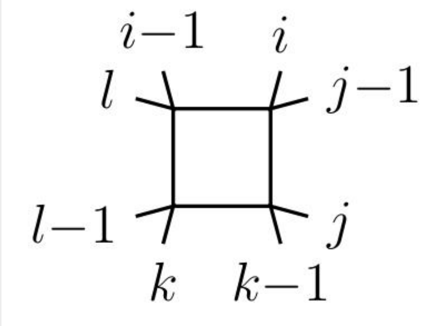


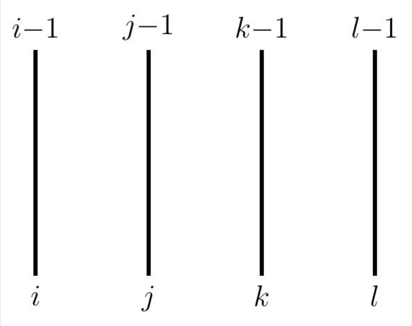


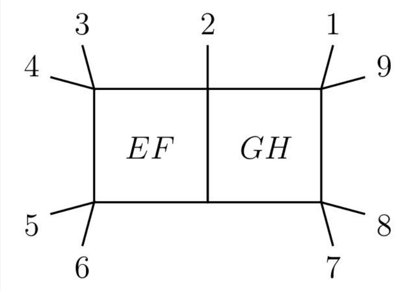


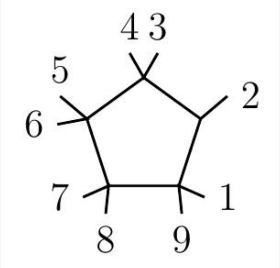


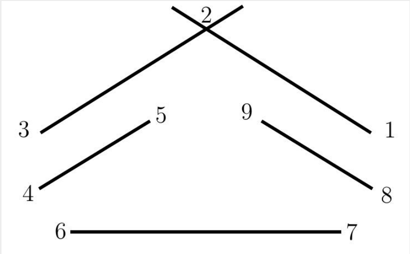
Dual Conformal Invariant





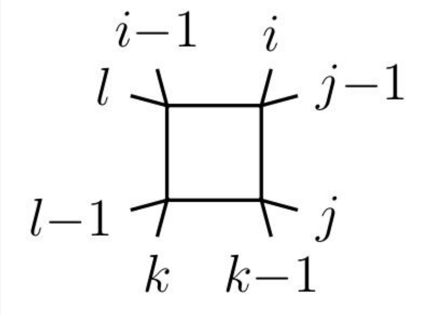






2.Twistor geometries from Leading Singularities (LS)

One-loop leading singularities: residue of the integrand under 4 on-shell conditions of propagators:



$$:= \int \mathrm{d}^4\ell \frac{(x_i-x_k)^2(x_j-x_l)^2}{(\ell-x_i)^2(\ell-x_j)^2(\ell-x_k)^2(\ell-x_l)^2}$$

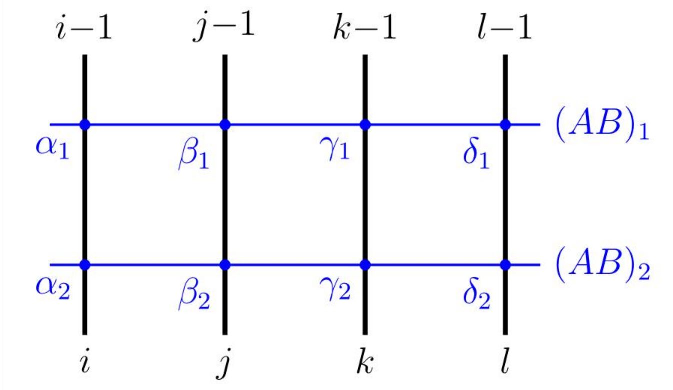
$$= \int_{AB} \frac{\langle i{-}1\, i\, k{-}1\, k\rangle\langle j{-}1\, j\, l{-}1\, l\rangle}{\langle AB\, i{-}1\, i\rangle\langle AB\, j{-}1\, j\rangle\langle AB\, k{-}1\, k\rangle\langle AB\, l{-}1\, l\rangle}$$

Two lines (AB) and (l-1 l) intersects

$$\langle AB\, i{-}1\, i\rangle = \langle AB\, j{-}1\, j\rangle = \langle AB\, k{-}1\, k\rangle = \langle AB\, l{-}1\, l\rangle = 0.$$

- D=4, loop momentum is fixed by the four conditions
- AB intersects with 4 external lines simultaneously (exactly two solutions in $\mathbb{P}^3$)
  [Schubert, 19th century; N. Arkani-Hamed et al, 1012.6032]

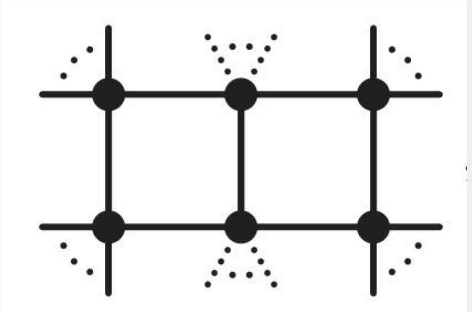
Schubert problem!

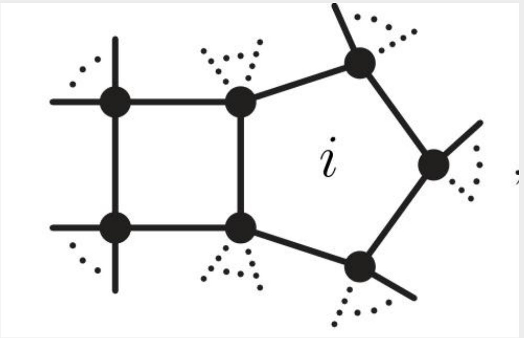


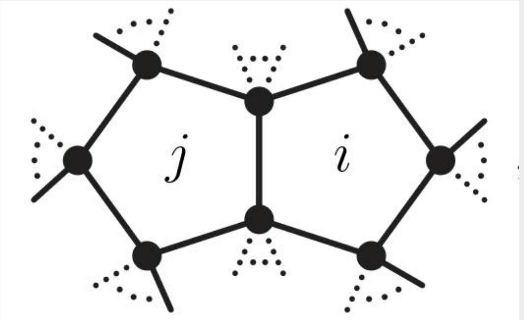
$$LS \propto \frac{1}{\Delta_{i,j,k,l}}$$

Two-loop Leading Singularities [1505.05886 by J.Bourjaily et al]

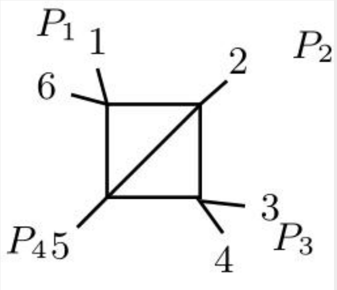
- Three types: composite, complete, overcomplete

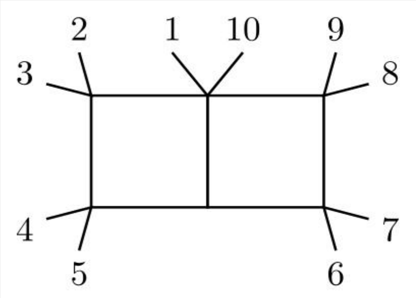


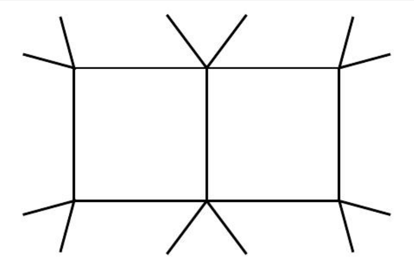


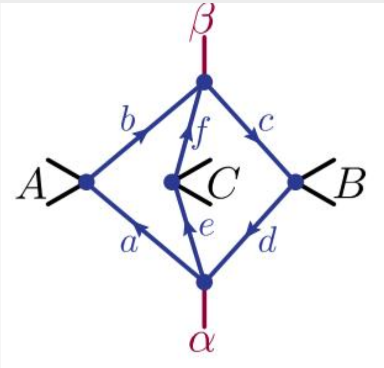


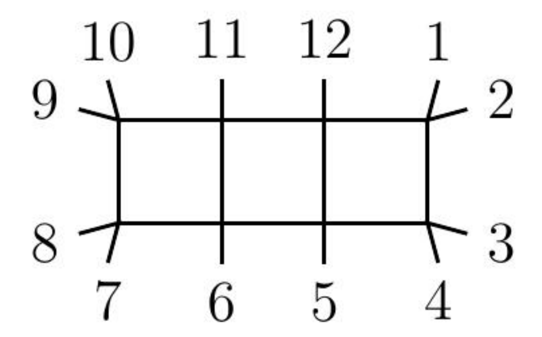
#propagator <4L =4L >4L

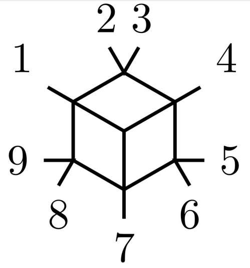






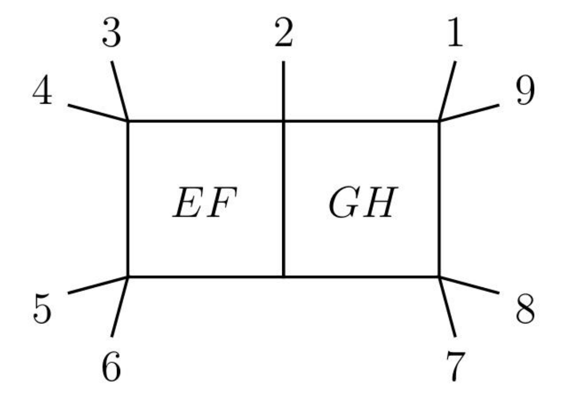






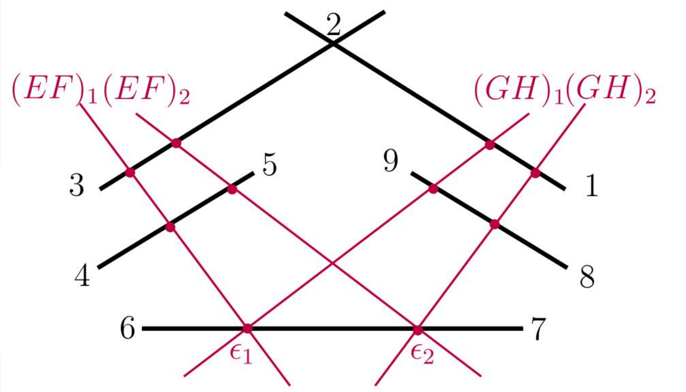
Composite LS

- # propagators< 4L: L loop momenta cannot be fixed by on-shell conditions (Jacobian)



Parametrizing two loop momenta with GL(2) gauge-fixing

$$E = \alpha_1 Z_5 + \beta_1 Z_6 + Z_4,\ \ F = \alpha_2 Z_5 + \beta_2 Z_6 + Z_7$$
$$G = \gamma_1 Z_7 + \delta_1 Z_8 + Z_9,\ \ H = \gamma_2 Z_7 + \delta_2 Z_8 + Z_6$$



Solving 7 on-shell conditions (linear in parameters)

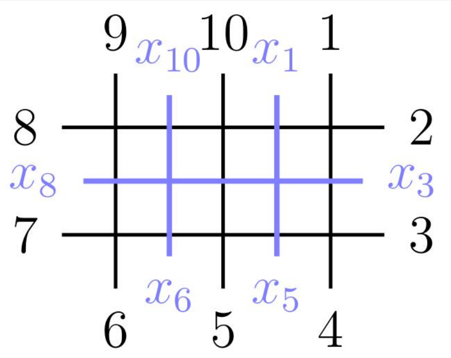
$$\langle 7\ (89)\cap(612)\ (645)\cap(623)\rangle - \langle 6\ (89)\cap(712)\ (745)\cap(723)\rangle\gamma_2^2$$
$$+(\langle 6\ (45)\cap(236)\ (789)\cap(712)\rangle - \langle 7\ (45)\cap(237)\ (689)\cap(612)\rangle)\gamma_2$$

Jacobian factor

Discriminant of Jacobian: Δ_9 (two-loop square root)

$\mathcal{J} = 0$

$$LS \propto \frac{1}{\Delta_9}$$



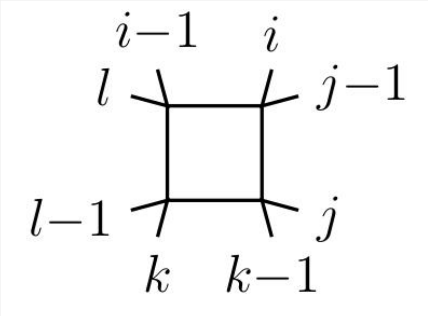
$\mathcal{J} = \sqrt{P(x)}$, with $P(x)$ a quartic curve

Elliptic function!

Elliptic LS: [2012.14438] by J.Bourjaily et al.

3.Intersections & letters from Schubert analysis [N. Arkani-Hamed, 21', QY 2203.16113]

After lines are determined by kinematics & LS, intersections are produced, from which we can construct symbol letters.



$$= \int_{AB} \frac{\langle i{-}1ik{-}1k\rangle\langle j{-}1jl{-}1l\rangle}{\langle ABi{-}1i\rangle\langle ABj{-}1j\rangle\langle ABk{-}1k\rangle\langle ABl{-}1l\rangle}$$

$$\frac{1}{2\Delta_{i,j,k,l}}\left(v \otimes \frac{z_{i,j,k,l}}{\bar{z}_{i,j,k,l}} + u \otimes \frac{1-\bar{z}_{i,j,k,l}}{1-z_{i,j,k,l}}\right)$$

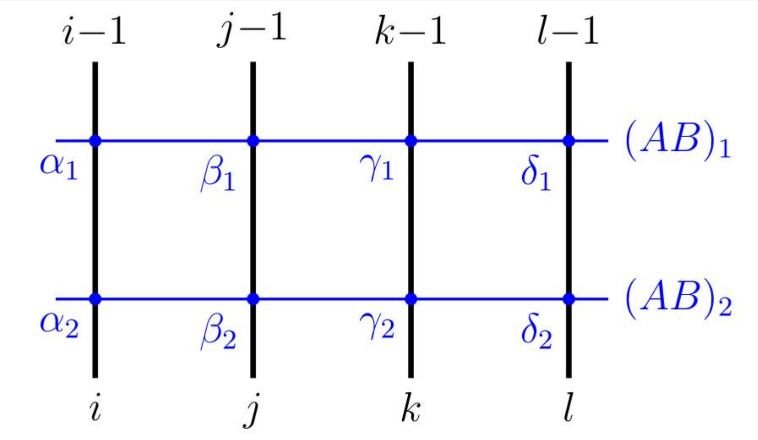
$$\Delta_{i,j,k,l} = \sqrt{(1{-}u{-}v)^2 - 4uv}$$

One-loop square root

$$z_{i,j,k,l}\bar{z}_{i,j,k,l} = u, \ \ (1{-}z_{i,j,k,l})(1{-}\bar{z}_{i,j,k,l}) = v.$$

$$u{=}\frac{\langle i{-}1ij{-}1j\rangle\langle k{-}1kl{-}1l\rangle}{\langle i{-}1ik{-}1k\rangle\langle j{-}1jl{-}1l\rangle}, \ \ v{=}\frac{\langle i{-}1il{-}1l\rangle\langle j{-}1jk{-}1k\rangle}{\langle i{-}1ik{-}1k\rangle\langle j{-}1jl{-}1l\rangle}$$

- From internal lines of the box, we consider the cross-ratios formed by the intersections



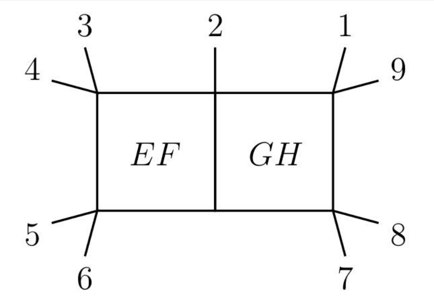
$$\frac{(\alpha_1,\beta_1)(\gamma_1,\delta_1)}{(\alpha_1,\gamma_1)(\beta_1,\delta_1)} = z_{i,j,k,l}, \ \frac{(\alpha_1,\delta_1)(\gamma_1,\beta_1)}{(\alpha_1,\gamma_1)(\beta_1,\delta_1)} = 1{-}z_{i,j,k,l}$$

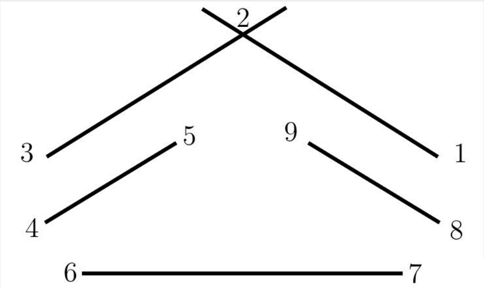
$$\frac{(\alpha_2,\beta_2)(\gamma_2,\delta_2)}{(\alpha_2,\gamma_2)(\beta_2,\delta_2)} = \bar{z}_{i,j,k,l}, \ \frac{(\alpha_2,\delta_2)(\gamma_2,\beta_2)}{(\alpha_2,\gamma_2)(\beta_2,\delta_2)} = 1{-}\bar{z}_{i,j,k,l}$$

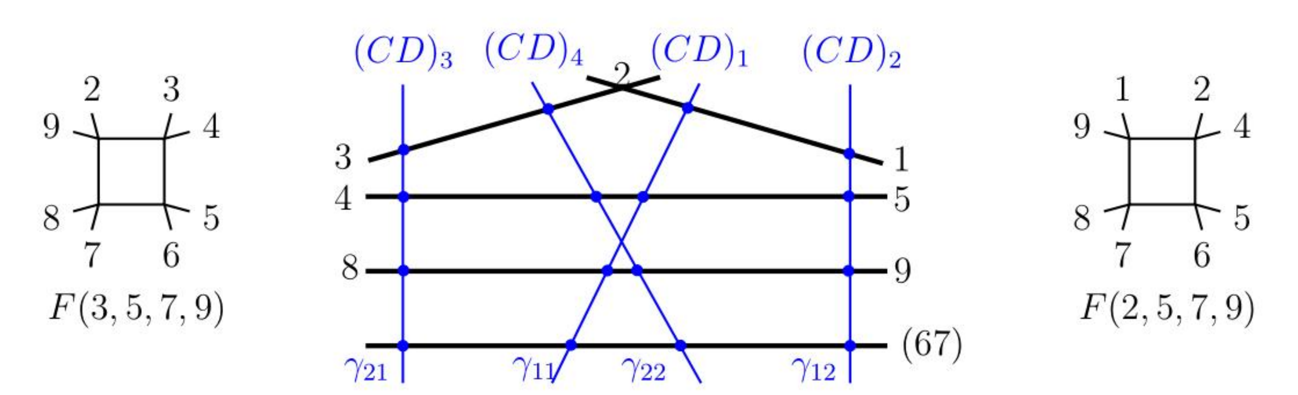
$$\frac{(X_1,X_3)(X_2,X_4)}{(X_1,X_4)(X_2,X_3)} := \frac{\langle X_1X_3I\rangle\langle X_2X_4I\rangle}{\langle X_1X_4I\rangle\langle X_2X_3I\rangle}$$

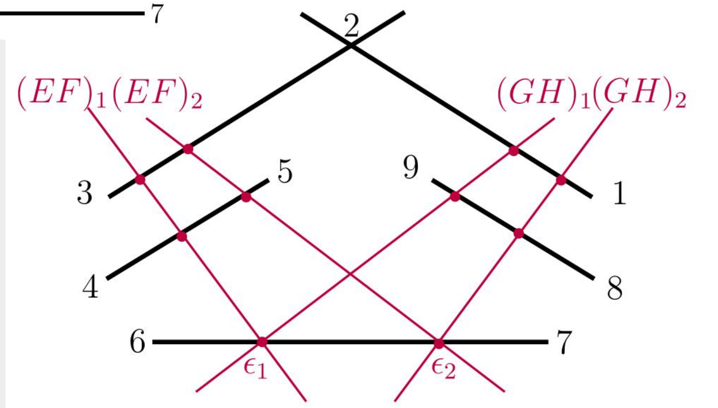
I: arbitrary reference line

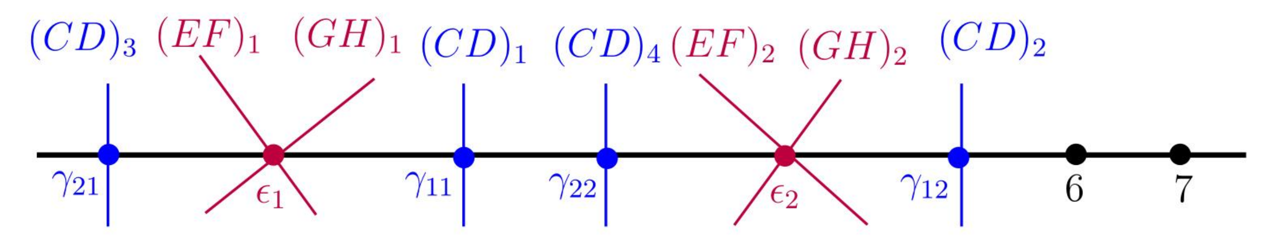
Two-loop example 1: 9pt double-box integral











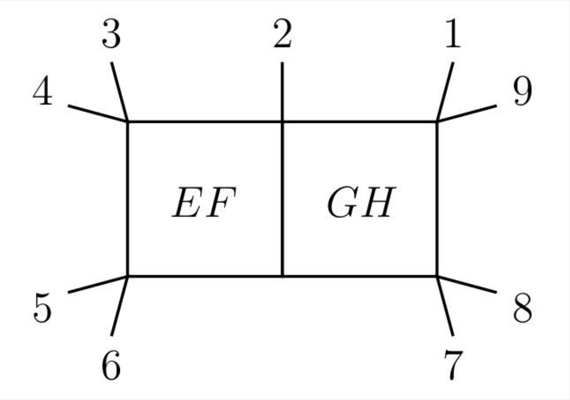
$$\frac{(\gamma_{11},\gamma_{21})(\gamma_{12},\gamma_{22})}{(\gamma_{11},\gamma_{22})(\gamma_{21},\gamma_{12})} = \frac{(z_{2,5,7,9}-z_{3,5,7,9})(\bar{z}_{2,5,7,9}-\bar{z}_{3,5,7,9})}{(z_{2,5,7,9}-\bar{z}_{3,5,7,9})(\bar{z}_{2,5,7,9}-z_{3,5,7,9})}$$

$$\frac{(\epsilon_1,\gamma_{11})(\epsilon_2,\gamma_{12})}{(\epsilon_2,\gamma_{11})(\epsilon_1,\gamma_{12})} = \frac{(1+az_{2,5,7,9})(1+b\bar{z}_{2,5,7,9})}{(1+a\bar{z}_{2,5,7,9})(1+bz_{2,5,7,9})},$$

- We call these configurations: combinations of Schubert problems on external lines

Two-loop example 1: 9pt double-box integral

9-point double-box integral (weight-4 MPL)



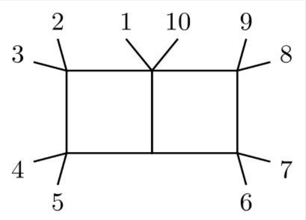
$$S(I_9) = \sum A_1 \otimes A_2 \otimes B \otimes C$$

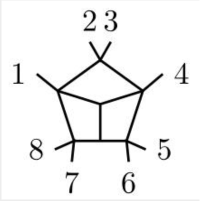
$A_1 \otimes A_2$ (first-two-entries) : letters from one-loop intersections on internal lines

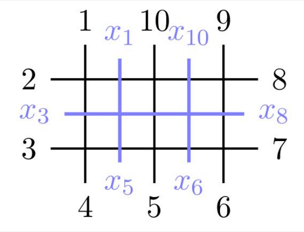
B (third entries): letters from one-loop intersections on external lines

C (last entries): letters from two-loop intersections combined with one-loop intersections

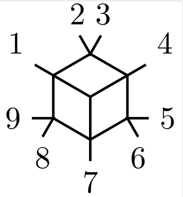
General structure for integrals with composite/complete LS

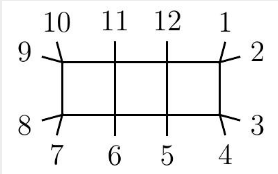






Guidance for symbology of higher loops/rigidity integrals



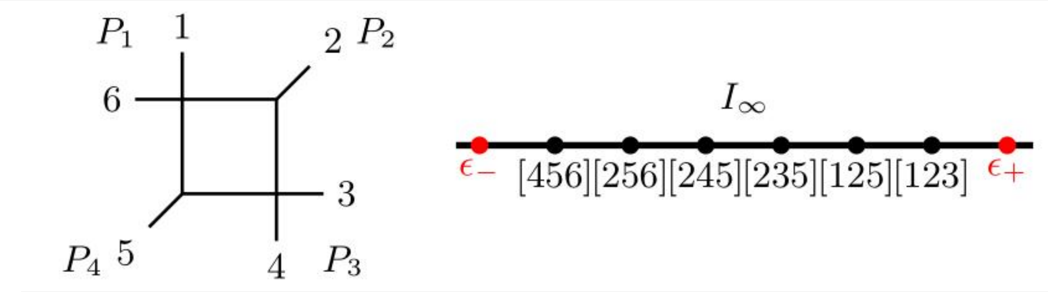


Two-loop example 2: 2-mass-easy box kinematics and its two-loop alphabet [2207.13482,w.S.He et al]

$$D = 4 - 2\epsilon$$

Black intersections: 4 one-loop triangles (A3)

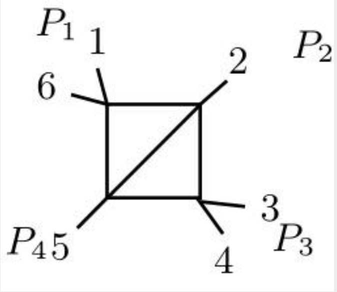
Red intersections: LS of two-loop slashed-box

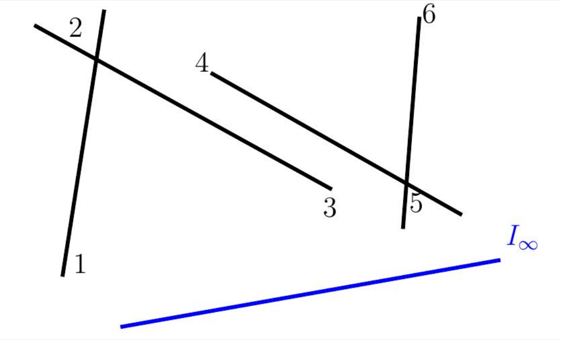


m_1^2, m_3^2, s, t $\quad s-m_1^2, t-m_1^2, s-m_3^2, t-m_3^2, m_1^2+m_3^2-s-t, st-m_1^2m_3^2.$

$$L_1 = \frac{s+t+\Delta_{nc}}{s+t-\Delta_{nc}}, L_2 = \frac{s-t+\Delta_{nc}}{s-t-\Delta_{nc}}, L_3 = \frac{-2m_1^2+s+t+\Delta_{nc}}{-2m_1^2+s+t-\Delta_{nc}}$$

$$\Delta_{nc} = \sqrt{(s+t)^2 - 4m_1^2m_3^2}$$

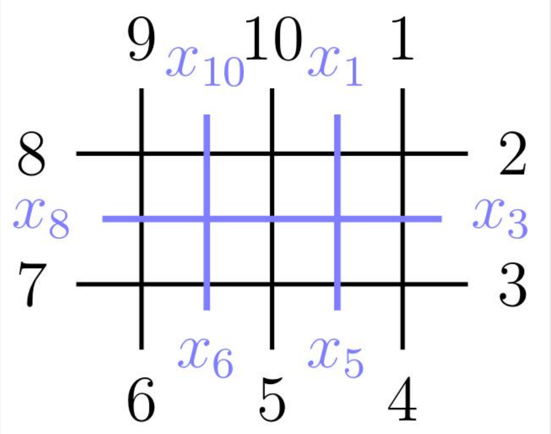


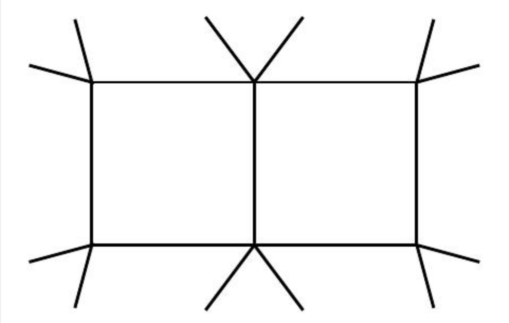


Following this procedure, we recover two-loop alphabets for non-DCI kinematics case by case, by going through all possible integrals···

(boxes, zero-, one-mass pentagon) (selection)

Schubert analysis beyond MPL





e.g. 10-pt double-box integral [2106.14902 by M.Wilhelm et al]

Elliptic "cross-ratios:"

a & b: two intersections from another Schubert problem

ω_1: elliptic period

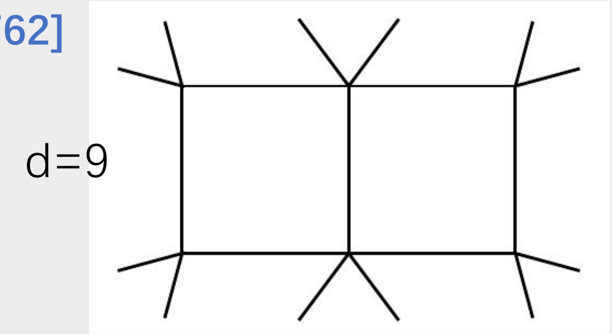
$$\frac{1}{\omega_1}\int_a^b \frac{\mathrm{d}x}{\sqrt{P(x)}}$$

Elliptic curve in $\mathbb{P}^3$

$$S(I_{10}) = \sum A_1 \otimes A_2 \otimes B \otimes \mathrm{D}$$

Successfully reproduce the last entries of 10-point db integrals and give a predictions of 12-point db integrals→ bootstrapping

Bootstrap 12-point double-box integral [w.M.Wilhelm et al 2212.09762]



- Alphabet from Schubert analysis:

9 first entries $\otimes$ 30+9 second entries $\otimes$ (30+105) third entries $\otimes$ (8+1) last entries

9 variables

Cross-ratios from internal lines of 15 boxes

Cross-ratios from external lines of 15 boxes

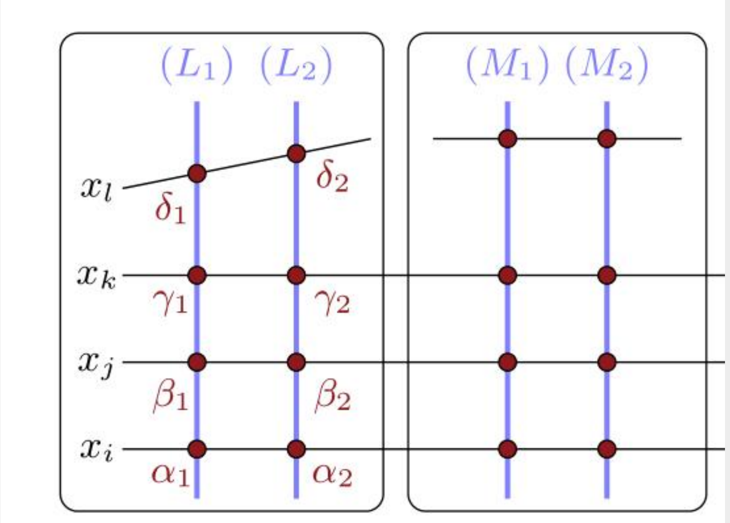
Elliptic Schubert analysis

Imposing integrability and Steinmann relations (of first-two entries), determining one single symbol $\Rightarrow$ **Symbol of the Result!**

Finally: 9 first entries $\otimes$ 39 second entries $\otimes$ 104 third entries $\otimes$ (8+1) last entries

Selection Rule

- Truth: 1. for two boxes sharing three external lines, cross-ratios from each external line (A_1) are the same
- 2. These cross-ratios account for all required third entries



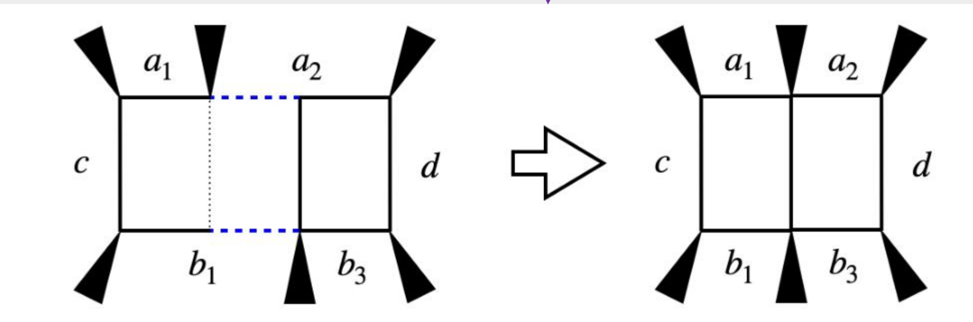
$$L([ad],[bc]) := \int_a^d \mathrm{d}\log\frac{(xc)}{(xb)} = \log\mathcal{U} = L([bc],[ad])$$

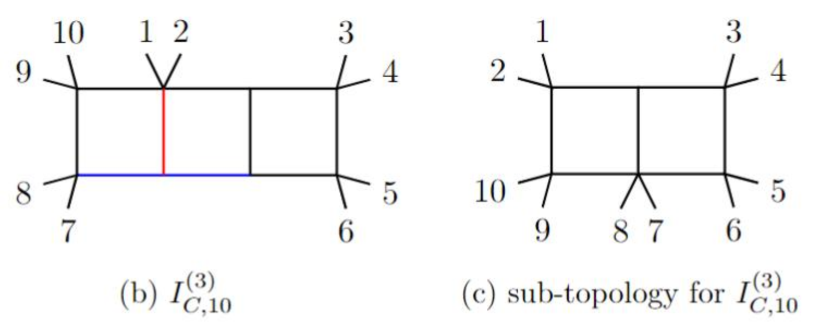
- Rules: we only consider those combinations of n Schubert problems, A_{2n-3} configurations from each shared external line are equivalent
- (checked for all known results/ one-loop letters of non-DCI kinematics…)

May still be redundant

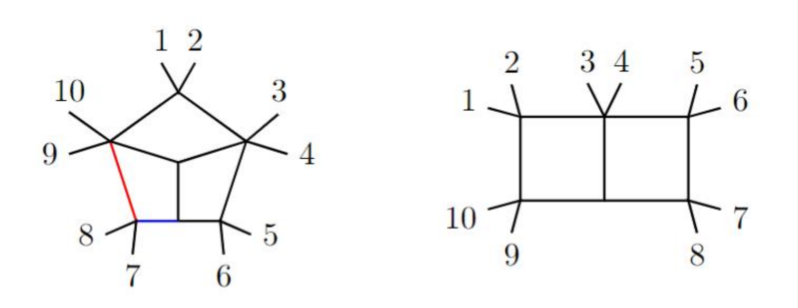
Higher-loop analysis

- Generally speaking, for an L-loop integral with composite LS, its alphabet consists of:
- 1. letters from all (L-1)-loop sub-topologies
- 2. letters from all (legal) combinations of two Schubert problems from two sub-topologies (after selections)
- 3.letters from all (legal) combinations of Schubert problem from LS with problems from one sub-topology (after selections, last entries)



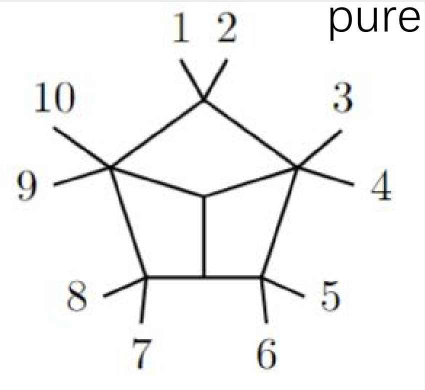


(b) $I^{(3)}_{C,10}$ (c) sub-topology for $I^{(3)}_{C,10}$



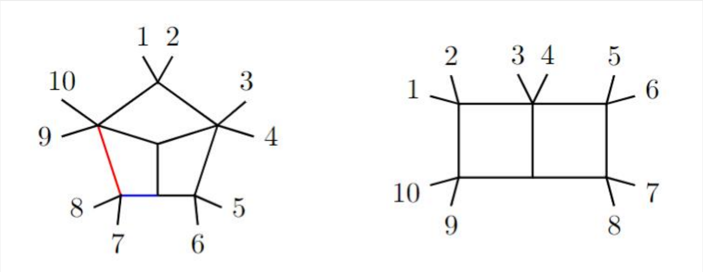
Exercise: a three-loop example

Weight-6
pure MPL



Let us construct its alphabet by Schubert analysis:

1.One-loop letters:
It has five one-loop sub-topologies (five 4m boxes)
5 1^{st}-entries + 15 2^{nd} entries from internal lines (u,v,z)
By combinations: 5 Deltas, 1 Gram and 10 mixed-algebraic letters from external lines (3rd entries)



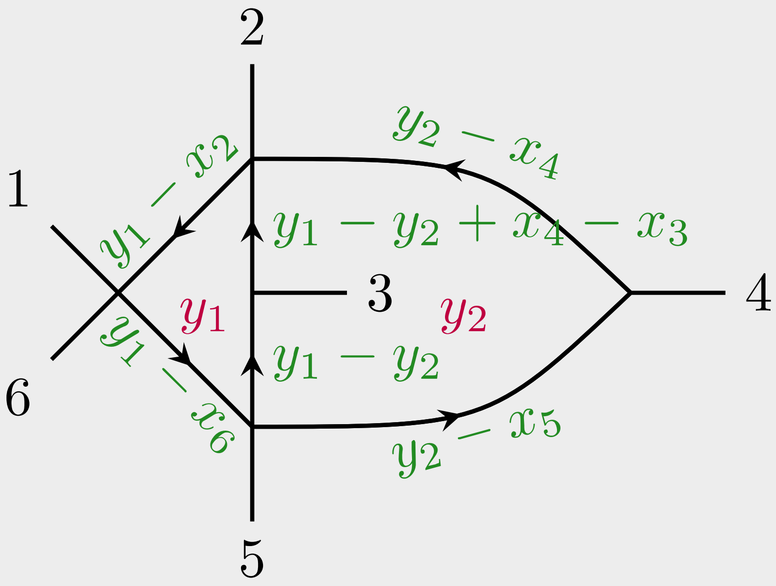
2.Two-loop letters:
It has 2 two-loop sub-topologies (10-point double-box)
Each contributes 5 new algebraic letters and 1 Delta (4^{th}&5^{th} entries)

3.Three-loop letters (last entries)
Five algebraic letters from combinations of three-loop intersections with one-loop intersections (and 1 delta)
Other cross-ratios have been <u>ruled out</u> by our selection

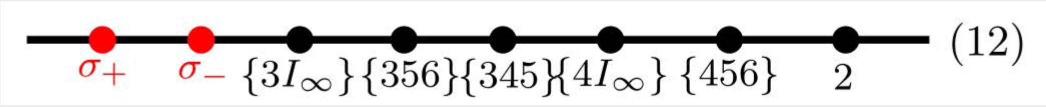
Integrability: only five possible functions
physical limits imposed: determined!

Non-planar Schubert analysis

- By expanding propagators on (y-x)^2, we can rewrite the integrand by twistors



[2107.14180] by S. Abreu et al



$$\Sigma_5 = (s_{12}s_{15} - s_{12}s_{23} - s_{15}s_{45} + s_{34}s_{45} + s_{23}s_{34})^2 - 4s_{23}s_{34}s_{45}(s_{34} - s_{12} - s_{15}) .$$

$$\begin{aligned}(y_1 - y_2 + x_4 - x_3)^2 &= (y_1 - y_2)^2 + (x_4 - x_3)^2 \\ &+ (y_1 - x_3)^2 - (y_1 - x_4)^2 \\ &- (y_2 - x_4)^2 + (y_2 - x_3)^2\end{aligned}$$

Constructing cross-ratios by two red points and any two black points, we recover the 5-dim space for algebraic letters with Σ_5

$$\langle AB12\rangle = \langle AB56\rangle = \langle CD34\rangle = \langle CD45\rangle = \langle ABCD\rangle = 0 \;\&$$
$$\langle ABI_\infty\rangle\langle CD\ \bar{3}\cap(3I_\infty)\rangle - (AB \leftrightarrow CD) = 0$$

Summary

- Twistor geometries provide a geometric explanations for the symbol letters of L-loop integrals with massless propagators in QCD, and especially for N=4 SYM

- Through this way, we can predict symbol letters for individual integrals quite precisely, and finally bootstrap them instead of direct computations

- This method can be generalized to elliptic cases, and proves to be useful for 10-/12-pt double-box integral

- More precise description for Integrals with overcomplete LSs ? non-planar integrals? Massive propagators? D=6?...
- Relation to Landau equations? Cut integrals? Why it works?
- K3 integrals?...

Thanks!