Mixed Hodge module and $\mathcal{N}=2$ Coulomb branch Solution

D. Xie & D.X. Zhang

Nov 29, 2020

USTC

Motivation

Seiberg-Witten solution (1994) of 4d $\mathcal{N}=2$ Coulomb branch is one of the most important result in the study of quantum field theory!

Seiberg-Witten solution provides crucial insights for following physical and mathematical questions:

a) Quark Confinement

b) Electri-magnetic duality

c) strong-coupled dynamics of qft.

d) Instanton, four-manifold invariants, etc

e) String / M theory

While there are huge number of further developments on Seiberg – Witten theory. (Citations: 2000 +).

I always find it disappointing that not many new physical results were found from SW solution!

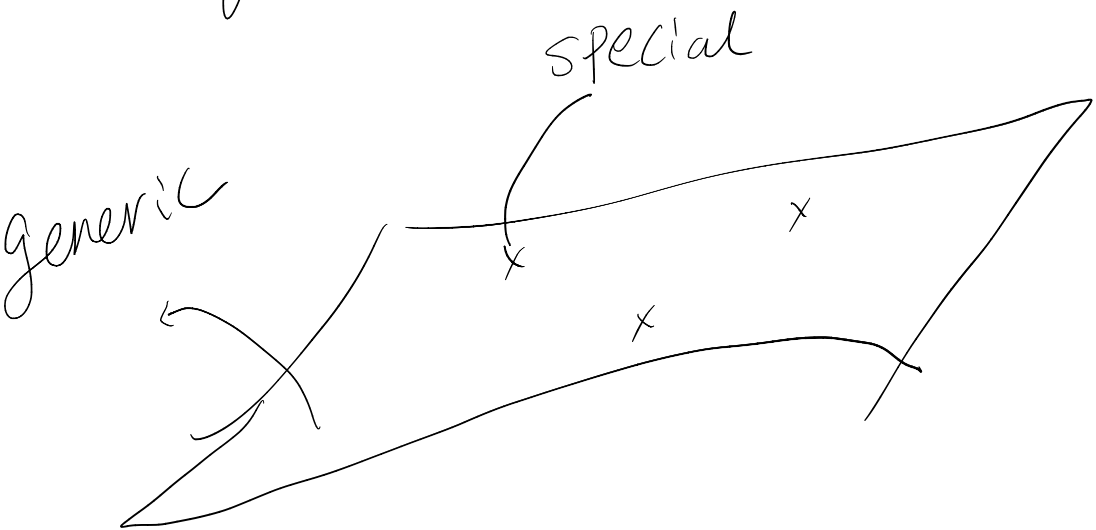
I will soon explain the difficulties in using SW solution to solve physical questions.

In the past year, together with D.X Zhang, we found a new formalism which will solve above problems!

Coulomb branch:

4d $\mathcal{N}=2$ theory has interesting moduli space of vacua

On the Coulomb branch, the low energy theory is rather interesting



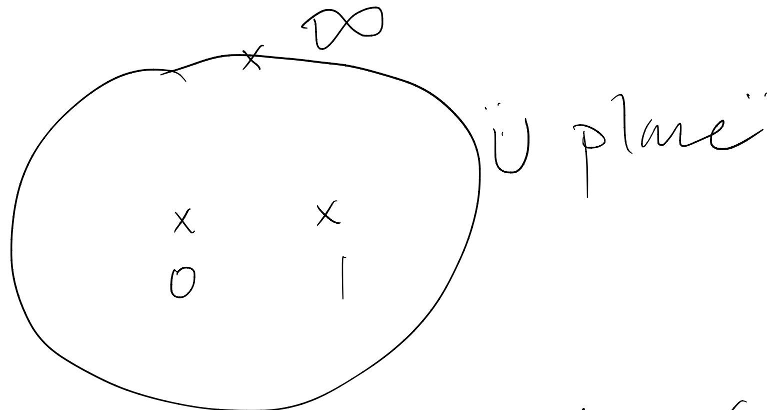
At generic point, the low energy theory is described by abelian gauge theory. (Reason it is called Coulomb branch)

At special point, the low energy theory is much more complicated: it could be an interacting SCFT, or IR gauge theory!

The goal of solving $\mathcal{N}=2$ Colomb branch is

- describe the low energy physics at every vacua!

<u>An example:</u> $N=2$ $SU(2)$ pure YM



1) Generic point: $U(1)$ gauge theory (Higgs mechanism)
one also need to determine the photon coupling

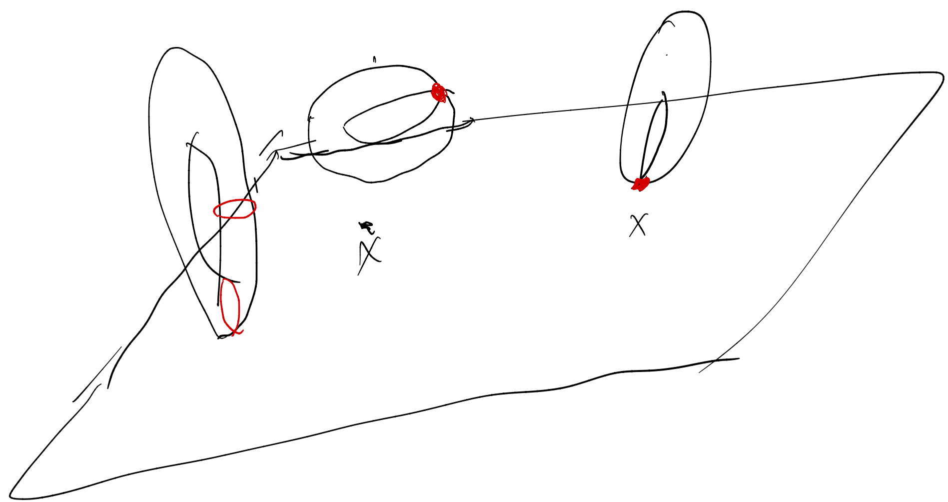
2) Special point:

At $u=0,1$: low energy theory is given by $U(1)$ with one hypermultiplet.

At $u=\infty$: it is given by the original $SU(2)$ gauge theory

The above picture of Coulomb branch is very non-trivial, as it involves strong coupled dynamics! It is given by the remarkable SW solution!

SW solution is given by attaching an extra curve fibered over the Coulomb branch.



The SW Solution for pure SU(2) SYM:

$$f(\Lambda, u) = y^2 + x + \frac{\Lambda}{x} + u = 0$$

1). if f is smooth, it describes the generic vacua, the photon coupling is given by the complex structure of the curve!

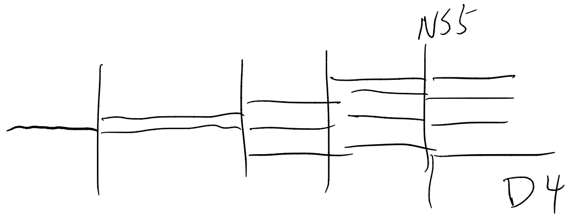
2). if f is singular, it describes the special vacua.

So SW tells us that to find the Coulomb branch solution for $\mathcal{N}=2$ solution. One just need to find a curve (or higher dimensional variety) fibered over Coulomb branch! (SW curve)

$$f(x, y, \lambda) = 0.$$

SW solutions for a large class of $\mathcal{N}=2$ theories have been found, mostly using the relation to

- string/M theory
- connection to integrable system



However, not many interesting physics are derived from these solutions:

1). At generic vacua, we do not know how to determine an abelian variety, whose complex structure is identified with the low energy couplings!

2) Related to point (1), we do not know how to distinguish the cycles which gives the electric-magnetic, and flavor central charges for BPS particles

3) There is no systematical tools to determine the physics for special vacua!

This is very disappointing, as the special values is the most interesting ones! (Confinement, SCFT)

4) In using SW solution, one needs a SW differential $\lambda_{SW}^{(\lambda)}$, and there is no method to determine it !!

We discover that Mixed Hodge structure & Mixed Hodge module will help us solving all these problems!

Hodge structure:

$(H_{\mathbb{Z}}, F^{\bullet})$. $\qquad H_{\mathbb{Z}}$ = lattice

1) $H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes \mathbb{C}$ is a complex vector space

2) $F^{\bullet}$ is an decreasing filtration (Hodge filtration)

$$F^0 \supset F^1 \supset F^2 \cdots$$

$$H_{\mathbb{C}} = F^p \oplus \overline{F^{n-p+1}}$$

for all p!

Hodge decomposition:

$$H_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q}$$

$$H^{p,q} = F^p \cap \overline{F^q}$$

An example: Cohomology of compact Kahler manifolds

$$H^k = \bigoplus_{p+q=k} H^{p,q}$$

$$F^p = \bigoplus_{r \geq p} H^{r,q}$$

Compact Riemann surface:

$$H^1(\Sigma) = H^{1,0} \oplus H^{0,1}$$

$H^{1,0}$ = holomorphic differential

This Hodge decomposition is important: it determines the complex structure!!

Polarized Hodge structure:

$$(H_{\mathbb{Z}}, F^{\cdot}, S)$$

S is a bilinear form, satisfy

$$\begin{cases} S(H^{p,q}, H^{p',q'}) = 0 & \text{if } (p',q') \neq (q,p) \\ i^{p-q} S(\psi, \overline{\psi}) > 0 & \text{for } \psi \in H^{p,q},\ \psi \neq 0 \end{cases}$$

(polarized) Mixed Hodge structure: $(H_{\mathbb{Z}}, F^{\bullet}, W_{\bullet}, S)$

$$\begin{cases} F^{\bullet}: \text{ Hodge filtration: } F^0 \supset F^1 \cdots \\ W_{\bullet}: \text{ Weight filtration } \quad W^0 \subset W^1 \subset \cdots \end{cases}$$

S: graded polarization

Solutions to four problems

1) & 2) : Given a SW solution,

$$f_\lambda(x,y)=0$$

at generic vacua, it is a smooth open variety!

Delign: Cohomology carries graded MHS

$H^1(f)$ has weight 1 piece,

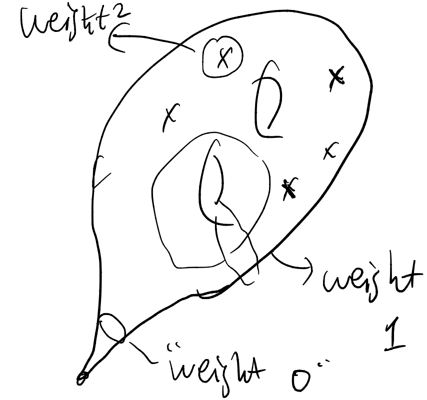
and weight 2 piece.

Weight 1: 1) Electric - magnetic charge

2) Abelian Variety

weight 2: flavor charge

3): For a Special vacua, there are two MHS



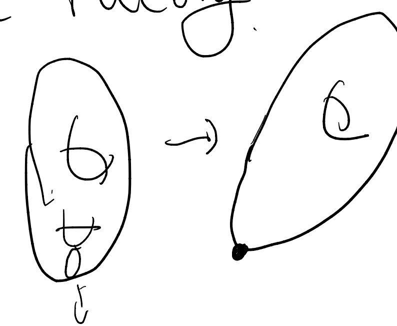
a). Deligne MHS for singular variety

weight 1: Electric-magnetic

weight 2 = flavor charge

weight 0 part = Coupled with singularity

b) The singularity is associated with the vanishing cycle, & gives the interacting SCFT or IR free gauge theory.



Associated with vanishing cycles, one can define a limiting mixed Hodge structure:

1) Hodge filtration (subtle)

2) weight filtration (monodromy)

The MHS associated with singular curve

& MHS associated with singularity

formed an exact sequence. physical

interpretation: Abelian gauge theory

Coupled with interacting theory!

An example: $f = x^3 + q^7 + 2y^3$

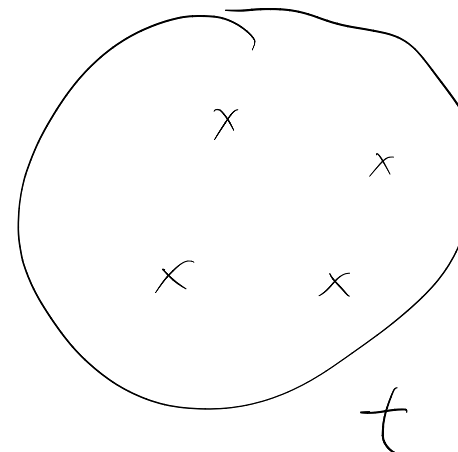
IR theory: $T \begin{array}{l} \nearrow 1 \\ \searrow 1 \end{array} \oplus U(1)^3$

Here T is an Argyres-Douglas theory with $SU(3)$ flavor symmetry

4) Seiberg-Witten differential.

MHs at $t=\infty$.

Given a SW solution,



one can consider $t \to \infty$ limit,

it also carries a MHS,

which will give us a

SW differential!

Mixed Hodge module

One can package above structures into a single object over Coulomb branch called Mixed Hodge module!!

Many unsolved problems can be solved using this formalism!! A lot of results will come soon!!

Thank you!