Exotic Consistent (1+1)D Anomalies : A Ghost Story

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Anomalies (Traditional)

- The study of anomalies has a long history.
- An important milestone: **Wess-Zumino consistency** ['71].

  — Anomalous transformation is compatible with the symmetry algebra.
- Together with the "local" properties of anomalies led to the descent equations and a cohomological classification of perturbative anomalies.

[Lectures: Stora'77, Stora'84, Zumino'85, Stora'86]
[TASI Lectures Harvey'03]

Anomalies (Traditional)

CONSEQUENCES OF ANOMALOUS WARD IDENTITIES

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Anomalies (Modern)

- Inflow paradigm: [Callan-Curtis-Harvey'85]
  - Anomaly in D dimensions is the inflow of a classical action in D+1 dimensions.
- Can we classify anomalies using inflow? [Wen'13, …]
  - Inadequate for e.g. Conformal/Weyl anomalies.
  - Assumed to be okay for 't Hooft anomalies (of internal symmetry or spacetime symmetry).

’t Hooft anomalies

- ’t Hooft anomaly is a controlled breaking of symmetries in QFT
- The partition function on a background Φ (e.g. metric, gauge field, …) transforms under the background transformation Λ (e.g. diffeomorphism, gauge, …) with an **anomalous phase**

$$Z[\Phi^\Lambda] = Z[\Phi] \; \exp(i\alpha[\Phi, \Lambda])$$

- Inflow paradigm: $\exists$ D+1 dimensional bulk classical action $S_{\text{bulk}}[\Phi]$

$$Z_{\text{bulk}}[\Phi] = \exp(-S_{\text{bulk}}[\Phi]) \; , \qquad Z_{\text{bulk}}[\Phi^\Lambda] = Z_{\text{bulk}}[\Phi] \; \exp(-i\alpha[\Phi, \Lambda])$$

- Bulk + boundary is free from anomaly.

Anomalies (Modern)

- Classification of D+1 dimensional bulk phases:
  - In $D+1=1,2,3$ dimensions, bulk phases are classified by group cohomology.

    [Chen-Gu-Liu-Wen'11, Hung-Wen'12, Wen'13, …]
  - In $D+1=1,\ \cdots,6$ dimensions, bulk phases are classified by cobordism (Some results assume bulk reflection-positivity).

    [Kapustin'14, Freed-Hopkins'16, Yonekura'18, …]
  - With topological order. (partition function transforms more generally)

    [Kong-Wen'14, Lan-Wang-Wen'14, Witten'15, Ji-Wen'19, …]

(1+1)D QFT with U(1) symmetry

- Restrict to QFTs defined on Riemannian geometry. (Non-spin QFT)
  Partition functions are scalars under the symmetries.

$$\Phi : \quad g,\, \omega,\, R, \quad A,\, F.$$

- (1+1)D anomalies are inflowed by bulk (2+1)D Chern-Simons actions
  - Gravitational anomaly: $S_{\text{bulk}} = \frac{i\kappa_{R^2}}{81\pi}\int CS(\omega)\,, \quad CS(\omega) = \omega d\omega + \frac{2}{3}\omega^3\,.$
  - U(1) anomaly: $S_{\text{bulk}} = \frac{i\kappa_{F^2}}{4\pi}\int CS(A)\,, \quad CS(A) = AdA + \frac{2}{3}A^3\,.$

(1+1)D QFT with U(1) symmetry

- For Chern-Simons actions to be well-defined, levels are quantized.
- Translates to quantization of anomaly coefficients in (1+1)D

$$\kappa_{R^2} \in 8\mathbb{Z} \quad \text{and} \quad \kappa_{F^2} \in 2\mathbb{Z}$$

- In CFT, the anomaly coefficients are related to $T \times T$ and $J \times J$ OPE coefficients

$$\kappa_{R^2} = c_- = c - \bar{c} \quad \text{and} \quad \kappa_{F^2} = k_- = k - \bar{k}$$

Holomorphic bc Ghosts

- Some basics of the holomorphic bc system:
  - Free Grassmann fields: b, c.
  - Action: $S = \frac{1}{2\pi} \int d^2 z \, b \bar{\partial} c.$
  - (Holomorphic) conformal weights: $h_b = \lambda, \quad h_c = 1 - \lambda.$
  - U(1) ghost number symmetry: $J =: bc :.$
  - Ghost number / U(1) charge: $q_b = -1, \quad q_c = +1.$
  - Stress tensor: $T = (1 - \lambda) : (\partial b) c : - \lambda : b \partial c :.$

Holomorphic bc Ghosts

- When $\lambda \in \mathbb{Z}$, b and c have integer spins, and can be defined on arbitrary Riemann surfaces (without specifying the spin structure).
- Central charge: $c_- = 1 - 3(2\lambda - 1)^2 \in -2 + 24\mathbb{Z}$.
- U(1) level: $k_- = 1$.
- Incompatible with the quantization of Chern-Simons levels. Recap:
  - $c_- = c - \bar{c} \in 8\mathbb{Z}$ and $k_- = k - \bar{k} \in 2\mathbb{Z}$.

Plan

1. Classify 't Hooft anomalies from purely (1+1)D perspective.
2. Verify that the holomorphic *bc* ghost's anomalies fit in.
3. Discussion & future directions

Let us go back to our traditional roots and study the Wess-Zumino consistency condition.

Finite Wess-Zumino Consistency

- Anomalous phase:

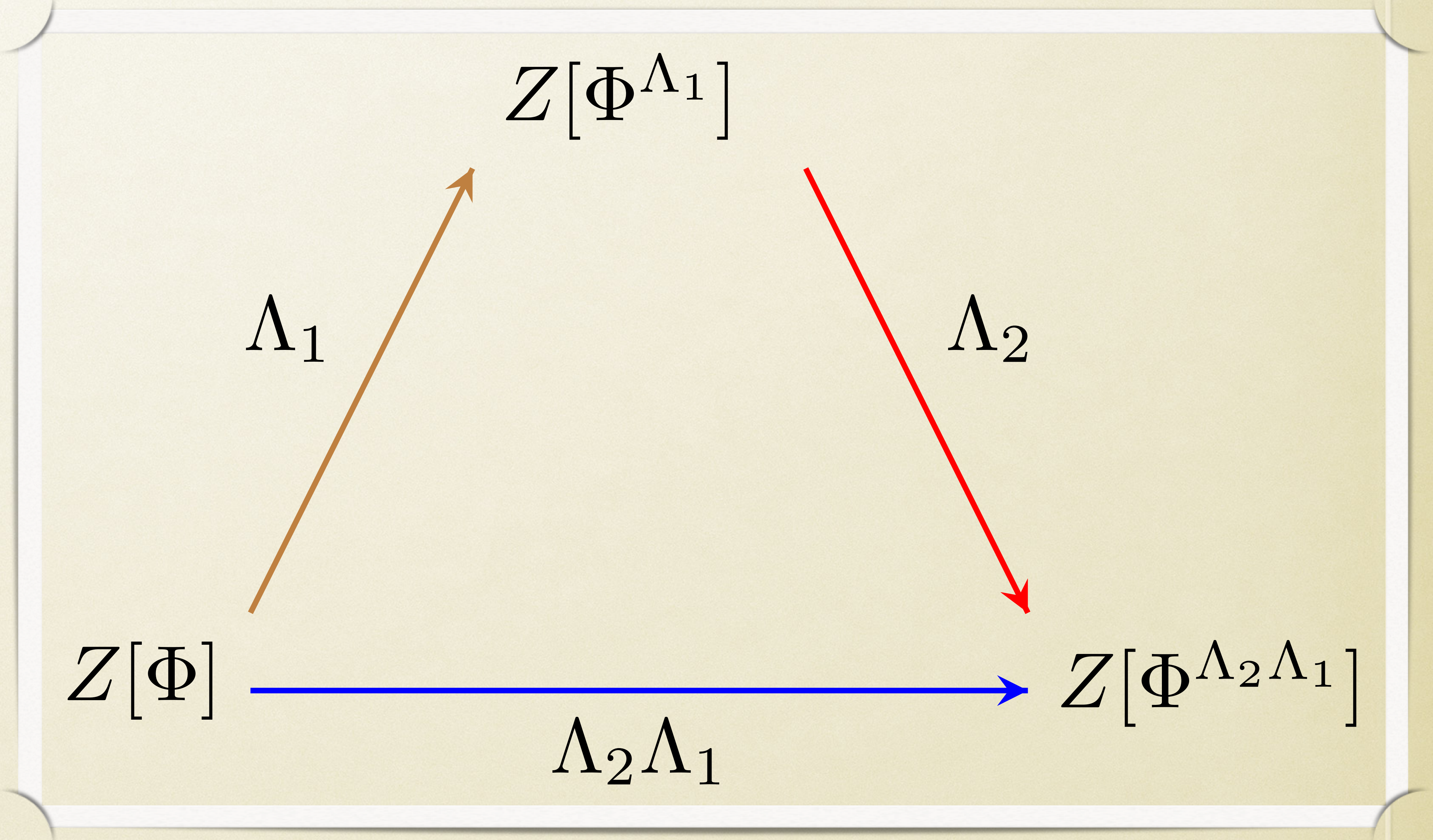
$$Z[\Phi^\Lambda] = Z[\Phi]\ \exp(i\alpha[\Phi,\Lambda]).$$

- Diagram on the right commutes:

$$\alpha[\Phi,\Lambda_2\Lambda_1] - \alpha[\Phi^{\Lambda_1},\Lambda_2] - \alpha[\Phi,\Lambda_1] \in 2\pi\mathbb{Z}.$$

- Infinitesimal Λ leads to original Wess-Zumino.

[Book: Azcarraga-Izquierdo'95]



Locality

1. The anomalous phase $\alpha[\Phi, \Lambda]$ is a local functional of Φ.
2. For infinitesimal Λ, the anomalous phase $\alpha[\Phi, \Lambda]$ is a local functional of both Φ and Λ, and vanishes on the trivial background.

- On trivial background, the current J^μ is conserved away from any other operator insertions. In correlation functions,

$$\langle \nabla_\mu J^\mu(x) \cdots \rangle = \text{contact terms.}$$

- Had locality been false, Ward identities would violate this structure.

Locality

- The two locality conditions can be stated more precisely as
  - $\mathscr{G}$: Space of background transformations Λ.
  - $\mathscr{G}_n$: Connected components, $\mathscr{G}_0$ contains the trivial transformation.
  - $\mathscr{A}_i\,[\Phi, \Lambda]$: Basis of local functionals that vanish when $\Phi = 0$.
  - The anomalous phase admits an expansion in the basis $\mathscr{A}_i\,[\Phi, \Lambda]$ as

$$\alpha[\Phi, \Lambda] = \sum_i \kappa_i(n)\,\mathscr{A}_i[\Phi, \Lambda] + \theta(n), \qquad \theta(0) = 0.$$

Gravitational Anomaly

- Consider CFT on flat torus. Large diffeomorphism: $SL(2,\mathbb{Z})$.
- By locality, the anomalous phases are constants. (No local functional $\mathscr{A}_i[\Phi, \Lambda]$ could be written down.)

$$Z\left(\frac{a\tau+b}{c\tau+d}, \frac{a\bar{\tau}+b}{c\bar{\tau}+d}\right) = Z(\tau,\bar{\tau})\ e^{i\theta(a,b,c,d)}.$$

- (Assume that the partition function does not vanish identically for all torus moduli τ.)

Gravitational Anomaly

- Solutions to finite Wess-Zumino Consistency

$$= \text{Group cohomology } H^1(PSL(2,\mathbb{Z}), \mathrm{U}(1)) \cong \mathbb{Z}_6.$$

[CC-Lin, also in Seiberg-Tachikawa-Yunikura'18]

- Solution corresponds to the generator of $\mathbb{Z}_6$:
  - General anomalous phases $\theta(a,b,c,d)$ are determined by θ_S and θ_T

$$Z\left(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}\right) = Z(\tau,\bar{\tau})\ e^{i\theta_S}\,, \quad Z\left(\tau+1, \bar{\tau}+1\right) = Z(\tau,\bar{\tau})\ e^{i\theta_T}$$

  - $\theta_S = \pi \mod 2\pi, \qquad \theta_T = \frac{\pi}{3} \mod 2\pi \quad \Rightarrow \quad c_- = -\frac{12}{\pi}\theta_T \in -4 + 24\mathbb{Z}\,.$

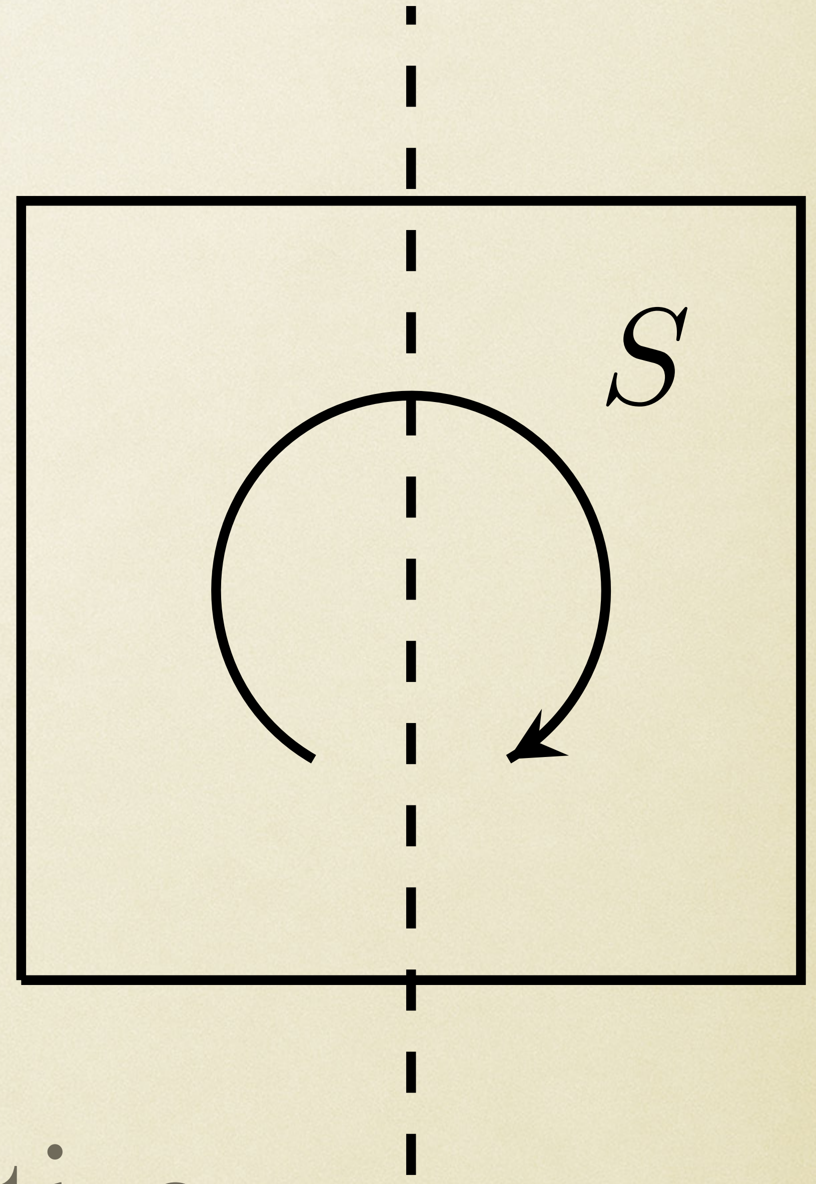
Gravitational Anomaly

- For any odd element of $\mathbb{Z}_6$

$$\theta_S \equiv \pi \mod 2\pi, \qquad c_- \in 8\mathbb{Z} + 4.$$

- This implies that at the S-invariant point (square torus)

$$Z(\tau = i, \bar{\tau} = -i) = 0.$$



- The square torus is reflection symmetric. If the CFT is reflective positivity, the partition function on square torus is positive. Hence,

$$c_- \in 8\mathbb{Z} \quad \text{for reflection positive CFT.}$$

Gravitational Anomaly

- What if the partition function vanishes identically for all τ? Consider torus one-point function instead.
- The solution to the finite Wess-Zumino condition is given by the group cohomology

$$H^1(\mathrm{SL}(2,\mathbb{Z}), \mathrm{U}(1)) \cong \mathbb{Z}_{12}.$$

- If at least one torus one-point function does not vanish identically, then

$$c_- \in 2\mathbb{Z}.$$

- For $c_- \in 4\mathbb{Z} + 2$, the CFT must contain Grasmann-valued operators.

U(1) Anomaly

- Consider CFT on flat torus.
- Space of U(1) gauge transformations λ has many connected components, labeled by the winding numbers around the non-contractible cycles $\mathscr{C}_i$

$$\vec{m}[\lambda] = \frac{1}{2\pi}\int_{\vec{\mathscr{C}}} d\lambda.$$

U(1) Anomaly

- By locality, the general form of the anomalous phase is

$$\alpha[A,\lambda] = -\frac{\kappa(\overrightarrow{m}[\lambda])}{4\pi}\int_\Sigma d\lambda\, A + \sum_i \frac{\kappa_i'(\overrightarrow{m}[\lambda])}{2\pi}\int_\Sigma f_i(\lambda)\, F + \theta(\overrightarrow{m}[\lambda]).$$

- $f_i(\lambda)$ is a basis of periodic functions

$$f_i(\lambda + 2\pi) = f_i(\lambda)\,.$$

- Restrict to flat gauge orbits, $F = 0$, so that κ_i' do not contribute.

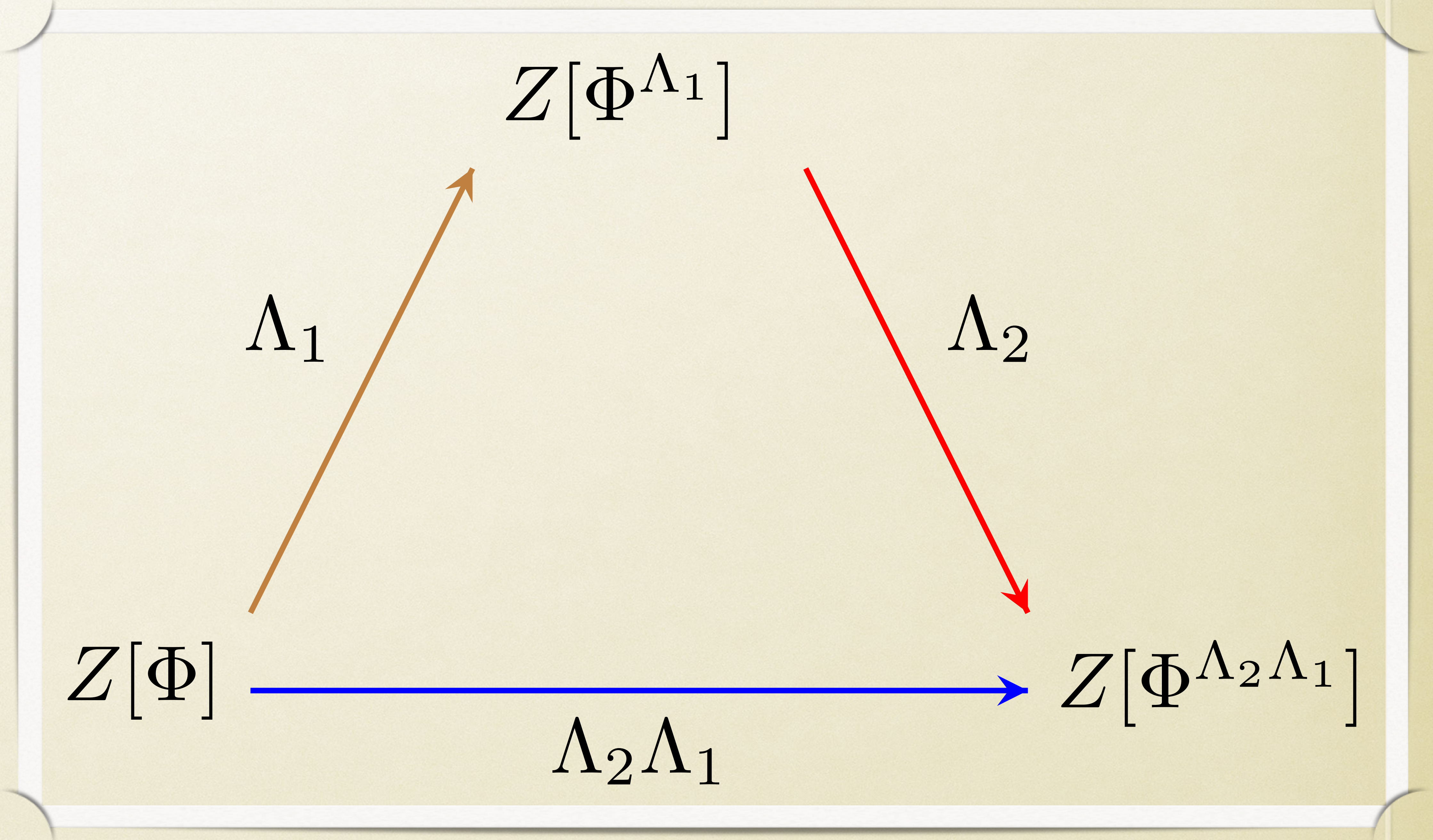
U(1) Anomaly

- Finite Wess-Zumino:

$$\left[-\frac{\kappa(\vec{m}_{12})}{4\pi}\int_\Sigma d(\lambda_1+\lambda_2)\,A+\theta(\vec{m}_{12})\right]$$

$$-\left[\frac{\kappa(\vec{m}_2)}{4\pi}\int_\Sigma d\lambda_2(A+d\lambda_1)+\theta(\vec{m}_2)\right]$$

$$-\left[\frac{\kappa(\vec{m}_1)}{4\pi}\int_\Sigma d\lambda_1\,A+\theta(\vec{m}_1)\right]\equiv 0 \mod 2\pi.$$



U(1) Anomaly

- A bit of manipulation:

$$\left[-\pi\kappa(\vec{m}_2)\,\vec{m}_1\cdot\Omega\cdot\vec{m}_2+\theta(\vec{m}_{12})-\theta(\vec{m}_1)-\theta(\vec{m}_2)\right]$$

$$-\left[\frac{\kappa(\vec{m}_{12})-\kappa(\vec{m}_1)}{4\pi}\int_\Sigma d\lambda_1\,A\right]$$

$$-\left[\frac{\kappa(\vec{m}_{12})-\kappa(\vec{m}_2)}{4\pi}\int_\Sigma d\lambda_2\,A\right]\equiv 0 \mod 2\pi.$$

- Ω is intersection matrix.
- We used

$$\frac{1}{4\pi^2}\int_\Sigma d\lambda_1 d\lambda_2=\vec{m}_1\cdot\Omega\cdot\vec{m}_2.$$

- A, λ_1, λ_2 are arbitrary, so 2nd and 3rd lines vanish.
- $\kappa(\vec{m})=\kappa_{F^2}$ is constant.

U(1) Anomaly

- We are left with

$$-\pi\kappa_{F^2}\,\vec{m}_1\cdot\Omega\cdot\vec{m}_2+\theta(\vec{m}_{12})-\theta(\vec{m}_1)-\theta(\vec{m}_2)\equiv 0 \mod 2\pi.$$

- On a torus,

$$\vec{m}=(m_a,m_b),\qquad \Omega=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

- This can be explicitly solved:

$$\theta(m_a,m_b)=\theta(1{,}0)m_a+\theta(0{,}1)m_b-\pi\kappa_{F^2}m_am_b,\qquad \kappa_{F^2}\in\mathbb{Z}\,.$$

- Mixed fWZ of U(1) + modular transforms gives $\theta(1{,}0)=\theta(0{,}1)=\pi\kappa_{F^2}$.

Concluding Remarks

Summary

- Inflow paradigm + Chern-Simons level quantization:

$$\kappa_{R^2} = c_- \in 8\mathbb{Z}, \qquad \kappa_{F^2} = k_- \in 2\mathbb{Z}\,.$$

- Finite Wess-Zumino:

$$\kappa_{R^2} = c_- \in 2\mathbb{Z}, \qquad \kappa_{F^2} = k_- \in \mathbb{Z}\,.$$

- Saturated by holomorphic *bc*. Can explicitly verify anomalous phases.
- CFT with $c_- \notin 4\mathbb{Z}$ must include ghosts, and with $c_- \notin 8\mathbb{Z}$ **cannot** be reflection positive.

New (2+1)D Classical Action?

- Is there a new (2+1)D classical action responsible for inflowing the anomalies $c_- = -2$ and $k_- = 1$?
- A classification of (2+1)D non-spin invertible topological order using braided fusion categories by Kong-Wen shows that $c_- \in 2\mathbb{Z}$.
- However, there is no know non-spin invertible topological order that realizes the minimal chiral central charge $c_- = \pm 2$.

[Kong-Wen'14]

Finite Wess-Zumino in Higher D

- T^D: mapping class group is $SL(D, \mathbb{Z})$. However, $H^1(SL(D, \mathbb{Z}), U(1))$ is trivial for all $D \geq 3$. No global gravitational anomaly. There could still be nontrivial global $U(1)$ anomaly.
- S^D: There exist large diffeomorphisms in $D \geq 6$. For example, the mapping class group is $\mathbb{Z}_{28}$ in $D = 6$. fWZ gives $H^1(\mathbb{Z}_{28}, U(1)) \cong \mathbb{Z}_{28}$ which agrees with the inflow by (6+1)D Chern-Simons [Witten'85]. Odd D? fWZ for global $U(1)$ anomaly?
- More general manifolds…

Mixed Gravitational Anomaly

- The ghost number current J is not conserved nontrivial backgrounds:

$$\langle \nabla^{\mu} J_{\mu}(x) \rangle \supset \kappa_{FR} R.$$

- Inflow by a mixed Chern-Simons action:

$$S_{\text{bulk}} = -\frac{2i\kappa_{FR}}{\pi} \int_{\mathscr{M}_3} A \wedge dA_R \ , \qquad A_R \Big|_{\mathscr{M}_2} = \frac{1}{4} \varepsilon^{ab} \omega_{ba} \ .$$

A_R: (2+1)D SO(2) gauge field.

- CS level quantization: $\kappa_{FR} \in \dfrac{\mathbb{Z}}{4}$, saturated by holomorphic *bc*. fWZ quantization?

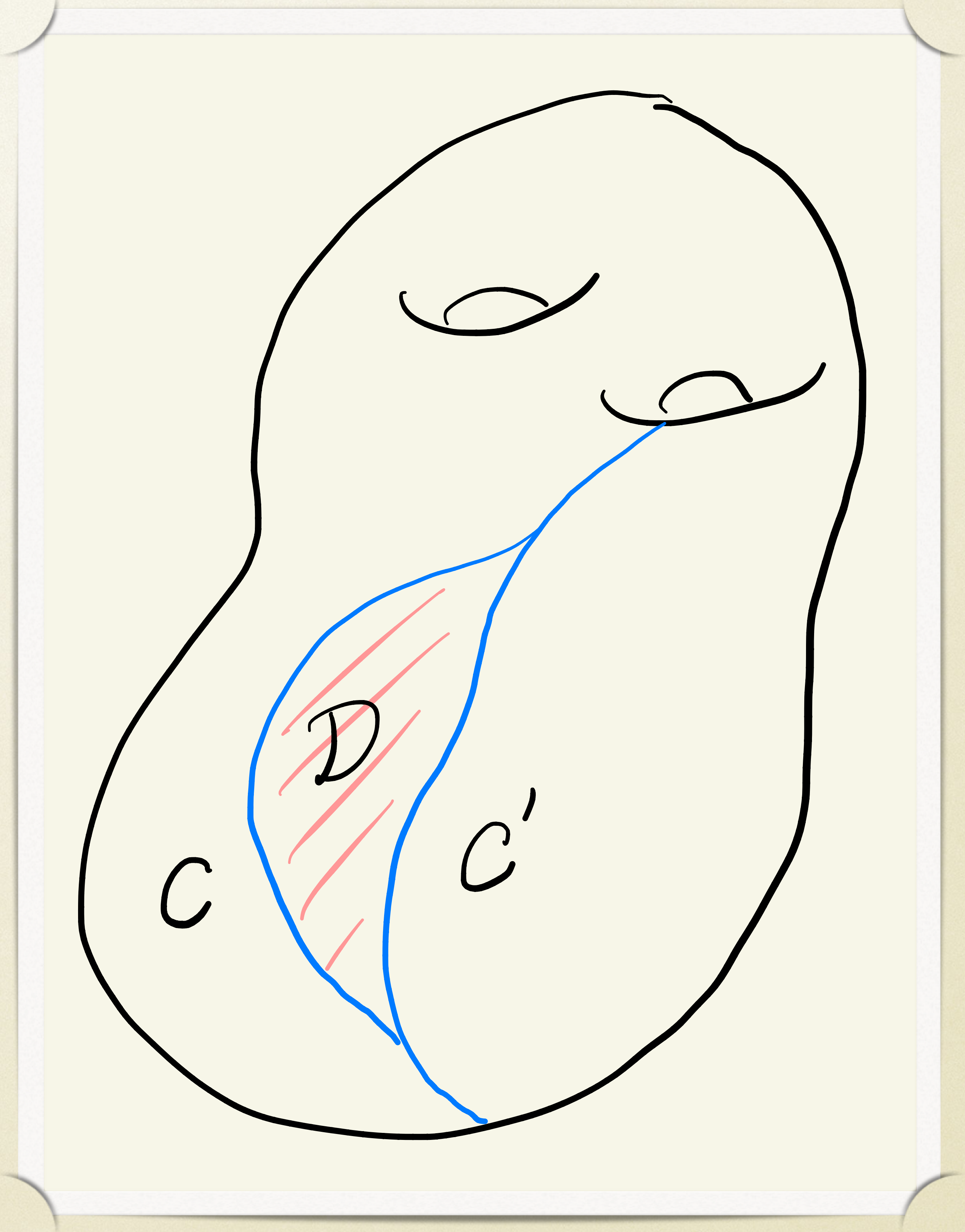
Thank you

Isotopy Anomaly

- From the current J^μ, construct U(1) symmetry defect by integration along a curve $\mathscr{C}$

$$\mathscr{L}_\eta(\mathscr{C}) \;=: \exp\left[i\eta \oint_{\mathscr{C}} ds\, n_\mu J^\mu \right]:.$$

- η labels U(1) elements, has periodicity 1.
- Deform $\mathscr{C} \to \mathscr{C}'$. Swipes over domain $\mathscr{D}$, $\partial\mathscr{D} = \mathscr{C}' - \mathscr{C}$.



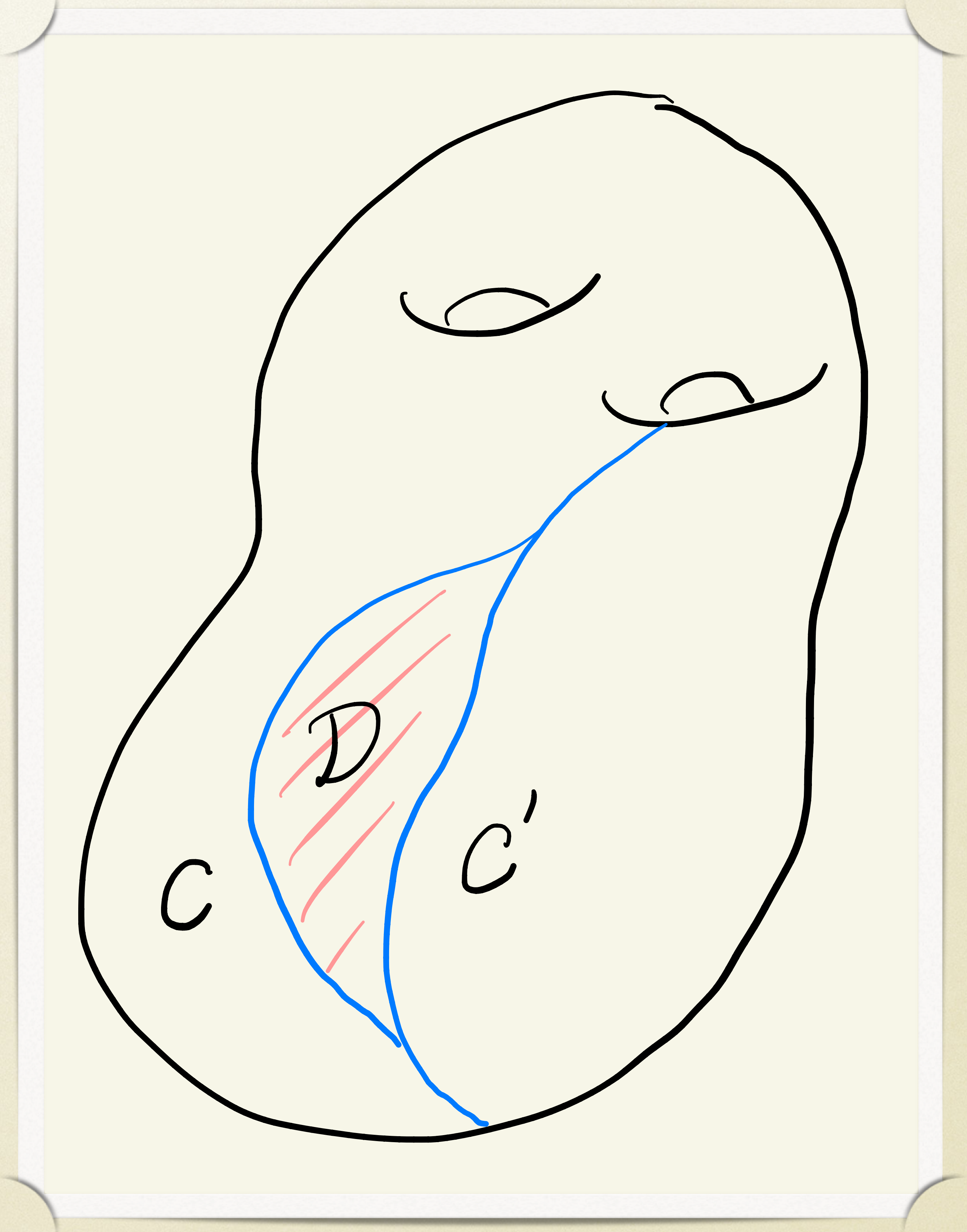
Isotopy Anomaly

- The deformation gives an anomalous phase:

$$: \exp\left[i\eta \oint_{\partial\mathscr{D}=\mathscr{C}'-\mathscr{C}} ds\, n_\mu J^\mu \right] :$$

$$= : \exp\left[i\eta \int_{\mathscr{D}} d^2x \sqrt{g}\, \nabla_\mu J^\mu \right] : \qquad \text{(Divergence theorem)}$$

$$= \exp\left[i\eta\kappa_{FR} \int_{\mathscr{D}} d^2x \sqrt{g} R \right] \qquad \text{(Anomalous conservation)}$$



Isotopy Anomaly

- Anomalous phase:

$$\exp\left[i\eta\kappa_{FR}\int_{\mathscr{D}} d^2x\sqrt{g}R\right].$$

- The defect line $\mathscr{L}_\eta$ is topological on flat space but not on curved space, due to mixed gravitational anomaly κ_{FR} .
- This is the isotopy anomaly.