On Galilean Conformal Bootstrap

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Based on the work with Peng-xiang Hao, Han Liu and Zhe-fei Yu, 2011.11092.

CFT

A conformal field theory (CFT) is a conformal invariant field theory quantum mechanically.

- RG-flow fixed points: phase transitions, ...
- AdS/CFT correspondence

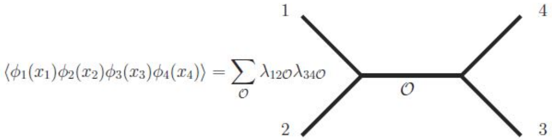
A CFT is defined by its spectrum and OPE coefficients

Primary operators: $\{\mathcal{O}_i\}$, and their scaling dimensions $\{\Delta_i\}$

OPE coefficients: $\mathcal{O}_i\mathcal{O}_j\mathcal{O}_k \sim c_{ijk}$

Conformal block

Consider a 4-point function in a CFT



$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}} \frac{G_{\mathcal{O}}(u,v)}{(x_{12})^{2\Delta}(x_{34})^{2\Delta}},$$

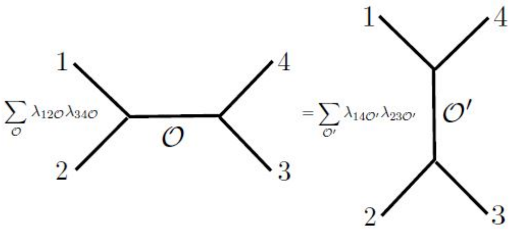
where u, v are the conformal invariant cross-ratios.
$G_{\mathcal{O}}(u,v)$ is called the conformal block (CB), determined by the conformal symmetry. In terms of

$$g(u,v) \equiv \sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}} G_{\mathcal{O}}(u,v),$$

the 4-point function can be written as

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{g(u,v)}{(x_{12})^{2\Delta}(x_{34})^{2\Delta}}$$

Crossing symmetry



The crossing equation

$$g(u,v) = \left(\frac{u}{v}\right)^{\Delta} g(v,u)$$

which could be written as

$$1 = \sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}} \left(\frac{v^{\Delta} G_{\mathcal{O}}(u,v) - u^{\Delta} G_{\mathcal{O}}(v,u)}{u^{\Delta} - v^{\Delta}} \right)$$

Conformal bootstrap

For what spectra and OPE coefficients can we find to satisfy the crossing equation?

The conformal bootstrap aims to constrain the CFT data by using the crossing symmetry and unitarity.

A completely nonperturbative tool to study field theories!

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The conformal bootstrap was proposed in 1970s. Ferrara et.al. (1973)Polyakov(1974)
It was successfully applied to 2D minimal models in 1980s. Belavin et.al. (1984)

Modern conformal bootstrap

The renaissance of conformal bootstrap in the past decade started from the numeric study of 3D Ising models. Rattazzi et.al. (2008)

Later on, it was applied to the study of AdS/CFT. Heemskerk et.al. (2009)

Analytic conformal bootstrap Fitzpatrick et.al. (2013), Alday et.al. (2013), Komargodski et.al. (2013)... Caron-Huot(2017)...

CFT in various dimensions, SCFT, ...

It would be interesting to extend conformal bootstrap program to field theories with other conformal-like symmetries.

Schrödinger symmetry W. Goldberger et.al. 1412.8507

Warped conformal symmetry in 2D, Anisotropic Galilean conformal symmetry in 2D, ...

In this talk, I would like to report our study of the Galilean conformal bootstrap in the past two years.

Multiplets in Galilean CFT

Galilean conformal symmetry

Typical feature: in any dimensions, it is generated by an infinite dimensional algebra, being called Galilean conformal algebra (GCA) Bagchi and Gopakumar 0902.1385

Global part: could be obtained by a non-relativistic contraction of the conformal symmetry M. Negro et.al. (1997), J. Lukierski et.al. 0511259

- Translations, Isotropic scaling, Galilean transformations
- Analogues of special conformal transformations,

The full GCA could be obtained by taking the non-relativistic limit of conformal Killing equations, and is the maximal subset of non-relativistic conformal isometries C. Duval and P. Horvathy 0904.0531, D. Martelli and Y. Tachikawa, 0903.5184

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Finite GCA appears in non-relativistic Navier-Stokes equation Fouxon and Oz (2008), Bhattacharyya et.al. (2008)
2D GCA in non-equilibrium statistical systems Henkel et.al. (2006)
In particular, 2D GCA is isotropic to BMS_3

$\hookrightarrow$ Flat holography Bagchi 1006.3354, · · ·

2D Galilean conformal symmetry

Symmetry:

$$x \to f(x), \qquad y \to f'(x)y.$$
$$x \to x, \qquad y \to y + g(x).$$

The symmetry is generated by the Galilean conformal algebra Bagchi et.al. 0912.1090

$$[L_n, L_m] = (n-m)L_{n+m} + C_T n(n^2-1)\delta_{n+m,0},$$
$$[L_n, M_m] = (n-m)M_{n+m} + C_M n(n^2-1)\delta_{n+m,0},$$
$$[M_n, M_m] = 0.$$

Global subalgebra: $\{L_{\pm 1}, L_0, M_{\pm 1}, M_0\}$

Cartan subalgebra: $\{L_0, M_0\}$

Non-relativistic limit

This algebra could be obtained by taking the limit of 2D CFT:

$$z = x + \epsilon y, \qquad \bar{z} = x - \epsilon y$$

with $\epsilon \to 0$. This is equivalent to the non-relativistic limit $v \sim \epsilon$. Consequently,

$$\mathcal{L}_n + \bar{\mathcal{L}}_n \to L_n, \qquad \epsilon(\mathcal{L}_n - \bar{\mathcal{L}}_n) \to M_n$$

where $\mathcal{L}_n, \bar{\mathcal{L}}_n$ are Virasoro generators, and

$$C_T = \frac{c + \bar{c}}{12}, \qquad C_M = \epsilon \frac{c - \bar{c}}{12}$$

In order to have nonzero C_T and C_M, the parent CFT must have very large central charges with opposite sign. This implies that the parent theory is not unitary!

Primary operators

The local operators in a GCFT$_2$ can be labelled by the eigenvalues (Δ, ξ) of the generators of the Cartan subalgebra (L_0, M_0)

$$[L_0, \mathcal{O}(0,0)] = \Delta\mathcal{O}(0,0), \quad [M_0, \mathcal{O}(0,0)] = \xi\mathcal{O}(0,0).$$

Δ: conformal weight $\qquad \xi$: charge

The highest weight representations require the primary operators satisfy

$$[L_n, \mathcal{O}(0,0)] = 0, \quad [M_n, \mathcal{O}(0,0)] = 0, \qquad \text{for } n > 0.$$

The tower of descendant operators can be got by acting L_{-n}, M_{-n} with $n > 0$ on the primary operators. The primary operator and its descendants form a module.

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The descendant states must have negative norm states, reflecting the fact that the theory is not unitary. For example, for the level-1 states $L_{-1}|\Delta, \xi\rangle, M_{-1}|\Delta, \xi\rangle$, their inner products matrix has determinant $-\xi^2$.

2- and 3-point functions of primary operators

Two-point function:

$$G_2(x_1, x_2, y_1, y_2) = d\delta_{\Delta_1,\Delta_2}\delta_{\xi_1,\xi_2}x_{12}^{-2\Delta_1}e^{\frac{2\xi_1 y_{12}}{x_{12}}}$$

Three-point function:

$$G_3(x_1, x_2, x_3, y_1, y_2, y_3) = c_{123}x_{12}^{\Delta_{123}}x_{23}^{\Delta_{231}}x_{31}^{\Delta_{312}}e^{-\xi_{123}\frac{y_{12}}{x_{12}}}e^{-\xi_{312}\frac{y_{31}}{x_{31}}}e^{-\xi_{231}\frac{y_{23}}{x_{23}}}$$

where d is the normalization factor of the two-point function, c_{123} is the coefficient of three-point function which depend on the details of the GCFT_2, and

$$x_{ij} \equiv x_i - x_j, \qquad \Delta_{ijk} \equiv -(\Delta_i + \Delta_j - \Delta_k),$$
$$y_{ij} \equiv y_i - y_j, \qquad \xi_{ijk} \equiv -(\xi_i + \xi_j - \xi_k).$$

4-point functions of primary operators

The four-point functions of primary operators read

$$G_4 = \langle \prod_{i=1}^{4} \mathcal{O}_i(x_i, y_i) \rangle = \prod_{i,j} x_{ij}^{\sum_{k=1}^{4} \frac{\Delta_{ijk}}{3}} e^{-\frac{y_{ij}}{x_{ij}} \sum_{k=1}^{4} \frac{\xi_{ijk}}{3}} \mathcal{G}(x, y)$$

where $\mathcal{G}(x, y)$ is called the stripped four-point function with x, y being the GCA cross ratios,

$$x \equiv \frac{x_{12}x_{34}}{x_{13}x_{24}}, \quad \frac{y}{x} \equiv \frac{y_{12}}{x_{12}} + \frac{y_{34}}{x_{34}} - \frac{y_{13}}{x_{13}} - \frac{y_{24}}{x_{24}}.$$

The function $\mathcal{G}(x, y)$ cannot be determined by the symmetries only.

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Crossing equation:

$$G_{34}^{21}(x, y) = G_{32}^{41}(1 - x, -y).$$

Q: Galilean conformal block?

Quasi-primary states

Hilbert space:

$$\mathcal{H} = \sum_{\Delta,\xi} \mathcal{H}_{\Delta,\xi},$$

where each module is composed of a primary state and its descendants. However such a classification is not suited to bootstrap:

1. The conformal bootstrap is based on the global symmetry, rather than the local one;
2. The explicit form of the local GCA block is unknown.

The Galilean conformal bootstrap is based on the global symmetry, generated by $L_{\pm 1}, L_0, M_{\pm 1}, M_0$. This means that we should start from "quasi-primary" operators. Actually this is feasible as the operators in GCFT_2 can be classified into different quasi-primary operators and their global descendants.

A: Global block of quasi-primaries!

Global block (of the singlet) Bagchi 1612.01730,1705.05890

The contribution of the primary operator and its global descendant operators (which can be got by acting L_{-1} and M_{-1}) to the stripped four-point function $G(x, y)$ could be written as

$$c_{12p}c_{34p}g_p(x, y)/d$$

where the indices $i = 1, 2, 3, 4$ label the operators $\mathcal{O}_i$ on the external legs, and the index p labels the propagating primary operator $\mathcal{O}_p$.
The function $g_p(x, y)$ is the global block (for identical $\mathcal{O}_i$), obeying the Casimir equations of the global algebra

$$\hat{C}_i g_p(x, y) = \lambda_i g_p(x, y), \quad i = 1, 2$$

where λ_i are the eigenvalues, and

$$\begin{aligned}
\hat{C}_1 &= M_0^2 - M_1 M_{-1}, \\
\hat{C}_2 &= 4L_0 M_0 - L_{-1}M_1 - L_1 M_{-1} - M_1 L_{-1} - M_{-1}L_1.
\end{aligned}$$

Solution:

$$g_p(x, y) = 2^{2\Delta_p - 2} x^{\Delta_p - 2\Delta}(1 + \sqrt{1-x})^{2-2\Delta_p} e^{\frac{-\xi_p y}{x\sqrt{1-x}} + 2\xi\frac{y}{x}}(1-x)^{-1/2}.$$

Subtlety

M_0 usually acts non-diagonally on these quasi-primary operators , even though L_0, M_0 act diagonally on the primary operators.

Consider the following level-2 descendant operators of a primary operators $\mathcal{O}$ with a weight Δ and a charge ξ

$$\mathcal{A} = L_{-2}\mathcal{O}, \quad \mathcal{B} = M_{-2}\mathcal{O}.$$

They are quasi-primary operators, on which M_0 acts as

$$M_0\mathcal{A} = \xi\mathcal{A} + 2\mathcal{B}, \quad M_0\mathcal{B} = \xi\mathcal{B}.$$

This phenomenon is typical in Galilean CFT, similar to Logarithmic CFT.

$\mathcal{A}$ and $\mathcal{B}$ share the same conformal dimension, and form a multiplet of rank 2.

A primary operator is referred to as a singlet, or a rank-1 multiplet.

Multiplet

Generically, M_0 acts block-diagonally, consisting of Jordan blocks,

$$M_0 \mathcal{O} = \tilde{\xi} \mathcal{O}$$

where $\mathcal{O}$ are the local operators in the theory, and $\tilde{\xi}$ is block-diagonalised,

$$\tilde{\xi} = \begin{pmatrix} \ddots & & & \\ & \tilde{\xi}_i & & \\ & & \tilde{\xi}_j & \\ & & & \ddots \end{pmatrix}$$

where $\tilde{\xi}_i$ can be written in the form of Jordan block,

$$\tilde{\xi}_i = \begin{pmatrix} \xi_i & & & \\ 1 & \xi_i & & \\ & \ddots & \ddots & \\ & & 1 & \xi_i \end{pmatrix}_{r \times r} .$$

The quasi-primary operators in the same Jordan block form a multiplet.

2-point function of multiplets

Since the $\tilde{\xi}$ can be block-diagonalised, the non-vanishing two-point functions are the ones of $\mathcal{O}_{k_1}, \mathcal{O}_{k_2}$ belonging to the same multiplet of rank r. Here we denote $\mathcal{O}_{k_i}$ as the (k_i+1)-th operator in the multiplet. The two-point functions can be determined by the Ward identities with respect to global symmetries,

$$\langle \mathcal{O}_{k_1}(x_1, y_1)\mathcal{O}_{k_2}(x_2, y_2)\rangle = \begin{cases} 0 & \text{for } q < 0 \\ d_r x_{12}^{-2\Delta_1} e^{2\xi_1 \frac{y_{12}}{x_{12}}} \frac{1}{q!}\left(\frac{2y_{12}}{x_{12}}\right)^q, & \text{otherwise} \end{cases}$$

where

$$q = k_1 + k_2 + 1 - r,$$

and d_r is the overall normalization of this rank-r multiplet.

3-point function of multiplets

The three-point functions of the multiplets read,

$$\langle \mathcal{O}_i \mathcal{O}_j \mathcal{O}_k \rangle = A_{ijk} B_{ijk} C_{ijk}$$

where

$$\begin{aligned} A_{ijk} &= \exp(-\xi_{123}\frac{y_{12}}{x_{12}} - \xi_{312}\frac{y_{31}}{x_{31}} - \xi_{231}\frac{y_{23}}{x_{23}}), \\ B_{ijk} &= x_{12}^{\Delta_{123}} x_{23}^{\Delta_{231}} x_{31}^{\Delta_{312}}, \\ C_{ijk} &= \sum_{a=0}^{r_1-1} \sum_{b=0}^{r_2-1} \sum_{c=0}^{r_3-1} c_{ijk}^{(abc)} \frac{(q_i)^a (q_j)^b (q_k)^c}{a!b!c!}, \end{aligned}$$

with

$$q_i = \partial_{\xi_i} \ln A_{ijk}$$

Note that $\mathcal{O}_i, \mathcal{O}_j, \mathcal{O}_k$ can belong to different multiplets of rank r_1, r_2, r_3 respectively. The coefficient $c_{ijk}^{(abc)}$ encodes the dynamical information of the theory.

4-point function

The 4-point function can be expanded in terms of the global blocks of all multiplets

$$G_4 = \sum_{\text{multiplets } p} \sum_{i,j} T_{ij} < \mathcal{O}\mathcal{O}\mathcal{O}_{p,i} >< \mathcal{O}_{p,j}\mathcal{O}\mathcal{O} >$$

where $\mathcal{O}_{p,i}$ is the $(i+1)$-th operator in the rank-r multiplet labelled by p, and T_{ij} is the element of the inverse of the Gram matrix

$$\sum_{i,j} T_{ij} < \mathcal{O}_{p,i} | \mathcal{O}_{p,j} >= 1.$$

Different from the case of a singlet, the global block of a multiplet is not

the eigenfunction of the Casimir operators. Instead, the Casimir operators act on the multiplet as follows,

$$(\hat{C}_i - \lambda_i)^r |\mathcal{O}_{r,k_i}\rangle = 0.$$

Striped 4-point function

$$\mathcal{G}(x,y) = \sum_{\mathcal{O}_r} \frac{1}{d_r} f\left[\mathcal{O}_r\right]$$

where the propagating quasi-primary operator $\mathcal{O}_r$ is a rank-r multiplet with an overall normalization d_r, and $f\left[\mathcal{O}_r\right]$ satisfy the following Casimir equations

$$(\hat{C}_i - \lambda_i)^r f\left[\mathcal{O}_r\right] = 0, \quad \text{for } i = 1, 2.$$

The solution reads

$$f\left[\mathcal{O}_r\right] = \sum_{s=0}^{r-1} A_s g^{(s)}_{\Delta_r,\xi_r}.$$

Here $g^{(s)}_{\Delta_r,\xi_r}, s = 0, \cdots r-1$ make up the global block for the multiplet,

$$g^{(s)}_{\Delta_r,\xi_r} = \partial^s_{\xi_r} g^{(0)}_{\Delta_r,\xi_r}$$

where $g^{(0)}_{\Delta_r,\xi_r}$ is the global block for the singlet.

Global block expansion

The global block expansion of the stripped four-point function in GCFT is

$$\mathcal{G}(x,y) = \sum_{\mathcal{O}_p|\xi_p=0} \frac{c_p^2}{d_p} k_p(x) + \sum_{\mathcal{O}_r|\xi_r\neq 0} \frac{1}{d_r} \sum_{s=0}^{r-1} \frac{1}{s!} \sum_{a,b|a+b+s+1=r} c_a c_b \partial_{\xi_r}^s g^{(0)}_{\Delta_r,\xi_r}.$$

Note that the first term on R.H.S comes from $\xi_p = 0$ sector, which has no multiplet and reduces to CFT_1

$$k_p(x) = x^{\Delta_p}\ {}_2F_1(\Delta_p, \Delta_p, 2\Delta_p; x).$$

Harmonic analysis

CPW expansion and inversion formula

A 4-point function admits global block expansion in which the expansion coefficients contain the data of the theory.

Actually, it admits an expansion in terms of a set of complete basis of conformal group as well, where the expansion coefficients can be obtained by using the inversion formula.
Such a set of compete basis is provided by so-called conformal partial waves (CPWs).

The two expansions are related by the contour deformation.

Harmonic analysis in CFT

The complete basis consists of the normalizable eigenfunctions of the Hermitian Casimir operators. Dobrev et.al. (1977)

What one needs to do is specifying the Hilbert space which makes Casimirs Hermitian, which requires:

1. specifying the inner product;
2. specifying the boundary condition.

Then using the boundary conditions, we can obtain the eigenfunctions of Casimirs, which include the principal series representations and possible discrete ones.

The so-called conformal partial waves (CPWs) corresponding to principal series representations and possible discrete ones are the expected complete basis.

Harmonic analysis on GCA

1. As the group generated by GCA is not semi-simple, we cannot apply the formal harmonic analysis for conformal symmetry group. We just follow the discussion on the SYK model. J. Maldacena and D. Stanford 1604.07818, J. Murugan et.al. 1706.05362

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4. The invariance under the exchange of $1 \leftrightarrow 2$ or $3 \leftrightarrow 4$ (just as in the SYK model) induce the symmetry

$$x \to \frac{x}{1-x}, \qquad y \to \frac{-y}{(1-x)^2},$$

which restrict the region

$$x \in [0, 2], \qquad y \in (-\infty, +\infty).$$

Galilean Conformal Partial Waves (GCPWs)

$$\Psi_{\Delta,\xi}(x,y) = \begin{cases} B(\Delta)(\chi'_{\Delta,\xi} + \chi'_{\Delta,-\xi}) + B(2-\Delta)(\chi'_{2-\Delta,\xi} + \chi'_{2-\Delta,-\xi}), 1 < x < 2 \\ \\ A(\Delta)(\chi_{\Delta,\xi} + \chi_{\Delta,-\xi}) + A(2-\Delta)(\chi_{2-\Delta,\xi} + \chi_{2-\Delta,-\xi}), 0 < x < 1 \end{cases}$$

where

$$\begin{aligned} A(\Delta) &= \sin\frac{\pi\Delta}{2} + \cos\frac{\pi\Delta}{2}, \\ B(\Delta) &= e^{\frac{i\pi}{2}(\Delta-1)} \end{aligned}$$

and

$$\begin{aligned} \chi_{\Delta,\xi} &= \frac{x^{\Delta}(1-\sqrt{1-x})^{2-2\Delta}}{\sqrt{1-x}} e^{\frac{\xi y}{x\sqrt{1-x}}}. \\ \chi'_{\Delta',\xi'} &= \frac{x^{\Delta'}(1-i\sqrt{x-1})^{2-2\Delta'}}{\sqrt{x-1}} e^{\frac{-i\xi' y}{x\sqrt{x-1}}}. \end{aligned}$$

Representations

Using normalizable condition, we can determine the possible values of ξ and Δ .

$$\xi \in ir, \qquad r \in \mathbb{R}.$$

Principal series representation:

$$\Delta = 1 + is, \qquad s \in \mathbb{R}$$

Discrete series:

$$\Delta = \frac{5}{2} + 2n \qquad \text{or} \qquad \Delta = -\frac{1}{2} - 2n, \qquad n = 0, 1, 2, \cdots .$$

There is a symmetry:

$$\xi \leftrightarrow -\xi, \qquad \Delta \leftrightarrow 2 - \Delta.$$

One can check the orthogonality and completeness of the basis.

Non-relativistic limit?

Many kinematic quantities can be obtained by taking the non-relativistic limit from the ones in parent CFT_2, including the Casimirs, the two-point functions, three-point functions, the null vectors and the global conformal blocks of singlets, etc.

However, GCPWs cannot be obtained in this way, as the Hilbert space in the GCA harmonic analysis can not be obtained from the one in 2d conformal harmonic analysis.

- Renormalizable conditions: quantum numbers are very different
- Boundary conditions cannot be fixed by taking the limit

GCPW expansion

1. The harmonic analysis include two regions, $0 < x < 1$ and $1 < x < 2$. One can actually use any one region to find the block expansion.

$$\begin{aligned}\mathcal{G}(x,y) &= \int_0^\infty dr \frac{1}{4\pi^2} \int_0^\infty ds \frac{1}{N} \Psi_{\Delta,\xi}(x,y)(\Psi_{\Delta,\xi},\mathcal{G}) \\ &\quad + \int_0^\infty dr \sum_n \frac{1}{N'} \Psi_{\Delta,\xi}(x,y)(\Psi_{\Delta,\xi},\mathcal{G}).\end{aligned}$$

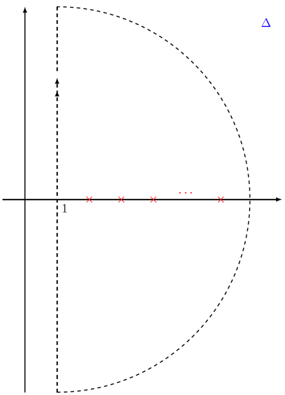
2. The $\xi = 0$ part should be treated separately, using CPWs in CFT_1.

3. The multiplets appear as the multiple poles in the inversion function.

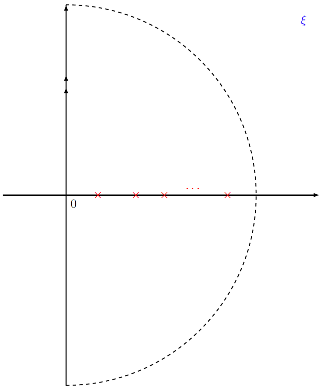
GCA inversion function

$$I(\Delta,\xi) = (\Psi_{\Delta,\xi},\mathcal{G}) \sim -\sum_{\Delta_m,\xi_l,k} \Gamma(k+1)\frac{2^{2\Delta_m-2}}{(\xi-\xi_l)^{k+1}}\frac{P_{\Delta_m,\xi_l,k+1}}{\Delta-\Delta_m}$$

where $\{\Delta_m, \xi_l\}$ are the physical poles.



(a) Δ-plane



(b) ξ-plane

Generalized Galilean free field theory

Generalized free theory

The generalized free field theory(GFT) or Mean Field Theory (MFT) plays an important role in analytic conformal bootstrap.

It provides the leading contribution to correlators at large spin. The data in GFT is the starting point for many computations.

Holographically it is the dual of free field theories in AdS.

By definition, correlators in GFT are simply sums of products of two-point functions.

Generalied Galilean free theory (GGFT)

We may start from the generalized Galilean free field theory (GGFT) which contains two fundamental scalar type operators $\mathcal{O}_1, \mathcal{O}_2$ with the conformal weights and the charges Δ_1, ξ_1 and Δ_2, ξ_2 respectively. The two-point functions read

$$\begin{aligned} \langle \mathcal{O}_1 \mathcal{O}_1 \rangle &= x_{12}^{-2\Delta_1} e^{2\xi_1 y_{12}/x_{12}}, \\ \langle \mathcal{O}_2 \mathcal{O}_2 \rangle &= x_{12}^{-2\Delta_2} e^{2\xi_2 y_{12}/x_{12}}, \\ \langle \mathcal{O}_1 \mathcal{O}_2 \rangle &= 0. \end{aligned}$$

The four-point function of $\mathcal{O}_1\mathcal{O}_1\mathcal{O}_2\mathcal{O}_2$ is simply the product of two-point functions.

We would like to study the spectrum and 3-pt coefficients in such GGFT.

Three different approaches

1. Operator construction: show "double trace" operator explicitly
2. Taylors expansion of 4-point function in terms of global block
3. Apply GCA inversion formula

They are consistent with each other.

Operator construction: Level-1 multiplet

Building blocks:

$$\mathcal{A}: \mathcal{O}_1 M_{-1} \mathcal{O}_2, \qquad \mathcal{B}: M_{-1} \mathcal{O}_1 \mathcal{O}_2$$

$$\mathcal{C}: \mathcal{O}_1 L_{-1} \mathcal{O}_2, \qquad \mathcal{D}: L_{-1} \mathcal{O}_1 \mathcal{O}_2$$

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Level-1 multiplet

$$\mathcal{P}_i = a_i \mathcal{A} + b_i \mathcal{B} + c_i \mathcal{C} + d_i \mathcal{D}$$

where $i = 0, 1$ and

$$\mathcal{P}_0 : a_0 = \xi_1, \qquad b_0 = -\xi_2, \qquad c_0 = d_0 = 0$$

$$\mathcal{P}_1 : a_1 = a, \quad b_1 = -\frac{a\xi_2}{\xi_1} - \frac{\Delta_2 \xi_1 - \Delta_1 \xi_2}{\xi_1}, \quad c_1 = \xi_1, \quad d_1 = -\xi_2$$

Note that there is still one undetermined parameter a in $\mathcal{P}_1$.

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It is straightforward to compute the 2-point $\langle \mathcal{P}_i \mathcal{P}_j \rangle$ and 3-point function $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{P}_i \rangle$.
Striped 4-point function ($\mathcal{O}_1 = \mathcal{O}_2$)

$$\mathcal{G}(x, y) = \Delta_{\mathcal{O}} g^{(0)}_{2\Delta_{\mathcal{O}}+1, 2\xi_{\mathcal{O}}} + \xi_{\mathcal{O}} g^{(1)}_{2\Delta_{\mathcal{O}}+1, 2\xi_{\mathcal{O}}} + \cdots$$

where $\cdots$ represents the contributions from higher levels.

Taylor expansion of GGFT

The s-channel conformal block expansion of four identical operators with the same weight and charge $(\Delta_{\mathcal{O}}, \xi_{\mathcal{O}})$ is

$$\sum_{\Delta} P_{\Delta} \chi_{\Delta} + \sum_{\Delta,\xi,k} P_{\Delta,\xi,k} \chi_{\Delta,\xi,k} = \frac{\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O} \rangle}{x^{-2\Delta_{\mathcal{O}}} \exp(\frac{2\xi_{\mathcal{O}} y}{x})}.$$

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Key point: the spectrum of ξ is localized at $2\xi_{\mathcal{O}}$ in GGFT.

- Operator construction: something like $\mathcal{O}\partial_x^a\partial_y^b\mathcal{O}$
- Representation: only in this case, there is no infinite-level multiplet.
- This fact can be understood in taking nonrelativistic limit of 2D GFT.

In radial coordinates, the block expansion has a simple form

$$\sum_{\Delta,k} P_{\Delta,k}\,\rho^{\Delta}(-\kappa)^{k} = \frac{1}{4}(1-\rho^{2})e^{\xi\kappa}\mathcal{G}.$$

Ansatz: double twist operators $\Delta_n = 2\Delta_{\mathcal{O}} + n$.
In t-channel:

$$\sum_{n,k} P^{t}_{n,k}\rho^{n}(-\kappa)^{k} = 2^{4\Delta_{\mathcal{O}}-2}(1-\rho^{2})\,(1-\rho)^{-4\Delta_{\mathcal{O}}}\exp\Big(\frac{-4\xi_{\mathcal{O}}\kappa\rho}{1-\rho}\Big)$$

Expanding the right hand side we get the coefficients,

$$P^{t}_{n,k} = \frac{2^{4\Delta_{\mathcal{O}}+2k-2}\xi^{k}_{\mathcal{O}}}{k!}A_{n,k}$$

where $A_{n,k}$ is a polynomial of $\Delta_{\mathcal{O}}$ of degree $n-k$

$$A_{n,k} = \frac{4\Delta_{\mathcal{O}}+2n-k-2}{4\Delta_{\mathcal{O}}+n-2}\binom{4\Delta_{\mathcal{O}}+n-2}{n-k}$$

At level 1, we have,

$$P^{t}_{1,0} = 2^{4\Delta_{\mathcal{O}}}\Delta_{\mathcal{O}}, \quad P^{t}_{1,1} = 2^{4\Delta_{\mathcal{O}}}\xi_{\mathcal{O}},$$

which matches the operator construction result.

Inversion function of GGFT

Inversion function: $I = (\Psi_{\Delta,\xi}, \mathcal{G})$
It is divergent with $\mathrm{Re}\xi_0 \neq 0$. However one may introduce a regularization, and analyticity ensures the results are cut-off free.
By deforming the contour of Δ and ξ, the nontrivial contribution comes from the integral over $D_0 = (0,1) \times (-\infty, 0)$

$$I_1 = A(2-\Delta)(\chi_{\Delta,\xi}, \mathcal{G}_t)_{D_0}.$$

Input the t-channel contribution $\mathcal{G}_t$, one can find

$$(\chi_{\Delta,\xi}, \mathcal{G}_t)_{D_0} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{-\Delta + 2\Delta_{\mathcal{O}} + n} \frac{1}{(\xi - 2\xi_{\mathcal{O}})^{k+1}} P_{n,k}^{t,\text{inversion}}$$

where,

$$P_{n,k}^{t,\text{inversion}} = k!\, P_{n,k}^{t}.$$

It matches with the coefficients obtained by the Taylor series expansion. The factorial $k!$ is due to reading out the block expansion from higher order poles.

Conclusions

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2. We developed harmonic analysis of GCA, which paves the way for further analytic study.
3. We studied GGFT in three different ways, and found consistent picture.

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1. The appearance of multiplets is an essential feature in Galilean conformal bootstrap.
2. We developed harmonic analysis of GCA, which paves the way for further analytic study.
3. We studied GGFT in three different ways, and found consistent picture.
4. We estimated the spectral density by using Hardy-Littlewood tauberian theorem.
5. We discussed shadow formalism and alpha space approach.

Outlook

More careful study! numerical method?

Other nonunitary CFT, say Log-CFT or warped CFT?

Higher dimensions?

Its implication in flat space holography?

Local GCA block?

Thanks for your attention!