有限域/块三角系统

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Feynman integrals reduction based on Laporta’s algorithm

Integration-by-parts: example

- A family of FIs: $F(n)=\int \frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{1}{(\ell^2-\Delta)^n}$

➢ Vanishing on the big hypersphere with radius R

Lagrange, Gauss, Green, Ostrogradski, 1760s-1830s

'‘t Hooft, Veltman, NPB (1972)

$$\int \frac{\mathrm{d}^D\ell}{(2\pi)^D}\frac{\partial}{\partial\ell^\mu}\left[\frac{\ell^\mu}{(\ell^2-\Delta)^n}\right] \overset{\Downarrow}{=} \int_\partial \frac{\mathrm{d}^{D-1}S_\mu}{(2\pi)^D}\left[\frac{\ell^\mu}{(\ell^2-\Delta)^n}\right] \overset{\Downarrow}{=} 0.$$

- Integrand: fixed power in R; Measure: R^{D-1}
- Thus vanishing in dimensional regularization

➢ Relations between FIs

$$0=\int_\ell\left[\frac{D}{(\ell^2-\Delta)^n}-n\int_\ell\frac{2(\ell^2-\Delta)+2\Delta}{(\ell^2-\Delta)^{n+1}}\right] \quad =(D-2n)F(n)-2n\Delta F(n+1)$$

$$F(n+1)=\frac{1}{-\Delta}\frac{n-\frac{D}{2}}{n}F(n)$$

- All FIs in this family can be expressed by $F(1)$

IBP equations

➢ **Dimensional regularization: vanish at boundary**

't Hooft, Veltman, NPB (1972)
Chetyrkin, Tkachov, NPB (1981)

$$\int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\partial}{\partial \ell_j^\mu} \left(q_k^\mu \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{\mathcal{D}_1^{\nu_1} \cdots \mathcal{D}_K^{\nu_K}} \right) = 0, \qquad \forall \vec{\nu}, j, k$$

$$\overrightarrow{q^\mu} = (\ell_1^\mu, \cdots, \ell_L^\mu, p_1^\mu, \cdots, p_E^\mu)$$

$$\Downarrow$$

- **Linear equation:** $\sum_{\vec{\nu}'} Q_{\vec{\nu}'}^{\vec{\nu}jk}(D, \vec{s}) I_{\vec{\nu}'}(D, \vec{s}) = 0$
- Q: **polynomials in** $D, \vec{s}$
- **Plenty of linear equations can be easily obtained by varying:** $\vec{v}, j, k$

Master integrals

➢ # of equations grows faster than # of FIs

Laporta, Remiddi, 9602417, Gehrmann, Remiddi, 9912329

- Let positive powers $d = \nu_{i_1} + \cdots + \nu_{i_z} - Z$, nonpositive $r = -(\nu_{i_{z+1}} + \cdots + \nu_{i_N})$,

  $N_{d,r} = C^{d}_{d+z-1} C^{r}_{r+N-z-1}$ is the # of FIs with fixed d, r
- # of equations (for seeds with fixed d, r) $= L(L+E) \times N_{d,r}$
- # of new FIs $= N_{d+1,r} + N_{d+1,r+1}$ ($\approx 2\, N_{d,r}$ for sufficient large d, r)
- Expectation: finite # of linearly independent FIs

➢ A family of FIs form a FINITE-dim. linear space

Proved by: Smirnov, Petukhov, 1004.4199

- Bases of the linear space called master integrals (MIs)
- IBPs reduce tens of thousands of FIs to much less MIs

IBP reduction

➢ Laporta's algorithm to do reduction

$$\sum_{\vec{\nu}'} Q_{\vec{\nu}'}^{\vec{\nu}jk}(D,\vec{s})I_{\vec{\nu}'}(D,\vec{s}) = 0$$

Laporta, 0102033

- Generate eqs for all $\vec{v}$ with $d \in [d_{min}, d_{max}], r \in [r_{min}, r_{max}]$
- Ordering: simpler FI has smaller z, then smaller d, then smaller r
- Solving linear eqs to eliminate more complicated FIs
- Eventually, all FIs are linear combinations of MIs

➢ Solving IBP eqs.: automatic, any-loop order

- Public codes: AIR, FIRE, LiteRed, Reduze, Kira, FiniteFlow, Blade, NeatIBP…
- Many more private codes
- Warning: time-consuming for complicated problems

Current status of integral reduction

➢ IBP is crucial

- Laporta algorithm — Laporta, 0102033

➢ Difficulties of IBP method

- Complicated intermediate expressions
- Too many auxiliary equations — E.g. Laporta 1910.01248

Hundreds GB RAM — E.g. J. Klappert et al., 2008.06494

Months of runtime using super computer — E.g. Davies, Herren, Steinhauser, 1911.10214 (wall time 860 days)

➢ Selected developments

- Finite field: solving intermediate express swell — Manteuffel, Schabinger, 1406.4513
- Syzygy equations: trimming IBP system — Gluza, Kajda, Kosower, 1009.0472; Larsen, Zhang, et. al., 1511.01071, 1805.01873, 2104.06866
- Block-triangular form: minimize IBP system (needs input) — Liu, YQM, 1801.10523, Guan, Liu, YQM, 1912.09294
- A better choice of basis: UT basis/ D-factorized — S. Abreu, et al., PRL (2019); Usovitsch, 2002.08173; A. V. Smirnov, V. A. Smirnov , 2002.08042

Finite field

Numerical sampling and reconstruction

➢ Functional reconstruction

- **Univariate polynomial: Newton formula**

$$f(z)=\sum_{r=0}^{R}a_r\prod_{i=0}^{r-1}(z-y_i)$$
$$=a_0+(z-y_0)\Big(a_1+(z-y_1)\Big(a_2+(z-y_2)\big(\cdots+(z-y_{R-1})a_R\big)\Big)\Big),$$

- **Multivariate polynomial: iterative Newton formula**

$$f(z_1,\ldots,z_n)=\sum_{r=0}^{R}a_r(z_2,\ldots,z_n)\prod_{i=0}^{r-1}(z_1-y_i).$$

- **Univariate rational function: Thiele formula**

$$f(z)=a_0+\cfrac{z-y_0}{a_1+\cfrac{z-y_1}{a_2+\cfrac{z-y_2}{\cdots+\cfrac{z-y_{N-1}}{a_N}}}}$$
$$=a_0+(z-y_0)\left(a_1+(z-y_1)\left(a_2+(z-y_2)\left(\cdots+\frac{z-y_{N-1}}{a_N}\right)^{-1}\right)^{-1}\right)^{-1},$$

- **Multivariate rational function**

T. Peraro, JHEP(2019)

J. Klappert, F. Lange, Comput. Phys. Commun(2019)

Finite field

➢ Problem to solve

- Big numbers encountered in numerical sampling

➢ Arithmetic

$$\phi_p(z) = z\ Mod\ p$$

$$\phi_p(1/b) = t\Big|_{t\,*\,b\ Mod\ p=1}$$

$$\phi_p(a/b) = \phi_p(a)\phi_p(1/b)$$

E.g. $1 = GCD(7,3) = 1 * 7 + (-2) * 3$

$$\Rightarrow \quad \phi_p(1/3) = (-2) mod\ 7 = 5$$

EEA algorithm

➢ Rational Reconstruction

Wang, Guy, Davenport, SIGSAM(1982)

g	a	b	$a-b[a/b]$	…	$g_{i-2}-g_{i-1}q_i$	0
s	1	0	1	…	$s_{i-2}-s_{i-1}q_i$	
t	0	1	$-[a/b]$	…	$t_{i-2}-t_{i-1}q_i$	
q			$[a/b]$	…	$q_i \equiv [g_{i-2}/g_{i-1}]$	

$$a\, s_i + b\, t_i = g_i$$

Let $a = p$:

$$g_i = b\, t_i \mod p \quad \Leftrightarrow \quad g_i/t_i \mod p = b$$

$$b \xrightarrow{Rational\ Reconstruct} g_i/t_i$$

When $g_i^2 \le p, t_i^2 \le p, GCD(g_j, t_j) = 1$

Rational Reconstruction

➢ Example

g	7	3	1	0
s	1	0	1	-3
t	0	1	-2	7
q			2	

$$3 \xrightarrow{\text{Rational Reconstruction}} -\frac{1}{2}$$

➢ Question

- What happens if p is not large enough?

Chinese remainder theorem

➢ Evaluation under different primes

- Obtain arbitrarily large $p = p_1 p_2 \dots p_k$

More in detail, given $a \in Z_n$, a set of pairwise co-prime numbers $n_1, \dots n_k$ such that $n = n_1 \cdots n_k$, and a set of congruences

$$a_i = a \bmod n_i, \tag{A.9}$$

a can be uniquely determined in $\mathbb{Z}_n$ as

$$a = \sum_i m_i a_i \bmod n, \tag{A.10}$$

where

$$m_i \equiv \left(\left(\frac{n}{n_i} \right)^{-1} \bmod n_i \right) \frac{n}{n_i}. \tag{A.11}$$

今有物不知其数，
三三数之剩二，
五五数之剩三，
七七数之剩二，
问物几何？

Chinese remainder theorem

```
In[26]:= Mod[46, 3]
         模余
         Mod[46, 5]
         模余
         Mod[46, 7]
         模余

Out[26]= 1

Out[27]= 1

Out[28]= 4
```

```
In[29]:= m1 = ModularInverse[105 / 5, 5] * 105 / 5
              模逆
Out[29]= 21

In[30]:= m2 = ModularInverse[105 / 7, 7] * 105 / 7
              模逆
Out[30]= 15

In[31]:= m3 = ModularInverse[105 / 3, 3] * 105 / 3
              模逆
Out[31]= 70

In[32]:= n = 3 * 5 * 7
Out[32]= 105

In[33]:= a = m1 * 1 + m2 * 4 + m3 * 1
Out[33]= 151

In[34]:= Mod[a, n]
         模余
Out[34]= 46
```

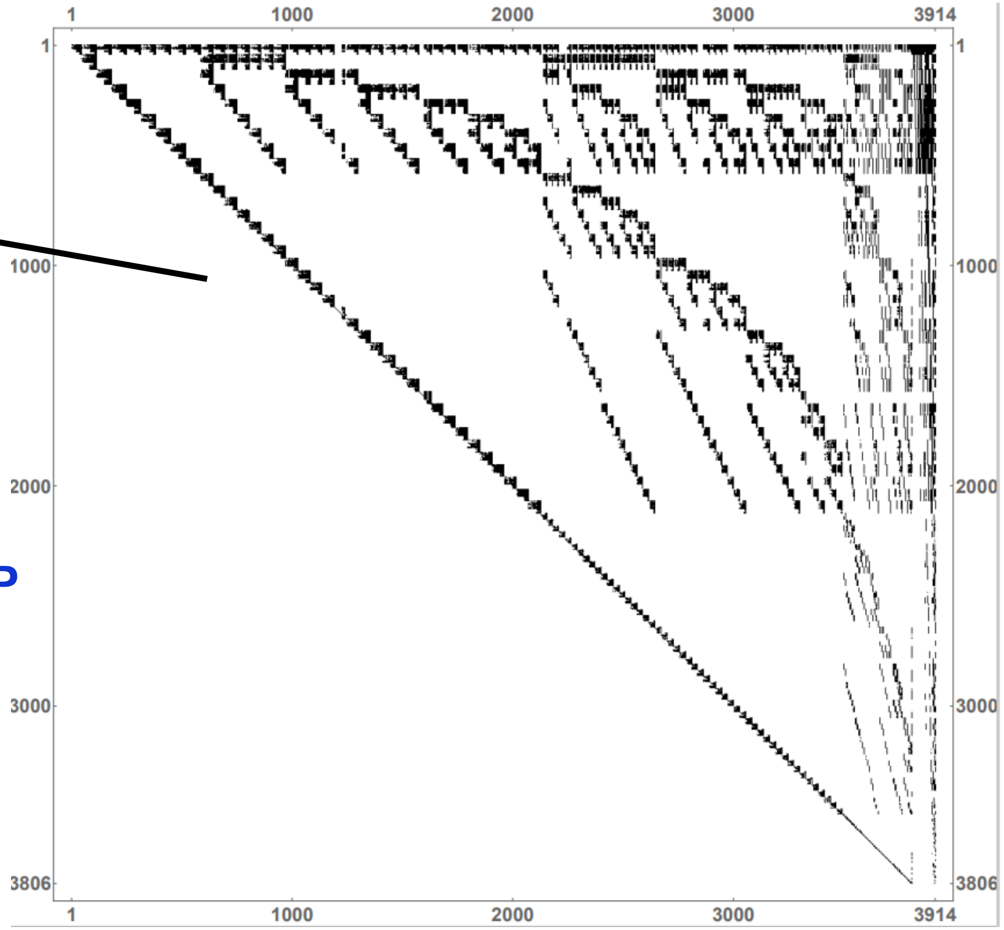
Block-triangular form

Block-triangular form

➢ Improved linear system

$$Q_{11}\, I_1 + Q_{12}\, I_2 + Q_{13}\, I_3 + Q_{14}\, I_4 + \ \dots + Q_{1N}\, I_N = 0$$
$$Q_{21}\, I_1 + Q_{22}\, I_2 + Q_{23}\, I_3 + Q_{24}\, I_4 + \ \dots + Q_{2N}\, I_N = 0$$
$$Q_{33}\, I_3 + Q_{34}\, I_4 + \ \dots + Q_{3N}\, I_N = 0$$
$$Q_{43}\, I_3 + Q_{44}\, I_4 + \ \dots + Q_{4N}\, I_N = 0$$
$$\dots$$

Liu, YQM, 1801.10523, Guan, Liu, YQM, 1912.09294



- **Simple relations among Feynman integrals**
- **Several orders of magnitude equations less than IBP**
- **Nice block-triangular structure, efficient for numerical sampling (finite field / floating number)**

Search algorithm

- **Decomposition of** $Q_i(\vec{s},\epsilon)$ $\quad \sum Q_i(\vec{s},\epsilon)\, I_i(\vec{s},\epsilon) = 0$

$$Q_i(\vec{s},\epsilon) = \sum_{\mu_0=0}^{\epsilon_{max}} \sum_{\mu} \tilde{Q}_i^{\mu_0\mu_1\ldots\mu_r} \epsilon^{\mu_0} s_1^{\mu_1} \ldots s_r^{\mu_r}$$

- $\tilde{Q}_i^{\mu_0\mu_1\ldots\mu_r}$ **are unknowns**
- $\mu_1 + \cdots + \mu_r = d_i$

- **Input from numerical IBP** $\quad I_i(\vec{s},\epsilon) = \sum_{j=1}^{n} C_{ij}(\vec{s},\epsilon) M_j(\vec{s},\epsilon)$

$$\Rightarrow \sum_{\mu_0,\mu} \sum_{j=1}^{n} \tilde{Q}_i^{\mu_0\ldots\mu_r} \epsilon^{\mu_0} s_1^{\mu_1} \ldots s_r^{\mu_r} C_{ij}(\vec{s},\epsilon) M_j(\vec{s},\epsilon) = 0$$

- **Linear equations:** $\quad \sum_{\mu_0,\mu} \tilde{Q}_i^{\mu_0\ldots\mu_r} \epsilon^{\mu_0} s_1^{\mu_1} \ldots s_r^{\mu_r} C_{ij}(\vec{s},\epsilon) = 0$

  - **With enough constraints** $\Rightarrow \tilde{Q}_i^{\mu_0\ldots\mu_r}$
  - **With finite field technique, equations can be efficiently solved**
  - **Relations among** $G \equiv \{I_1, I_2, \ldots I_N\}$ **can be determined**

Reduction

- **With $G = G_1 \cup G_2$, satisfy**
  - G_1 is more complicated than G_2
  - G_1 can be reduced to G_2

$$
\begin{aligned}
Q_{11} I_1 + Q_{12} I_2 + Q_{13} I_3 + Q_{14} I_4 + \ldots + Q_{1N} I_N &= 0 \\
Q_{21} I_1 + Q_{22} I_2 + Q_{23} I_3 + Q_{24} I_4 + \ldots + Q_{2N} I_N &= 0 \\
Q_{33} I_3 + Q_{34} I_4 + \ldots + Q_{3N} I_N &= 0 \\
Q_{43} I_3 + Q_{44} I_4 + \ldots + Q_{4N} I_N &= 0 \\
\ldots
\end{aligned}
$$

- **Algorithm** *Search for efficient relations*

  1. Set degree bound
  2. Search relations among G
  3. If obtained relations are enough to determine G_1 by G_2, stop;
     else, increase degree bound and go to step 2

- **Conditions for G_1 and G_2**

  1. Relations among G_1 and G_2 are not too complicated: easy to find
  2. $\# G_1$ is not too large: numerically diagonalize relations easily

Adaptive search strategy

➢ Semi-analytic

- The number of unknowns of full-analytic block-triangular form may be too large
- Keep a subset of variables analytic ⇒ easy to search
- The integral set is the same ⇒ still very efficient
- More than one block-triangular form is needed

$$Q_i(\vec{z}) = \sum_{\mu} \tilde{Q}_i^{\mu_1 \dots \mu_r} z_1^{\mu_1} \dots z_r^{\mu_r}$$

$$Q_i(z_{1,0}, \dots z_{r-1,0}, z_r) = \sum_{\mu_r} \tilde{Q}_i^{\mu_0} z_r^{\mu_r}$$

$$Q_i(z_{1,0}, \dots z_{r-2,0}, z_{r-1}, z_r) = \sum_{\mu_{r-1},\mu_r} \tilde{Q}_i^{\mu_{r-1}\mu_r} z_{r-1}^{\mu_{r-1}} z_r^{\mu_r}$$

… …

➢ Adaptive search

1. $n = 1$
2. Search n-variable block-triangular form within time limit T
3. If search succeed, $n++$ and go to step 2, otherwise go to step 4
4. Perform reduction by solving the most efficient linear system(i-variable)

➢ Exploit full potential of block-triangular form

Package: Blade

Guan, Liu, Ma, Wu, 230x.xxxx

➢ Download

Link: https://gitlab.com/multiloop-pku/blade

B

Blade

Project ID: 42197761

75 Commits **1** Branch **0** Tags **2.4 MB** Project Storage

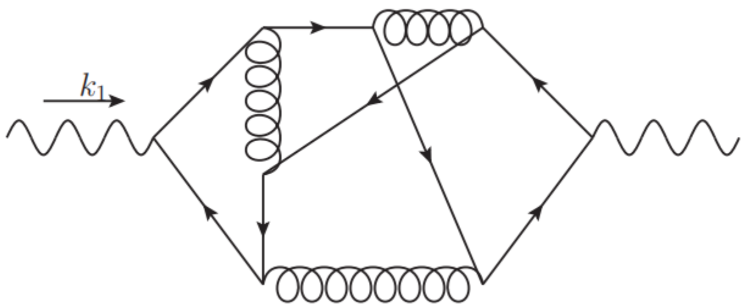
Block-triangular form improved Feynman integral decomposition .

➢ Description

- **Carefully designed integral set**
- **Flexible polynomial ansatz**
- **Automatic reduction**
- **Usually improve the efficiency of IBP reduction by 1-2 orders**

A four loop example

➢ Forward scattering with massive internal line



- **Functions of ϵ and m_t^2 ($k1^2 \to 1$)**
- **Feynman integrals up to degree 4**
- **Applied to $e^+e^- \to \gamma^* \to t\,\bar{t}$ at $\mathrm{N^3LO_{QCD}}$**

Chen, Guan, He, Liu, YQM 2209.14259

➢ Comparison

# int.	# MIs	t_{search}/h	t_{IBP}/s	t_{solve}/s	# IBP	# sample	# primes
43788	369	8	432	4.5	64	4555	7

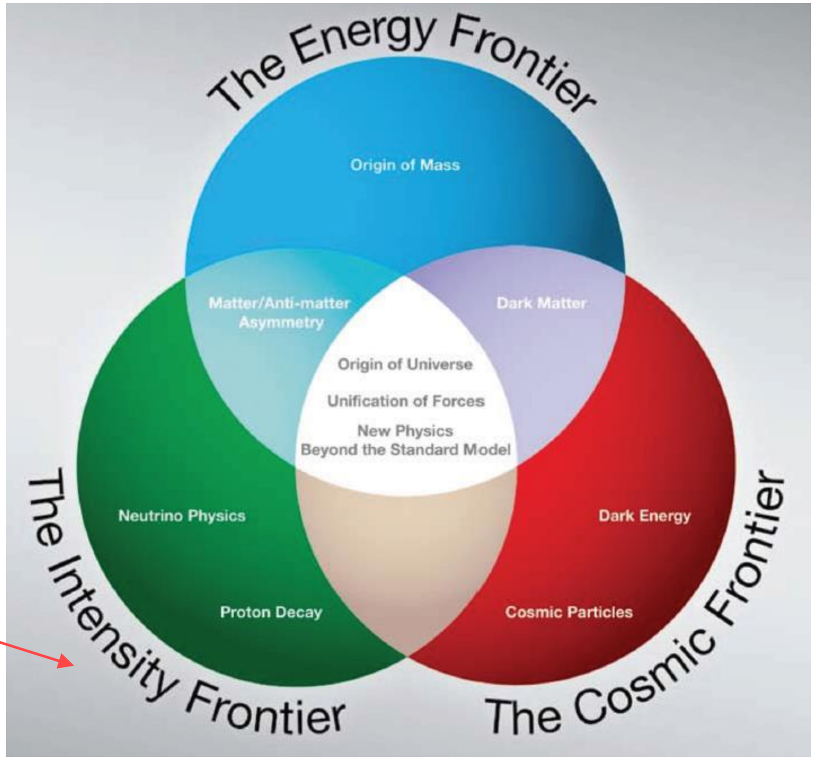
- **About two orders of magnitude faster than plain IBP**

Outline

Precision: gateway to discovery

➢ Discovery via precision

- (HL-)LHC, BELLII, EIC, CEPC/ILC/FCC-ee
- Search anomalous deviations from theory
- Interplay between exp. and th.

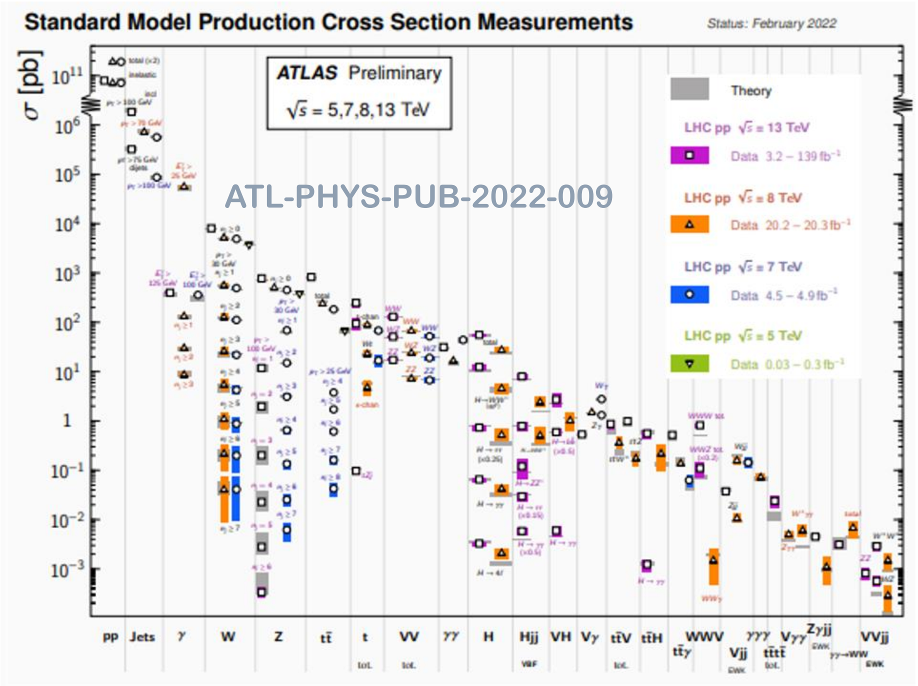


Era of precision physics at the LHC

➢ High-precision data

- Many observables probed at precent level precision

- At least NNLO QCD and NLO EW corrections generally required (plus parton shower, resummation, etc.)



Automatic NNLO perturbative calculation is highly demanded

➢ A "billion-dollar project"

- Halving total uncertainty ≈ building another LHC
- Note: LHC cost about 10 billion

Perturbative QFT

1. Generate Feynman amplitudes

- Feynman diagrams and Feynman rules (New developments: unitarity, recurrence relation, CHY, …)
- Express amplitudes as linear combinations of FIs with rational coefficients

2. Calculate Feynman loop integrals (FIs)

- Integral reduction + Master integrals calculation

3. Perform phase-space integrations

- Monte Carlo simulation with IR subtractions
- Relating to loop integrals via reverse unitarity (if no jet)

$$\int \frac{\mathrm{d}^D p}{(2\pi)^D}(2\pi)\delta_+(p^2) = \int \frac{\mathrm{d}^D p}{(2\pi)^D}\left(\frac{\mathrm{i}}{p^2+\mathrm{i}0^+} + \frac{-\mathrm{i}}{p^2-\mathrm{i}0^+}\right)$$

Fully automatic calculation: packages ABC

	Generate amplitudes	Manipulate amplitudes	Integral reduction	Master integrals calculation
Package used	FeynArts or qgraf	CalcLoop	Blade	AMFlow
Notes	https://feynarts.de/ http://cfif.ist.utl.pt/~paulo/qgraf.html	https://gitlab.com/yqma/CalcLoop	https://gitlab.com/multiloop-pku/blade	https://gitlab.com/multiloop-pku/amflow

Implementing block-triangular form, usually improves efficiency by $O(10^2)$

- **Fully automatic, valid to any-loop order**
- **The key: AMFlow**
- **Main challenge: integral reduction is time/resource consuming**

The dawn of automatic multi-loop calculation!

Automatic NLO correction obtained more than 10 years ago: MadGraph, Helac, FDC, etc

Outline

I. Introduction

II. Auxiliary mass flow method

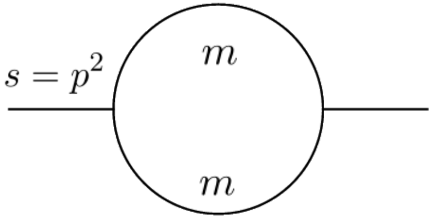
III. Block-triangular form

IV. CalcLoop

V. Summary and outlook

Differential equations: example

➢ Due to IBP: DEs of MIs



$$I_{\nu_1\nu_2} = \int \frac{\mathrm{d}^D\ell}{\mathrm{i}\pi^{D/2}} \frac{1}{(\ell^2 - m^2)^{\nu_1}[(\ell+p)^2 - m^2]^{\nu_2}}$$

$$\begin{cases} \dfrac{\partial}{\partial m^2} I_{11} = I_{21} + I_{12} \overset{\text{IBP}}{=} \dfrac{2(D-3)}{4m^2 - s} I_{11} - \dfrac{D-2}{m^2(4m^2 - s)} I_{10} \\ \dfrac{\partial}{\partial m^2} I_{10} = I_{20} \overset{\text{IBP}}{=} \dfrac{D-2}{2m^2} I_{10} \end{cases}$$

$$\begin{cases} \dfrac{\partial}{\partial s} I_{11} = \dfrac{p^\mu}{2s} \dfrac{\partial}{\partial p^\mu} I_{11} = -\dfrac{1}{2s} \displaystyle\int \frac{\mathrm{d}^D\ell}{\mathrm{i}\pi^{D/2}} \frac{2(\ell+p)\cdot p}{(\ell^2 - m^2)[(\ell+p)^2 - m^2]^2} \\ \qquad = -\dfrac{sI_{12} + I_{11} - I_{02}}{2s} \overset{\text{IBP}}{=} a_{11} I_{11} + a_{10} I_{10} \\ \dfrac{\partial}{\partial s} I_{10} = 0 \end{cases}$$

➢ Boundary Condition

$$\begin{cases} I_{11}|_{m^2 \to 0} = (-s)^{D/2-2} \Gamma(2 - D/2) \dfrac{\Gamma(D/2-1)^2}{\Gamma(D-2)} \\ I_{10} \end{cases}$$

DEs method

✓ Step 1: Set up $\vec{s}$-DEs of MIs

- Differentiate MIs w.r.t. invariants $\vec{s}$, such as $m^2, p \cdot q$ Kotikov, PLB(1991)
- IBP relations result in: $\frac{\partial}{\partial s_i}\vec{I}(D,\vec{s}) = A_i(D,\vec{s})\vec{I}(D,\vec{s})$

? Step 2: Calculate boundary condition

- Calculate integrals at special value of m^2, p^2 (strategy of region)
- Case by case, not systematic enough, maybe still hard!

✓ Step 3: Solve DEs

- Analytical : if ϵ-form exists, but not always Henn, 1304.1806
- Numerical: efficient, systematic (explain later)

Since 90s’

Using IBPs: express any FI as linear combination of MIs, also setup DEs for MIs

FIs $\triangleq$ Linear algebra $\oplus$ Master integrals

↓

Input:

The same kinematics

The same spacetime dimension

The same number of loops

DEs method: needs BCs

Current status: master integrals calculation

➢ Main methods

- **Sector decomposition** — Hepp, (1966); Binoth, Heinrich, 0004013; F. Feng, Z. Li, …
- **Mellin-Barnes representation** — Usyukina (1975); Smirnov, 9905323; J. Wang, …
- **Difference equations** — Laporta, 0102033; Lee, 0911.0252
- **Differential equations** — Kotikov, PLB(1991)

Analytical : if ϵ-form exists

Henn, 1304.1806
L.B Chen, L.L. Yang, G. Yang, Y. Zhang, …

Numerical: general and efficient

X. Liu, YQM, C. Y. Wang, 1711.09572
Hidding, 2006.05510
X. Liu, YQM, 2201.11669

See also Prof. Yang's talk

Powered by auxiliary mass flow

Auxiliary mass terms

➢ Auxiliary FIs

X. Liu, YQM, C. Y. Wang, 1711.09572

$$I_{\vec{\nu}}^{\mathrm{aux}}(D,\vec{s},\eta) = \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{(\mathcal{D}_1 - \lambda_1 \eta + \mathrm{i}0^+)^{\nu_1} \cdots (\mathcal{D}_K - \lambda_K \eta + \mathrm{i}0^+)^{\nu_K}}$$

- $\lambda_i \geq 0$ (typically 0 or 1), an auxiliary mass if $\lambda_i > 0$
- Analytical function of η
- Physical result obtained by (causality)

$$I_{\vec{\nu}}(D,\vec{s}) \equiv \lim_{\eta \to \mathrm{i}0^-} I_{\vec{\nu}}^{\mathrm{aux}}(D,\vec{s},\eta)$$

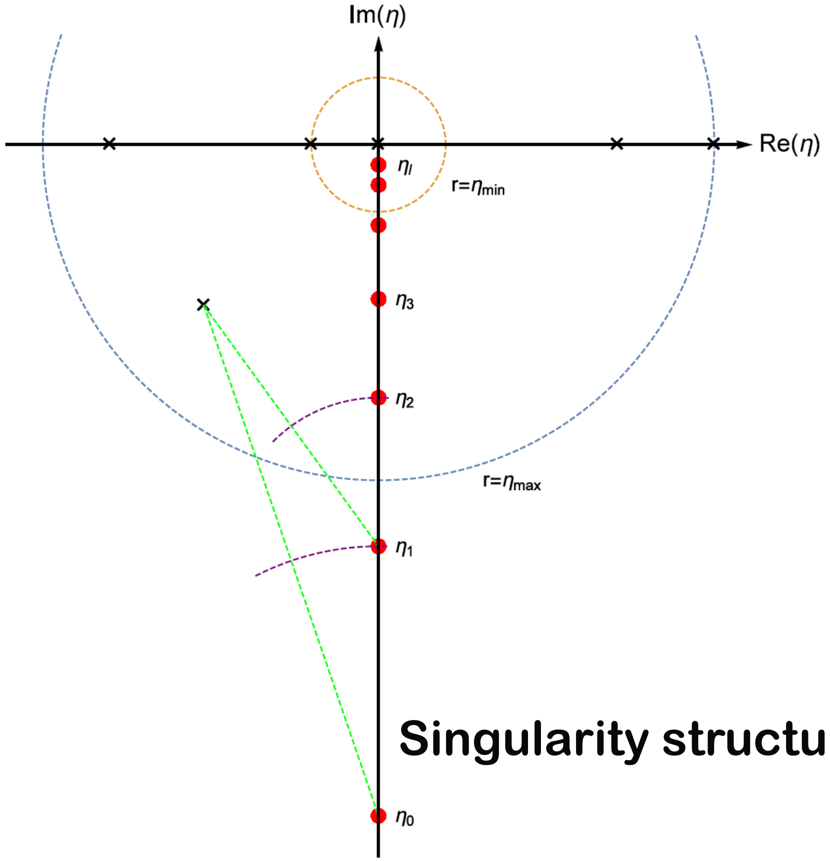
- 1) Setup η-DEs; 2) Calculate boundary conditions; 3) Solve η-DEs

➢ η-DEs for MIs in auxiliary family using IBP

$$\frac{\partial}{\partial \eta} \vec{I}^{\mathrm{aux}}(D,\vec{s},\eta) = A(D,\vec{s},\eta) \vec{I}^{\mathrm{aux}}(D,\vec{s},\eta)$$

Flow of auxiliary mass

➢ Solve ODEs of MIs



Singularity structure

$$\frac{\partial}{\partial\eta}\vec{I}^{\mathrm{aux}}(D,\vec{s},\eta) = A(D,\vec{s},\eta)\vec{I}^{\mathrm{aux}}(D,\vec{s},\eta)$$

- If $\vec{I}^{aux}(D,\vec{s},\infty)$ is known , solving ODEs numerically to obtain $\vec{I}^{aux}(D,\vec{s},\mathrm{i}0^{-})$
- A well-studied mathematical problem

Step1: Asymptotic expansion at $\eta=\infty$
Step2: Taylor expansion at analytical points
Step3: Asymptotic expansion at $\eta=0$

- Efficient to get high precision :
  ODEs, known singularity structure

Boundary values at $\eta \to \infty$

- **Nonzero integration regions as $\eta \to \infty$**
  - Linear combinations of loop momenta: $\mathcal{O}(\sqrt{|\eta|})$ or $\mathcal{O}(1)$

Beneke, Smirnov, 9711391
Smirnov, 9907471

- **Simplify propagators at $\eta \to \infty$**
  - ℓ_L is the $\mathcal{O}(\sqrt{|\eta|})$ part of loop momenta
  - ℓ_S is the $\mathcal{O}(1)$ part of loop momenta
  - p is linear combination of external momenta

$$\frac{1}{(\ell_{\mathrm{L}}+\ell_{\mathrm{S}}+p)^2-m^2-\kappa\,\eta} \sim \frac{1}{\ell_{\mathrm{L}}^2-\kappa\,\eta}$$

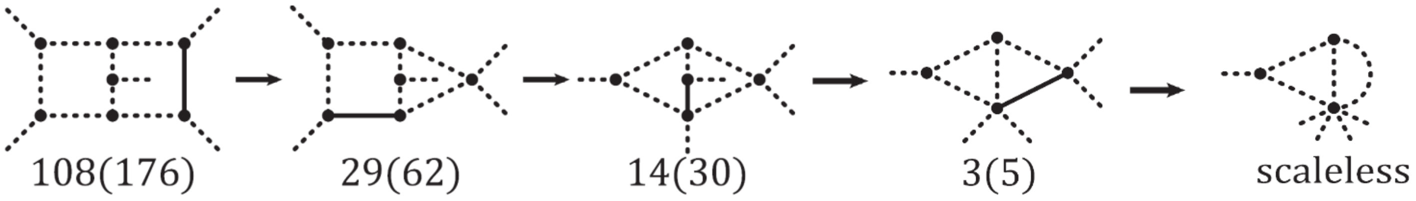
  - Unchange if $\ell_L = 0$ and $\kappa = 0$

- **Boundary FIs after simplification**
  1. Vacuum integrals
  2. Simpler FIs with less denominators, if all loop momenta are $\mathcal{O}(1)$

Iterative strategy

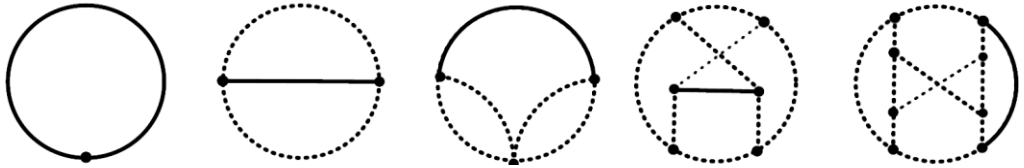
- **For boundary FIs with less denominators:**
  - Calculate them again use AMF method, even simpler boundary FIs as input (besides vacuum integrals)

X. Liu, YQM, 2107.01864



  - Eventually, leaving only (single-mass) vacuum integrals as input
  - Kinematic information can be recovered by linear algebra!

- **Typical single-mass vacuum MIs**



Baikov, Chetyrkin, 1004.1153
Lee, Smirnov, Smirnov, 1108.0732
Georgoudis, et. al., 2104.08272

  - Much simpler to be calculated
  - The same number of loops and spacetime dimensions

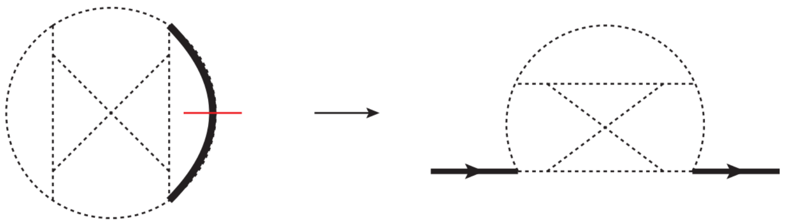
From vacuum integrals to p-integrals

➢ A family of single-mass vacuum integrals

$$I_{\vec{\nu}}(D,m^2) = \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{(\mathcal{D}_1 + \mathrm{i}0^+)^{\nu_1} \cdots (\mathcal{D}_K + \mathrm{i}0^+)^{\nu_K}}$$

$$\mathcal{D}_1 = \ell_1^2 - m^2 + \mathrm{i}0^+$$

- m^2: the only scale. Can choose $m^2 = 1$



➢ Propagator (p-)integrals

$$\widehat{I}_{\vec{\nu}'}(\ell_1^2) = \int \left(\prod_{i=2}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \right) \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{\mathcal{D}_2^{\nu_2} \cdots \mathcal{D}_K^{\nu_K}}$$

$$\vec{\nu}' = (\nu_2, \cdots, \nu_N)$$

$$\nu = \sum_{i=1}^{N} \nu_i$$

- As ℓ_1^2 is the only scale: $\widehat{I}_{\vec{\nu}'}(\ell_1^2) = (-\ell_1^2)^{\frac{(L-1)D}{2} - \nu + \nu_1} \widehat{I}_{\vec{\nu}'}(-1)$
- L-loop single-mass vacuum integral expressed by $(L-1)$-loop p-integral

$$I_{\vec{\nu}} = \int \frac{\mathrm{d}^D \ell_1}{\mathrm{i}\pi^{D/2}} \frac{(-\ell_1^2)^{\frac{(L-1)D}{2} - \nu + \nu_1}}{(\ell_1^2 - 1 + \mathrm{i}0^+)^{\nu_1}} \widehat{I}_{\vec{\nu}'}(-1) = \frac{\Gamma(\nu - LD/2)\Gamma(LD/2 - \nu + \nu_1)}{(-1)^{\nu_1}\Gamma(\nu_1)\Gamma(D/2)} \widehat{I}_{\vec{\nu}'}(-1)$$

From p-integrals to vacuum integrals

Z. F. Liu, YQM, 2201.11637

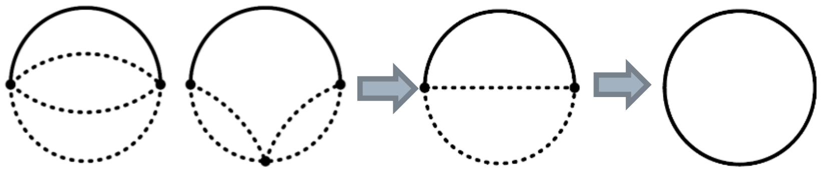
➢ Apply AMF method on $(L-1)$-loop p-integral

1) IBP to setup η-DEs

2) Single-mass vacuum integrals no more than $(L-1)$ loops as input

Single-mass vacuum integrals with L loops are determined by that with no more than $(L-1)$ loops (besides IBP)

- Boundary: 0-loop p-integrals equal 1



➢ Only IBPs are needed to determine FIs!

FIs ≜ Linear algebra

Package: AMFlow

➢ Download

X. Liu, YQM, 2201.11669

Link: https://gitlab.com/multiloop-pku/amflow

Name	Last commit	Last update
diffeq_solver	update	5 months ago
examples	update	3 months ago
ibp_interface	fix_a_bug_for_mpi_version	1 week ago
AMFlow.m	fix mass mode	2 months ago
CHANGELOG.md	update changelog	1 week ago
FAQ.md	update	6 months ago
LICENSE.md	test	7 months ago
README.md	update	3 months ago
options_summary	update	3 months ago

Sang	2202.11615
Tao	2204.06385
Chen	2204.13500
Armadillo	2205.03345
Chaubey	2205.06339
Zhang	2205.06124
Abreu	2206.03848
Bonciani	2206.10490
Feng	2207.14259
Feng	2208.04302
Chaubey	2208.05837
Sang	2208.10118
Tao	2209.15521
Sang	2210.02979
Henn	2210.13505
Badger	2210.17477
Jakubčík	2211.08446
Abreu	2211.08838
Wang	2211.13713

➢ Description

- The first (method and) package that can calculate any FI (with any number of loops, any D and $\vec{s}$, or linear propagators) to arbitrary precision, *given sufficient resource*
- Integral reduction is the bottleneck

Outline

Outline

I. Introduction

II. Auxiliary mass flow method

III. Block-triangular form

IV. CalcLoop

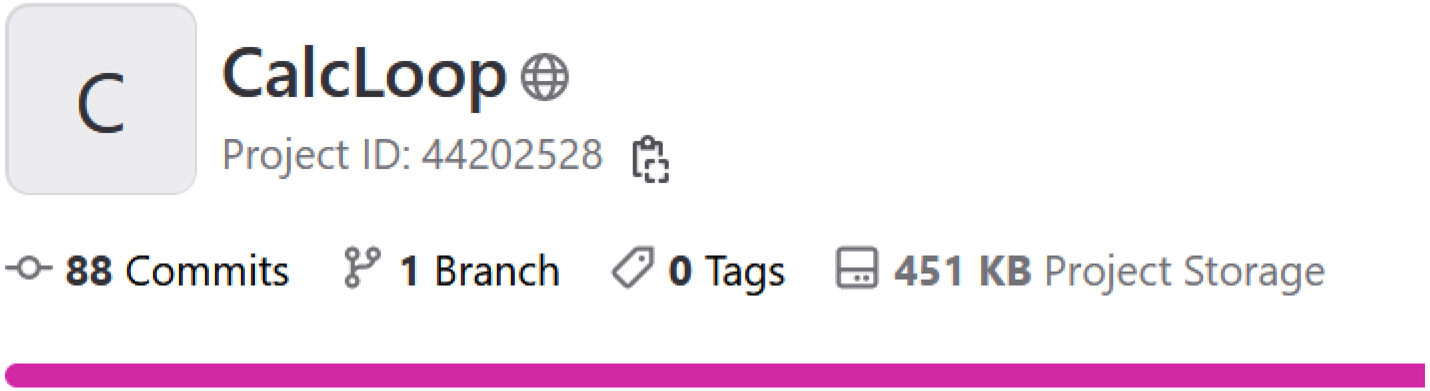
V. Summary and outlook

Package: CalcLoop

YQM, To appear

➢ Download (to be finished)

Link: https://gitlab.com/yqma/CalcLoop



➢ Description

- Automatic high order perturbative calculation

Example

- E.g. $e^+e^- \to Q\bar{Q}$ at NNLO QCD

```
files={"twoloop","tree"};
dir=FileNameJoin[{filename,"integrals",StringJoin@@files}];
```

```
ampAssoc=AmplitudeSquared[dir,amps@files[[1]],amps@files[[2]]]//CLTiming;
```

```
fiAssoc=FamilyDecomposition[FileNameJoin[{dir,dirXsection}],ampAssoc@"Amplitude",
    "CutInformation"->ampAssoc@"PhaseSpace"]//CLTiming;
```

```
resLoop2=RunAMFlow[FileNameJoin[{dir,dirXsection}],kinematics]//Expand//CLTiming;
```

Summary

- AMFlow: in principle any FI can be calculated
- Blade: improve the efficiency of IBP reduction significantly
- CalcLoop: towards fully automatic high-order perturbative calculation
- Integral reduction: bottleneck of most current cutting-edge problems, stay tuned

Thank you!

Adaptive search strategy

➢ Semi-analytic

- The number of unknowns of full-analytic block-triangular form may be too large
- Keep a subset of variables analytic ⇒ easy to search
- The integral set is the same ⇒ still very efficient
- More than one block-triangular form is needed

$$Q_i(\vec{z}) = \sum_{\mu} \tilde{Q}_i^{\mu_1 \cdots \mu_r} z_1^{\mu_1} \dots z_r^{\mu_r}$$

$$Q_i(z_{1,0}, \dots z_{r-1,0}, z_r) = \sum_{\mu_r} \tilde{Q}_i^{\mu_0} z_r^{\mu_r}$$

$$Q_i(z_{1,0}, \dots z_{r-2,0}, z_{r-1}, z_r) = \sum_{\mu_{r-1},\mu_r} \tilde{Q}_i^{\mu_{r-1}\mu_r} z_{r-1}^{\mu_{r-1}} z_r^{\mu_r}$$

… …

➢ Adaptive search

1. $n = 1$
2. Search n-variable block-triangular form within time limit T
3. If search succeed, $n++$ and go to step 2, otherwise go to step 4
4. Perform reduction by solving the most efficient linear system(i-variable)

➢ Exploit full potential of block-triangular form