Integral equations of the Heisenberg chain with a finite temperature

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Outline

- Introducation
- Exact solution of the Heisenberg chain at $T = 0$
  - Bethe ansatz solution.
  - Ground state.
- Thermodynamics of the Heisenberg Spin Chain
  - String hypothesis and TBA.
  - Quantum Transfer Matrix Method.
- Non-linear Integral equation for the partition function
  - T-W relation .
  - Non-linear Integral equation.
- Conclusion and Comments

Quantum integrable systems have many applications in

- String/ Gauge theories: AdS/CFT, Super-symmetric Yang-Mills theories...
- Statistical mechanics: The Ising model, the six-vertex models...
- Condensed Matter Physics: The super-symmetric $t - J$ Model, the Hubbard model...
- Mathematics: Quantum group, Representation theory, Algebraic Topology, ...

There are many methods to solve quantum integrable systems (The case of $T = 0$):

- The Coordinate Bethe Ansatz method (H. Bethe 1931)
- The Baxter's $T - Q$ relation method (R. Baxter 1970s)
- The Quantum Inverse Scattering (or Algebraic Bethe Ansatz) method (L. Faddeev's School 1979s) and its generalizations
- The off-diagonal Bethe Ansaz method (2013s)

There are many methods to solve quantum integrable systems (With a finite T):

- The Thermodynamical Bethe Ansatz method (C.N Yang & C.P. Yang 1969)
- TBA based on the string hypothesis (M. Takahashi 1971s)
- The Quantum transfer matrix method (A. Klumper et al 1992s)
- A generalized version of QTM

The Hamiltonian of the closed Heisenberg chain is

$$H = \sum_{k=1}^{L} \left(\sigma_k^x \, \sigma_{k+1}^x + \sigma_k^y \, \sigma_{k+1}^y + \sigma_k^z \, \sigma_{k+1}^z \right),$$

where

$$\sigma_{L+1}^{\alpha} = \sigma_1^{\alpha}, \quad \alpha = x, \, y, \, z.$$

The system is **integrable**, i.e., there exist enough conserved charges

$$i\hbar \frac{\partial}{\partial t} h_i = [H, h_i] = 0, \qquad i = 1, \ldots .$$

and

$$[h_i, h_j] = 0.$$

It is convenient to introduce a generation function of these charges, the so-called transfer matrix

$$t(u) = \sum_{i=0} h_i u^i.$$

Then

$$[t(u),\ t(v)] = 0, \qquad H = 2\eta \frac{\partial}{\partial u} \ln t(u)|_{u=0} - L,$$

or

$$H \propto h_0^{-1} h_1 + const,$$
$$h_0\, \sigma_i^{\alpha}\, h_0^{-1} = \sigma_{i+1}^{\alpha}.$$

The eigenstates and the corresponding eigenvalues can be obtained by Quantum Inverse Scattering Method (QISM). In the framework of QISM, the monodromy matrix $T(u)$

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix},$$

has played a central role. It is built from the six-vertex R-matrix of

$$T_0(u) = R_{0L}(u) \ldots R_{01}(u),$$

where the well-known six-vertex R-matrix is given by

$$R(u) = \frac{1}{\eta} \begin{pmatrix} u+\eta & & & \\ & u & \eta & \\ & \eta & u & \\ & & & u+\eta \end{pmatrix}.$$

The transfer matrix is $t(u) = trT(u) = A(u) + D(u)$.

The R-matrix satisfies the Yang-Baxter equation (YBE)

$$R_{12}(u-v)\,R_{13}(u)\,R_{23}(v) = R_{23}(v)\,R_{13}(u)\,R_{12}(u-v). \tag{1}$$

The above fundamental relation leads to the following so-called RLL relation between the monodromy matrix

$$R_{0\,0'}(u-v)\,T_0(u)\,T_{0'}(v) = T_{0'}(v)\,T_0(u)\,R_{0\,0'}(u-v). \tag{2}$$

This leads to

$$[t(u), t(v)] = 0, \tag{3}$$

which ensures the integrability of the Heisenberg chain with periodic boundary condition.

The eigenvalue $\Lambda(u)$ of the transfer matrix $t(u)$ can be parameterized by some parameters $\{\lambda_1, \cdots, \lambda_M | M = 0, \cdots, L\}$ as follows (H. Bethe, Z. Phys. 71, 205 (1931)):

$$\Lambda(u) = a(u)\frac{Q(u-\eta)}{Q(u)} + d(u)\frac{Q(u+\eta)}{Q(u)}, \tag{4}$$

where

$$a(u) = (u+\eta)^L = d(u+\eta), \quad Q(u) = \prod_{j=1}^{M}(u - \lambda_j + \frac{\eta}{2}),$$

the parameters $\{\lambda_i\}$ should satisfy Bethe ansatz equations (BAEs),

$$\prod_{k\neq j}^{M} \frac{\lambda_j - \lambda_k + \eta}{\lambda_j - \lambda_k - \eta} = \frac{(\lambda_j + \frac{\eta}{2})^L}{(\lambda_j - \frac{\eta}{2})^L}, \qquad j = 1, \ldots, M. \tag{5}$$

For the ground state the corresponding Bethe roots to the BAEs (5) (with $M = \frac{L}{2}$) are all real with $\lambda_{j+1} - \lambda_j \sim (1/L)$, which allows to define the density $\rho_g(\lambda)$ of the distribution of Bethe roots. The density satisfy the linear integral equation

$$\rho_g(\lambda) = a_1(\lambda) - \int_{-\infty}^{\infty} a_2(\lambda - u)\rho_g(u)\, du, \quad a_n(\lambda) = \frac{1}{2\pi} \frac{n}{\lambda^2 + \frac{n^2}{4}}, \text{ for } n > 0. \tag{6}$$

The solution to the above equation is

$$\rho_g(\lambda) = \frac{1}{2\cosh(\pi\lambda)}. \tag{7}$$

Then the energy density of the ground state is give

$$e_g = -2\pi \int_{-\infty}^{\infty} 2a_1(\lambda)\rho_g(\lambda)\, d\lambda + 1 = 1 - 4\ln 2. \tag{8}$$

e_g is also the energy density of the Heisenberg chain at $T = 0$.

Besides of the real solution of the BAEs (5), there exist many complex solution which form some n-strings

$$\lambda_{j,\alpha}^{(n)} = \lambda_{\alpha}^{(n)} - \frac{i}{2}(n+1-2j) + O(e^{-\delta L}), \quad j = 1, 2, \cdots, n; \quad n = 1, 2 \cdots, \tag{9}$$

where $\lambda_{\alpha}^{(n)}$ is real, which corresponds to the n-string position.

- At zero temperature, the system stays at the ground state. There are only 1-strings whose density is e_g.
- At a finite temperature T, the system stays at all the states with a possibility. This means that all n-strings are excited with a density $(\rho_n(\lambda), \rho^{(h)}(\lambda))$.

At the thermodynamics equilibrium state with a finite temperature T, the resulting functions $\{\eta_n(\lambda) = \frac{\rho^{(h)}(\lambda)}{\rho(\lambda)} | n = 1, 2, \cdots\}$ satisfy the associated thermodynamic Bethe ansatz (TBA) equations:

$$\ln\{1 + \eta_n(\lambda)\} = -2\pi\beta a_n(\lambda) + \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} A_{m,n}(\lambda - u) \ln\{1 + \eta_m^{-1}(u)\} \, du, \; n = 1, 2, \cdots, \quad (10)$$

with an asymptotical behavior $\lim_{n\to\infty} \frac{\ln \eta_n(\lambda)}{n} = 0$. The free energy per site is then given by

$$f(\beta) = e_g - \beta \int_{-\infty}^{\infty} \rho_g(\lambda) \ln\{1 + \eta_1(\lambda)\} \, d\lambda.$$

The partition function $Z(\beta)$ of the Heisenberg chain at a temperature T is given by

$$\begin{aligned} Z(\beta) &= \lim_{L\to\infty} tr_{1,\cdots,L}\left\{e^{-\beta H}\right\} = \lim_{L\to\infty} tr_{1,\cdots,L}\left\{e^{-\beta(H+L-L)}\right\} \\ &= e^{\beta L}\lim_{L\to\infty} tr_{1,\cdots,L}\left\{e^{-\beta(H+L)}\right\} \\ &= e^{\beta L}\lim_{L\to\infty} tr_{1,\cdots,L}\left\{\lim_{N\to\infty}\left\{1-\frac{2\beta}{N}(H+L)+O(\frac{1}{N^2})\right\}^{\frac{N}{2}}\right\} \\ &= e^{\beta L}\lim_{L\to\infty}\lim_{N\to\infty} tr_{1,\cdots,N}\left\{\left\{t^{(Q)}(0)\right\}^{L}\right\} \\ &= e^{\beta L}\lim_{L\to\infty}\lim_{N\to\infty}\left\{\Lambda^{(Q)}(0)_{max}\right\}^{L} \\ &= e^{\beta L}\lim_{L\to\infty}\lim_{N\to\infty}\left\{\Lambda^{(Q)}(0)_{max}\right\}^{L}, \end{aligned} \tag{11}$$

where N is a large even integer.

The corresponding quantum transfer matrix $t^{(Q)}(u)$ of the Heisenberg chain at a temperature T is given by a inhomogeneous quantum chain

$$\begin{aligned} t^{(Q)}(u) &= tr_0\left\{(R_{0\,N}(u-\frac{2\eta\beta}{N})\,R_{0\,N-1}(u+\frac{2\eta\beta}{N}-\eta))\ldots\right. \\ &\left.\times\,(R_{0\,2}(u-\frac{2\eta\beta}{N})\,R_{0\,1}(u+\frac{2\eta\beta}{N}-\eta))\right\}. \end{aligned} \quad (12)$$

The quantum transfer matrix $t^{(Q)}(u)$ can be obtained by the algebraic Bethe Ansatz method, where $\Lambda^{(Q)}(u)$ is given in terms of a homogeneous $T-Q$ relation, namely,

$$\Lambda^{(Q)}(u) = a^{(Q)}(u)\frac{Q(u-\eta)}{Q(u)} + d^{(Q)}(u)\frac{Q(u+\eta)}{Q(u)}, \quad (13)$$

$$a^{(Q)}(u) = \left\{u-\frac{2\eta\beta}{N}+\eta\right\}^{\frac{N}{2}}\left\{u+\frac{2\eta\beta}{N}\right\}^{\frac{N}{2}} = d^{(Q)}(u+\eta), \quad (14)$$

$$Q(u) = \prod_{j=1}^{M}(u-\lambda_j), \quad M=0,\cdots,N.$$

The Bethe roots (i.e., roots of $Q(u)$) satisfy the Bethe ansatz equations:

$$\frac{d^{(Q)}(\lambda_j)}{a^{(Q)}(\lambda_j)} = -\frac{Q(\lambda_j - \eta)}{Q(\lambda_j + \eta)}, \quad j = 1, \cdots, M; \quad M = 0, \cdots, N. \tag{15}$$

It was shown that the largest eigenvalue $|\Lambda^{(Q)}(0)|_{max}$ belongs to the sector of $M = \frac{N}{2}$ with all the Bethe roots being real, and that it has a finite gap different from the other eigenvalues in the limit $N \to \infty$.

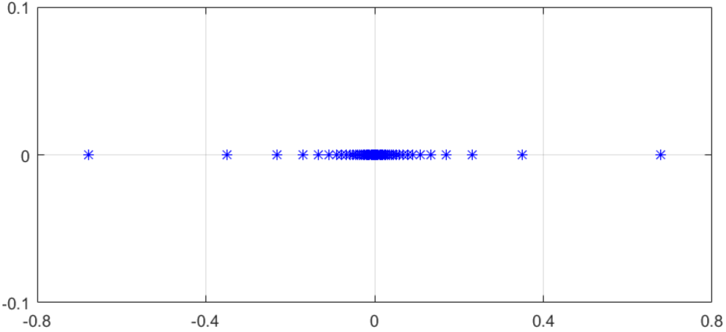

Figure 1: Bethe roots

Then the free energy per site $f(\beta)$ is given in terms of $\bar{\Lambda}^{(Q)}(u)$ by

$$\begin{aligned} f(\beta) &= -\frac{1}{\beta}\lim_{L\to\infty}\lim_{N\to\infty}\frac{1}{L}\left(\ln Z(\beta)\right) \\ &= -1-\frac{1}{\beta}\lim_{L\to\infty}\lim_{N\to\infty}\left\{\ln\Lambda^{(Q)}(0)_{max}\right\}. \end{aligned} \tag{16}$$

Basing on the distribution of the Bethe roots for the state corresponding to $\Lambda^{(Q)}_{max}(0)$ (see Fig.1), Klumper et al developed a method which needs to introduce two auxiliary functions to be satisfy two nonlinear integral equations.

It can be proven that the transfer matrix $t(u)$ satisfies the relations:

$$t^{(Q)}(u)\, t^{(Q)}(u-\eta) = a^{(Q)}(u)\, d^{(Q)}(u-\eta) \times \mathrm{id} + d^{(Q)}(u)\, \mathbb{W}(u), \qquad (17)$$

$$t^{(Q)}(u) = 2\, u^N \times \mathrm{id} + \cdots, \quad u \to \infty, \qquad (18)$$

where $\mathbb{W}(u)$, as a function of u, is a operator-valued polynomial of degree N, which actually is some fused transfer matrix of the fundamental one. The transfer matrices $t^{(Q)}(u)$ and $\mathbb{W}(u)$ commute with each other, namely,

$$[t^{(Q)}(u),\, t^{(Q)}(v)] = [\mathbb{W}(u),\, \mathbb{W}(v)] = [t(u),\, \mathbb{W}(v)] = 0.$$

Let $|\Psi\rangle$ be a common eigenstate of the transfer matrices with the eigenvalues $\Lambda^{(Q)}(u)$ and $W(u)$, namely,

$$t^{(Q)}(u)\, |\Psi\rangle = \Lambda^{(Q)}(u)\, |\Psi\rangle, \quad \mathbb{W}(u)\, |\Psi\rangle = W(u)\, |\Psi\rangle.$$

The relation (17) gives rise to that the corresponding eigenvalues satisfy the t-W relation

$$\Lambda^{(Q)}(u)\,\Lambda^{(Q)}(u-\eta) = a^{(Q)}(u)\,d^{(Q)}(u-\eta) + d^{(Q)}(u)\,W(u). \tag{19}$$

The polynomials $\Lambda^{(Q)}(u)$ and $W(u)$ have the decompositions

$$\Lambda^{(Q)}(u) = 2\prod_{j=1}^{N}(u-z_j), \tag{20}$$

$$W(u) = 3\prod_{j=1}^{N}(u-w_j). \tag{21}$$

Taking u at the $2N$ points $\{z_j|j=1,\cdots,N\}$ and $\{w_j|j=1,\cdots,N\}$ gives rise to the Bethe-Ansatz-Like equations (BAEs)

$$a^{(Q)}(z_j)\,d^{(Q)}(z_j-\eta) = -d^{(Q)}(z_j)\,W(z_j), \quad j=1,\cdots,N, \tag{22}$$

$$a^{(Q)}(w_j)\,d^{(Q)}(w_j-\eta) = \Lambda^{(Q)}(w_j)\Lambda(w_j-\eta), \quad j=1,\cdots,N. \tag{23}$$

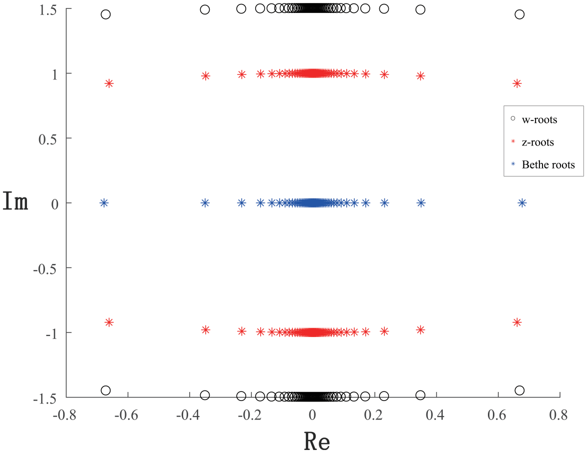

Figure 2: Various roots of the state with $|\Lambda^{(Q)}_{max}(0)|$

Numerical study with some N (up to 100) shows that the roots of $\Lambda^{(Q)}(u)$ for the state with the maximus $\Lambda^{(Q)}(0)$ has the special distribution as Fig. 2 (or for $W(u)$). For a convenience, let us introduce a normalized eigenvalue $\bar{\Lambda}^{(Q)}(u)$

$$\begin{aligned}\bar{\Lambda}^{(Q)}(u) &= \frac{\Lambda^{(Q)}(u)}{(u-\eta\tau+\eta)^M(u+\eta\tau-\eta)^M} \\ &= 2\,\frac{\prod_{j=1}^{M}(u-u_j^{(+)}-\eta)\,(u-u_j^{(-)}+\eta)}{(u+\eta\tau-\eta)^M\,(u-\eta\tau+\eta)^M}, \qquad (24)\end{aligned}$$

where $\tau=\frac{\beta}{M}=\frac{1}{MT}=\frac{2\beta}{N}$. Meanwhile, the corresponding $W(u)$ has decomposition

$$W(u)=3\prod_{j=1}^{M}(u-w_j^{(+)}-2\eta)(u-w_j^{(-)}+\eta). \qquad (25)$$

Here $\mathrm{Im}(u_j^{(\pm)})\sim 0$ and $\mathrm{Im}(w_j^{(\pm)})\sim 0$ for a large N.

Then the t-W relation (19) for the state with the maximus $\Lambda^{(Q)}(0)$ becomes

$$\bar{\Lambda}^{(Q)}(u+\frac{\eta}{2})\bar{\Lambda}^{(Q)}(u-\frac{\eta}{2})$$

$$=4\frac{\prod_{j=1}^{M}(u-u_j^{(-)}+\frac{3}{2}\eta)(u-u_j^{(+)}-\frac{\eta}{2})(u-u_j^{(-)}+\frac{\eta}{2})(u-u_j^{(+)}-\frac{3}{2}\eta)}{(u-\eta\tau+\frac{3}{2}\eta)^M(u+\eta\tau-\frac{\eta}{2})^M(u-\eta\tau+\frac{\eta}{2})^M(u+\eta\tau-\frac{3}{2}\eta)^M}$$

$$=\frac{(u+\eta\tau+\frac{\eta}{2})^M(u-\eta\tau-\frac{\eta}{2})^M}{(u-\eta\tau+\frac{\eta}{2})^M(u+\eta\tau-\frac{\eta}{2})^M}$$

$$+3\frac{\prod_{j=1}^{M}(u-w_j^{(+)}-\frac{3}{2}\eta)(u-w_j^{(-)}+\frac{3}{2}\eta)}{(u+\eta\tau-\frac{3}{2}\eta)^M(u-\eta\tau+\frac{3}{2}\eta)^M}$$

$$=e^{\frac{2\beta}{u^2+\frac{1}{4}}}+3\bar{w}(u)+O(\frac{1}{N})\overset{\text{def}}{=}q(u)+3\bar{w}(u)+O(\frac{1}{N}). \quad (26)$$

The functions $q(u)$ and $\bar{w}(u)$ are

$$q(u) = \lim_{M\to\infty} \frac{(u+\eta\tau+\frac{\eta}{2})^M(u-\eta\tau-\frac{\eta}{2})^M}{(u-\eta\tau+\frac{\eta}{2})^M(u+\eta\tau-\frac{\eta}{2})^M} = e^{\frac{2\beta}{u^2+\frac{1}{4}}}, \quad (27)$$

$$\bar{w}(u) = \lim_{M\to\infty} \frac{\prod_{j=1}^{M}(u-w_j^{(+)}-\frac{3}{2}\eta)(u-w_j^{(-)}+\frac{3}{2}\eta)}{(u+\eta\tau-\frac{3}{2}\eta)^M(u-\eta\tau+\frac{3}{2}\eta)^M} \stackrel{\text{def}}{=} e^{-\beta\,\bar{\epsilon}(u)}. \quad (28)$$

The function $\bar{\epsilon}(u)$ satisfies the analytic and asymptotic properties:

$$\bar{\epsilon}(u) \text{ is analytic except some singularities on the axis } \mathrm{Im}(u) = \pm\frac{3}{2}, \quad (29)$$

$$\lim_{u\to\infty} \bar{\epsilon}(u) = 0. \quad (30)$$

The decomposition (24) and the very $t - W$ relation (26) allow us to give an integral representation of $\bar{\Lambda}^{(Q)}(u)$

$$\ln \bar{\Lambda}^{(Q)}(u) = \ln 2 + \frac{1}{2\pi i}\oint_{\mathcal{C}_1} dv\, \frac{\ln\left((q(v) + 3e^{-\beta\bar{\epsilon}(v)})/4\right)}{u - v - \frac{\eta}{2}}$$

$$+\frac{1}{2\pi i}\oint_{\mathcal{C}_2} dv\, \frac{\ln\left((q(v) + 3e^{-\beta\bar{\epsilon}(v)})/4\right)}{u - v + \frac{\eta}{2}}, \qquad (31)$$

where the closed integral contour $\mathcal{C}_1$ is surrounding the axis of $\mathrm{Im}(v) = \frac{1}{2}$, while $\mathcal{C}_2$ is surrounding the axis of $\mathrm{Im}(v) = -\frac{1}{2}$. This leads to an integral equation of the function $\bar{\epsilon}(u)$

$$\ln(q(u) + 3e^{-\beta\bar{\epsilon}(u)}) = 2\ln 2 + \frac{1}{2\pi i}\oint_{\mathcal{C}_1} dv\Big(\frac{1}{u-v} + \frac{1}{u-v-\eta}\Big)\ln\Big((q(v) + 3e^{-\beta\bar{\epsilon}(v)})/4\Big)$$

$$+\frac{1}{2\pi i}\oint_{\mathcal{C}_2} dv\Big(\frac{1}{u-v+\eta} + \frac{1}{u-v}\Big)\ln\Big((q(v) + 3e^{-\beta\bar{\epsilon}(v)})/4\Big). \qquad (32)$$

Due to the fact the roots and the poles of $\bar{\Lambda}^{(Q)}(u)$ locate nearly on the two lines with imaginary parts close to $\pm\eta$ (see the decomposition (24)), we can use the Fourier transformation to obtain another integral representation of $\bar{\Lambda}^{(Q)}(u)$

$$\ln \bar{\Lambda}^{(Q)}(u) = \int_{-\infty}^{+\infty} \frac{dv}{2\cosh \pi(u-v)} \left\{ \ln \bar{\Lambda}^{(Q)}(v+\frac{\eta}{2}) + \ln \bar{\Lambda}^{(Q)}(v-\frac{\eta}{2}) \right\}$$

$$= \int_{-\infty}^{+\infty} \frac{dv}{2\cosh \pi(u-v)} \left\{ \frac{2\beta}{v^2+\frac{1}{4}} + \ln(1+3q^{-1}(v)\bar{w}(v)) \right\}.$$

Let us introduce the dressing energy function $\epsilon(u)$

$$\epsilon(u) = -\frac{1}{\beta} \ln \left(q^{-1}(u)\bar{w}(u)\right) = \frac{2}{u^2+\frac{1}{4}} + \bar{\epsilon}(u), \quad \lim_{u\to\infty} \epsilon(u) = 0. \tag{33}$$

Finally we can obtain the free energy of the Heisenberg chain

$$
\begin{aligned}
f(\beta) &= 1 - \frac{1}{\beta} \ln \bar{\Lambda}^{(Q)}(0) \\
&= 1 - \int_{-\infty}^{+\infty} \frac{dv}{\cosh \pi v} \frac{1}{v^2 + \frac{1}{4}} - \frac{1}{\beta} \int_{-\infty}^{+\infty} \frac{dv}{2 \cosh \pi v} \ln \left(1 + 3e^{-\beta \epsilon(v)}\right) \\
&= \ e_g - \frac{1}{\beta} \int_{-\infty}^{+\infty} \frac{dv}{2 \cosh \pi v} \ \ln \left(1 + 3e^{-\beta \epsilon(v)}\right), \qquad (34)
\end{aligned}
$$

where e_g is the energy density (8) of the ground state for the Heisenberg chain at $T = 0$.

The method can also apply to the Heisenberg chain with a uniform field described by the Hamiltonian

$$H = \sum_{n=1}^{L} (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \sigma_n^z \sigma_{n+1}^z) + \frac{h}{2} \sum_{j=1}^{L} \sigma_j^z . \qquad (35)$$

The corresponding quantum transfer matrix becomes

$$\begin{aligned} t^{(Q)}(u) \quad &= \quad tr_0 \left\{ e^{\frac{h\beta}{2}\sigma_0^z} (R_{0\,N}(u - \frac{2\eta\beta}{N})\, R_{0\,N-1}(u + \frac{2\eta\beta}{N} - \eta)) \ldots \right. \\ &\quad \left. \times\, (R_{0\,2}(u - \frac{2\eta\beta}{N})\, R_{0\,1}(u + \frac{2\eta\beta}{N} - \eta)) \right\} . \qquad (36) \end{aligned}$$

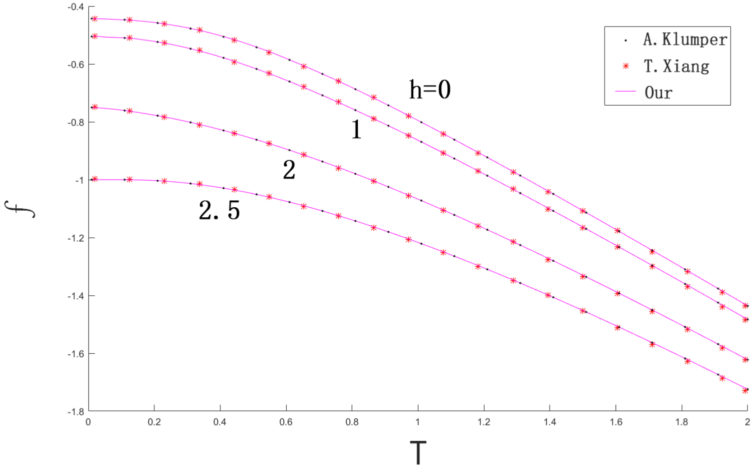

Figure 3: Free energy with different methods

Conclusion and comments

So far, we have developed a method to derive non-linear integral equation of the partition function for a quantum integrable model:

- The Heisenberg chain and its anisotropic generalizations (such as the XXZ and XYZ chains).
- The multi-components generalizations (such as $su(n)$, $so(n)$ and $sp(2n)$).
- The supersymmetric t-J model.
- The Hubbard model.

Thank for your attentions