# SCFT and VOA

1st Symposium on Field Theory and String Theory

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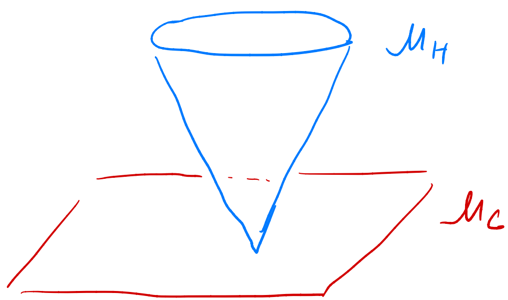
Ref: P. Shan, D. Xie, WY work in progress

also: L. Fredrickson, D. Pei, WY, K. Ye 1701.08782
L. Fredrickson, A. Neitzke 1709.06142
C. Kozcaz, S. Shakirov, WY 1801.08316

# Background

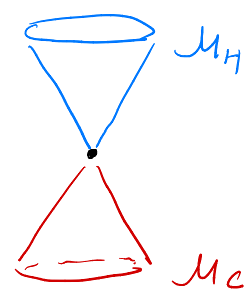
4d $\mathcal{N}=2$ Superconformal Field Theory (SCFT)

- Higgs branch: $\langle \hat{B}_R \rangle$, hyper-Kähler cone
  no quantum correction
  e.g.: $\langle q^i_\alpha \tilde{q}^\alpha_j \rangle$.
- Coulomb branch $\langle \hat{\mathcal{E}}_r \rangle$, LEFT: $U(1)^r$ gauge
  freely generated, quantum correction
  e.g.: $\langle tr\,\phi^2 \rangle$ ... SW-geometry



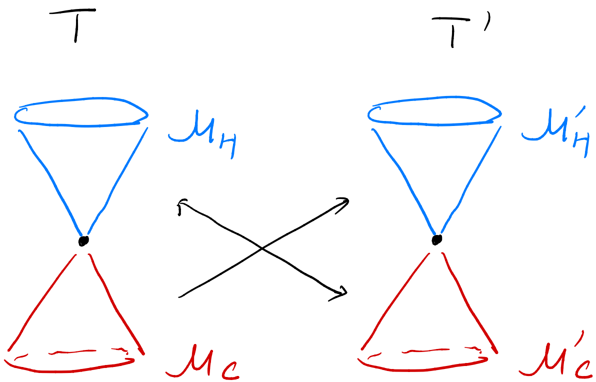

3d $\mathcal{N}=4$ SCFT

- Higgs branch: HK cone
  no quantum correction
- Coulomb branch: HK cone
  quantum correction



# 3d Mirror

3d $\mathcal{N}=4$ SCFT (IS96, HW97, ...)




Example:

$$\textcircled{1} - \boxed{N}$$

$\mathcal{M}_H$ : $\overline{\mathcal{O}}_{min}$ of $SU(N)$

$\mathcal{M}_C$ : $\mathbb{C}^2/\mathbb{Z}_N$

$$\boxed{1} - \underbrace{\textcircled{1} - \textcircled{1} - \cdots - \textcircled{1}}_{N-1} - \boxed{1}$$

$$\mathcal{M}'_H = \mathcal{M}_C$$

$$\mathcal{M}'_C = \mathcal{M}_H \quad (U(1)^{N-1} \to SU(N))$$

S-duality in IIB

# Dimension Reduction on $S^1_R$

$$
\begin{tikzcd}
4d \ \mathcal{N}=2 \arrow[rr, "\substack{\text{compactified on } S^1_R \\ R \to 0}"] & & 3d \ \mathcal{N}=4 \\
\mathcal{M}_H^{4d} & = & \mathcal{M}_H^{3d} \\
\mathcal{M}_C^{4d} & & \mathcal{M}_C^{3d} \arrow[ll, "\text{Elliptic Fibration}"'] \arrow[d, leftrightarrow] \\
& & 3d \ \text{Mirror}
\end{tikzcd}
$$

# Mirror Symmetry in 4d ?

$$4d\ \mathcal{N}=2 \xrightarrow{R\to 0} 3d\ \mathcal{N}=4$$

$$\mathcal{M}_H^{4d} \qquad = \qquad \mathcal{M}_H^{3d}$$

$$\mathcal{M}_C^{4d} \xleftarrow{\text{fiber}} \mathcal{M}_C^{3d}$$

$$?\ \updownarrow \qquad\qquad\qquad \updownarrow\ \text{mirror}$$

$$4d\ ? \xrightarrow{R\to 0} 3d\ \mathcal{N}=4$$

Obstruction:

$\mathcal{M}_C$ in general not HK, diff. from $\mathcal{M}_H$

Our Observation

- We can read some information on $\mathcal{M}_H$ from $\mathcal{M}_C$ for 4d SCFT
- Hitchin System $\longleftrightarrow$ VOA

# Our Tools I: 4d SCFT / 2d VOA (BLLPRvR)

| 4d SCFT | 2d VOA |
| --- | --- |
| Flavor Group $G_F$ | AKM Subalgebra $V^k(g_F)$ |
| $c_{4d}$ $\quad$ $k_{4d}$ | $c_{2d} = -12 c_{4d}$, $\quad$ $k_{2d} = -k_{4d}$ |
| Schur Index $I(q)$ | Character of Vacuum Module |
| $I(q)$ w. Surface Defects | Characters of Other Module (CGS) |
| $\mathcal{M}_H$ | Associated Variety (SXY, BR, Ara) |
| 4d Duality | Algebra Isomorphism (XY, AM, $\cdots$) |
| | Collapsing Level |
| | Quasi-Lisse VOA |

# Our Tools II: Hitchin System (Hitch 87)

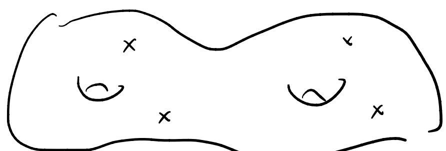
$4d\ \mathcal{N}=2$ class-$S$ SCFT

Gaiotto, GMN, ...

$6d\ (2,0)$ SCFT of Type ADE

$\mathbb{R}^{3,1} \times \Sigma_{g,n}$

$\parallel$



e.g. $SU(N)$ with $N_F = 2N$

$T_N$ theory

Argyres-Douglas (AD) theory

Witten 97, GMN 09, ...

$4d$ class-$S$: $\mathbb{R}^3 \times S^1_R$ $\quad \mathcal{M}_C = \mathcal{M}_{\text{Hitchin}}$

Hitchin on $\Sigma_{g,n}$, $A$, $\varphi$ (ADE)

$$\begin{cases} F_A + R^2 [\varphi, \varphi] = 0 \\ \bar{\partial}_A \varphi = 0 \end{cases}$$ integrable

$\mathcal{M}_{\text{Hitchin}} := \{\text{solutions}\}/\sim$, HK

Boundary Condition

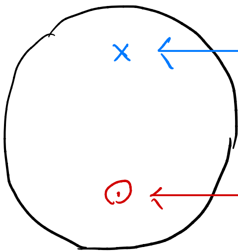
$$\begin{cases} A \sim \alpha\, dz \\ \varphi \sim \left( \frac{u_n}{z^n} + \frac{u_{n-1}}{z^{n-1}} + \cdots + \text{reg} \right) dz \end{cases}$$

$n=1$. reg. classification nilpotent orbits

$n>1$ irreg

# Model: (Generalized) Argyres-Douglas Theory (AD, CNV, Xie, XW, ...)

6d (2,0) $\mathfrak{g}$ SCFT on a sphere



reg $f$ $\quad \mathfrak{g} = ADE$

$f$ = nilpotent orbit

Irreg. $n$ $\quad T(\mathfrak{g}, f, n)$

$\updownarrow$ $\mathbb{R}^3 \times S^1$

$\mathcal{M}_{\text{Hitchin}}$ on a sphere

$\circlearrowright$

$U(1)$ : $U(1)_r$ symm.

(Boalch)

4d / 2d

$\longleftrightarrow$

CS, BN, Song

XYY, SXY

$(XY)^3$, ...

?

## VOA : W-algebra

$$W^k(\mathfrak{g}, f) \xleftarrow{\text{qDS}} V^k(\mathfrak{g})$$

$$\shortparallel$$

$$W^k(\mathfrak{g}, \text{trivial})$$

(FKW, KRW, KW, Ara, AM, AvE, ...)

$k$: boundary admissible

$$k = -h^\vee + \frac{h^\vee}{q} \qquad q \in \mathbb{Z}_{\geq 1}$$

$$(q, h^\vee) = 1$$

# Results

| Hitchin on a sphere $(\mathfrak{g}, n, f)$ | | VOA : W-algebra |
|---|---|---|
| $\mathcal{M}_{\mathrm{Hitchin}} \circlearrowleft U(1)$ : $U(1)_r$ symm. | $\longleftrightarrow$ | $W^k(\mathfrak{g}, f) \xleftarrow{qDS} V^k(\mathfrak{g})$ <br> $k = -h^\vee + \frac{h^\vee}{q}$ $(q, h^\vee) = 1$, $q \in \mathbb{Z}_{\geq 1}$ |

$U(1)$ Fixed Points $\xleftrightarrow{1-1}$ Simple modules

Moment Map $\mu_{U(1)}$ $\xleftrightarrow{\text{prop}}$ Conformal dim of H. W. S.

Remark:

- FPYY: $\mathfrak{g} = A_N$, $f$ = trivial  FN: $\mathfrak{g} = A_N$, $f$ = reg.
- Relates: VOA $\longleftrightarrow$ Hitchin. F. P relation proved by us using some math tools.
- Simple Module: Surface Defects $\langle \hat{B}_R \rangle$ non constant vev

# Example

$\mathfrak{g} = A_1 \quad f = \text{trivial}$

AD: $(A_1, D_{2N+1})$ $\quad$ W-alg: $su(2)_{-2+\frac{2}{2N+1}}$

H.W. $\quad \left(-2 + \frac{2j+2}{\delta}\right)\Lambda_0 - \frac{2j}{\delta}\Lambda_1 \qquad 0 \le j \le 2N$

$$h_j = -\frac{j}{2} + \frac{j^2}{2(2N+1)}$$

FP: $\quad \varphi \sim \frac{1}{2}\begin{pmatrix} 0 & z^j \\ z^{2N+3-j} & 0 \end{pmatrix} dz + \cdots$

$$\mu_j = h_j + \frac{2N+1}{8}$$

$\mathfrak{g} = A_1 \quad f = \text{reg}$

AD: $(A_1, A_{2N})$ $\quad$ W-alg: $(2, 2N+3)$ minimal model

HW $\quad (j, 1) \qquad 1 \le j \le N+1$

$$h_j = \frac{(j-1)(2N+2-j)}{2(2N+3)}$$

FP $\quad \varphi = \begin{pmatrix} 0 & z^{j-1} \\ z^{2N+2-j} & 0 \end{pmatrix} dz + \cdots$

$$\mu_j = h_j + \frac{(2N+1)^2}{8(2N+3)}$$

# Example

$\mathfrak{g} = A_2 \qquad f = \text{minimal} \qquad W^{-3+\frac{3}{q}}(sl_3, \min)$

contains a $U(1)_{-3+\frac{3}{q}}$ subalgebra

e.g. $q = 4$, Logrithmic $B_4$ Model

$\#$ of F. P. / S. M $= \frac{1}{2} q(q-1)$

# Formula for # of F.P.

$f \longleftrightarrow \mathfrak{l} \oplus \mathfrak{p} \subset \mathfrak{g}$ , $r$ : rank of $\mathfrak{g}$

$W_{\mathfrak{l}}$ : Weyl group of $\mathfrak{l}$, $\{h_1, h_2 \cdots h_j\}$ : Exponents of $\mathfrak{l}$

$$\boxed{\#\text{ of F.P.}/\text{S.M.} = \frac{1}{|W_{\mathfrak{l}}|}\, q^{r-j} \prod_{i=1}^{j} (q - h_i)}$$

# Mirror?

$$\mathcal{M}_{\text{Hitchin}} = \mathcal{M}_H \; T^{3d}_{\text{mirror}}$$

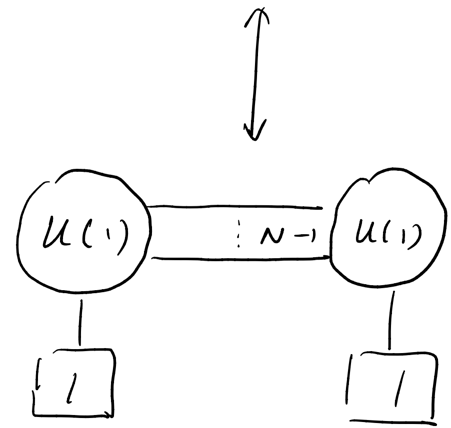
$$\uparrow R \to 0$$

$$\mathcal{M}_H \; T^{4d}_{\text{mirror}}$$

F. P. $\longleftrightarrow$ massive Higgs vacua of a mirror theory

e.g. (Xie)

$(A_1, D_{2N})$



# Back to $\mathbb{R}^{3,1}$

You may think $\mathbb{R}^3 \times S^1$ is cheating

4d $\mathcal{M}_C$ : $\langle \hat{\mathcal{E}}_{r_i} \rangle$, $1 \le i \le \ell$, Freely Gen.

$1, \ \hat{\mathcal{E}}_{r_i}, \ \hat{\mathcal{E}}_{r_i}^2, \ \hat{\mathcal{E}}_{r_i}^3 \ \cdots\cdots \qquad \Delta = r$

Counting:

$$I(t) = \prod_{i=1}^{\ell} \frac{1}{(1 - t^{\Delta_i})}$$

$$= \sum_{F.P} \frac{t^{\mu_i}}{\prod_{j=1}^{\dim \mathcal{M}_c} (1 - t^{\lambda_{ij}})(1 - t^{1-\lambda_{ij}})}$$

Example:

$(A_1, A_2) \longleftrightarrow$ Lee - Yang model
(2, 5) minimal model

$\Delta = \frac{6}{5}$ C B operator

$$I(t) = \frac{1}{1 - t^{6/5}}$$

$$= \frac{1}{(1-t^{2/5})(1-t^{3/5})} + \frac{t^{1/5}}{(1-t^{6/5})(1-t^{-1/5})}$$

# Summary

**Hitchin / VOA**

$\boxed{\text{Fixed Points} \longleftrightarrow \text{Simple Module}}$

Moment Map $\longleftrightarrow$ Conformal Dim

Hitchin Cha $\longleftrightarrow$ Modular Property (FP44, KS4)

$\boxed{\text{Fiber Dim} \longleftrightarrow \text{Dim Zhu's alg (SXY)}}$

## Questions

Fusion Rule $\longrightarrow$ F.P. ?

qDS reduction $\longrightarrow$ Hitchin ?

Equivalence at Categorical Level ?

More general case ?

- evidence for non-admissible, $T_N$, other class-S ...

# Summary

$\mathcal{M}_C$ | Mirror

T.P. $\longleftrightarrow$ massive Higgs Vacua

Hitchin Cha $\longleftrightarrow$ Higgs branch Index

## Questions

How to formulate precisely ?

Brane / M-theory understand ?

Higher Dimension ?

Thank You!