BCJ duality and worldsheet quantum algebra

Chih-Hao Fu
Shaanxi Normal University

USTC Hefei 2020

Based on work in collaboration with Pierre Vanhove and Yihong Wang
arXiv:1806.09584 JHEP09(2018)141
arXiv:2005.05177 JHEP accepted
& work in progress

special thanks to Kirill Krasnov & Yongshi Wu

Outline

- **A very brief introduction to quantum algebras**

  - FRT construction (motivating q-deformation)

- **BCJ Kinematic algebra**

  - A field theory problem

  - String theory explanation

- **algebra structure**

  - representation and root systems

Quantum groups

= Hopf algebra + quasi-triangular/RTT

$$(\mathcal{H},\, \cdot,\, I,\, \Delta,\, \epsilon,\, S)$$

$$(g_1 \cdot g_2)\,|\phi\rangle = g_1 \cdot (g_2\,|\phi\rangle)$$ multiplication (group representation)

$$(g_1 + g_2)\,|\phi\rangle = g_1\,|\phi\rangle + g_2\,|\phi\rangle$$ superposition

$$[J_x, J_y] = i\,J_z$$
$$\vdots$$

algebra

---

coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$

bialgebra

$$\Delta(g_1 \cdot g_2) = \Delta(g_1) \cdot \Delta(g_2)$$

multiparticle representation

$$|\phi\rangle \to |\phi_1\rangle \otimes |\phi_2\rangle \qquad J \xrightarrow{\Delta} J_1 \otimes I_2 + I_1 \otimes J_2$$

$$e^{i\theta J} \xrightarrow{\Delta} e^{i\theta J_1} \otimes e^{i\theta J_2}$$

Yangian

$$t^{(r)}_{i\,j} = \sum_{s=1}^{r} t^{(s)}_{i\,k} \otimes t^{(r-s)}_{k\,j}$$

---

antipode $S : \mathcal{H} \to \mathcal{H}$

Hopf algebra

$$m(S \otimes id)\Delta(g) = \epsilon(g)\,I$$

$$S(g) \cdot g = I$$

$$S(g) = g^{-1}$$ (generalised inverse)

Quantum Groups

satisfies RTT and Yang-Baxter eqn

$$R\,T_1 T_2 = T_2 T_1\, R$$

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

Л. Д. Фаддеев Н. Ю. Решетихин Л. А. Тахтаджян

[Faddeev, Reshetikhin, Takhtadzhyan 90]

$$R(h) = I + h\, P_{12}$$

$$= I + h \sum_{i,j=1}^{N} e_{i\,j} \otimes e_{j\,i}$$

(Yang) universal R-matrix

$$T(h) = \begin{bmatrix} t_{11}(h) & t_{12}(h) & \dots \\ t_{21}(h) & \ddots & \\ \vdots & & t_{NN}(h) \end{bmatrix}$$

$$\longrightarrow \quad [t_{i\,j}(h),\, t_{k\,\ell}(h)] \sim 0 + \mathcal{O}(h)$$

$$t_{i\,j}(h) = t^{(0)}_{i\,j} + h\, t^{(1)}_{i\,j} + h^2\, t^{(2)}_{i\,j} + \dots$$

(FRT construction)

$$e_{i\,j} \cdot v_j = v_i$$

$$e_{i\,j} \cdot v_k = 0 \quad (j \neq k)$$

$$e_{i\,j} \cdot e_{j\,k} = e_{i\,k}$$

$$T_1 T_2 = e_{i\,j} \otimes e_{k\,\ell} \otimes t_{i\,j} t_{k\,\ell}$$

$$T_2 T_1 = e_{i\,j} \otimes e_{k\,\ell} \otimes t_{k\,\ell} t_{i\,j}$$

Yangian (Янгиан) Symmetry

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v)$$

$$R(u) = I + u^{-1}P_{12}$$

В. Г. Дри́нфельд
Vladimir Drinfeld

Level 0: $$J_a^{(0)} = \sum_{i=1}^{n} J_{i,a}^{(0)}$$

total mtm/anglr mtm
= sum of mtm/anglr mtm
of each particle

$$[J_a^{(0)}, J_b^{(0)}\} = f_{ab}{}^c J_c^{(0)}$$

superconformal symmetry
SU(2,2|4)

Level 1: $$J_a^{(1)} = f_a{}^{bc} \sum_{1\le i<j\le n} J_{i,b}^{(0)} J_{j,c}^{(0)}$$

$$[J_a^{(1)}, J_b^{(0)}\} = f_{ab}{}^c J_c^{(1)}$$

Level 2: $\vdots$

Level 0: $J_a^{(0)}\mathcal{A}(1234) = 0$

mtm/anglr mtm conservation ⟷ transltion/roation invariance

Level 1: $J_a^{(1)}\mathcal{A}(1234) = 0$

$\vdots$ $\vdots$

$\mathcal{N}=4$ SYM ✓

pure YM ✗

[Drummond, Henn, Plefka 09]
[Chicherin, Derkachov, Kirschner 14]

Quantum Groups

satisfies RTT and Yang-Baxter eqn

$$R\,T_1 T_2 = T_2 T_1\,R$$

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

Ex1. quantum matrix group $SL_q(2)$ $\quad q = e^h$

$$R^{ab}{}_{cd} = \begin{array}{c} \\ 11 \\ 12 \\ 21 \\ 22 \end{array}\begin{array}{cccc} 11 & 12 & 21 & 22 \\ \end{array}\begin{bmatrix} q & & & \\ & 1 & & \\ & q-q^{-1} & 1 & \\ & & & q \end{bmatrix}$$

$$T(u) = T = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

$$R^{ab}{}_{cd} \xrightarrow{q \to 1} \delta^a{}_c \delta^b{}_d$$

$\longrightarrow$

$$\alpha\beta = q\,\beta\alpha \qquad \beta\gamma = \gamma\beta$$

$$\alpha\gamma = q\,\gamma\alpha \qquad \alpha\delta - \delta\alpha = (q - q^{-1})\beta\gamma \qquad q \in \mathbb{C}$$

$$\beta\delta = q\,\delta\beta$$

$$\gamma\delta = q\,\delta\gamma$$

(FRT construction)

[Faddeev, Reshetikhin, Takhtadzhyan 90]

Quantum Groups

satisfies RTT and Yang-Baxter eqn

$$R\,T_1 T_2 = T_2 T_1\,R$$

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

Ex2. quantum enveloping algebra $U_q(sl(2))$

T q-Lie group

L q-Lie algebra

$$R^{ab}{}_{cd} = \begin{array}{c} 11 \\ 12 \\ 21 \\ 22 \end{array}\!\!\overset{\begin{array}{cccc} 11 & 12 & 21 & 22 \end{array}}{\begin{bmatrix} q & & & \\ & 1 & & \\ & q-q^{-1} & 1 & \\ & & & q \end{bmatrix}}$$

$$L^+ = \begin{bmatrix} K^{-1} & q^{\frac{-1}{2}}(q-q^{-1})X^+ \\ & K \end{bmatrix}$$

$$L^- = \begin{bmatrix} K & \\ -q^{\frac{1}{2}}(q-q^{-1})X^- & K^{-1} \end{bmatrix}$$

satisfies RLL and Yang-Baxter eqn (same R)

$$RL_1^{\pm}L_2^{\pm} = L_2^{\pm}L_1^{\pm}R$$

$$RL_1^{+}L_2^{-} = L_2^{-}L_1^{+}R$$

(FRT construction)

[Faddeev, Reshetikhin, Takhtadzhyan 90]

Quantum Groups

satisfies RTT and Yang-Baxter eqn

$$R\,T_1T_2 = T_2T_1\,R$$

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

Ex2. quantum enveloping algebra $U_q(sl(2))$

$$[H,\,X^{\pm}] = \pm 2X^{\pm}$$

$$[X^+,\,X^-] = \frac{K - K^{-1}}{q - q^{-1}}$$

$$K = q^H$$

$h \to 0$

$q = e^h \to 1$

"classical" limit

$$[H,\,X^{\pm}] = \pm 2X^{\pm}$$

$$[X^+,\,X^-] = H$$

$sl(2)$ Lie algebra

T q-Lie group

↕

L q-Lie algebra

$$L^+ = \begin{bmatrix} K^{-1} & q^{\frac{-1}{2}}(q - q^{-1})X^+ \\ & K \end{bmatrix}$$

$$L^- = \begin{bmatrix} K & \\ -q^{\frac{1}{2}}(q - q^{-1})X^- & K^{-1} \end{bmatrix}$$

(FRT construction)

[Faddeev, Reshetikhin, Takhtadzhyan 90]

Quantum Groups

satisfies RTT and Yang-Baxter eqn

$$R\,T_1 T_2 = T_2 T_1\, R$$

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}$$

Ex2. quantum enveloping algebra $U_q(sl(2))$

$$[H,\, X^{\pm}] = \pm 2 X^{\pm}$$

$$[X^+,\, X^-] = \frac{K - K^{-1}}{q - q^{-1}}$$

$$K = q^H$$

$$\Delta(X^{\pm}) = X^{\pm} \otimes K^{\frac{1}{2}} + K^{-\frac{1}{2}} \otimes X^{\pm}$$

$$\Delta(K) = K \otimes K$$

$$S(X^{\pm}) = -q^{\pm 1} X^{\pm}$$

$$S(K) = K^{-1}$$

Hopf algebra

T q-Lie group

↕

L q-Lie algebra

$$L^+ = \begin{bmatrix} K^{-1} & q^{\frac{-1}{2}}(q - q^{-1})X^+ \\ & K \end{bmatrix}$$

$$L^- = \begin{bmatrix} K & \\ -q^{\frac{1}{2}}(q - q^{-1})X^- & K^{-1} \end{bmatrix}$$

(FRT construction)

[Faddeev, Reshetikhin, Takhtadzhyan 90]

BCJ duality

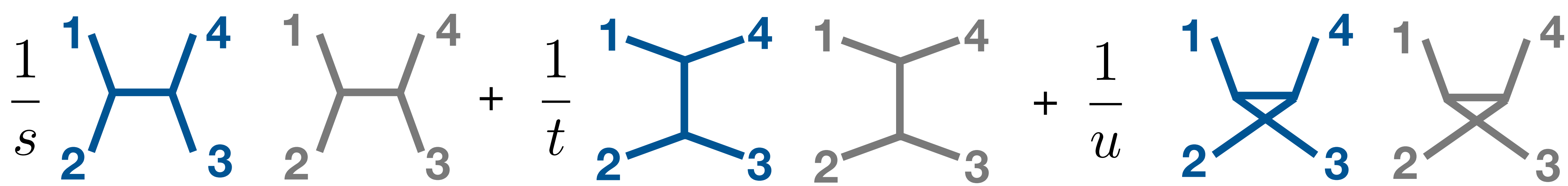
Bern, Carrasco, Johansson:
"What is the simplest QFT?"

Ans: (bi-adj) ϕ^3 theory

Z. Bern J. J. M. Carrasco H. Johansson

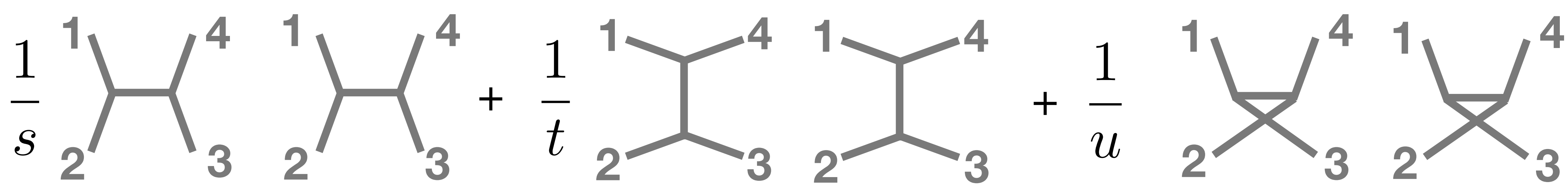
$$L \sim \phi^{a,b'} \partial^2 \phi^{a,b'} + f^{abc} \tilde{f}^{a'b'c'} \phi^{a,a'} \phi^{b,b'} \phi^{c,c'}$$

$\mathcal{A}_{YM}(1234)$ =

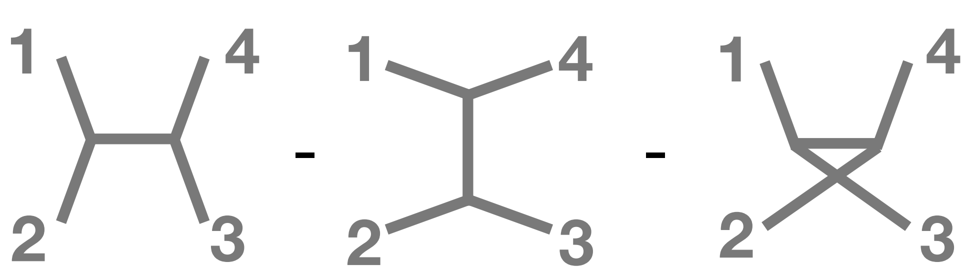


$SU(N)$ color mtm/helicity dep

$\mathcal{A}_{GR}(1234)$ =



(cf. Gang Yang's talk)



= 0 ⟷ $f^{12e} f^{e34} + f^{23e} f^{e14} + f^{31e} f^{e24} = 0$

[Bern, Carrasco, Johansson 08]

Vertex operator algebra

Classical

J_-

J_z

coproduct

$\Delta J = J \otimes 1 + 1 \otimes J$

counit

$\epsilon(J) = 0$

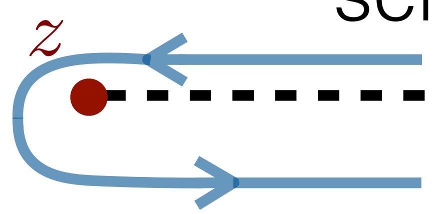
$\epsilon(1) = 1$

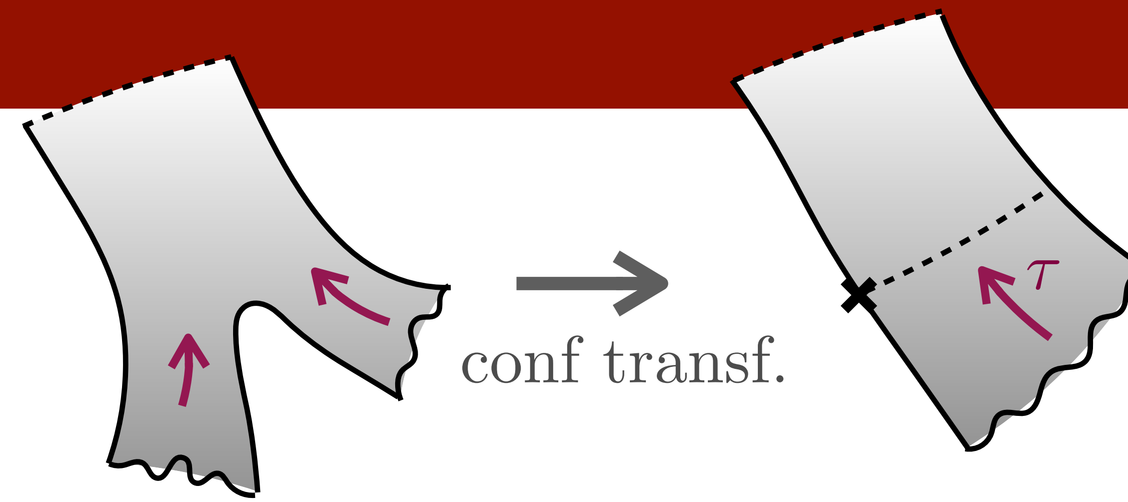
$\epsilon(e^{i\theta J}) = 1$

Quantum/Strings

$F_i = \int_C dt\, \tilde{S}_i(t)$

screening op





$|1\rangle \otimes |2\rangle$ $\qquad V_1(z)\,|2\rangle$

$H_i = \oint k_i \cdot \partial X$

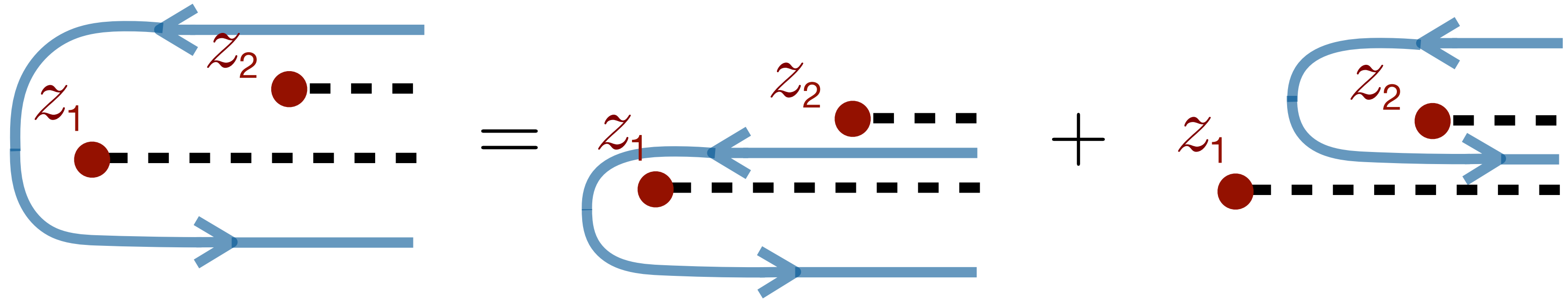
$\tilde{S}_i(t) = e^{ik_i \cdot X(t)},\ \epsilon_i \cdot \dot{X} e^{ik_i \cdot X(t)},\ \ldots$

$q = e^{-\pi i \alpha'}$

braiding relation

$\tilde{S}_i(t) V(z) = q^{k_i \cdot k_v} V(z) \tilde{S}_i(t)$

$\Delta F = F \otimes 1 + K^{-1} \otimes F$



$\epsilon(F) = 0$

$\epsilon(1) = 1$

destroys all contours

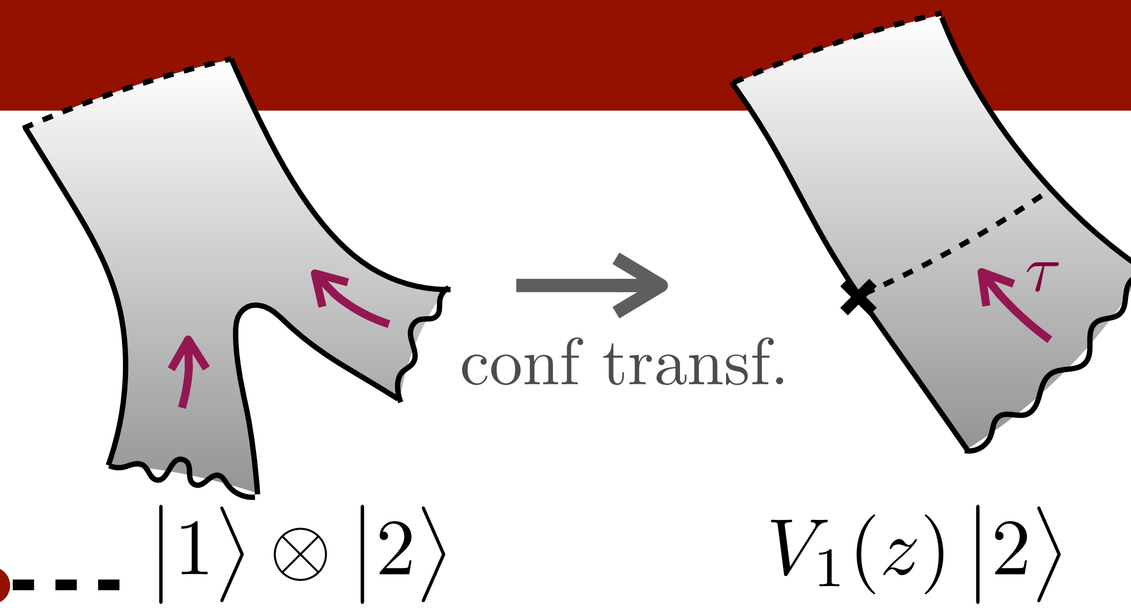
[Dotsenko, Fateev 84,85]

[Feigin, Fuks 82]

[Wakimoto 86]

Vertex operator algebra

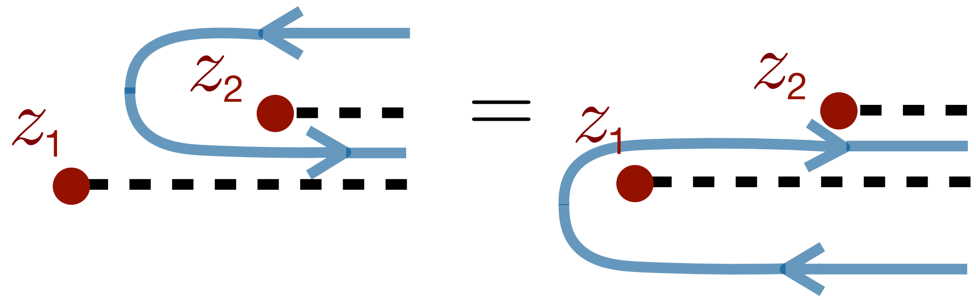
Classical | **Quantum/Strings**



antipode

$$S(J) = -J$$
$$S(e^{i\theta J}) = e^{-i\theta J}$$

$$S(F) = -KF$$



universal R-matrix

[Dotsenko, Fateev 84,85]

[Feigin, Fuks 82]

[Wakimoto 86]

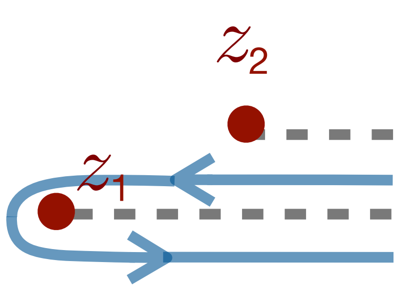
$$|1\rangle \otimes |2\rangle \xrightarrow{\sigma} |2\rangle \otimes |1\rangle \xrightarrow{\mathcal{R}} |1\rangle \otimes |2\rangle$$

$$R = \mathcal{R} \circ \sigma$$

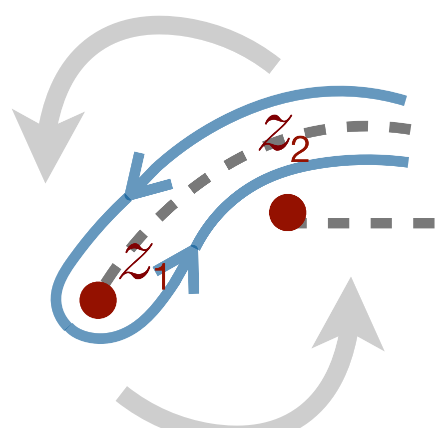
$$R = e^{\frac{H_i \otimes H_i}{2}}(1 \otimes 1 + (q_i - q_i^{-1})E_i \otimes F_i + \ldots)$$

Yang-Baxter eqn guaranteed by braiding

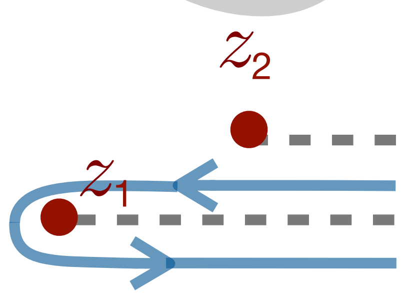
$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$



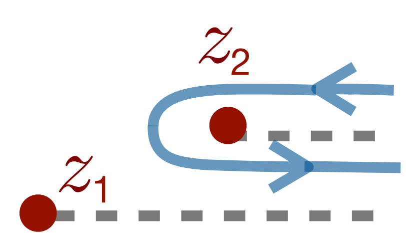
$\xrightarrow{\sigma}$



$=$



$-$



$$(e^{i\pi\alpha' s\cdot k_1} - e^{-i\pi\alpha' s\cdot k_1})$$

“What is the meaning of a polarisation in this pic?”

$$\left[\epsilon_1 \cdot \dot{X} e^{ik_1\cdot X(t_1)}\right]\left[e^{ik_2\cdot X(t_2)}\right] = q^{k_1\cdot k_2}\left[e^{ik_2\cdot X(t_2)}\right]\left[\epsilon_1 \cdot \dot{X} e^{ik_1\cdot X(t_1)}\right]$$

И. Б. Френкель   В. Г. Кац

[Frenkel, Kac 80]

affine Lie algebra $\quad g \to \tilde{g}$

$$e_\alpha, f_\alpha, h_\alpha \;\to\; e_{\alpha,n}, f_{\alpha,n}, h_{\alpha,n} \qquad n\in\mathbb{Z}$$

$$[x_n, y_m] := [x,y]_{m+n} + n\langle x, y\rangle \delta_{n,-m}1$$

repsn of affine Lie algebra

$$e^{ik\cdot X(z)} = \sum_{n\in\mathbb{Z}} e_{k,n} z^{-n}$$

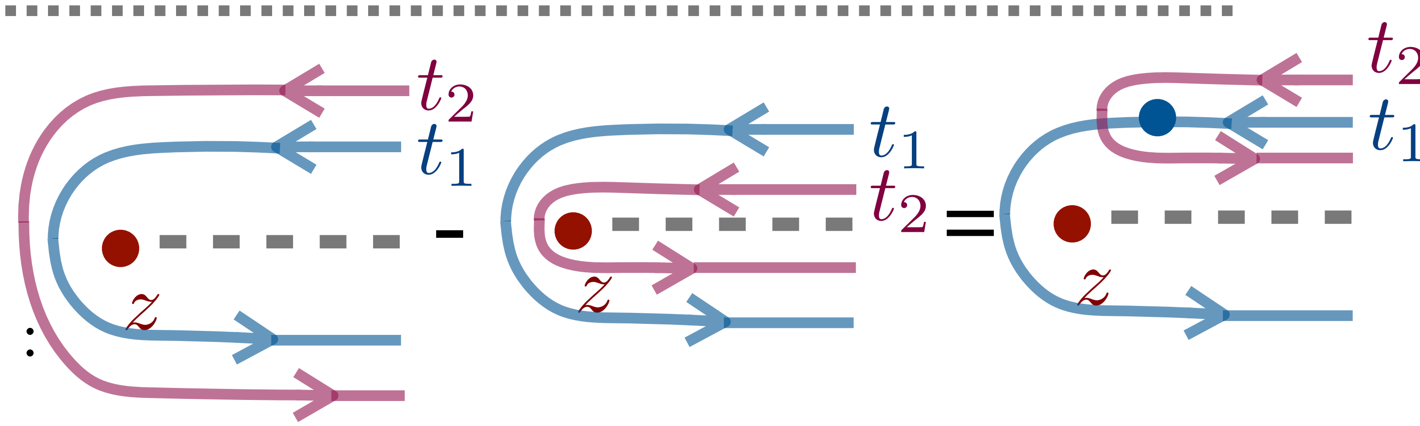
collections of α_n ‘s

$$k\cdot\dot{X}(z) = \sum_{n\in\mathbb{Z}} h_{k,n} z^{-n} = \sum_{n\in\mathbb{Z}} k\cdot\alpha_n z^{-n}$$

Cartan subalgebra

$$[e_{k_2,0}, e_{k_1,0}] = \int_{C_1} dt_1 \int_{C_2} dt_2 e^{ik_1\cdot X(t_1)} e^{ik_2\cdot X(t_2)}$$

$$= \int dt_1 \int_{C_2} dt_2 (t_1 - t_2)^{k_1\cdot k_2} : e^{ik_1\cdot X(t_1)} e^{ik_2\cdot X(t_2)} :$$



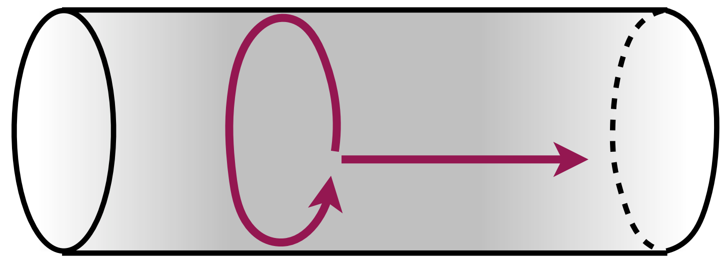
$t_2 \to t_1$ Residue theorem

$k_1\cdot k_2 = -1$ $\quad e^{i(k_1+k_2)\cdot X(t_1)}$ $\quad$ tachyon $\quad (k_1+k_2)^2 = 2-2+2 = 2$

$k_1\cdot k_2 = -2$ $\quad k_2\cdot\partial X e^{i(k_1+k_2)\cdot X(t_1)}$ $\quad$ gluon $\quad (k_1+k_2)^2 = 2-4+2 = 0$

$$k_2\cdot(k_1+k_2) = -2+2 = 0$$

“What is the meaning of a polarisation in this pic?”



И. Б. Френкель   В. Г. Кац

[Frenkel, Kac 80]

$k_i \cdot k_j$	all integers	colour $SU(N)$	$N = 1, 2, 3, \ldots$
	all non-integers	tachyons	
	some integers, some not	gluons, …	

RNS string

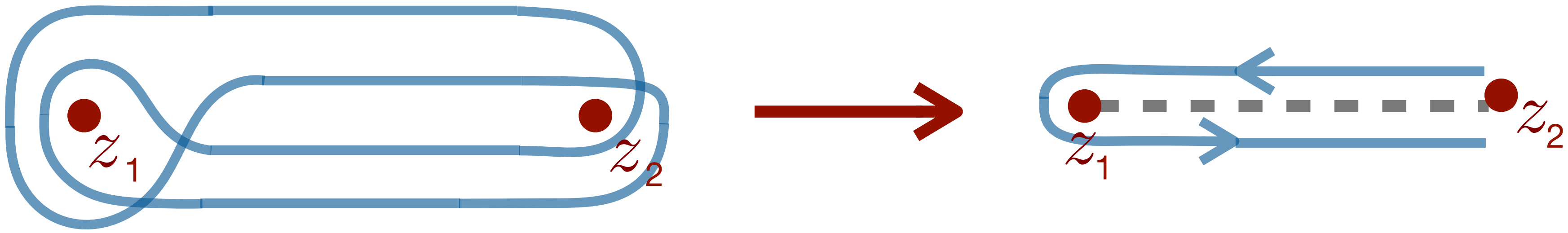
$$[e_{k_2,0}, e_{k_1,0}]_+ = \int_{C_1} dt_1 \int_{C_2} dt_2\, k_1 \cdot \psi e^{ik_1 \cdot X(t_1)}\, k_2 \cdot \psi e^{ik_2 \cdot X(t_2)}$$

$$= \int dt_1 \int_{C_2} dt_2 (t_1 - t_2)^{k_1 \cdot k_2} : \left(\frac{k \cdot k_2}{t_1 - t_2} + k_1 \cdot \psi\, k_2 \cdot \psi \right) e^{ik_1 \cdot X(t_1)} e^{ik_2 \cdot X(t_2)} :$$

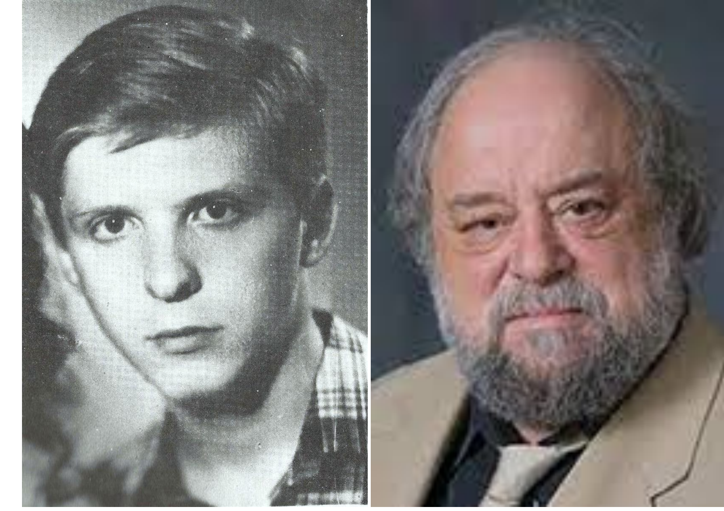
$k_1 \cdot k_2 = -1$ $(-k_2 \cdot \partial X + k_1 \cdot \psi\, k_2 \cdot \psi) e^{i(k_1 + k_2) \cdot X(t_1)}$

gluon   massless $k^2 = (k_1 + k_2)^2 = 1 - 2 + 1 = 0$

gauge inv $\epsilon \cdot k = k_2 \cdot (k_1 + k_2) = -1 + 1 = 0$



Pochhammer double loop



В. Г. Кни́жник А. Б. Замоло́дчиков

[Knizhnik, Zamolodchikov 84]

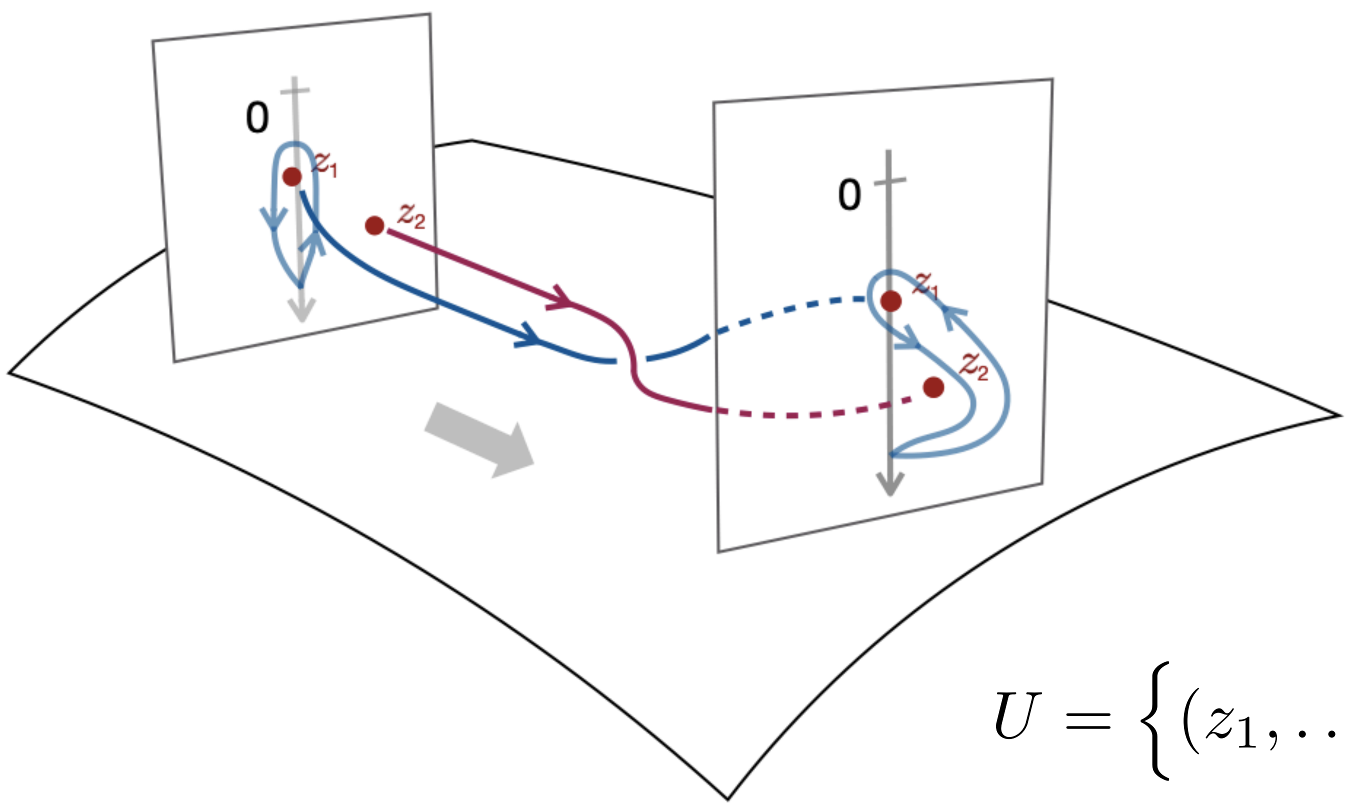
[Tsuchiya, Ueno, Yamada 92]

[Frenkel, Reshetikhin, Yu 92]

$$J_e(z) = \beta(z)$$
$$J_h(z) = -2 : \gamma(z)\beta(z) : +\sqrt{\kappa}\alpha(z)$$
$$J_f(z) = - : \gamma^2(z)\beta(z) : +\sqrt{\kappa}\alpha(z)\gamma(z) + k\frac{d}{dz}\gamma(z)$$

$$U(t) = e^{-i\alpha\cdot X(t)}\beta(t)$$
$$\Phi^m(z) := \int e^{i\mu\cdot X(z)}\exp(-\gamma(z)\otimes e)U(t_1)\dots U(t_m)\, dt_1\wedge\dots\wedge dt_m$$

$$I = \langle v^*_{\lambda_0}, \Phi^{m_1}(z_1)\dots\Phi^{m_N}(z_N)v_\lambda\rangle \qquad \in \mathbb{C}[z_i, z_i^{-1}]\otimes V_{\mu_1}\otimes\dots\otimes V_{\mu_N}$$



$$\frac{\partial}{\partial z_i} I = \frac{1}{\kappa}\sum_{i\neq j}\frac{\Omega_{i,j}}{z_i - z_j} I$$

Gauss-Manin connection

↓

K-Z connection $V_{\mu_1}\otimes\dots\otimes V_{\mu_N}$

$$U = \left\{(z_1,\dots,z_N)\in\mathbb{C}\,\middle|\, z_i\neq z_j\right\}$$

Summary & open problems

- **Screening vertex algebra**
  - algebra structure
  - relation to the KZ equations
- **Calculating hypergeometric function efficiently**
  - fractional derivatives
- **Homological structure**
  - resolution, syzygies
- **Loop level story**
  - Knizhnik–Zamolodchikov–Bernard equations
  - vertex algebra structure
- **Field theory picture**


Thank you!
спасибо большое

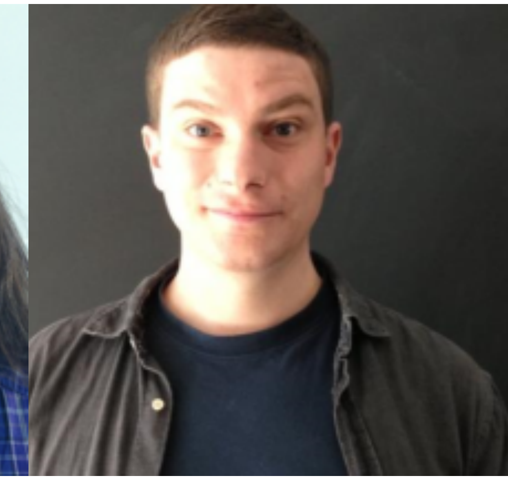
Key Point: **MANY Theories are Double Copies**

Bi-Adjoint Scalar: Bern, de Freitas, Wong ('99); Bern, Dennen, Huang; Du, Feng, Fu; Bjerrum-Bohr, Damgaard, Monteiro, O'Connell	color $\otimes$ color
(S)YM (…(S)QCD…): BCJ ('08) Bjerrum-Bohr, Damgaard, Vanhove; Steiberger; Feng et al; Mafra, Schlotterer, ('08-'11); Johansson, Ochirov	color $\otimes$ spin-1
(S)Gr (…(S)Einstein-YM…): KLT('86); BCJ ('08); Chiodaroli, Gunaydin, Johansson, Roiban; Johansson, Ochirov; Johansson, Kälin, Mogull	spin-1 $\otimes$ spin-1
NLSM / Chiral Lagrangian: Chen, Du '13 Cachazo, He, Yuan '14 Cheung, Shen '16	"color" $\otimes$ even-spin-0
(S)Born-Infeld: Cachazo, He, Yuan '14	spin-1 $\otimes$ even-spin-0
Special Galileon: Cachazo, He, Yuan '14 Cheung, Shen '16	even-spin-0 $\otimes$ even-spin-0
Open String: Broedel, Schlotterer, Stieberger	α' $\otimes$ spin-1
Closed String: Broedel, Schlotterer, Stieberger;	spin-1 $\otimes$ α' corrected spin-1
Z-theory: Broedel, Schlotterer, Stieberger; JJMC, Mafra, Schlotterer	α' $\otimes$ "color"

BCJ duality - the settings

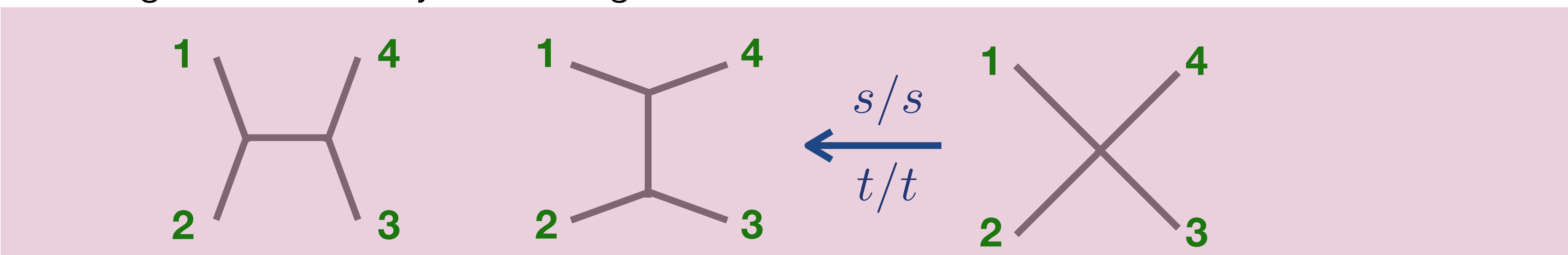
Z. Bern

J. J. M. Carrasco



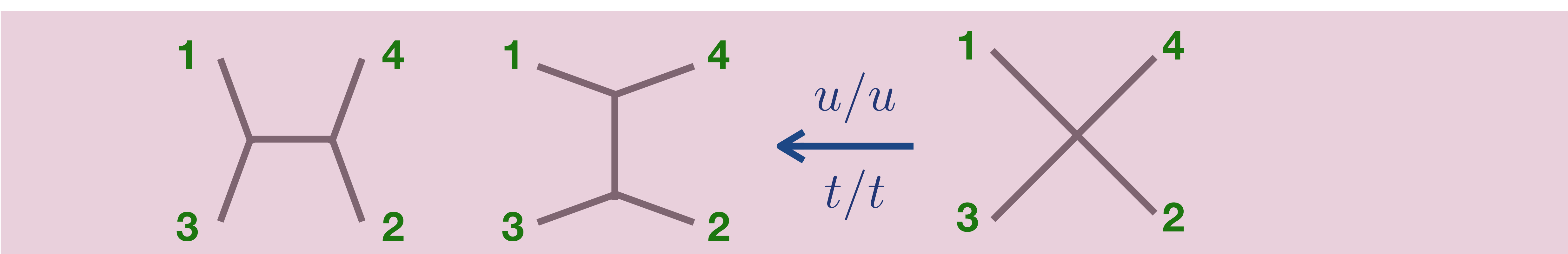
H. Johansson

Defining numerators by absorbing contact terms



$$A(1234) = \frac{n_s}{s} - \frac{n_t}{t}$$

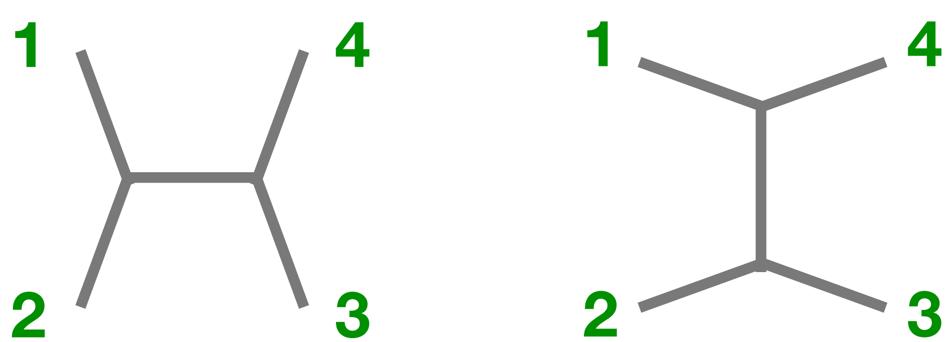
$$A(1324) = -\frac{n_u}{u} + \frac{n_t}{t}$$



BCJ duality - the settings

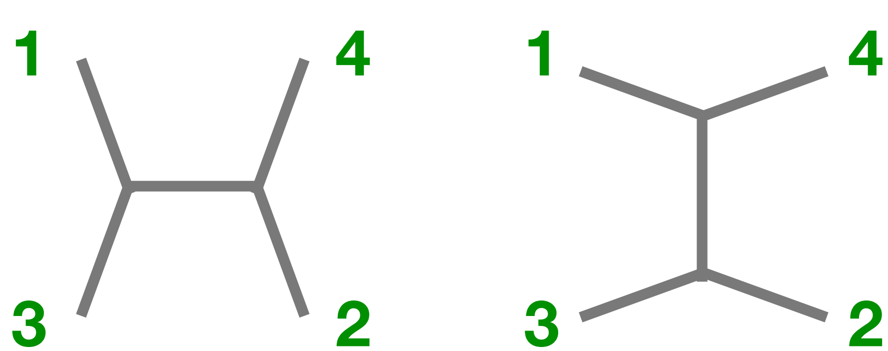
Z. Bern J. J. M. Carrasco H. Johansson

Tri-valent description of YM



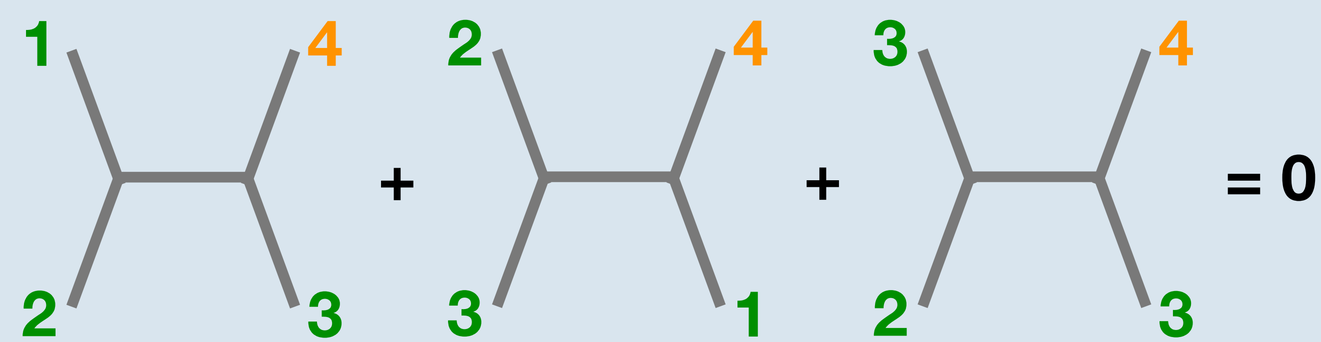
$$A(1234) = \frac{n_s}{s} - \frac{n_t}{t}$$

$$A(1324) = -\frac{n_u}{u} + \frac{n_t}{t}$$



Jacobi identity (& anti-symmetry)

$$n_s + n_t + n_u = 0$$



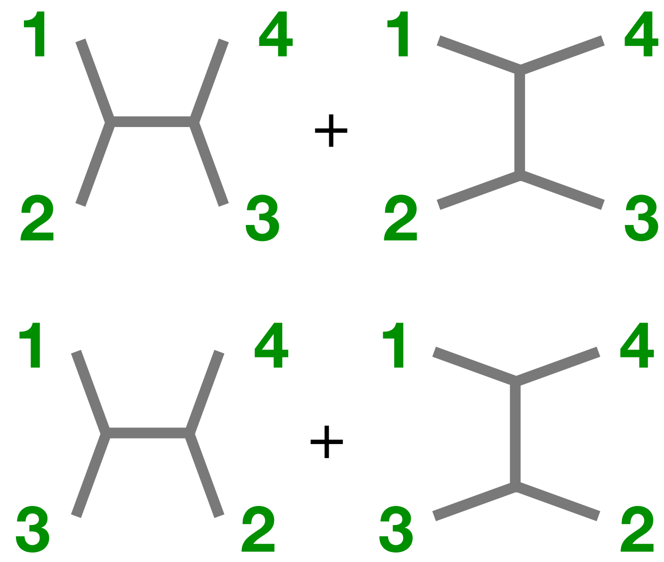
$$f^{12e} f^{e34} + f^{23e} f^{e14} + f^{31e} f^{e24} = 0$$

[Bern, Carrasco, Johansson 08]

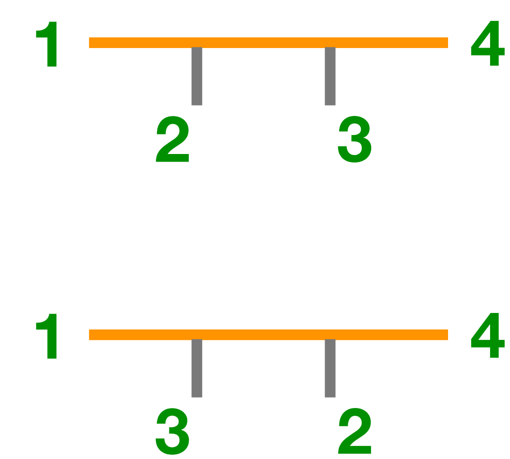
Ex: 4 pts

$A(1234) = \; + \;$

$A(1324) = \; + \;$



$$\begin{bmatrix} A(1234) \\ A(1324) \end{bmatrix} = \begin{bmatrix} \frac{1}{s_{12}} + \frac{1}{s_{23}} & \frac{-1}{s_{23}} \\ \frac{-1}{s_{23}} & \frac{1}{s_{13}} + \frac{1}{s_{23}} \end{bmatrix} \begin{bmatrix} n_{1234} \\ n_{1324} \end{bmatrix}$$



Off-shell

nonsingular

solve by taking inverse

$$n_{1\alpha n} = \frac{1}{k_n^2} S[\alpha^T|\beta]\, A(1\beta n)$$

Momentum kernel

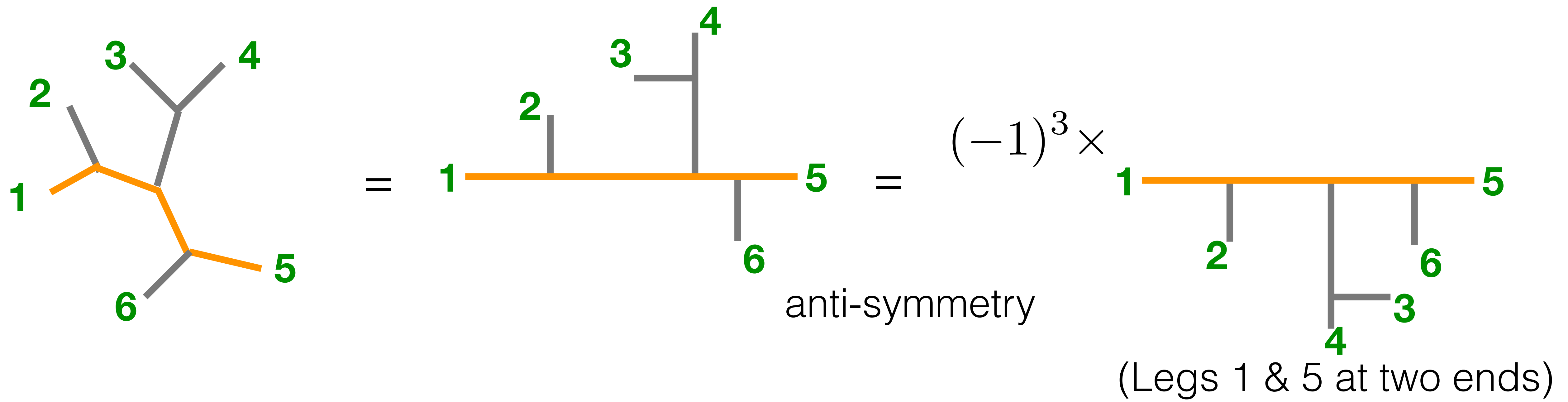
On-shell

$$\begin{bmatrix} A's \\ \\ \end{bmatrix} = \begin{bmatrix} 0 & \\ & \lambda \end{bmatrix} \begin{bmatrix} n's \\ \\ \end{bmatrix}$$

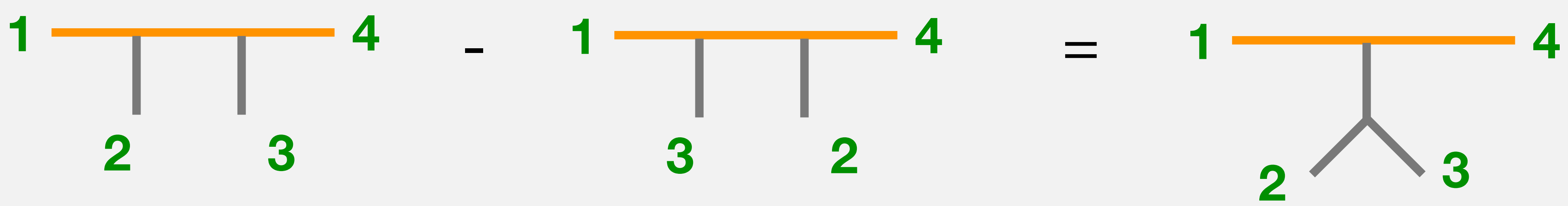
$$s_{21} A(1234) + (s_{21} + s_{23}) A(1324) = 0$$

kernel → **BCJ amplitude relations**

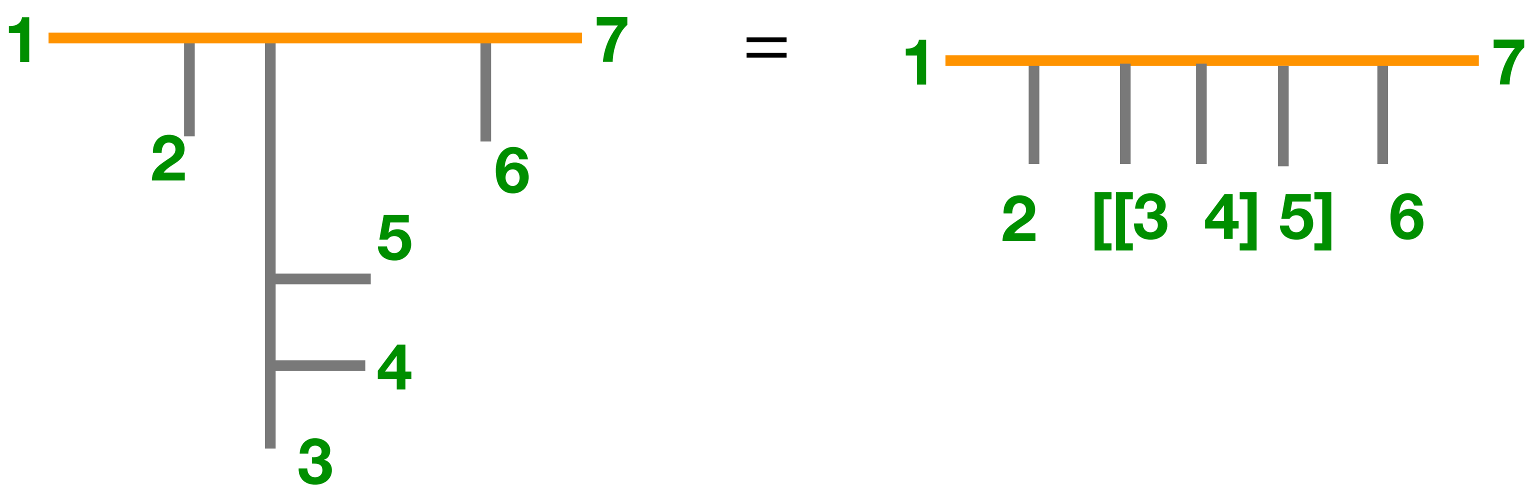
- **A basis for all cubic graphs**



- **Jacobi identity**



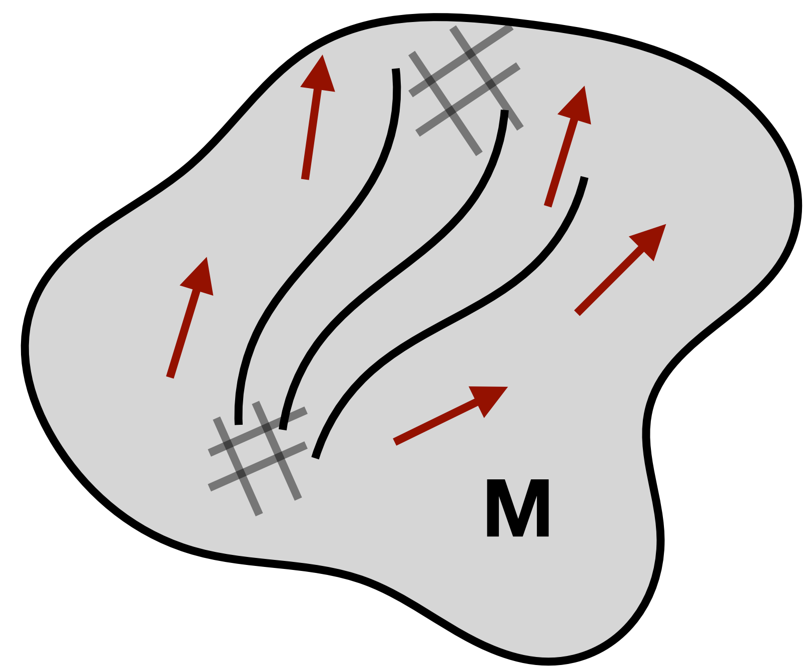
- **Generically any cubic graph can be spanned by (n-2)! half-ladders**



understanding the kinematic algebra

SDYM ⟶ diffeomorphism algebra

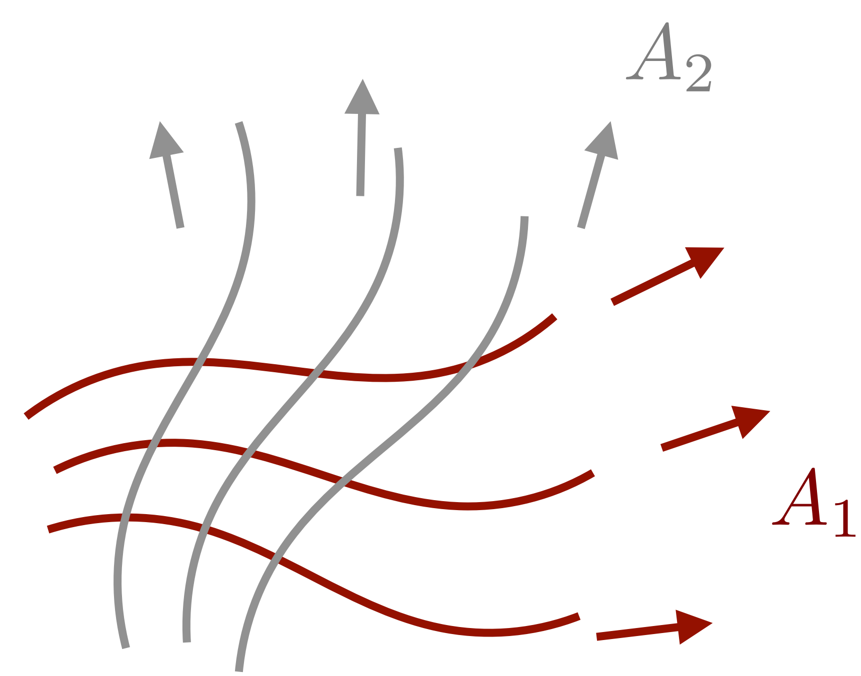
[Bjerrum-Bohr, Damgaard, Monteiro, O'Connell 13]
[Monteiro, O'Connell 11]
[Du, Feng, CF 12]
[CF, Krasnov 16]
[He, Schlotterer, Zhang 18]



Lie group manifold

group multiplication ⇄ diffeo

vector field



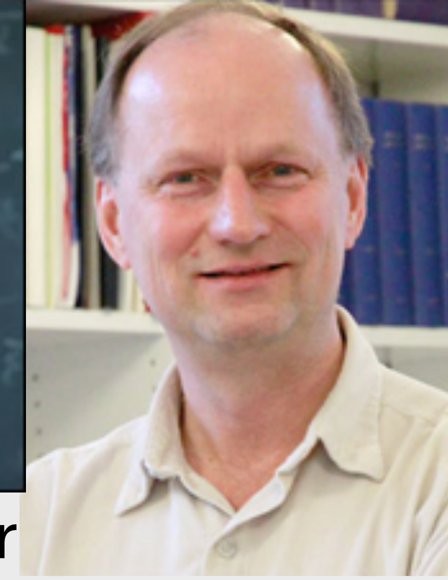
$$v_3^{\mathrm{YM}}(A_1, A_2, A_3) = [A_1, A_2]A_3 + \mathrm{cyclic}$$

$$[A_1, A_2] = (A_1^\mu \partial_\mu A_2^\nu - A_2^\mu \partial_\mu A_1^\nu)\partial_\nu$$

↳ f^{123}

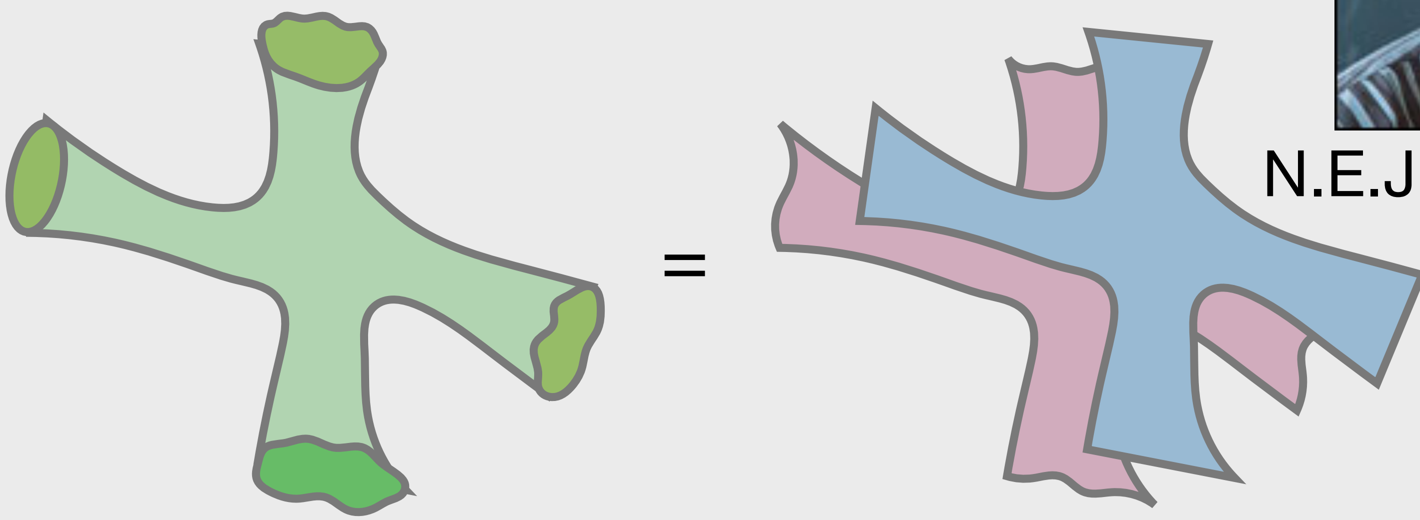
String KLT & monodromy

N.E.J.Bjerrum-Bohr



P.H.Damgaard

P.Vanhove



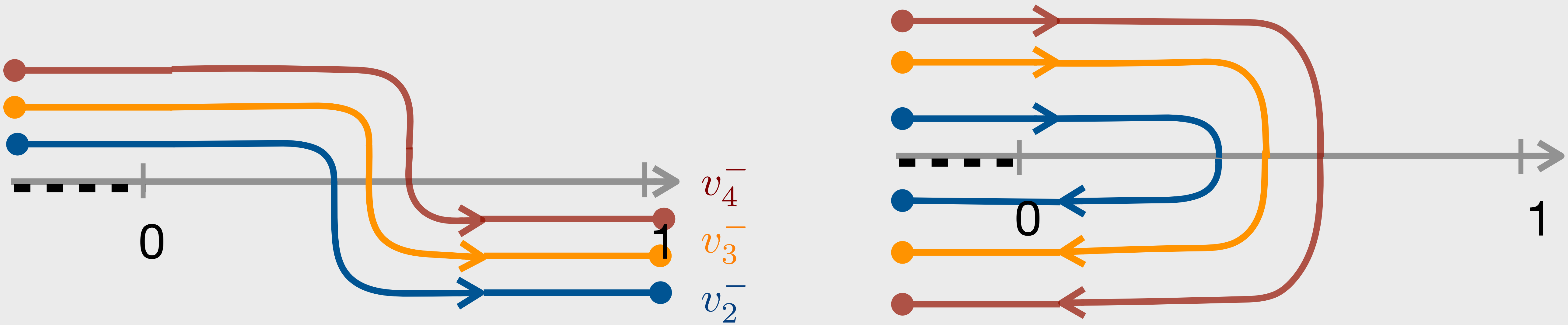
close string   open string x open string

$$\mathcal{M}_n = \sum_{\sigma} \tilde{A}_n(1,\sigma(2,\ldots,n-2),n-1,n) \times \mathcal{I}(1,\sigma(2,\ldots,n-2),n-1,n)$$

$$= \sum_{\sigma,\gamma\in S_{n-3}} \tilde{A}_n(1,\sigma(2,\ldots n-2),n-1,n) \times \mathcal{S}_{\alpha'}[\sigma^T|\gamma] \times \mathcal{A}(n-1,n,\gamma(2,\ldots,n-2),1)$$

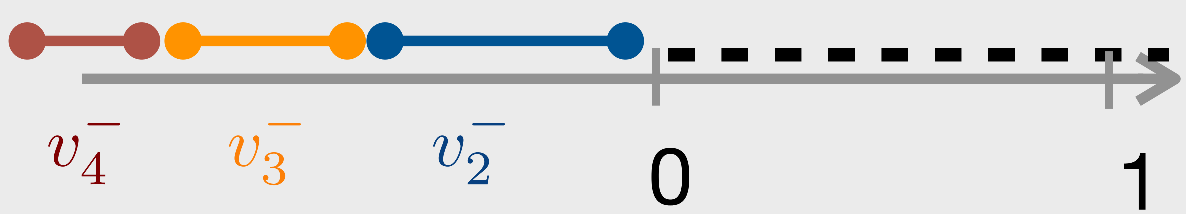
$$\mathcal{I} = \int_{-\infty}^{\infty} \prod_{i=2}^{n-2} dv_i^-(v_i^-)^{\alpha' k_i\cdot k_1}(1-v_i^-)^{\alpha' k_{n-1}\cdot k_i} \prod_{j>i}(v_j^- - v_i^-)^{\alpha' k_j\cdot k_i} f(v^-)$$

[Bjerrum-Bohr, Damgaard, Vanhove 11]



$$\int_{C_2} dv_2^- (v_2^-)^{\alpha' k_1\cdot k_2}(1-v_2^-)^{\alpha' k_{n-1}\cdot k_2}\prod_{j>2}(v_j^- - v_2^-)^{\alpha' k_j\cdot k_2} f(v^-)$$

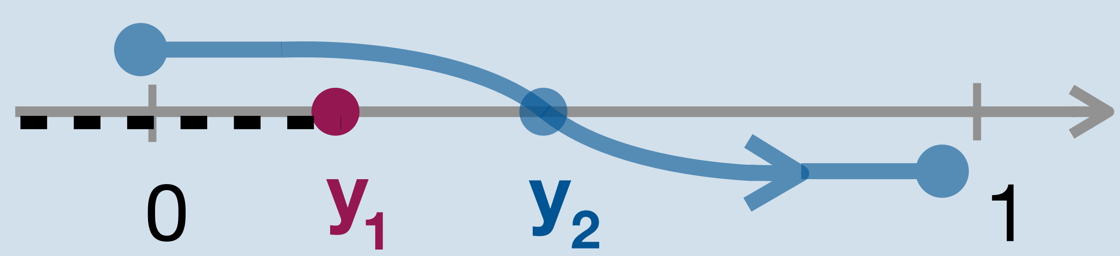
$$= 2i\sin(\pi\alpha' k_1\cdot k_2)\int_{-\infty}^{0} dv_2^- (-v_2^-)^{\alpha' k_1\cdot k_2}(1-v_2^-)^{\alpha' k_{n-1}\cdot k_2}\prod_{j>2}(v_j^- - v_2^-)^{\alpha' k_j\cdot k_2} f(v^-)$$

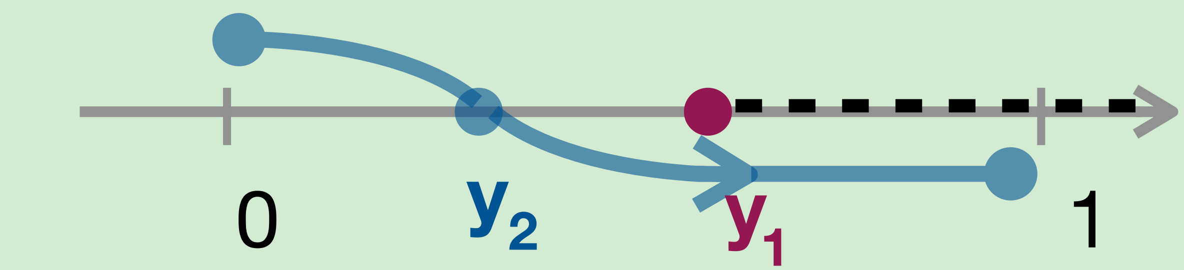


$$\int_{C_3} dv_3^- (v_3^-)^{\alpha' k_1\cdot k_3}(v_3^- - v_2^-)^{\alpha' k_3\cdot k_2}\dots$$

$$= 2i\sin(\pi\alpha' k_1\cdot k_3)\int_{v_2^-<v_3^-<0} dv_2^- (-v_3^-)^{\alpha' k_1\cdot k_3}(v_3^- - v_2^-)^{\alpha' k_3\cdot k_2}\dots$$

$$+2i\sin(\pi\alpha'(k_1+k_2)\cdot k_3)\int_{v_3^-<v_2^-} dv_2^- (-v_3^-)^{\alpha' k_1\cdot k_3}(v_2^- - v_3^-)^{\alpha' k_3\cdot k_2}\dots$$





$$\Big(\int_0^1 dy_2 \frac{V(y_2)}{y_2}\Big)\,\frac{V(y_1)}{y_1}$$

$$\sim e^{\alpha' k_1\cdot k_2\, ln\, y_2} e^{-\alpha' \sum_1^\infty \frac{1}{n}\left(\frac{y_1}{y_2}\right)^n} : V(y_1)V(y_2) :$$

$$\sim e^{\alpha' k_1\cdot k_2\, ln(y_2 - y_1)}$$

$$\longrightarrow \quad (y_2 - y_1)^{\alpha' k_1\cdot k_2}$$

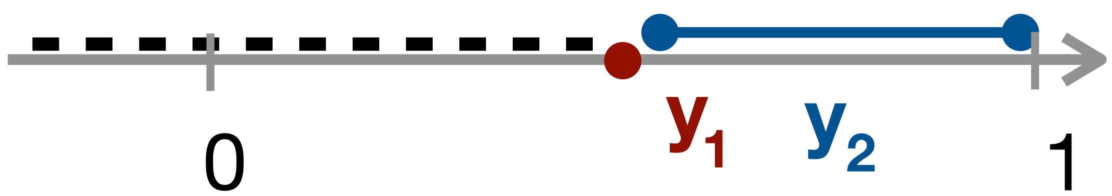
$$\frac{V(y_1)}{y_1}\,\Big(\int_0^1 dy_2 \frac{V(y_2)}{y_2}\Big)$$

$$\sim e^{\alpha' k_1\cdot k_2\, ln\, y_1} e^{-\alpha' \sum_1^\infty \frac{1}{n}\left(\frac{y_2}{y_1}\right)^n} : V(y_1)V(y_2) :$$

$$\sim e^{\alpha' k_1\cdot k_2\, ln(y_1 - y_2)}$$

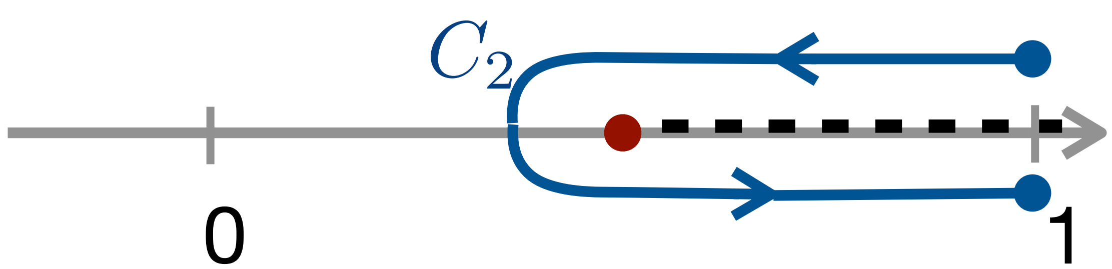
$$\longrightarrow \quad (y_1 - y_2)^{\alpha' k_1\cdot k_2}$$

[CF, Wang, Vanhove 18]



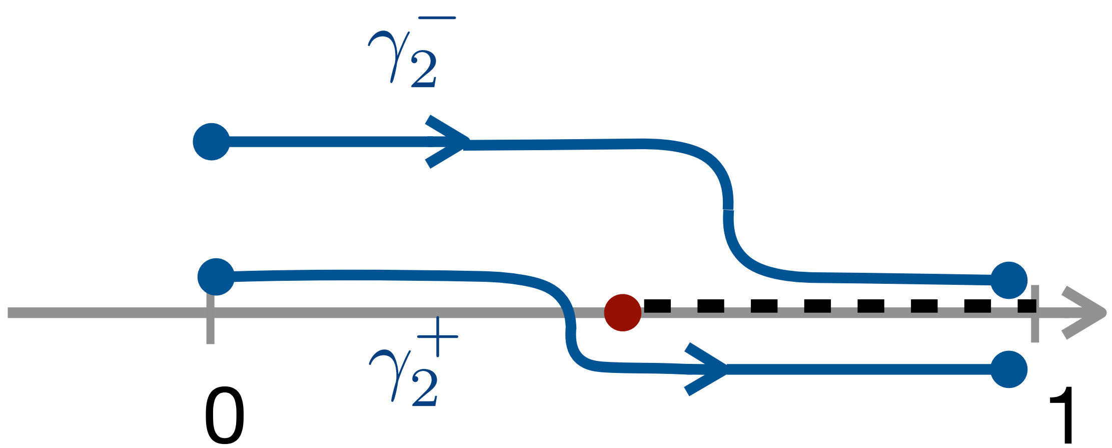
$$2i\sin(\alpha'\pi k_1\cdot k_2)\,A(123)$$

$$\sim 2i\sin(\alpha'\pi k_1\cdot k_2)\int_{y_1<y_2}\frac{dy_2}{y_2}(y_2-y_1)^{\alpha' k_1\cdot k_2}$$



$$=\int_{C_2}\frac{dy_2}{y_2}(y_1-y_2)^{\alpha' k_1\cdot k_2}$$

[CF, Wang, Vanhove 18]



$$=\int_{\gamma_2^+}-\int_{\gamma_2^-}$$

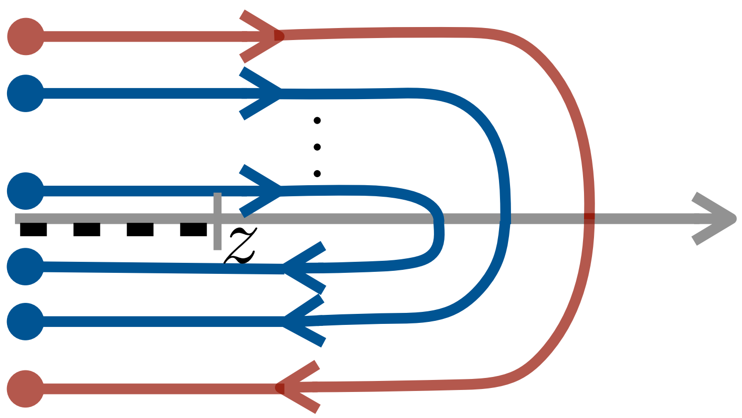
$$=\int_{\gamma_2^+}\frac{dy_2}{y_2}(y_1-y_2)^{\alpha' k_1\cdot k_2}-e^{-i\pi\alpha' k_1\cdot k_2}\int_{\gamma_2^-}\frac{dy_2}{y_2}(y_2-y_1)^{\alpha' k_1\cdot k_2}$$

$$\sim\int_0^1\frac{dy_2}{y_2}\,\frac{1}{y_1}\,V(y_1)V(y_2)-e^{-i\pi\alpha' k_1\cdot k_2}V(y_2)V(y_1)$$

$$[T_1,T_2]_{\alpha'}=T_1T_2-e^{-i\pi\alpha' k_1\cdot k_2}T_2T_1$$

Screening Vertex Operators & Nichols Algebra

Coulomb gas



$F_i \, e_{\{i_1, i_2, \ldots, i_r\}}(z)$

$$S_i(z_1) S_j(z_2) = q^{\Omega_{i\,j}} S_j(z_2) S_i(z_1)$$
$$S_i(z_1) V_a(z_2) = q^{\Omega_{i\,a}} V_a(z_2) S_i(z_1)$$

[Feigin, Fuks 82]

$$e_{\{i_1, i_2, \ldots, i_r\}}(z) = \int_{C_1} dt_1 S_{i_1}(t_1) \ldots \int_{C_r} dt_r S_{i_r}(t_r) \, V_1(z)$$
$$= F_{i_1} \ldots F_{i_r} e_1(z)$$

$$e_1(z) = V_1(z)$$

screen**ed** vertex operator

$$F_i = \int_C dt \, S_i(t)$$

screening

$$H_i = \oint \partial X_i$$

charge (momentum)

$$[H_i, \, F_j] = -\Omega_{i\,j} F_j$$

$$K_i = q^{H_i}$$

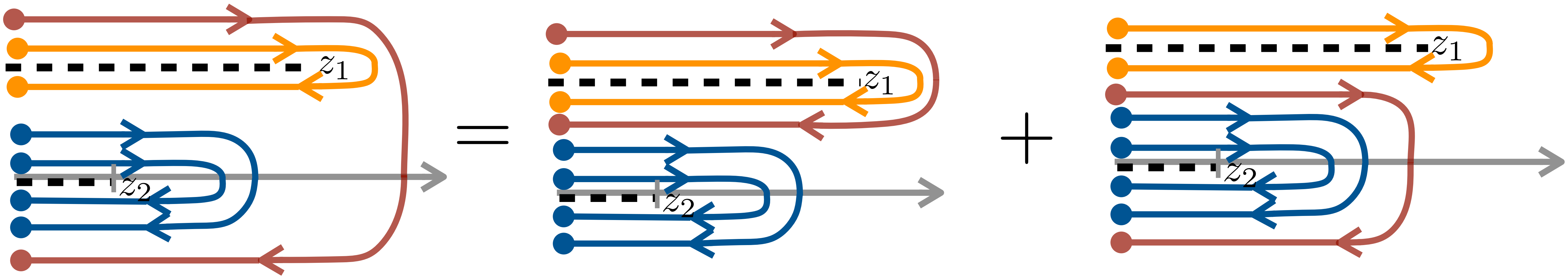
$$K_i F_j = q^{-\Omega_{i\,j}} F_j K_i$$

$$\Omega_{i\,j} = (\alpha_i, \, \alpha_j)$$

$$K_i K_j = K_j K_i$$

Screening Vertex Operators & Nichols Algebra

Coulomb gas



coproduct
2-particle representation

$$\Delta F_i \left[e_{i_1 \dots i_r}(z_1)\, e_{j_1 \dots j_s}(z_2) \right] = F_i \left[e_{i_1 \dots i_r}(z_1) \right] e_{j_1 \dots j_s}(z_2)$$

$$+ q^{\Omega_{i,\{i_1,\dots,i_r\}}} e_{i_1 \dots i_r}(z_1)\, F_i \left[e_{j_1 \dots j_s}(z_2) \right]$$

$$\longrightarrow \quad \Delta F_i = F_i \otimes I + K_i^{-1} \otimes F_i$$

$$\Delta K_i = K_i \otimes K_i$$

Screening Vertex Operators & Nichols Algebra

$$[H_i,\, H_j] = 0$$

$$[H_i,\, X_j^{\pm}] = \pm(\alpha_i,\, \alpha_j)\, X_j^{\pm}$$

$$[X_i^{+},\, X_j^{-}] = \delta_{i\,j}\, \frac{q^{H_i} - q^{-H_i}}{q - q^{-1}}$$

$\longleftrightarrow$

$$F_i = X_i^{-} q^{-H_i/2}$$

$$E_i = X_i^{+} q^{H_i/2}$$

$$K_i = q^{H_i}$$

$$\sum_{k=0}^{m} (-1)^k \begin{pmatrix} m \\ k \end{pmatrix}_q q_i^{-k(m-k)/2} \left(X_i^{\pm}\right)^k X_j^{\pm} \left(X_i^{\pm}\right)^{m-k} = 0$$

q-Serre relation

$m = 1 - A_{i\,j}$

$$\Delta(H_i) = H_i \otimes I + I \otimes H_i$$

$$\Delta(X_i^{\pm}) = X_i^{\pm} \otimes q^{-H_i/2} + q^{H_i/2} \otimes X_i^{\pm}$$

$$S(H_i) = -H_i$$

$$S(X_i^{\pm}) = q^{-\sum_i H_i} X_i^{\pm} q^{\sum_i H_i}$$

If only 1 tachyon screening

$\longrightarrow \quad U_q(sl(2))$

Quantum Groups

satisfies RTT and Yang-Baxter eqn

$$R\,T_1T_2 = T_2T_1\,R$$

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

Ex2. quantum enveloping algebra $U_q(sl(2))$

T q-Lie group

L q-Lie algebra

$$[H,\, X^{\pm}] = \pm 2X^{\pm}$$

$$[X^{+},\, X^{-}] = \frac{K - K^{-1}}{q - q^{-1}}$$

$$K = q^{H}$$

$q \to 1$

“classical” limit

$$L^{+} = \begin{bmatrix} K^{-1} & q^{\frac{-1}{2}}(q - q^{-1})X^{+} \\ & K \end{bmatrix}$$

$$L^{-} = \begin{bmatrix} K & \\ -q^{\frac{1}{2}}(q - q^{-1})X^{-} & K^{-1} \end{bmatrix}$$

$$[H,\, X^{\pm}] = \pm 2X^{\pm}$$

$$[X^{+},\, X^{-}] = H$$

$sl(2)$ Lie algebra

(FRT construction)

[Faddeev, Reshetikhin, Takhtadzhyan 90]

Monodromy relations & Yangian

quantum determinant

$$qdet\, T(u) = \overbrace{R_{n-1,n} \dots (R_{2n} \dots R_{23} R_{23})\,(R_{1n} \dots R_{13} R_{12})}^{\text{momentum kernel}}\, T_1 T_2 \dots T_n$$

$$= \epsilon^q_{\sigma_1 \sigma_2 \dots \sigma_n} T_{\sigma_1,1} T_{\sigma_2,2} \dots T_{\sigma_n,n}$$

$$= d^{(0)} + u^{-1} d^{(1)} + u^{-2} d^{(2)} \dots$$

known to generate the center of full Yangian $qdet\, T \to ZY(g)$

Ex. $U_q(sl_2)$ (1 tachyon only)

$$\tilde{L}^+(u) = L^+ - u^{-1} L^-$$

$$\tilde{L}^-(u) = L^- + u\, L^+$$

$$L^+ = \begin{bmatrix} K^{-1} & q^{\frac{-1}{2}}(q - q^{-1}) X^+ \\ & K \end{bmatrix}$$

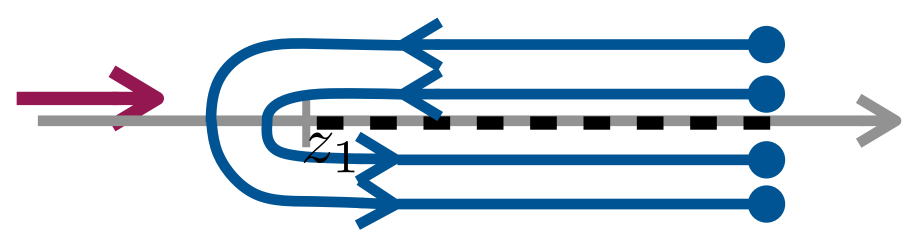
$$L^- = \begin{bmatrix} K & \\ -q^{\frac{1}{2}}(q - q^{-1}) X^- & K^{-1} \end{bmatrix}$$

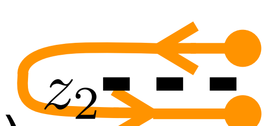
center:

$$d^{(1)} = \frac{q^{H-1} - q^{-H-1}}{q - q^{-1}} + X^+ X^-$$

$$d^{(0)} = d^{(2)} = I$$

$X^+ X^-$





world sheet symmetries
(perhaps no filed theory interpretation)

off-shell continuation

string analogue of current

$$
\begin{aligned}
\mathcal{J}(123\ldots n) \quad &:= \quad \alpha'^{n-3} g^{n-2} \frac{1}{\hat{k}_n^2} \Big\langle \hat{f} \Big| V_{n-1}(1) \frac{1}{L_0 - I} V_{n-2}(1) \ldots \frac{1}{L_0 - I} V_2(1) \Big| \hat{i} \Big\rangle \\
&= \quad \alpha'^{n-3} g^{n-2} \Big\langle \hat{f} \Big| \frac{1}{L_0 - I} V_{n-1}(1) \frac{1}{L_0 - I} V_{n-2}(1) \ldots \frac{1}{L_0 - I} V_2(1) \Big| \hat{i} \Big\rangle
\end{aligned}
$$

$$
\begin{aligned}
\sim \ \Big\langle \tilde{f} \Big| \Big(z_{n-1}^{L_0 - I} V_{n-1}(1) z_{n-1}^{-(L_0 - I)} \Big) \Big((z_{n-1} z_{n-2})^{L_0 - I} V_{n-2}(1) (z_{n-1} z_{n-2})^{-(L_0 - I)} \Big) \ldots \\
\ldots \Big((z_2 z_3 \ldots z_{n-1})^{L_0 - I} V_2(1) (z_2 z_3 \ldots z_{n-1})^{-(L_0 - I)} \Big) \Big| i \Big\rangle
\end{aligned}
$$

$$
\sim \quad \int_{0<y_2<y_3<\cdots<y_{n-1}<1} \prod_{i=2}^{n-1} dy_i \Big\langle \hat{f} \Big| \frac{V_{n-1}(y_{n-1})}{y_{n-1}} \ldots \frac{V_3(y_3)}{y_3} \, \frac{V_2(y_2)}{y_2} \Big| \hat{i} \Big\rangle
$$

explicit generators

$$n(123) = \lim_{k_3^2,\alpha'\to 0} sin(\pi\alpha' k_2 \cdot k_1)\left\langle \tilde{f}\right| : \int_0^1 \frac{dy_2}{y_2}\epsilon_2 \cdot \dot{X}(y_2)e^{ik_2\cdot X} : \left| i\right\rangle$$

$$\sim \; V_3^{YM} \int_0^1 dy_2 \frac{1}{y_2} e^{\alpha' k_1 \cdot k_2 \, ln \, y_2}$$

$$\int_0^1 dy_2 (y_2)^{\alpha' k_1 \cdot k_2 - 1} = \frac{1}{\alpha' k_1 \cdot k_2}$$

explicit generators

$$V(y) = \sum_{n=-\infty}^{\infty} e^{ik\cdot x} a_n e^{\ell k\cdot p \, ln \, y}$$

$$\int_0^1 \frac{dy}{y} V(y) = \sum_{n=-\infty}^{\infty} e^{ik\cdot x} a_n \frac{1}{k\cdot\alpha_o + n}$$

$$\begin{aligned} a_0 \quad = \quad & \cdots + \epsilon\cdot\alpha_1\Big(\ell k\cdot\alpha_{-1} - \ell^2 \sum_{n=1}^{\infty}\left(\frac{k\cdot\alpha_{-(n+1)}}{n+1}\right)\left(\frac{k\cdot\alpha_n}{n}\right) + \mathcal{O}(\ell^3)\Big) \\ & + \epsilon\cdot\alpha_0\Big(1 - \ell^2 \sum_{n=1}^{\infty}\left(\frac{k\cdot\alpha_{-n}}{n}\right)\left(\frac{k\cdot\alpha_n}{n}\right) + \mathcal{O}(\ell^3)\Big) \\ & + \epsilon\cdot\alpha_{-1}\Big(-\ell k\cdot\alpha_1 - \ell^2 \sum_{n=1}^{\infty}\left(\frac{k\cdot\alpha_{-n}}{n}\right)\left(\frac{k\cdot\alpha_{n+1}}{n+1}\right) + \mathcal{O}(\ell^3)\Big) + \dots \end{aligned}$$