# 4D differential equations for the leading divergences of dimensionally-regulated loop integrals

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16. May. 2023

This talk is about our recent project JHEP 03(2023)162 with

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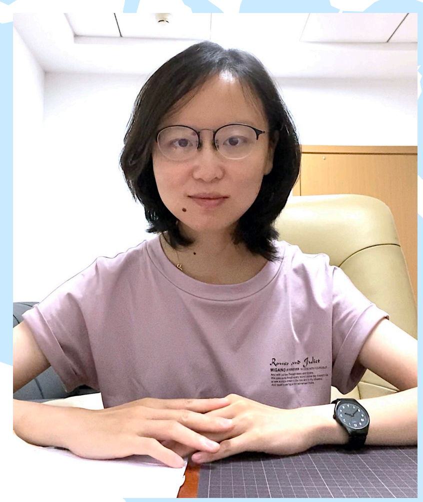
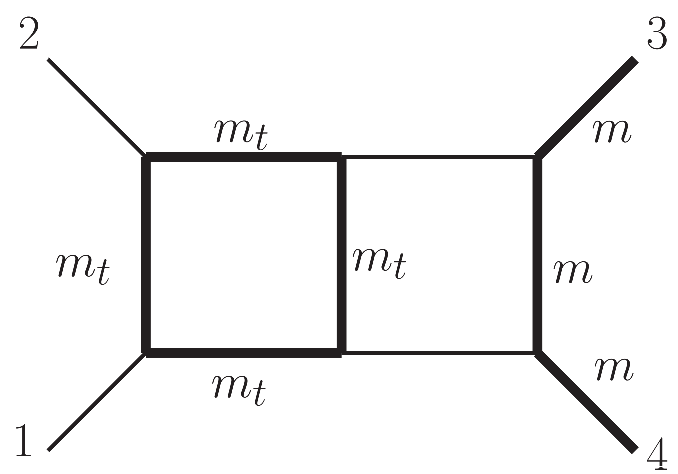
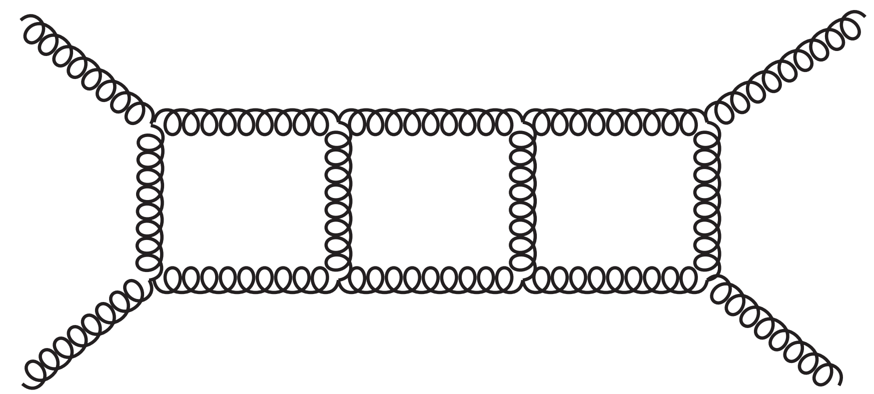



# Feynman integral evaluation



Multi-loop Feynman integrals are the hardcore objects for a perturbative QFT computation

- Important for high-energy phenomenology
- Theoretically important

# Mainstream Analytic Methods

## Integration by parts (IBP)

[Analytic Tool for Feynman Integrals] *Smirnov*

For the integrals $F(a) = \int \frac{d^dk}{(k^2 - m^2)^a}$

The IBP identity $\int d^dk \frac{\partial}{\partial k_\mu} \cdot k_\mu \frac{1}{(k^2 - m^2)^a} = 0$

$$(d - 2a)F(a) - 2am^2 F(a + 1) = 0$$

Use this set of relations between Feynman integrals
in order to solve the reduction problem

# Mainstream Analytic Methods

Differential equation for Feynman integrals

$d = 4 - 2\epsilon$ $\qquad\qquad \overrightarrow{f}\,(\overline{x}, \epsilon)$ the set of convenient basis integral

Canonical Differential equation

$$d\overrightarrow{f}\,(\overline{x}, \epsilon) = \epsilon[d\tilde{A}\,(\overline{x})]\overrightarrow{f}\,(\overline{x}, \epsilon)$$

$$d\tilde{A}\,(\overline{x}) = \sum_k a_k \mathrm{dlog}(\alpha_{\mathrm{k}})$$ (Henn 2013)

certain constant matrices

letters
alphabet

$$\overrightarrow{f}\,(\overline{x}) = P\mathrm{exp}\left(\epsilon \int d\tilde{A}\right)\overrightarrow{f}\,(\overline{x_0})$$

Chen's (陈国才) iterated integrals, to all orders in the $\varepsilon$ expansion.

It may take a lot of efforts to find a complete UT basis…

# Motivation

For many applications, it is not necessary to evaluate all MIs or all order of $\varepsilon$.

Can we simplify the DE in these cases?

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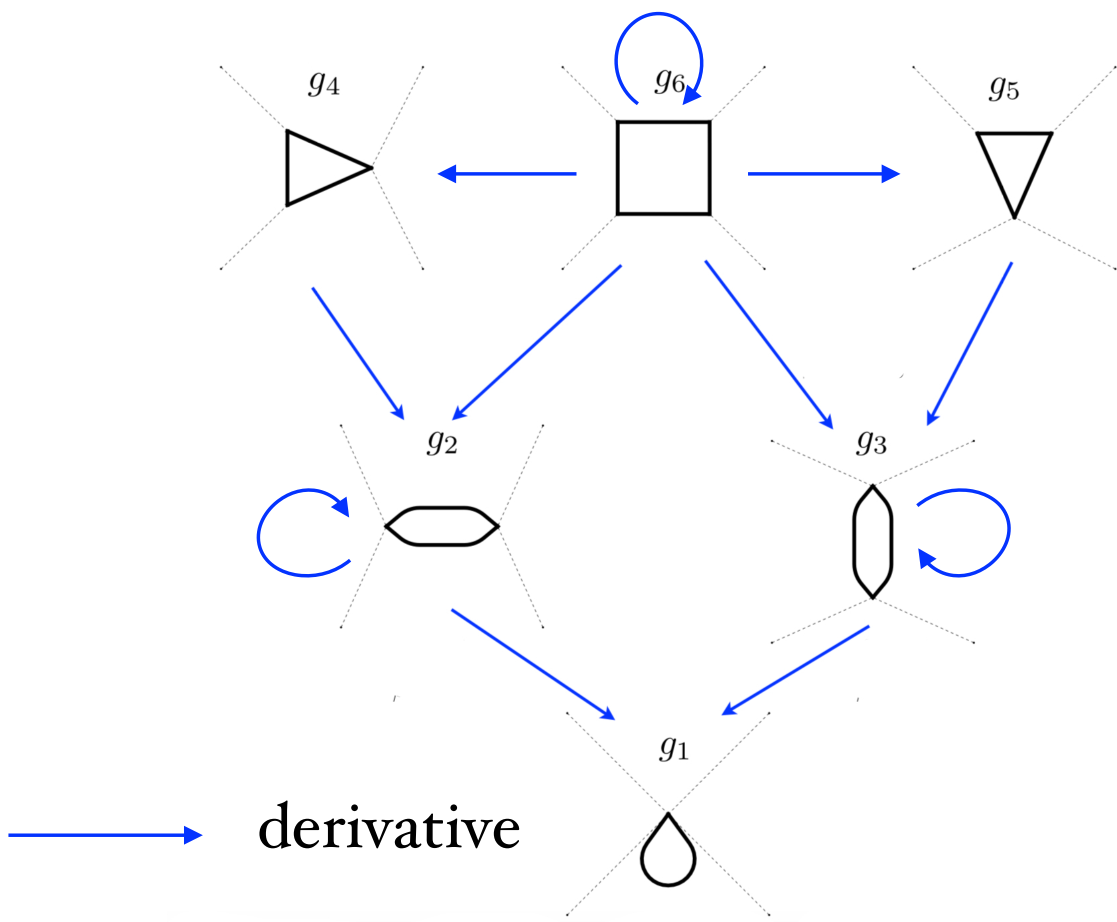
Can we simplify the DE in these cases?

Example: canonical DE for Finite integrals




Simon Caron-Huot, Johannes Henn 2014

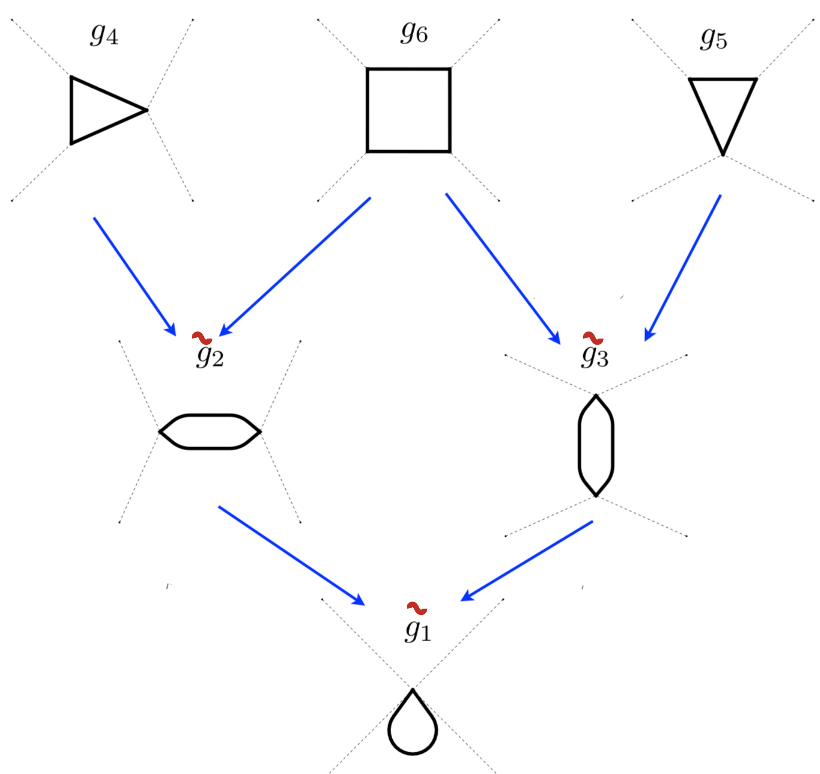
$$
\begin{aligned}
g_1 &= 2m^2 G[0,0,0,3]/\epsilon^2 \\
g_2 &= -\sqrt{s(s-4m^2)}G[1,0,2,0]/\epsilon \\
g_3 &= -\sqrt{t(t-4m^2)}G[0,1,0,2]/\epsilon \\
g_4 &= sG[1,1,1,0] \\
g_5 &= tG[1,1,0,1] \\
g_6 &= \frac{1}{2}\sqrt{st(st-4m^2(s+t))}G[1,1,1,1]
\end{aligned}
$$

$$d\vec{g} = \epsilon(dA)\vec{g}$$

# canonical DE

$$
\varepsilon \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
\log(\frac{\beta_u-1}{\beta_u+1}) & \log(\frac{u}{1+u}) & 0 & 0 & 0 & 0 \\
\log(\frac{\beta_v-1}{\beta_v+1}) & 0 & \log(\frac{v}{1+v}) & 0 & 0 & 0 \\
0 & -\log(\frac{\beta_u-1}{\beta_u+1}) & 0 & 0 & 0 & 0 \\
0 & 0 & -\log(\frac{\beta_v-1}{\beta_v+1}) & 0 & 0 & 0 \\
0 & \log(\frac{\beta_{uv}-\beta_u}{\beta_{uv}+\beta_u}) & \log(\frac{\beta_{uv}-\beta_v}{\beta_{uv}+\beta_v}) & \log(\frac{\beta_{uv}-1}{\beta_{uv}+1}) & \log(\frac{\beta_{uv}-1}{\beta_{uv}+1}) & \log(\frac{u+v}{1+u+v})
\end{pmatrix}
$$

# canonical DE for the finite integrals




$$\tilde{g}_1 = 2m^2 G[0,0,0,3]/\cancel{\epsilon^2}$$

$$\tilde{g}_2 = -\sqrt{s(s-4m^2)}G[1,0,2,0]/\cancel{\epsilon}$$

$$\tilde{g}_3 = -\sqrt{t(t-4m^2)}G[0,1,0,2]/\cancel{\epsilon}$$

$$g_4 = sG[1,1,1,0]$$

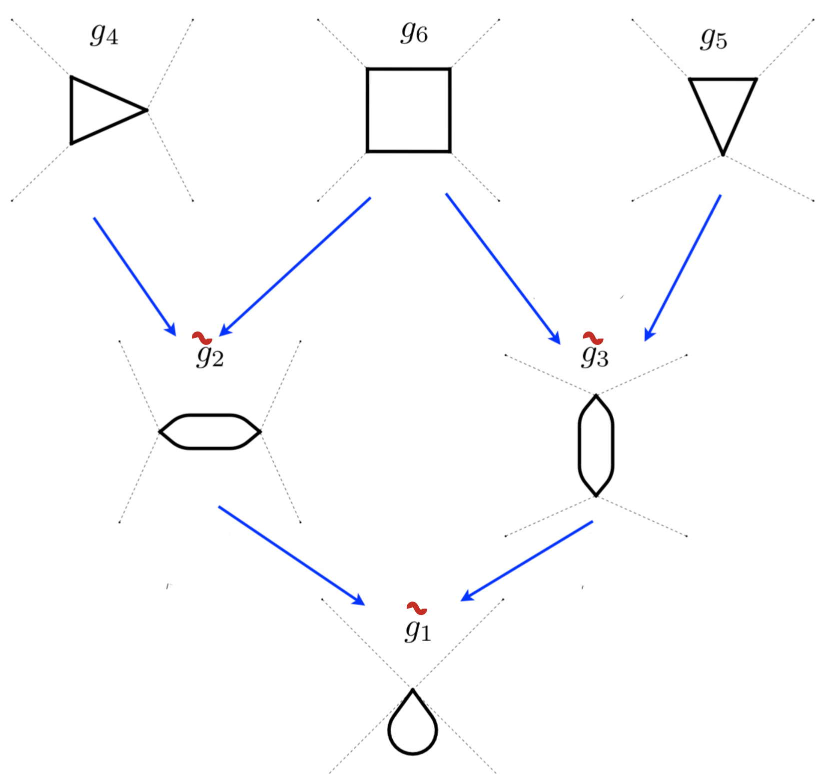
$$g_5 = tG[1,1,0,1]$$

$$g_6 = \frac{1}{2}\sqrt{st(st-4m^2(s+t))}G[1,1,1,1]$$

# canonical DE

$$
\varepsilon \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
\log(\frac{\beta_u-1}{\beta_u+1}) & \log(\frac{u}{1+u}) & 0 & 0 & 0 & 0 \\
\log(\frac{\beta_v-1}{\beta_v+1}) & 0 & \log(\frac{v}{1+v}) & 0 & 0 & 0 \\
0 & -\log(\frac{\beta_u-1}{\beta_u+1}) & 0 & 0 & 0 & 0 \\
0 & 0 & -\log(\frac{\beta_v-1}{\beta_v+1}) & 0 & 0 & 0 \\
0 & \log(\frac{\beta_{uv}-\beta_u}{\beta_{uv}+\beta_u}) & \log(\frac{\beta_{uv}-\beta_v}{\beta_{uv}+\beta_v}) & \log(\frac{\beta_{uv}-1}{\beta_{uv}+1}) & \log(\frac{\beta_{uv}-1}{\beta_{uv}+1}) & \log(\frac{u+v}{1+u+v})
\end{pmatrix}
$$

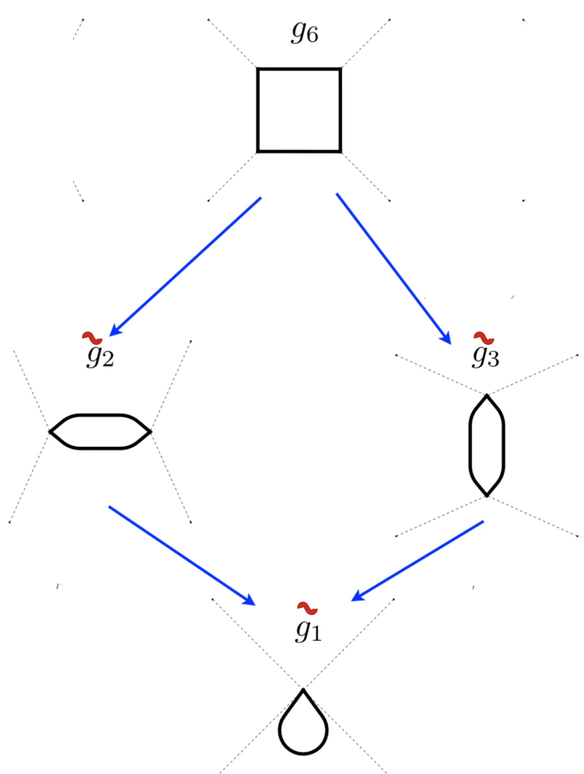
# canonical DE for the finite integrals



## canonical DE

$$
\varepsilon \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
\log(\frac{\beta_u-1}{\beta_u+1}) & \log(\frac{u}{1+u}) & 0 & 0 & 0 & 0 \\
\log(\frac{\beta_v-1}{\beta_v+1}) & 0 & \log(\frac{v}{1+v}) & 0 & 0 & 0 \\
0 & -\log(\frac{\beta_u-1}{\beta_u+1}) & 0 & 0 & 0 & 0 \\
0 & 0 & -\log(\frac{\beta_v-1}{\beta_v+1}) & 0 & 0 & 0 \\
0 & \log(\frac{\beta_{uv}-\beta_u}{\beta_{uv}+\beta_u}) & \log(\frac{\beta_{uv}-\beta_v}{\beta_{uv}+\beta_v}) & \log(\frac{\beta_{uv}-1}{\beta_{uv}+1}) & \log(\frac{\beta_{uv}-1}{\beta_{uv}+1}) & \log(\frac{u+v}{1+u+v})
\end{pmatrix}
$$

## canonical DE for the finite integrals




6 ➡ 4

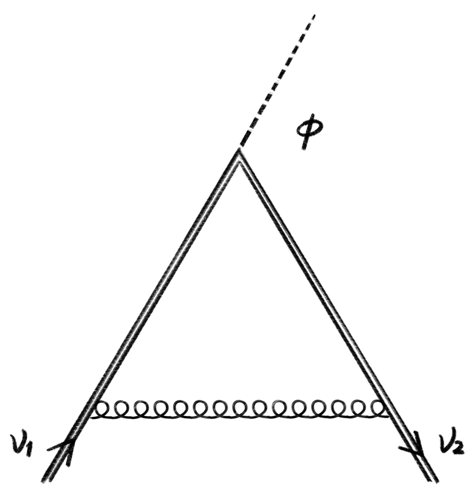
$$
\tilde{A} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
\log(\frac{\beta_u-1}{\beta_u+1}) & 0 & 0 & 0 & 0 & 0 \\
\log(\frac{\beta_v-1}{\beta_v+1}) & 0 & 0 & 0 & 0 & 0 \\
0 & \log(\frac{\beta_u-1}{\beta_u+1}) & 0 & 0 & 0 & 0 \\
0 & 0 & \log(\frac{\beta_v-1}{\beta_v+1}) & 0 & 0 & 0 \\
0 & \log(\frac{\beta_{uv}-\beta_u}{\beta_{uv}+\beta_u}) & \log(\frac{\beta_{uv}-\beta_v}{\beta_{uv}+\beta_v}) & 0 & 0 & 0
\end{pmatrix}
$$

All diagonal elements vanish
and the DE can be solved recursively.

# Why divergent Feynman Integral

Consider the renormalization of quantum filed theory, $\beta$ and $\gamma$ function are simply related to the coefficients of the divergent logarithms of counterterm

In HQET, the coefficient of the 1/ε pole is the angle-dependent cusp anomalous dimension




$$I^{(1)}(\phi,\epsilon) \sim \frac{1}{\epsilon}\phi\cot\phi + \mathcal{O}(\epsilon^0)$$

where $\cos\phi = \dfrac{v_1 \cdot v_2}{\sqrt{v_1^2 v_2^2}}$

……

- have to deal with divergent Feynman integrals (UV/IR)
- have no general method to compute them using IBPs

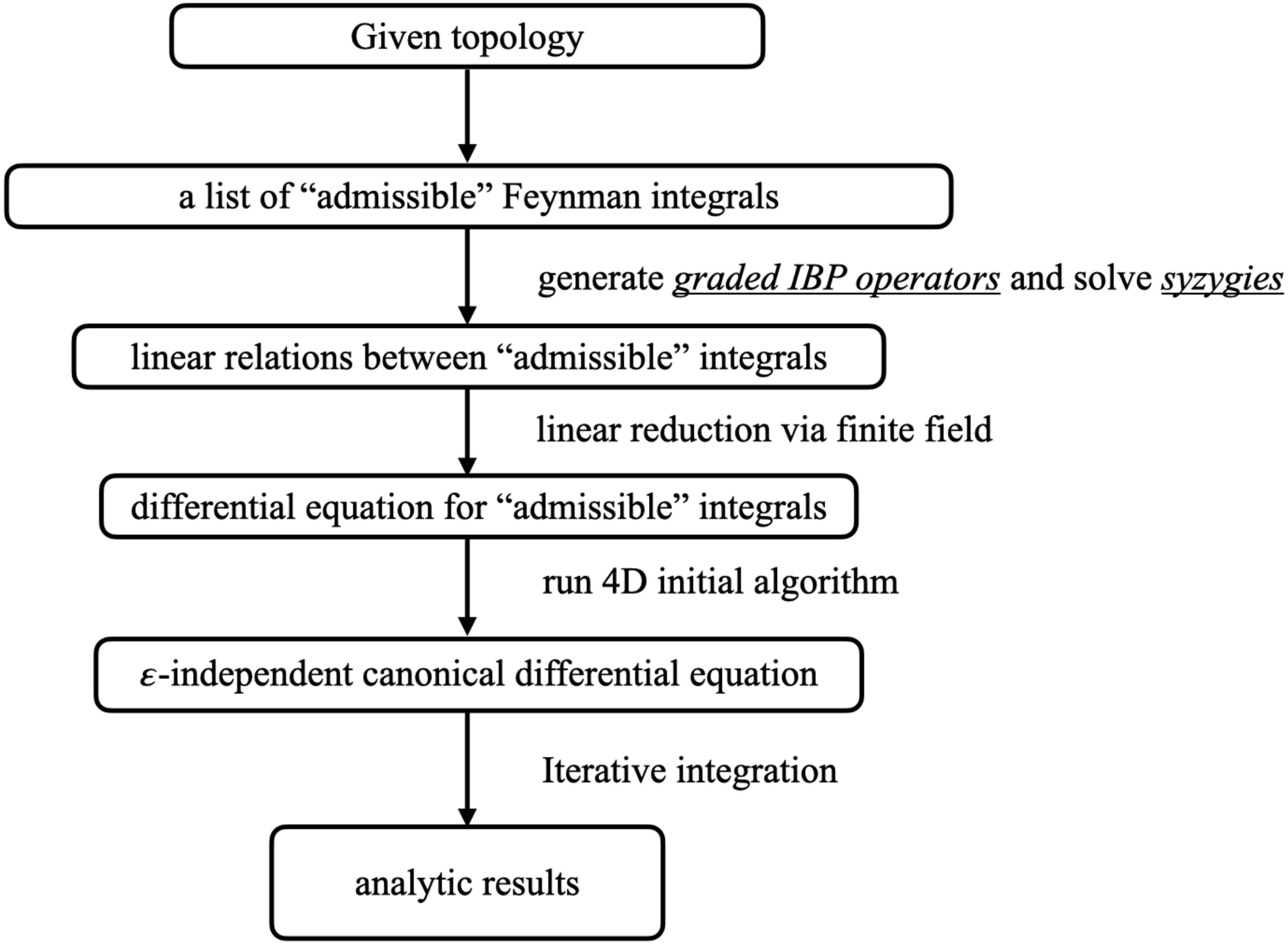
Can we simplify the canonical DE of divergent integrals?

If we only need the divergent part…

# Our New Method

## Canonical DE of the divergent part

- Generate admissible integrals
- Search for an IBP system with only admissible integrals
  (with **graded syzygy** approach/cross loop order relations)
- 4D **initial** algorithm
  the original initial algorithm is proposed by Dlapa, Henn, Kai (2020)

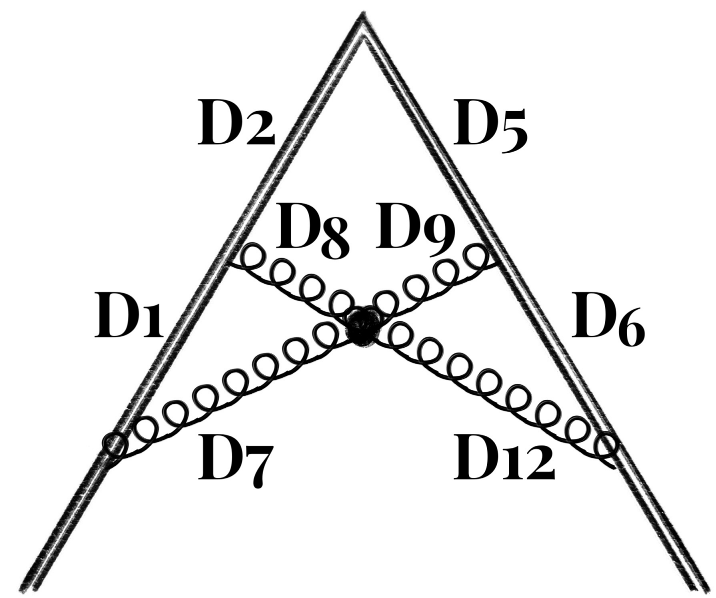

Given topology
a list of "admissible" Feynman integrals
generate *graded IBP operators* and solve *syzygies*
linear relations between "admissible" integrals
linear reduction via finite field
differential equation for "admissible" integrals
run 4D initial algorithm
$\varepsilon$-independent canonical differential equation
Iterative integration
analytic results

# Example: 3-loop HQET

Admissible: only one region of divergence, overall UV

The coefficient of the 1/$\varepsilon$ pole is the angle-dependent cusp anomalous dimension

$$D_1 = -2k_1v_1 + \delta, \quad D_2 = -2k_2v_1 + \delta, \quad D_3 = -2k_3v_1 + \delta$$
$$D_4 = -2k_1v_2 + \delta, \quad D_5 = -2k_2v_2 + \delta, \quad D_6 = -2k_3v_2 + \delta$$
$$D_7 = -k_1^2, \quad D_8 = -(k_1-k_2)^2, \quad D_9 = -(k_2-k_3)^2$$
$$D_{10} = -(k_1-k_3)^2, \quad D_{11} = -k_2^2, \quad D_{12} = -k_3^2$$




power counting zero
free of UV sub-divergences
free of soft divergence

divergence of k3 divergence of k3

admissible $G[2,1,\underline{-1},0,1,\underline{1},1,1,\underline{1},0,0,\underline{1}] - G[2,1,\underline{-1},0,1,\underline{1},1,1,\underline{1},\underline{1},0,0]$

$$\frac{1}{k_3^2} - \frac{1}{(k_1-k_3)^2} = \frac{k_1^2}{k_3^2(k_1-k_3)^2} - \frac{2k_1k_3}{k_3^2(k_1-k_3)^2}$$

reducible SP $\quad D_{13} \equiv k_1 \cdot k_2 \quad D_{14} \equiv k_1 \cdot k_3, \quad D_{15} \equiv k_2 \cdot k_3$

0 superficial degree of divergence

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + 2a_7 + 2a_8 + 2a_9 + 2a_{10} \\ +2a_{11} + 2a_{12} + 2a_{13} + 2a_{14} + 2a_{15} = 12$$

free of sub-divergences

$$a_1 + a_4 + 2a_7 + 2a_8 + 2a_{10} + a_{13} + a_{14} > 4,$$
$$a_2 + a_5 + 2a_8 + 2a_9 + 2a_{11} + a_{13} + a_{15} > 4,$$
$$a_3 + a_6 + 2a_9 + 2a_{10} + 2a_{12} + a_{14} + a_{15} > 4,$$

$$a_1 + a_2 + a_4 + a_5 + 2a_7 + 2a_8 + 2a_9 + 2a_{10} + a_{11} + 2a_{13} + a_{14} + a_{15} > 8,$$
$$a_1 + a_2 + a_4 + a_6 + 2a_7 + 2a_8 + 2a_9 + 2a_{10} + 2a_{12} + a_{13} + 2a_{14} + a_{15} > 8,$$
$$a_2 + a_3 + a_5 + a_6 + 2a_8 + 2a_9 + 2a_{10} + 2a_{11} + a_{12} + a_{13} + a_{14} + 2a_{15} > 8.$$

free of soft divergence

$$a_7, a_8, a_9, a_{10}, a_{11}, a_{12} < 2$$

it's easy to list admissible integrals by simple computer program

# IBP operator via Graded Syzygy

Graded — ensure overall UV divergence, control UV sub-divergence

$$\deg_i(k_i) = 1, \deg_i(\frac{\partial}{\partial k_i}) = -1$$

each term have the same degree

Solved by linear algebra

Syzygy — make sure free of soft divergence

$$OD_k = g_k D_k$$

$g_k$ is a polynomial

$$O\mathscr{F} \equiv \frac{\partial}{\partial k_1^\mu}\Big(\big(v_1^\mu(k_1\cdot k_2) - k_2^\mu(k_1\cdot v_1) + k_1^\mu(k_2\cdot v_1)\big)\mathscr{F}\Big) + \frac{\partial}{\partial k_2^\mu}\Big(v_1^\mu(k_2\cdot k_2)\mathscr{F}\Big) + \frac{\partial}{\partial k_3^\mu}\Big(\big(v_1^\mu(k_2\cdot k_3) + k_3^\mu(k_2\cdot v_1) - k_2^\mu(k_3\cdot v_1)\big)\mathscr{F}\Big)$$

$$O\ D_7 = 2(d+2)(k_2\cdot v_1)D_7$$

$$
O\mathscr{F} \equiv \frac{\partial}{\partial k_1^\mu}\left(\big(v_1^\mu (k_1\cdot k_2) - k_2^\mu (k_1\cdot v_1) + k_1^\mu (k_2\cdot v_1)\big)\mathscr{F}\right)
$$
$$
+\frac{\partial}{\partial k_2^\mu}\left(v_1^\mu (k_2\cdot k_2)\mathscr{F}\right) + \frac{\partial}{\partial k_3^\mu}\left(\big(v_1^\mu (k_2\cdot k_3) + k_3^\mu (k_2\cdot v_1) - k_2^\mu (k_3\cdot v_1)\big)\mathscr{F}\right)
$$

$$
O\ G[1,1,0,0,2,1,1,1,1,0,0,1]
$$

$$
0 = -2\left(x+\frac{1}{x}\right)G(1,1,0,0,3,1,1,1,1,0,-1,1) - \frac{1}{2}G(1,0,0,0,2,1,1,1,1,0,0,1)
$$
$$
-\frac{1}{2}G(1,1,-1,0,1,2,1,1,1,0,0,1) - 2G(1,2,0,0,2,1,1,1,1,0,-1,1)
$$
$$
+2g[2,1,0,0,2,1,1,1,1,0,0,1,-1,0,0]
$$
$$
+\left(x+\frac{1}{x}\right)g[1,1,0,0,2,2,1,1,1,0,0,1,0,0,-1]
$$

IBP with only
admissible integrals

$$O\mathscr{F} \equiv \frac{\partial}{\partial k_1^\mu}\left(\left(v_1^\mu(k_1\cdot k_2) - k_2^\mu(k_1\cdot v_1) + k_1^\mu(k_2\cdot v_1)\right)\mathscr{F}\right)$$
$$+\frac{\partial}{\partial k_2^\mu}\left(v_1^\mu(k_2\cdot k_2)\mathscr{F}\right) + \frac{\partial}{\partial k_3^\mu}\left(\left(v_1^\mu(k_2\cdot k_3) + k_3^\mu(k_2\cdot v_1) - k_2^\mu(k_3\cdot v_1)\right)\mathscr{F}\right)$$

$$\frac{\partial}{\partial k_1^\mu}\left(v_1^\mu(k_1\cdot k_2)\right) \quad G[1,1,0,0,2,1,1,1,1,0,0,1]$$

$$-2\frac{\partial}{\partial k_1^\mu}\left(v_1^\mu(k_1\cdot v_1)\right)\, g[2,1,0,0,2,1,1,1,1,0,0,1,-1,0,0]$$

zero overall degree

admissible integral

Get IBP with only admissible integrals

safely set $\varepsilon$ and $\delta$ to zero!

# "4D" Initial Algorithm

"One UT integral rules all" — Dlapa, Henn, Kai (2020)

$(f_1, \ldots, f_n)$ is a master integral basis, where $f_1$ is UT but the others are not.
Try to find a UT basis $(f_1, g_2 \ldots g_n)$ which is UT.

$$\begin{pmatrix} f_1 \\ f_1^{[1]} \\ \ldots \\ f_1^{[n-1]} \end{pmatrix} = \Psi \begin{pmatrix} f_1 \\ f_2 \\ \ldots \\ f_n \end{pmatrix} \qquad \begin{pmatrix} f_1 \\ f_1^{[1]} \\ \ldots \\ f_1^{[n-1]} \end{pmatrix} = \Phi \begin{pmatrix} f_1 \\ g_2 \\ \ldots \\ g_n \end{pmatrix}$$

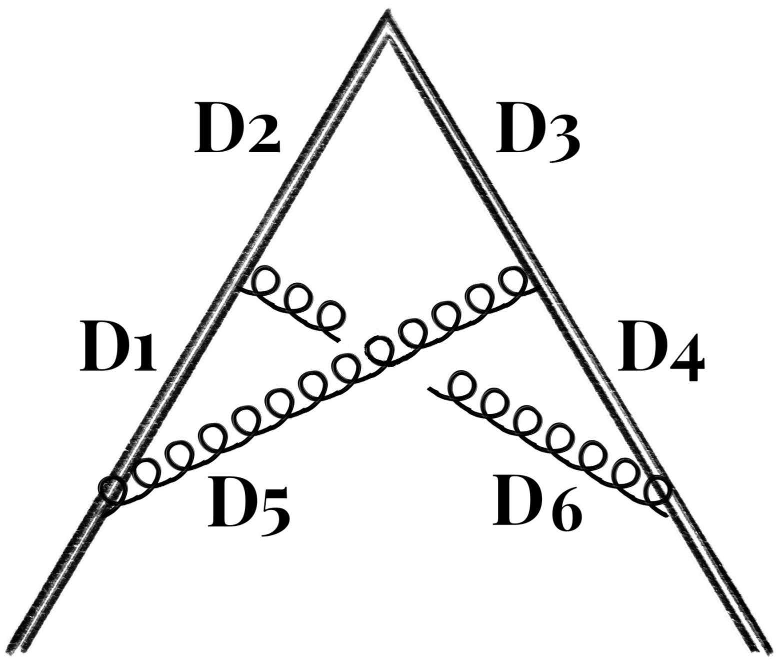
known ($\Psi$) unknown ($\Phi$)

$\Phi$ is derived from a canonical DE with undermined coefficients

$$\sum_{i=1}^{k} a_i \partial_x \log(W_i) \qquad v_0 \Psi^{-1} \Phi = v_0, \quad v_0 = (1, 0 \ldots 0)$$

For 4D, ansatz $\{a_i\}$ is a strictly upper triangular matrix with rational number entries

# Toy example: 2-loop HQET integral




$$2v_1v_2 \equiv x + \frac{1}{x}, \quad v_1^2 = 1, \quad v_2^2 = 1$$

$$\mathbf{d} \equiv \frac{x^2-1}{2}\frac{d}{dx} \equiv \frac{(v_1 \cdot v_2)v_1^\mu - v_1^2 v_2^\mu}{\sqrt{v_1^2\, v_2^2}} \frac{\partial}{\partial v_1^\mu}$$

$$T^{(2)}[a_1, \cdots, a_7] = \int \frac{d^D k_1}{i\pi^{D/2}} \frac{d^D k_2}{i\pi^{D/2}} \frac{D_7^{-a_7}}{D_1^{a_1} \cdots D_6^{a_6}}$$

$$D_1 = -2k_1 \cdot v_1 + \delta,\ D_2 = -2(k_1+k_2)\cdot v_1 + \delta,\ D_3 = -2(k_1+k_2)\cdot v_2 + \delta,$$

$$D_4 = -2k_2 \cdot v_2 + \delta,\ D_5 = -k_1^2,\ D_6 = -2k_2^2,\ D_7 = k_1 \cdot k_2$$

reducible SP $\quad D_8 \equiv -2k_1 \cdot v_2 \quad D_9 \equiv -2k_2 \cdot v_1$

$T^{(2)}[1,1,1,1,1,1,0]$ is admissible

All its derivatives in *x* are admissible

# Toy example: 2-loop HQET integral

Admissible Integrals

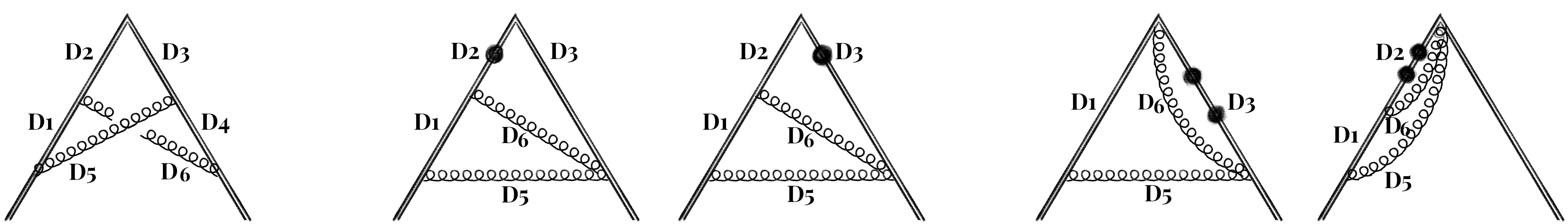
$$A_1 = T^{(2)}[1,1,1,1,1,1,0]\,,\, A_2 = t_{(2)}[2,1,1,1,1,1,0,-1,0]\,,$$
$$A_3 = t^{(2)}[1,1,1,2,1,1,0,0,-1]\,,\, B_1 = T^{(2)}[1,2,1,0,1,1,0]\,,$$
$$B_2 = T(2)[1,1,2,0,1,1,0]\,,\, B_3 = T^{(2)}[1,2,0,1,1,1,0]\,,$$
$$C_1 = T^{(2)}[1,0,3,0,1,1,0]\,,\, C_2 = T^{(2)}[1,3,0,0,1,1,0]\cdots$$

Admissible IBP vectors

$$\mathbf{P_1} \equiv \partial_{1\mu} k_{1\nu}\,,$$
$$\mathbf{P_2} \equiv \partial_{1\mu} k_{1\nu}\, v_1^{[\mu} v_2^{\nu]}\,,$$
$$\mathbf{P_3} \equiv (\partial_{1\mu} k_{1\mu} + \partial_{2\mu} k_{2\nu})\, v_1^{[\mu} v_2^{\nu]} \cdots$$

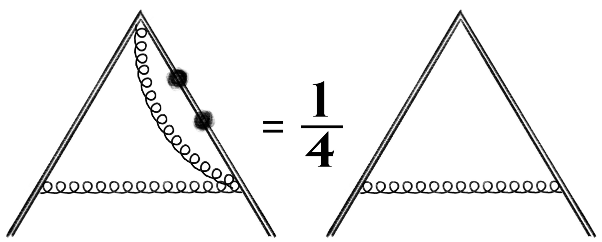
IBPs with only admissible integrals

$$\mathbf{P_2} A_1 = 0 \to 2A_2 + B_3 - 2B_1 - 2(v_1\cdot v_2)B_2 = 0$$
$$\mathbf{P_3} A_1 = 0 \to A_3 - A_2 = 0$$
$$\mathbf{P_1} B_3 = 0 \to B_3 - 2C_1 = 0 \cdots$$



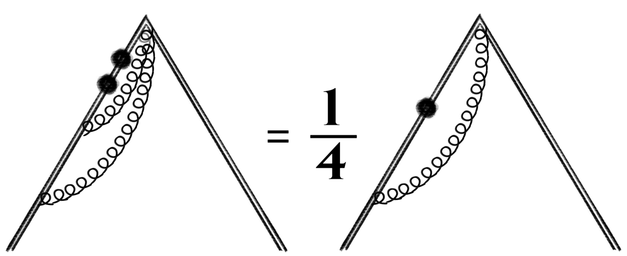
# Toy example: 2-loop HQET integral

## Cross loop order relation




$$\int \frac{d^{4-2\epsilon}k_1}{i\pi^{2-\epsilon}} \frac{1}{(-2k_1\cdot v_1+\delta)^{a_1}(-(k_1-k_2)^2)} = \frac{1}{(a_1-1)(a_1-2)}\frac{1}{(-2k_2 v_1+\delta)^{a_1-2+2\epsilon}}$$

$$\int \frac{d^{4-2\epsilon}k_1}{i\pi^{2-\epsilon}} \frac{1}{(-2k_1\cdot v_1+\delta)^{a_1}(-(k_1-k_2)^2)} \to \frac{L-1}{L}\frac{1}{(a_1-1)(a_1-2)}\frac{1}{(-2k_2 v_1+\delta)^{a_1-2}}$$




admissible differential equation for our IBP system by Initial

$$\frac{d}{dx}\begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix} = \begin{pmatrix} 0 & \frac{2}{x} & 0 & 0 \\ & & \frac{1+x^2}{x(1-x)(1+x)} & 0 \\ & & & \frac{4}{x} \\ & & & 0 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix}$$

$$\{F_1, F_2, F_3, F_4, F_5\} = \{\frac{(x^2-1)^2}{2\,x^2}A_1, \frac{x^2-1}{x}(B_1+\frac{1+x^2}{2\,x}B_2) \frac{2(x^2-1)}{x}C_1, C_2, \frac{(x^2-1)^2}{2\,x^2}B_2\}$$

$$\frac{d}{dx}\begin{pmatrix} F_5 \\ F_3 \\ F_4 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{x} & 0 \\ & & \frac{4}{x} \\ & & 0 \end{pmatrix} \begin{pmatrix} F_5 \\ F_3 \\ F_4 \end{pmatrix}$$

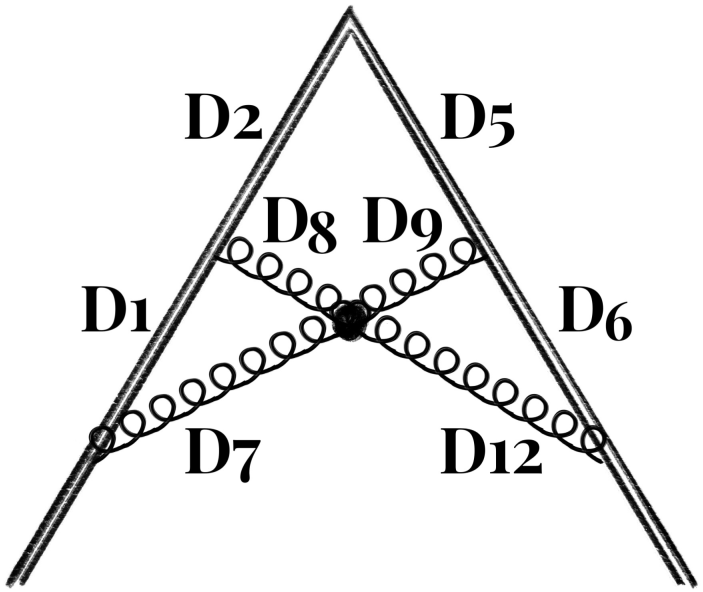
strictly upper triangular matrix

easy to solve with boundary condition

$$F_1 = \frac{1}{2}\left[\frac{1}{3}\ln^3 x + \ln x \,\mathrm{Li}_2(x^2) - \mathrm{Li}_3(x^2) + \zeta_2 \ln x + \zeta_3\right]$$

$$F_5 = \frac{1}{4}\ln^2 x$$

# 3-loop HQET




$$G[1,1,0,0,1,1,1,1,1,0,0,1]$$

10503 admissible integrals
23575 4D IBPs

cross loop order relation act on
the sector (1,0,0,0,1,1,0,1,1,0,0,1)
and the sector (1,0,0,0,1,1,1,0,1,1,0,0)

This linear system is very sparse,
solve with *FiniteFlow (Peraro 2019)* in 2mins with 1 core

13 irreducible integrals

$$\{I_1,\ldots,I_{13}\} = \Big\{ g[2,2,0,0,1,1,1,1,1,0,0,1,-1,0,0], \quad G[1,1,-1,0,1,2,1,1,1,0,0,1], \\ G[1,1,0,0,1,1,1,1,1,0,0,1], \quad g[3,1,0,0,2,2,0,1,1,0,0,1,0,0,-1], \\ G[3,1,-1,0,1,2,0,1,1,0,0,1], \quad g[1,0,0,0,3,2,1,1,1,0,0,1,0,0,-1], \\ G[1,0,-1,0,2,2,1,1,1,0,0,1], G[1,-1,0,0,3,1,1,1,1,0,0,1], G[1,0,0,0,2,1,1,1,1,0,0,1], \\ H_2[2,0,1,1,1,0,1], \quad H_2[1,0,2,1,1,0,1], \quad H_1[1,1,1], \quad H_1[0,2,1] \Big\}$$

# 13*13 DE matrix

$$
\begin{pmatrix}
-\frac{2(1+x^2)}{(-1+x)x(1+x)} & 0 & 0 & 0 & 0 & \frac{4}{(-1+x)(1+x)} & 0 & 0 & 0 & \frac{2(1+x^2)}{3(-1+x)x(1+x)} & 0 & -\frac{1}{3(-1+x)(1+x)} & 0 \\
-\frac{2(-1+x)(1+x)}{x^2} & 0 & -\frac{2}{(-1+x)(1+x)} & 0 & 0 & 0 & 0 & 0 & \frac{1+x^2}{(-1+x)x(1+x)} & \frac{4(-1+x)(1+x)}{3x^2} & 0 & 0 & 0 \\
0 & \frac{2}{(-1+x)(1+x)} & -\frac{2(1+x^2)}{(-1+x)x(1+x)} & 0 & 0 & 0 & 0 & 0 & \frac{2}{(-1+x)(1+x)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{2(1+x^2)}{(-1+x)x(1+x)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{x^2}{3(-1+x)^3(1+x)^3} & \frac{x(1+x^2)}{6(-1+x)^3(1+x)^3} \\
0 & 0 & 0 & -\frac{2(-1+x)(1+x)}{x^2} & 0 & 0 & 0 & 0 & 0 & -\frac{2}{3(-1+x)(1+x)} & \frac{1+x^2}{3(-1+x)x(1+x)} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2(-1+x)(1+x)} & \frac{1}{2(-1+x)(1+x)} & \frac{-1-x^2}{2(-1+x)x(1+x)} & -\frac{2}{3(-1+x)(1+x)} & \frac{-1-x^2}{3(-1+x)x(1+x)} & \frac{-1-x^2}{12(-1+x)x(1+x)} & \frac{1}{2(-1+x)(1+x)} \\
0 & 0 & 0 & 0 & 0 & -\frac{4(-1+x)(1+x)}{x^2} & 0 & \frac{2(1+x^2)}{(-1+x)x(1+x)} & -\frac{4}{(-1+x)(1+x)} & 0 & 0 & \frac{(-1+x)(1+x)}{3x^2} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{2(-1+x)(1+x)}{x^2} & \frac{1+x^2}{(-1+x)x(1+x)} & \frac{1+x^2}{(-1+x)x(1+x)} & -\frac{(1+x^2)^2}{(-1+x)x^2(1+x)} & 0 & \frac{2(-1+x)(1+x)}{3x^2} & -\frac{(-1+x)(1+x)}{6x^2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{(-1+x)(1+x)} & \frac{4}{(-1+x)(1+x)} & -\frac{3(1+x^2)}{(-1+x)x(1+x)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2(1+x^2)}{(-1+x)x(1+x)} & 0 & \frac{1}{(-1+x)(1+x)} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{(-1+x)(1+x)} & -\frac{1+x^2}{(-1+x)x(1+x)} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1+x^2}{(-1+x)x(1+x)} & \frac{2}{(-1+x)(1+x)} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

4D initial algorithm

$$
\begin{pmatrix}
0 & -\frac{2}{(-1+x)(1+x)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{x} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{x} & \frac{2x}{(-1+x)(1+x)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{x} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{x} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{x} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{x} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

8*8 strictly upper triangular DE matrix

$$\begin{pmatrix} 0 & -\frac{2}{(-1+x)(1+x)} & 0 & 0 & 0 & 0 & 0 & 0 \\ & & \frac{1}{x} & 0 & 0 & 0 & 0 & 0 \\ & & & \frac{1}{x} & \frac{2x}{(-1+x)(1+x)} & 0 & 0 & 0 \\ & & & & & \frac{1}{x} & 0 & 0 \\ & & & & & & \frac{1}{x} & 0 \\ & & & & & & & \frac{1}{x} \\ & & & & & & & -\frac{1}{x} \\ & & & & & & & 0 \end{pmatrix}$$

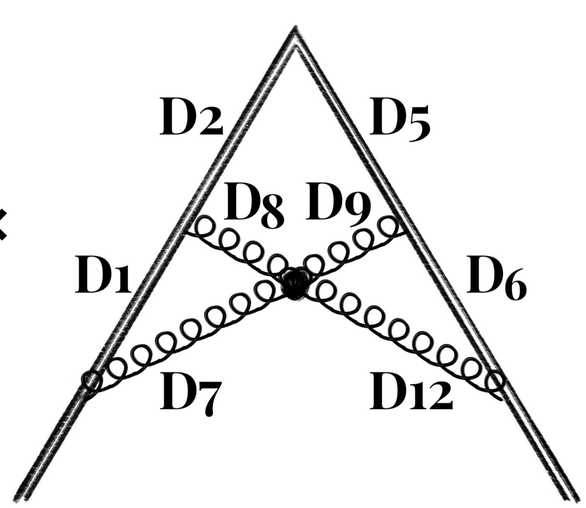
we can integrate the UT basis easily

$$J_7 = -J_6 = \frac{8}{3}\ln x\,,\ J_5 = -J_4 = \frac{4}{3}\ln^2 x$$

reduce system dimension from 8 to 6

with the boundary condition at x=1

$\varepsilon$ ×




$$J_1 = \frac{8}{3}H_{-1,-2,0,0} - \frac{8}{3}H_{-1,2,0,0} + \frac{8}{3}H_{1,-2,0,0} - \frac{8}{3}H_{1,2,0,0} - \frac{8}{3}H_{-1,0,0,0,0} - \frac{8}{3}H_{1,0,0,0,0} - \zeta_4\ \ln(1-x) + \zeta_4\ \ln(1+x) + \frac{4}{3}\zeta_3\ \ln(1-x)\ln x - \frac{4}{3}\zeta_3\ \ln x\ln(1+x) + \frac{8}{3}\zeta_3\,\mathrm{Li}_2(x) - \frac{2}{3}\zeta_3\,\mathrm{Li}_2(x^2)$$

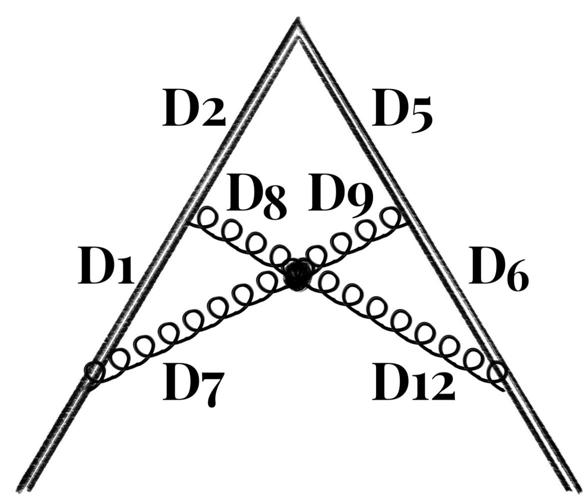
our method

6 master integrals to get $J_1$

arXiv[1510.07803]

Grozin, Heen, Korchemsky and Marquard

39 master integrals to get $J_1$



# Implementation notes

syzygy generation : A simple Mathematica code

Similar to Schabinger, JHEP 01(2012)001

IBP reduction: *FiniteFlow*

find UT : *4D Initial* algorithm in Mathematica + *FiniteFlow*

# Summary

- Simplified canonical DE for divergent part of Feynman integrals
- Graded Syzygy + Integrating out bubble + 4D Initial
- First application: 3-loop HQET integrals for cusp anomalous dimension
- More generalisation + applications to come ...
- Feynman integral with subdivergences

Thanks