Quantum Periods and TBA-like Equations for a Class of Calabi-Yau Geometries

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Bao-ning Du, MH, arXiv:2009.07009, accepted by JHEP;
MH, arXiv:2006.08860.

Introduction

- Period integrals are ubiquitous in mathematics and physics. For example, they are essential ingredients in classic mirror symmetry, in the solutions of Seiberg-Witten gauge theories, and are also frequently related to appealing mathematical objects like modular forms.

- Bethe Ansatz has served as a cornerstone in the developments of exactly solvable many-body quantum systems for almost a century. We will consider a type of difference equations similar to those that appear in the Thermodynamic Bethe Ansatz (TBA).

- Quantum periods have been a useful tool in recent studies of topological string theory and related topics. We will consider local Calabi-Yau geometries which can be described by a complex one-dimensional mirror curve with complex coordinates (x, p). Promoting the coordinates to canonical quantum position and momentum operators, we can then compute the quantum periods as integrals of a quantum corrected differential one-form over cycles.

- We will study the novel relation between quantum periods and TBA-like equations proposed in Y. Hatsuda, M. Marino, S. Moriyama, and K. Okuyama, arXiv:1306.1734 . In particular, we focus on the parts concerning quantum A-periods which can be determined by a residue calculation at finite Planck constant $\hbar$.

- The original proposal is for a particular local $\mathbb{P}^1 \times \mathbb{P}^1$ Calabi-Yau geometry related to the ABJM (Aharony-Bergman-Jafferis-Maldacena) theory. We will further explore the relation and generalize to a large class of local Calabi-Yau geometries.

- The classical mirror curves of local Calabi-Yau geometries can be quantized $[\hat{x}, \hat{p}] = i\hbar$. We have a quantum spectral problem

$$\rho(\hat{x}, \hat{p})|\psi\rangle = e^{-E}|\psi\rangle, \tag{1}$$

where the energy E is related to the only dynamical complex structure modulus. For the purpose of deriving the corresponding TBA-like equation, it needs to have the following form

$$\rho(\hat{x}, \hat{p}) = u(\hat{x})^{-\frac{1}{2}}[2\cosh(\hat{p})]^{-1}u^*(\hat{x})^{-\frac{1}{2}}, \tag{2}$$

- In the 1990's, it was conjectured in A. B. Zamolodchikov, arXiv:hep-th/9409108 and proved in C. A. Tracy and H. Widom, arXiv:solv-int/9509003 that the spectral equation (1) is related to some TBA-like equations which first appeared in the context of two-dimensional $\mathcal{N} = 2$ supersymmetric theories.

- We will use the TBA-like difference equation for the function $\eta(X)$ after these manipulations

$$1 + z[\eta(qX) + \eta(X)][\eta(q^{-1}X) + \eta(X)]|u(q^{\frac{1}{2}}X)u(q^{-\frac{1}{2}}X)| = \eta(X)^2, \tag{3}$$

where we use the exponentiated parameters $X = e^x, q = e^{i\hbar}$.

- The case of local $\mathbb{P}^1 \times \mathbb{P}^1$ model, which has the mirror curve

$$e^x + e^p + z_1 e^{-x} + z_2 e^{-p} = 1. \tag{4}$$

Promote the x, p coordinates to quantum operators and consider a particular case $z_1 = q^{-\frac{1}{2}} z, z_2 = q^{\frac{1}{2}} z$ which corresponds to the ABJM theory.

- We make a change of variables

$$\widehat{x} \to \frac{\widehat{x}}{2} - \widehat{p} - \frac{i\hbar}{4} - E, \quad \widehat{p} \to \frac{\widehat{x}}{2} + \widehat{p} + \frac{i\hbar}{4} - E, \quad z \to e^{-2E}, \tag{5}$$

which preserve the same canonical commutation relation. Acting the mirror curve on a quantum state $|\phi\rangle$ we get

$$4 \cosh(\frac{\widehat{x}}{2}) \cosh(\widehat{p}) |\phi\rangle = e^E |\phi\rangle. \tag{6}$$

- So we have a spectral equation in the form of (1, 2), with the function $u(x) = e^{\frac{x}{2}} + e^{-\frac{x}{2}}$.

- For the general case $z_1 = e^{\tilde{m}} z, z_1 = e^{-\tilde{m}} z$, it seems difficult to write the spectral equation in the form of (1, 2) by simple manipulations of elementary functions. Instead, we will need to use Faddeev's quantum dilogarithm function Φ_b. Using the results of Kashaev et al

$$\rho = (e^{\widehat{x}} + e^{\widehat{p}} + e^{\tilde{m}} e^{-\widehat{x}} + e^{-\tilde{m}} e^{-\widehat{p}})^{-1} = f(\widehat{x'})[2\cosh(\widehat{p'})]^{-1} f^*(\widehat{x'}), \quad (7)$$

where the redefined variables are $\widehat{x'} = \widehat{x} - \widehat{p} - \tilde{m}, \quad \widehat{p'} = \frac{1}{2}(\widehat{x} + \widehat{p} - \tilde{m})$, which satisfy the same canonical commutation relation. The function $f(x)$ is defined in terms of the quantum dilogarithm

$$f(x) = e^{\frac{x}{4}} \Phi_b(\frac{x - \tilde{m}}{2\pi b} + i\frac{b}{4}) \Phi_b(\frac{x + \tilde{m}}{2\pi b} - i\frac{b}{4})^{-1}, \quad (8)$$

with the parameter definition $b = \sqrt{\frac{\hbar}{\pi}}$. Using the well known functional relations of quantum dilogarithm, it is straightforward to calculate

$$|f(x + \frac{i\hbar}{2}) f(x - \frac{i\hbar}{2})|^{-2} = e^x + e^{-x} + e^{\tilde{m}} + e^{-\tilde{m}}. \quad (9)$$

So we can now derive the TBA-like equation for $\mathbb{P}^1 \times \mathbb{P}^1$ model with general mass parameter

$$1 + z[\eta(qX) + \eta(X)][\eta(q^{-1}X) + \eta(X)](X + 1/X + e^{\tilde{m}} + e^{-\tilde{m}}) = \eta(X)^2, \quad (10)$$

The $\mathcal{O}_{m,1}$ operators: first method

- We consider a class of three-term quantum operators of the form

$$\mathcal{O}_{m,n} = e^{\widehat{x}} + e^{\widehat{p}} + e^{-m\widehat{x}-n\widehat{p}}, \tag{11}$$

  where m, n are natural numbers and in this section we will focus on the case of $n = 1$ where the corresponding TBA-like equation can be derived.

- The case of $m = n = 1$ corresponds to the well studied local $\mathbb{P}^2$ Calabi-Yau geometry, while the case of $(m, n) = (2, 1)$ corresponds to a subfamily of local Hirzebruch $\mathbb{F}_2$ Calabi-Yau geometry. In general, the operator corresponds to a $\mathbb{C}^3/\mathbb{Z}_{m+2}$ resolved orbifold Calabi-Yau space. For $m > 2$, the operator has multiple complex deformations which correspond to different dynamical Hamiltonians, we will consider a one-parameter subfamily of such deformations.

- Quantum period: the $\mathcal{O}_{m,1}$ operator can be parametrized as

$$e^{\widehat{x}} + e^{\widehat{p}} + ze^{-m\widehat{x}-\widehat{p}} = 1. \tag{12}$$

  We act the quantum curve on a wave function $\psi(x)$ and denote $V(x) = \frac{\psi(x)}{\psi(x-i\hbar)}$, then we have a difference equation

$$X - 1 + \frac{1}{V(X)} + \frac{zV(qX)}{q^{\frac{m}{2}}X^m} = 0, \tag{13}$$

  where again the notation is $X = e^x, q = e^{i\hbar}$.

- The difference equation for $V(X)$ can then be solved perturbatively as a power series in z, and for example up to order z we have the expression

$$V(X) = \frac{1}{1-X} + \frac{q^{-\frac{m}{2}}X^{-m}z}{(1-X)^2(1-qX)} + \mathcal{O}(z^2). \tag{14}$$

- We consider the residue around $X = 0$

$$\Pi_m = \text{Res}_{X=0} \frac{\log[V(X)]}{X} = \text{Res}_{X=0} \frac{\log[(1-X)V(X)]}{X}. \tag{15}$$

- This is the quantum period (up to a possible logarithmic term). For example

$$\begin{aligned}\Pi_1 &= \frac{(1+q)z}{\sqrt{q}} + \frac{[2(1+q^4)+7(q+q^3)+12q^2]z^2}{2q^2} + [3(1+q^9) \\ &\quad + 9(q+q^8) + 36(q^2+q^7) + 88(q^3+q^6) + 144(q^4+q^5)]\frac{z^3}{3q^{9/2}} + \mathcal{O}(z^4) \\ \Pi_2 &= \frac{(1+q+q^2)z}{q} + [\frac{27}{2} + 10(q+q^{-1}) + \frac{13}{2}(q^2+q^{-2}) + 2(q^3+q^{-3}) \\ &\quad + q^4 + q^{-4}]z^2 + \mathcal{O}(z^3),\end{aligned} \tag{16}$$

- To derive the corresponding TBA-like equation, we change variables

$$\widehat{x} \to \widehat{x}, \quad \widehat{p} \to -m\frac{\widehat{x}}{2} + \widehat{p} - \frac{im\hbar}{4} - E, \quad z \to e^{-2E}. \tag{17}$$

  Acting the mirror curve on a quantum state $|\phi\rangle$, we find

$$2\cosh(\widehat{p})|\phi\rangle = e^{\frac{m\widehat{x}}{2}}(1 - e^{\widehat{x}})e^{E}|\phi\rangle. \tag{18}$$

  After a simple redefinition of the quantum state, we can now write the spectral equation in the form of (1, 2) with the function $u(x) = e^{-\frac{mx}{2}}(1 - e^{x})^{-1}$.

- We should note that unlike the examples in literature, in our case the integral $\int_{-\infty}^{\infty} |u(x)|^{-1} dx$ is actually divergent, which implies the corresponding integral kernel (2) may not be a trace class operator. However this is not really an issue since we are not studying the spectral theory, but just use it as a formal trick to derive the TBA-like equations. Our end result should justify ignoring such subtleties in the process.

- So we arrive at a TBA-like equation for the $\mathcal{O}_{m,1}$ operator

$$1 + z[\eta_m(qX) + \eta_m(X)][\eta_m(q^{-1}X) + \eta_m(X)]/[X^{m+1}(X + X^{-1} - q^{\frac{1}{2}} - q^{-\frac{1}{2}})]$$
$$= \eta_m(X)^2. \tag{19}$$

We take the residue and check up to the first few orders that it is indeed simply related to the residue (16) in quantum periods

$$\mathrm{Res}_{X=0}\frac{1}{X}\eta_m(X,q,z) = 1 + 2\theta_z\Pi_m, \tag{20}$$

where $\theta_z \equiv z\partial_z$.

- The case $m = 2$ is somewhat interesting, as it is known that the $\mathbb{F}_2$ model is related to the $\mathbb{F}_0 \equiv \mathbb{P}^1 \times \mathbb{P}^1$ model by a reparametrization. After identifying the parameters, we have another TBA-like equation for the $m = 2$ case as

$$1 + z^{\frac{1}{2}}[\tilde{\eta}(qX) + \tilde{\eta}(X)][\tilde{\eta}(q^{-1}X) + \tilde{\eta}(X)](X + 1/X) = \tilde{\eta}(X)^2. \qquad (21)$$

  The perturbative solution for $\tilde{\eta}$ has half integer powers of z. However after taking residue, only the integer powers survive. We check that it is indeed simply related to η_2 by the following intriguing equation

$$\text{Res}_{X=0}\frac{1}{X}[2\eta_2(X, q, z) - \tilde{\eta}(X, q, z)] = 1. \qquad (22)$$

- Surprisingly, without concerning about Calabi-Yau conditions and mirror symmetry, the formalism can be actually applied to a much more general curve

$$e^{\widehat{p}} + ze^{-m\widehat{x}-\widehat{p}} = r(X), \tag{23}$$

where the function $r(X)$ can be a general rational function of $X = e^{\widehat{x}}$.

- It is more convenient to directly normalize $V(X)$ and also define a function $s(X)$ as

$$\tilde{V}(X) \equiv r(q^{-\frac{1}{2}}X)V(q^{-\frac{1}{2}}X), \quad s(X) \equiv 1/[X^m r(q^{\frac{1}{2}}X)r(q^{-\frac{1}{2}}X)]. \tag{24}$$

Here we also shift X by a factor of $q^{-\frac{1}{2}}$ in the definition of $\tilde{V}(X)$, so that the difference equation looks much simplified

$$\frac{1}{\tilde{V}(X)} + \tilde{V}(qX)s(X)z = 1. \tag{25}$$

- We can recursively compute the small z expansion of $\tilde{V}(X)$ and its logarithm. The explicit expression up to a few orders is

$$\log(\tilde{V}(X)) = s(X)z + \frac{1}{2}s(X)[s(X) + 2s(qX)]z^2 + \frac{1}{3}s(X)[s(X)^2 + 3s(X)s(qX) + 3s(qX)^2 + 3s(qX)s(q^2X)]z^3 + \mathcal{O}(z^4). \tag{26}$$

  The residue is then defined by

$$\Pi_{s(X)} = \mathrm{Res}_{X=0}\frac{\log[\tilde{V}(X)]}{X}. \tag{27}$$

- On the other hand, the derivation of the TBA-like equation for a function $\eta_{s(X)}$ also goes through smoothly for this class of curves, we have

$$1 + s(X)[\eta_{s(X)}(qX) + \eta_{s(X)}(X)][\eta_{s(X)}(q^{-1}X) + \eta_{s(X)}(X)]z = \eta_{s(X)}(X)^2. \tag{28}$$

  We can also explicitly compute the small z expansion up to a few orders

$$\eta_{s(X)}(X) = 1 + 2s(X)z + 2s(X)[s(X) + s(q^{-1}X) + s(qX)]z^2 + \mathcal{O}(z^3). \tag{29}$$

- We can now check that the following relation is still valid as a formal z power series for various functions $s(X)$

$$\text{Res}_{X=0}\frac{1}{X}\eta_{s(X)}(X,q,z) = 1 + 2\theta_z \Pi_{s(X)}. \tag{30}$$

  This is true for any function $s(X)$ as long as it has a Laurent expansion around $X \sim 0$.

- It is not difficult to explicitly check the relation (30) up to a finite order in z expansion. We note for any function $f(X)$ with a Laurent expansion around $X \sim 0$, the constant term remains the same with a scaling of X by any constant a, i.e. we have $\text{Res}_{X=0}\frac{1}{X}[f(aX) - f(X)] = 0$. We can then for example show that the relation (30) holds up to second order using e.g. $\text{Res}_{X=0}\frac{1}{X}s(X)[s(qX) - s(q^{-1}X)] = 0$.

The $\mathcal{O}_{m,1}$ operators: second method

- we provide another method to derive the TBA-like equation for the $\mathcal{O}_{m,1}$ operator. This approach use the formulas for writing the $\mathcal{O}_{m,n}^{-1}$ operator in terms of Faddeev's quantum dilogarithm, in similar fashion as in (1, 2). The notable difference is that the momentum operator is shifted by a constant.

- After some transformations, the integral kernel becomes the following expression

$$\langle x_1|\rho_{m,n}|x_2\rangle = \frac{u_{m,n}(x_1)^{-\frac{1}{2}}u_{m,n}(x_2)^{-\frac{1}{2}}}{2(m+n+1)\hbar\cosh(\frac{\pi(x_1-x_2)}{(m+n+1)\hbar}+\frac{i(m-n+1)\pi}{2(m+n+1)})}, \tag{31}$$

where the potential $u_{m,n}(x)$ can be described by Faddeev's quantum dilogarithm

$$u_{m,n}(x) = e^{-\frac{m}{m+n+1}x}|\Phi_b(\frac{x-i\hbar(m+1)/2}{2\pi b})|^2, \tag{32}$$

with the parameter $b := \sqrt{\frac{(m+n+1)\hbar}{2\pi}}$.

- The derivations of the TBA-like equations with this shift have been worked out in K. Okuyama and S. Zakany, arXiv:1512.06904. We focus on the $\mathcal{O}_{m,1}$ case where the TBA equation can be solved perturbatively. The key idea to to separate the generating function of integral kernel into $m+2$ parts, generalizing Tracy and Widom. We skip the technical details here.

- After some calculations, we find the following TBA-like difference equation for a properly defined function $\eta_m(X)$ with only elementary functions

$$(1-\eta_m(X))\prod_{i=-m}^{0} B(q^i X) = -zX^{-m}\prod_{k=-m/2}^{m/2}(1+q^k X)\prod_{i=-(m+1)}^{0} A(q^i X),$$

$$\text{with} \quad A(X) := \sum_{i=0}^{m+1}\eta_m(q^i X), \quad B(X) := 1+\sum_{i=0}^{m}\eta_m(q^i X). \tag{33}$$

- We check perturbatively the relation with quantum periods

$$\mathrm{Res}_{X=0}\frac{1}{X}\eta_m(X,q,z) = 1+(m+2)\theta_z\Pi_m. \tag{34}$$

Discussion and Conclusion

- In the WKB expansion of small $\hbar$ parameter, the quantum corrections to classical periods can be expressed exactly as differential operators acting on the classical periods. In the classical limit $\hbar \to 0$, the TBA-like equation in the first method becomes a simple quadratic equation with no linear term, which is simpler than the analogous equation in the calculations of the classical period. One simplification is that it is symmetric with $\hbar \to -\hbar$, so the WKB expansion has only even powers of $\hbar$. On the other hand, the WKB expansion of quantum wave function $\log(V(X))$ defined in (13) does have odd powers of $\hbar$, which turn out to be total derivatives and only vanish after taking residue.

- However, the TBA-like equations (33) of the second method are more complicated. In the classical limit, it becomes a degree $m+2$ polynomial equation for η_m. For $m = 1, 2$ one can still have analytic solution. However it does not generically have an algebraic solution for $m > 2$. So it seems that the TBA-like equations (33) are only suitable for calculations in small z perturbation, but not much in small $\hbar$ perturbation.

- It is well known that many 5d supersymmetric gauge theories are geometrically engineered by toric Calabi-Yau geometries. In the 4d limit, the quantum operator reduces to that of a non-relativistic particle with the standard quadratic kinetic term $\widehat{p}^2$. The connections between quantum periods of the 4d Seiberg-Witten-like curves and certain TBA-like equations have appeared many times in the literature.

- In our case, the appearance of the factor of $\cosh(\widehat{p})$ in the quantum operator e.g. (2) is crucial, and we are also taking residue with the exponentiated parameter $X = e^x$. So it seems the relation studied here is intrinsically 5 dimensional, and we are not aware of a simple 4d limit. It would be interesting to study this issue further.

- We should mention that in both 4d and 5d supersymmetric gauge theories, there is yet another method to determine the quantum mirror maps by a calculation of the vacuum expectation value of 4d chiral operators or 5d Wilson loop operators in the NS limit of the Ω background. It would be interesting to study the potential relation of the present work to this different method.

- The most intriguing feature of this paper is that the two different approaches give rise to entirely different TBA-like equations and perturbative solutions for the class of Calabi-Yau geometries. However, in both cases, the residues of the perturbative solutions of TBA-like equations are related to the same quantum period.

- In one particular case, namely the geometry of $\mathcal{O}_{2,1}$ operator, we even have three different TBA-like equations due to a geometric equivalence of the local $\mathbb{F}_0$ and $\mathbb{F}_2$ Calabi-Yau models.

- In this sense, our paper provides multiple different realizations of the same geometry. It would be interesting to study how the residues in these different looking TBA-like equations may be directly related without using quantum periods as a connecting hub.

- While the first method applies more generally to a larger class of curves, the second method appears to work more specifically for the $\mathcal{O}_{m,1}$ operators. Certainly, it would be interesting to further generalize the results to more Calabi-Yau geometries and consider a bigger moduli space instead of the one-parameter space in this paper.

- It may be interesting to follow the steps in J. Kallen and M. Marino, arXiv:1308.6485 to provide a more rigorous derivation of the relation between quantum periods and TBA-like equations discussed here. However, for the first method, the spectral theory is merely used as a formal trick, and may not be well behaved since the integral kernel may not be of trace class. So it seems unlikely at least in this case that one may establish a rigorous link using the spectral theory, and some new and more unifying approaches may be needed.

Thank You