海森堡自旋链中的
量子临界性

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摘要

- 海森堡自旋链
- 严格解
- 量子临界性
- 结论与展望

海森堡自旋链

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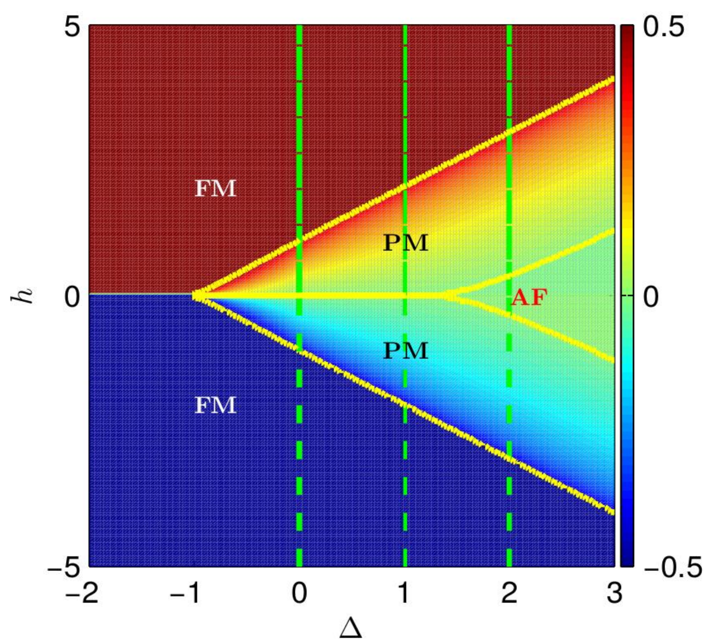
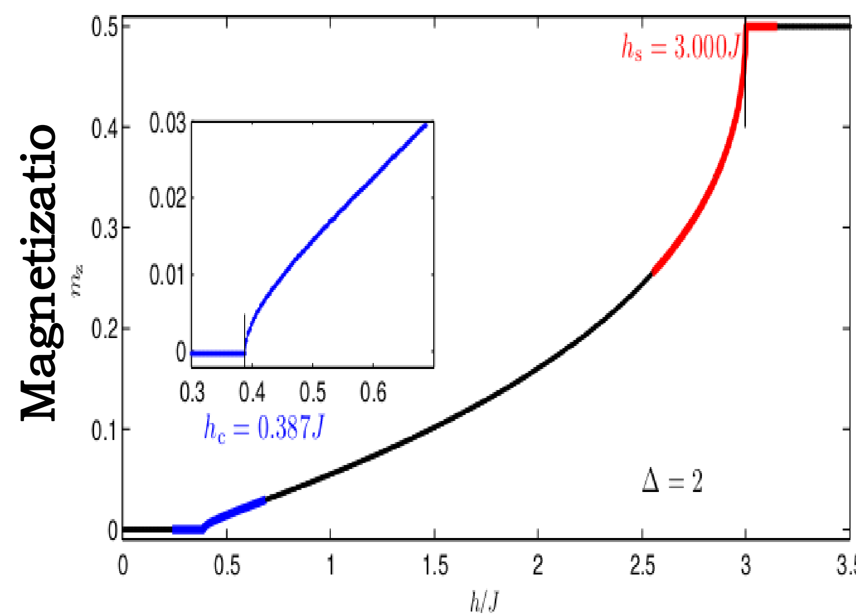
H Bethe, Z Phys 71, 205 (1931)

$$H = J\sum_{j=1}^{N}\left[\hat{s}_j^{\mathrm{x}}\hat{s}_{j-1}^{\mathrm{x}} + \hat{s}_j^{\mathrm{y}}\hat{s}_{j-1}^{\mathrm{y}} + \Delta\hat{s}_j^{\mathrm{z}}\hat{s}_{j-1}^{\mathrm{z}}\right] - h\hat{M}_{\mathrm{z}}$$

- 相图





海森堡自旋链

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$$H = J\sum_{j=1}^{N}\left[\hat{s}_j^{\mathrm{x}}\hat{s}_{j-1}^{\mathrm{x}} + \hat{s}_j^{\mathrm{y}}\hat{s}_{j-1}^{\mathrm{y}} + \Delta\hat{s}_j^{\mathrm{z}}\hat{s}_{j-1}^{\mathrm{z}}\right] - h\hat{M}_{\mathrm{z}}$$

- 通过Jordan-Wigner变换，模型等价与无自旋近邻相互作用费米子

$$H_{XXZ} = \frac{1}{2}\sum_j\left(c_j^\dagger c_{j+1} + \mathrm{H.c.}\right) + \Delta\sum_j\left(n_j - \frac{1}{2}\right)\left(n_{j+1} - \frac{1}{2}\right)$$

- TLL 理论

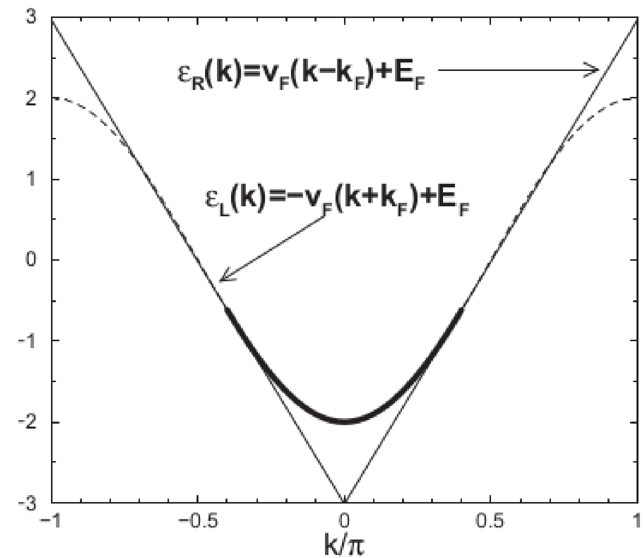
$$\mathcal{H}(\Delta) = \frac{v}{2}\int dx\left[K(\partial_x\theta)^2 + \frac{1}{K}(\partial_x\phi)^2\right] + \mathcal{H}_{\mathrm{irr}}.$$

TLL相区自由能的普适规律

$$F(T)/L \approx E_0 - \frac{\pi C(k_{\mathrm{B}}T)^2}{6\hbar v_{\mathrm{s}}}$$

- 相变点附近的量子临界性

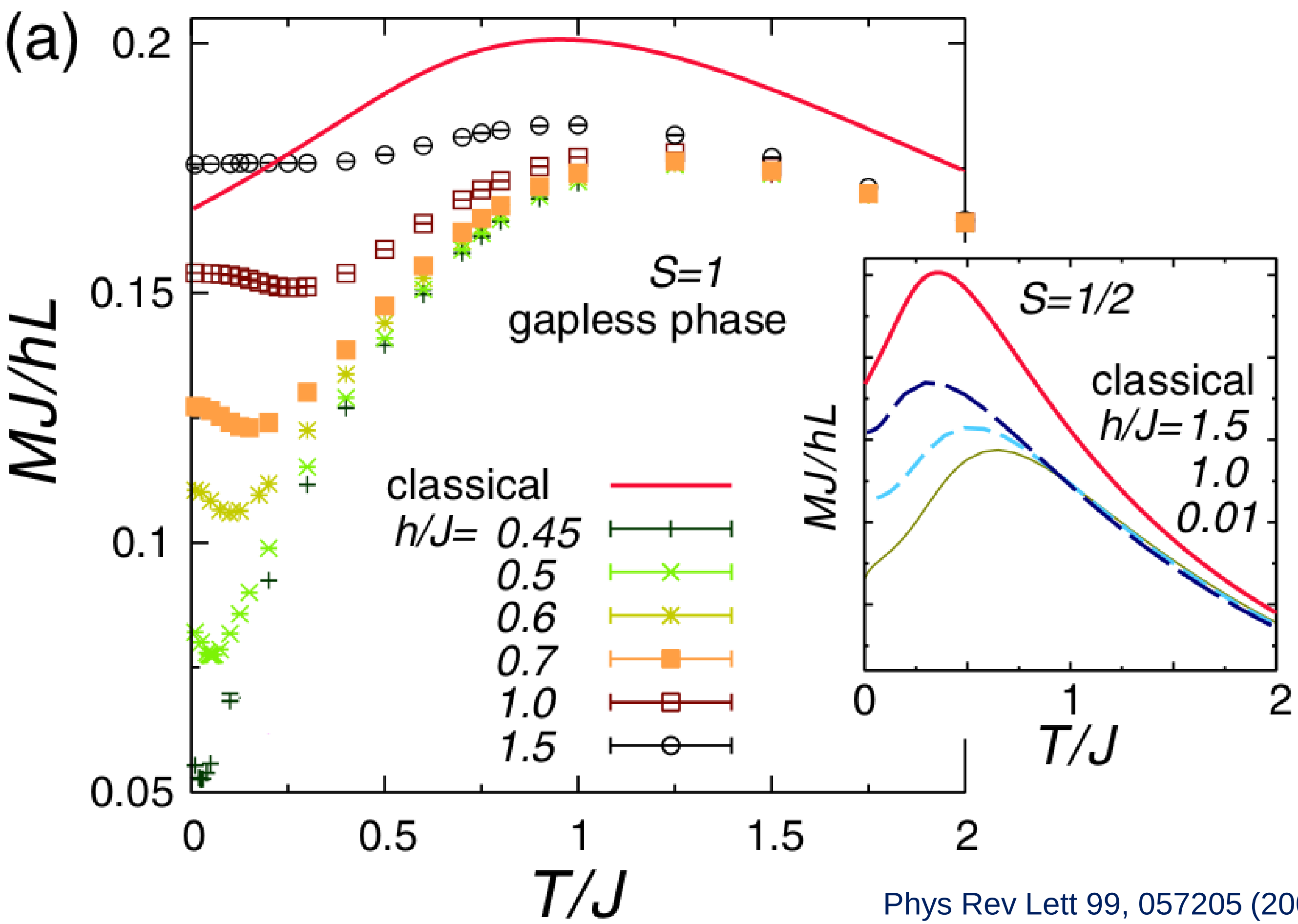
$$n(T,\mu) - n_0(T,\mu) \approx \tilde{T}^{\frac{d}{z}+1-\frac{1}{\nu z}}\mathcal{F}\left(\frac{\mu-\mu_c}{T^{\frac{1}{\nu z}}}\right)$$



Ann. Phys. 7, 225-306 (1998)
Braz. J. of Phys., 33, 3 (2003)

量子临界性

- 海森堡自旋链的磁性质可以用来观测量子临界性
  磁化强度随着温度变化会出现极值点



Phys Rev Lett 99, 057205 (2007)

量子临界性

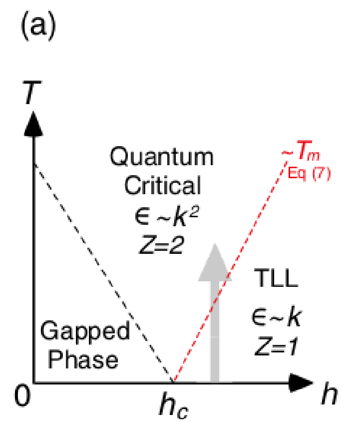
- 海森堡自旋链的磁性质可以用来观测量子临界性

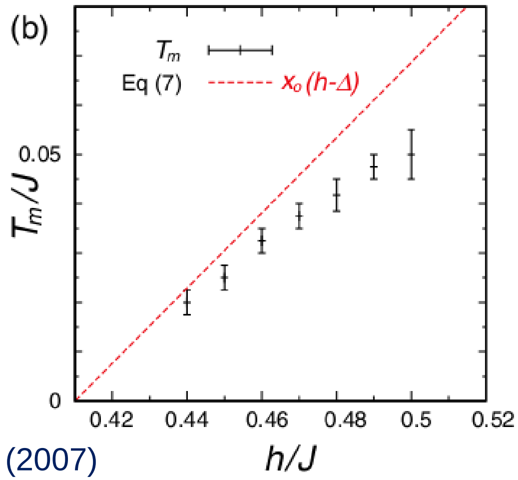
自旋1自旋链有能隙-无能隙相变
“自由费米子”量子临界性

$$\frac{M}{L} = -\sqrt{\frac{m}{2\pi\beta}}\mathrm{Li}_{n=1/2}(-e^{\beta(h-\Delta)}),$$

磁化强度极值点的标度规律

$$T_m = x_0(h - \Delta)$$
$$x = x_0 \sim 0.76238$$





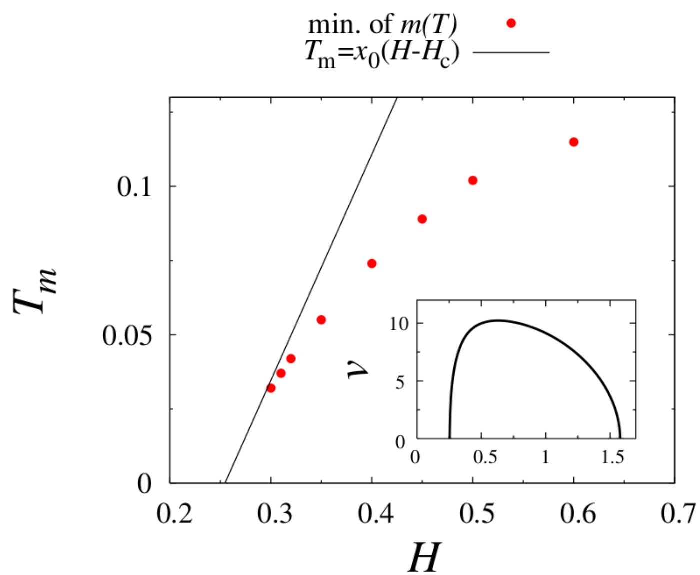
Phys Rev Lett 99, 057205 (2007)

XXZ海森堡自旋链的有能隙-无能隙相变是同样的标度规律吗？

$$\mathcal{H} = J\sum_i (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z) - g\mu_B H \sum_i S_i^z,$$

$$T_m = x_0(H - H_c)\ (x_0 \sim 0.76238).$$

?



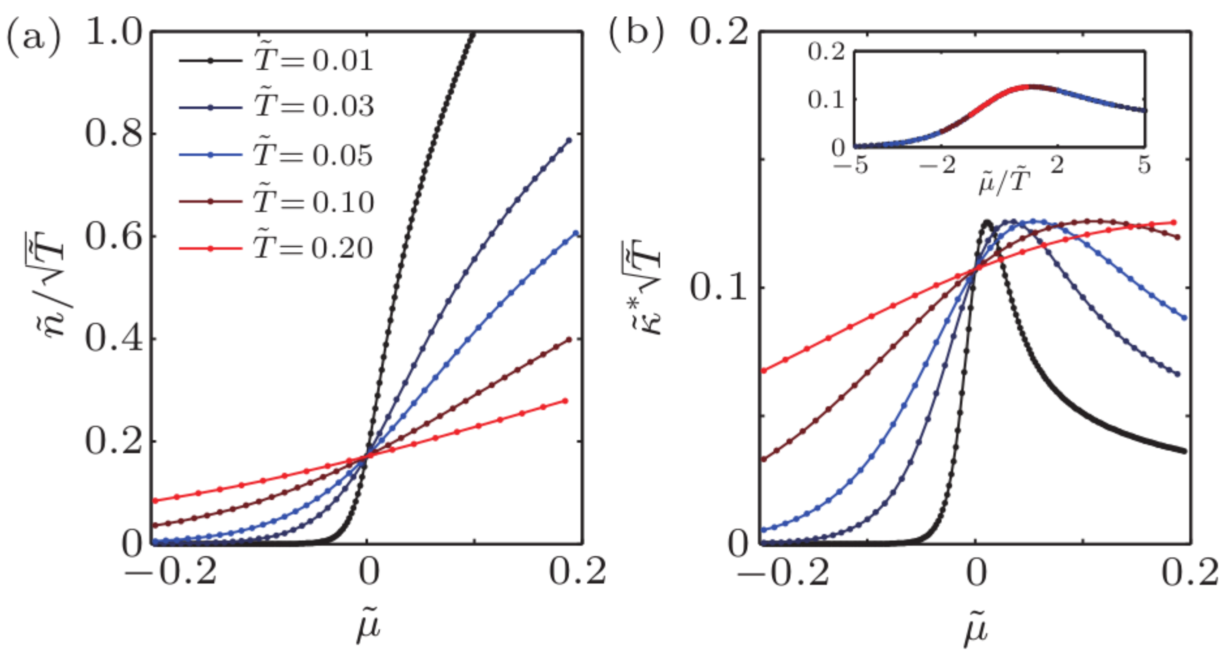
JPSJ 77, 074717 (2008)

量子临界性

- 利用量子可积模型，可以得到解析的相变临界区标度函数

$$n(T,\mu) - n_0(T,\mu) \approx \tilde{T}^{\frac{d}{z}+1-\frac{1}{\nu z}} \mathcal{F}\left(\frac{\mu-\mu_c}{T^{\frac{1}{\nu z}}}\right)$$

$$\mathcal{F}(x) = -A\,\mathrm{Li}_{1/2}(-\mathrm{e}^{rx}) = A\,\mathrm{F}_{-1/2}(rx)$$



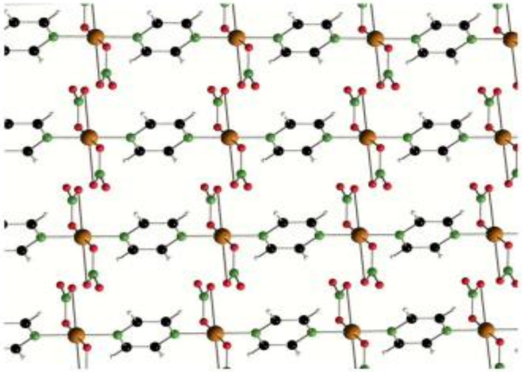
Phys. Rev. A 84, 023616 (2011)
J. Phys. A 44 102001 (2011)
... ...

自旋链系统的实验观测

- “一维”磁性材料

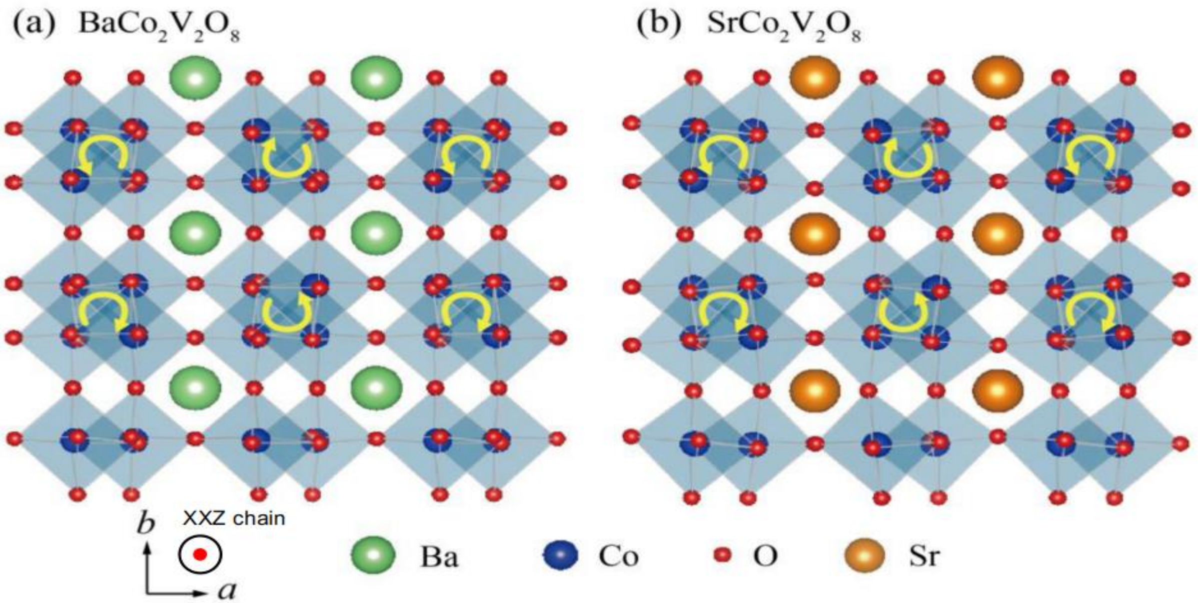
  $\Delta = 1$ XXX 海森堡自旋链

CuPZn



Phys. Rev. B 59 1008 (1999)

$\Delta = 2$



Physics Procedia 75, 779 (2015)

实验观测量子临界性

- 用磁性质观测量子临界性

$\Delta = 1$ 时的铁磁顺磁相变，

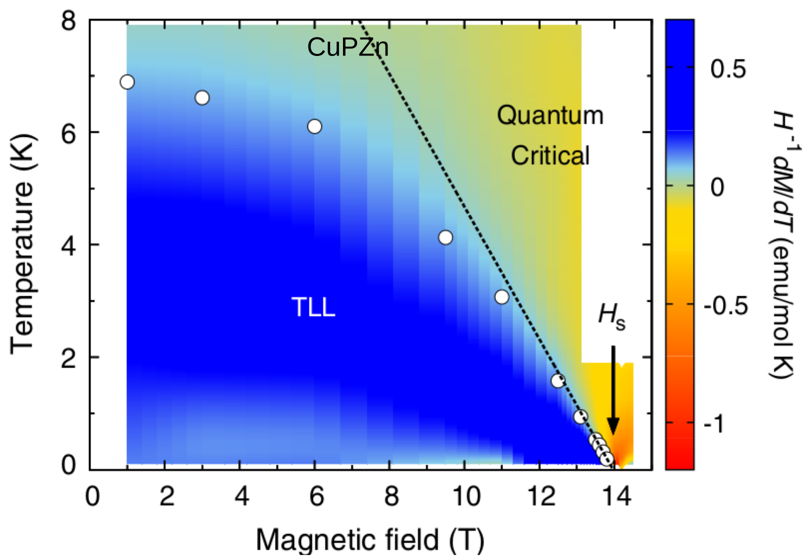
可以用“自由费米子”相变标度规律描述

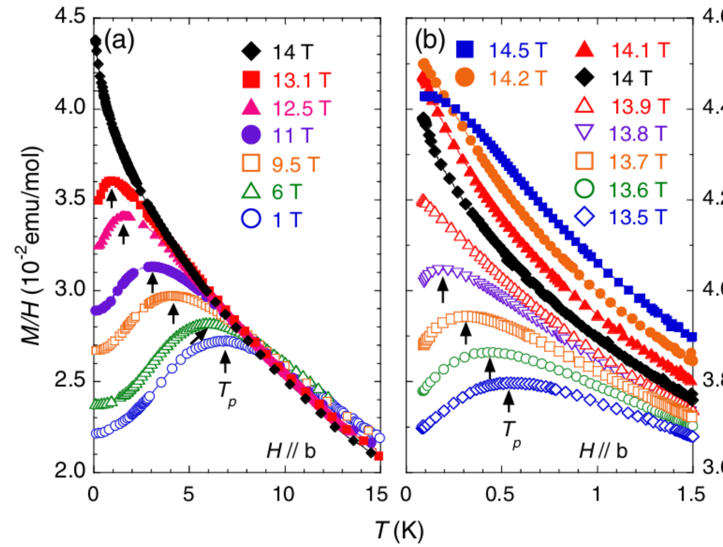
$$k_B T_{\text{ex}} = 0.76238\mu,$$

磁化强度极值点的标度规律

$$T_m = x_0(h - \Delta)$$

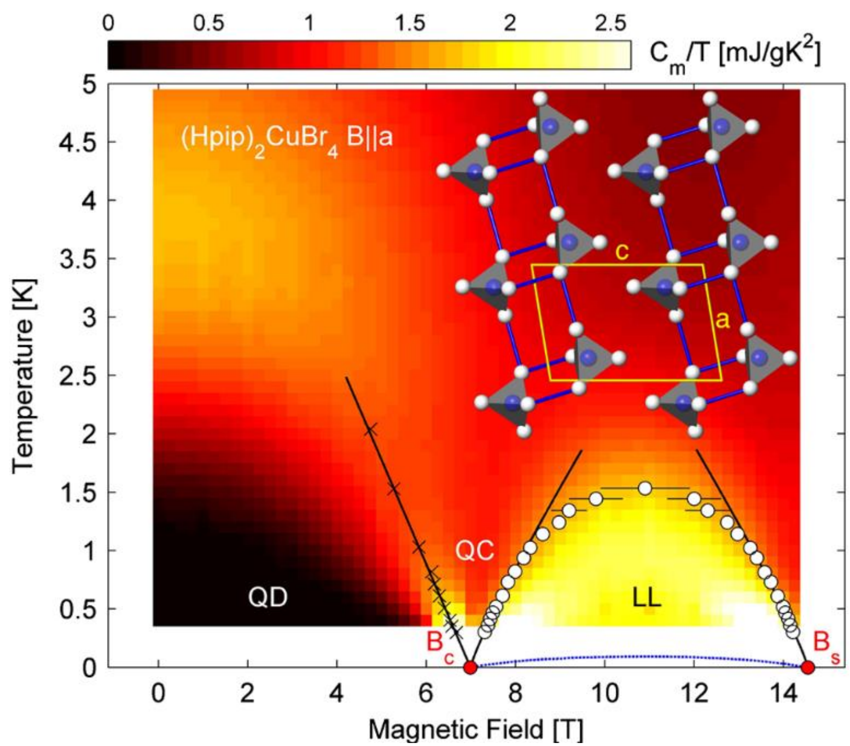
$$x = x_0 \sim 0.76238$$





Phys Rev Lett 114, 037202 (2015)

实验观测量子临界性



Phys Rev Lett 101, 247202 (2008)

用磁热性质观测
自旋梯子的
量子临界性

- 在有限温度时，如何来界定相变的临界区?
- 能否解析的求解相变的标度函数?

严格解

- 海森堡自旋链

H Bethe, Z Phys 71, 205 (1931)

$$H = J\sum_{j=1}^{N}\left[\hat{s}_j^{\mathrm{x}}\hat{s}_{j-1}^{\mathrm{x}} + \hat{s}_j^{\mathrm{y}}\hat{s}_{j-1}^{\mathrm{y}} + \Delta\hat{s}_j^{\mathrm{z}}\hat{s}_{j-1}^{\mathrm{z}}\right] - h\hat{M}_{\mathrm{z}}$$

- 量子可积性

Yang-Baxter equation

$$L_{bc}(\lambda)L_{ac}(\lambda+\mu)L_{ab}(\mu) = L_{ab}(\mu)L_{ac}(\lambda+\mu)L_{bc}(\lambda).$$

六顶角解

$$L_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a(\lambda) & b(\lambda) & 0 \\ 0 & b(\lambda) & a(\lambda) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

严格解

- **本征态波函数写为平面波叠加的形式（Bethe ansatz）**

$$\psi(\boldsymbol{x}) = \sum_{\mathcal{P}} A(\mathcal{P}) \mathrm{e}^{\mathrm{i}\hat{\mathcal{P}}\boldsymbol{k}\cdot\boldsymbol{x}} \qquad A(\mathcal{P}) \propto (-1)^{\mathcal{P}} \prod_{l<j} \left[\mathrm{e}^{\mathrm{i}(k_{\mathcal{P}_l}+k_{\mathcal{P}_j})} - 2\Delta \mathrm{e}^{\mathrm{i}k_{\mathcal{P}_l}}\right]$$

$$|\Psi\rangle_M = \hat{B}(\mu_1)\hat{B}(\mu_2)\cdots\hat{B}(\mu_M)|0\rangle$$

- **Bethe ansatz方程**

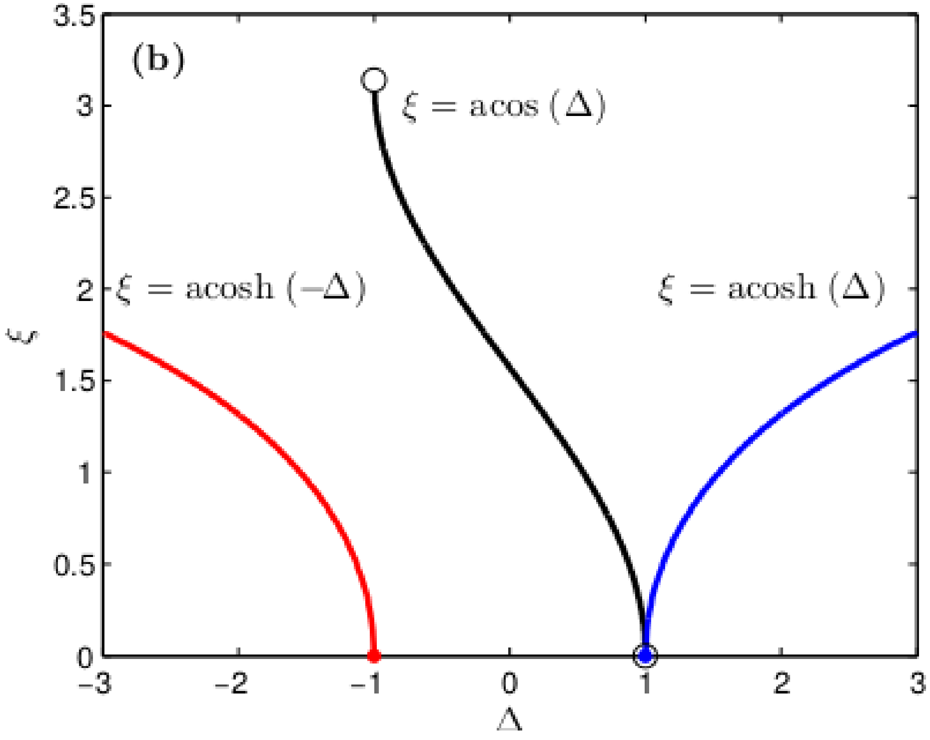
$$\mathrm{e}^{\mathrm{i}k_j} = \frac{\sinh(\mu_j - \eta/2)}{\sinh(\mu_j + \eta/2)} \qquad \Delta = \cosh\eta$$

$$\frac{\sinh^N(\mu_j + \frac{\eta}{2})}{\sinh^N(\mu_j - \frac{\eta}{2})} = -\prod_{l=1}^{M} \frac{\sinh(\mu_j - \mu_l + \eta)}{\sinh(\mu_j - \mu_l - \eta)}$$

由严格解出发，计算系统的物理性质，热力学量、关联函数 . . .

严格解

- **相互作用重新参数化**



$$e^{ik_j} = \alpha \frac{\phi(\varphi_j - i\xi/2)}{\phi(\varphi_j + i\xi/2)}.$$

Bethe ansatz 方程

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$\Delta \geq 1$ $\varphi = i\mu$, $\eta = \xi$, $\Delta = \mathrm{ch}\xi$

$$\frac{\sin^N(\varphi_j + i\xi/2)}{\sin^N(\varphi_j - i\xi/2)} = -\prod_{l=1}^{M} \frac{\sin(\varphi_j - \varphi_l + i\xi)}{\sin(\varphi_j - \varphi_l - i\xi)}$$

$\Delta \leq -1$ $\varphi = i\mu - \frac{\pi}{2}$, $\eta = \xi \pm i\pi$, $\Delta = -\mathrm{ch}\xi$

$$\frac{\sin^N(\varphi_j + i\xi/2)}{\sin^N(\varphi_j - i\xi/2)} = (-)^{N+1} \prod_{l=1}^{M} \frac{\sin(\varphi_j - \varphi_l + i\xi)}{\sin(\varphi_j - \varphi_l - i\xi)}$$

$|\Delta| \leq 1$ $\varphi = \mu$, $\eta = i\xi$, $\Delta = \cos\xi$

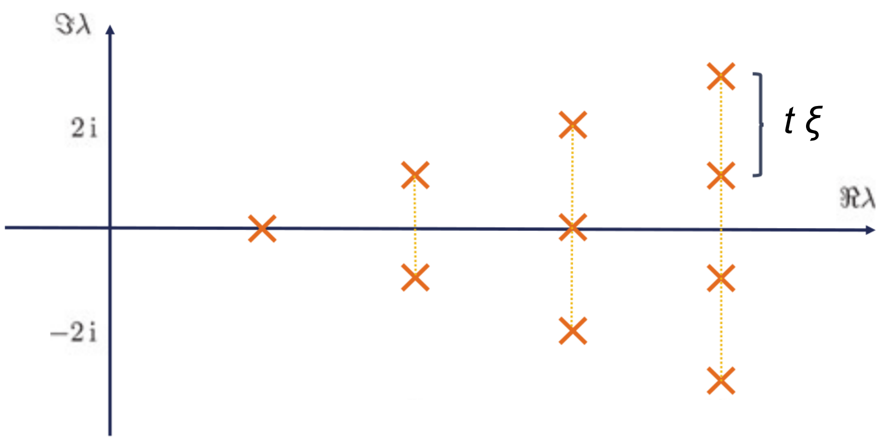
$$\frac{\sinh^N(\varphi_j + i\xi/2)}{\sinh^N(\varphi_j - i\xi/2)} = -\prod_{l=1}^{M} \frac{\sinh(\varphi_j - \varphi_l + i\xi)}{\sinh(\varphi_j - \varphi_l - i\xi)}$$

热力学极限

- **|Δ|≥1 时的弦假设**

$$\varphi^n_{\alpha,t} = \varphi^n + (n+1-2t)\mathrm{i}\xi/2 + \mathcal{O}(\mathrm{e}^{-\delta N}),$$

$$t = 1, 2, \cdots, n, \quad n = 1, 2, \cdots .$$



M Gaudin, Phys Rev Lett 26 1301 (1971)
M Takahashi, Pro. Theo. Phys. 48 2187 (1972)

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$$t = 1, 2, \cdots, n, \quad n = 1, 2, \cdots.$$

- **弦假设下的BAE**

$$\frac{\sin^N(\varphi^n_\alpha - \frac{n}{2}\mathrm{i}\xi)}{\sin^N(\varphi^n_\alpha + \frac{n}{2}\mathrm{i}\xi)} = (-1)^{n(\beta+1)+1}\prod_{m,\gamma}\mathbb{A}_{n,m}(\varphi^n_\alpha - \varphi^m_\gamma);$$

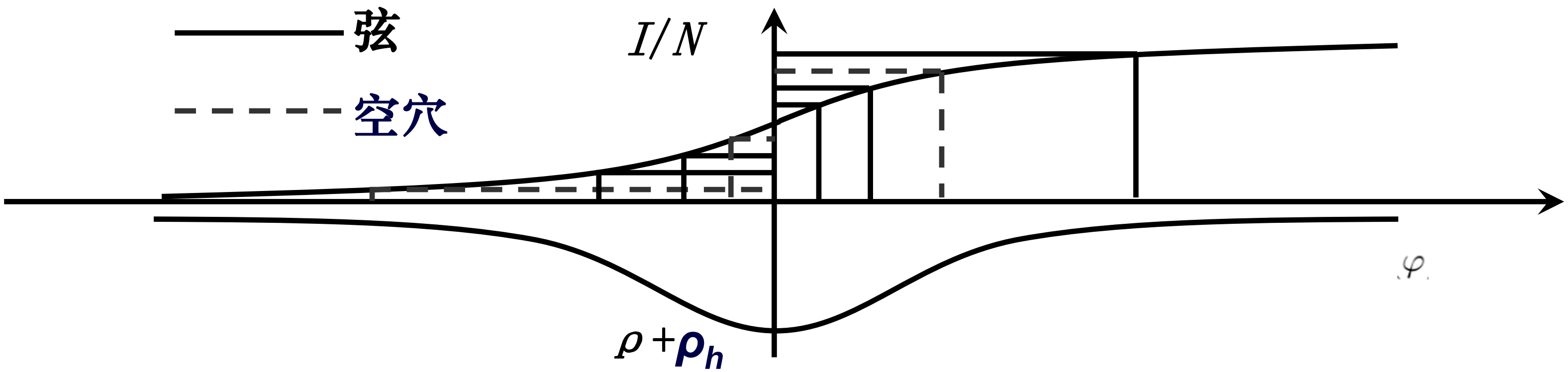
$$\mathbb{A}(x) = \frac{\sin(x - \frac{|n+m|}{2}\mathrm{i}\xi)}{\sin(x + \frac{|n+m|}{2}\mathrm{i}\xi)}\left[\frac{\sin(x - \frac{|n+m|-2}{2}\mathrm{i}\xi)}{\sin(x + \frac{|n+m|-2}{2}\mathrm{i}\xi)}\right]^2 \cdots \left[\frac{\sin(x - \frac{|n-m|+2}{2}\mathrm{i}\xi)}{\sin(x + \frac{|n-m|+2}{2}\mathrm{i}\xi)}\right]^2\left[\frac{\sin(x - \frac{|n-m|}{2}\mathrm{i}\xi)}{\sin(x + \frac{|n-m|}{2}\mathrm{i}\xi)}\right]^{1-\delta_{n,m}}$$

M Gaudin, Phys Rev Lett 26 1301 (1971)
M Takahashi, Pro. Theo. Phys. 48 2187 (1972)

热力学极限

- **对数化与量子数**

$$I^j(\varphi) = N\frac{1}{2\pi}\theta(\varphi, \kappa^{v_j}_{n_j/2}) - \frac{1}{2\pi}\sum_{l\beta}\bar{A}_{jl}(\varphi - \varphi^l_\beta),$$



- **热力学Bethe ansatz 方程 (TBAE)**

$$\rho_n(\varphi) + \rho^{\mathrm{h}}_n(\varphi) = a_{n/2}(\varphi) + \sum_m \hat{A}_{nm} * \rho_m(\varphi)$$

$$A_{nm}(x) = a_{\frac{|n+m|}{2}}(\varphi) + 2a_{\frac{|n+m|-2}{2}}(\varphi) + \quad \cdots + 2a_{\frac{|n-m|+2}{2}}(\varphi) + (1-\delta_{nm})a_{\frac{|n-m|}{2}}(\varphi)$$

热力学极限

- **热平衡的缀饰能量**

$$\varepsilon(\varphi) = T\ln[\rho^{\mathrm{h}}(\varphi)/\rho(\varphi)].$$

- **热平衡的熵**

$$s = \frac{1}{T}\sum_j \int \mathrm{d}\varphi\left[\rho_j(\varphi)\varepsilon_j^+(\varphi) - \rho_j^{\mathrm{h}}(\varphi)\varepsilon_j^-(\varphi)\right],$$

系统的自由能密度 $f = E/N - Ts,$

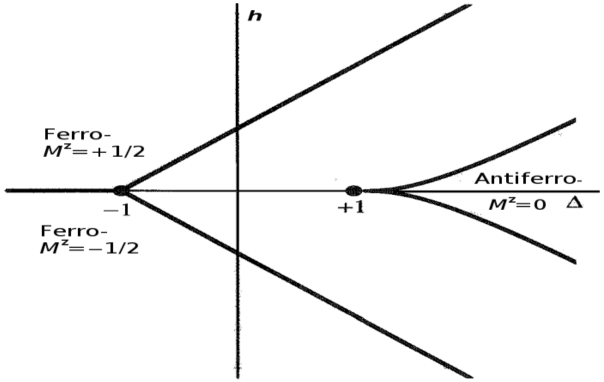
- **热平衡态的TBAE**

$$\varepsilon_n(\varphi) = \varepsilon_n^0(\varphi) + \sum_m \hat{A}_{nm} * \varepsilon_m^-(\varphi)$$

$$\varepsilon_n^0(\varphi) = -2\pi J \sinh\xi a_{n/2}(x) + nh,$$

$$\varepsilon^-(\varphi) = -T\ln\left(1 + \mathrm{e}^{-\varepsilon(\varphi)/T}\right)$$

$$f = \sum_j \int \mathrm{d}\varphi s_j a(\varphi,\kappa_j)\varepsilon_j^-(\varphi) + \frac{\Delta J}{4} - \frac{h}{2}.$$



热力学极限

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$$\varepsilon(\varphi) = T\ln[\rho^{\mathrm{h}}(\varphi)/\rho(\varphi)].$$

- **热平衡的熵**

$$s = \frac{1}{T}\sum_j \int \mathrm{d}\varphi \left[\rho_j(\varphi)\varepsilon_j^+(\varphi) - \rho_j^{\mathrm{h}}(\varphi)\varepsilon_j^-(\varphi)\right],$$

系统的自由能密度 $f = E/N - Ts,$

- **热平衡态的TBAE**

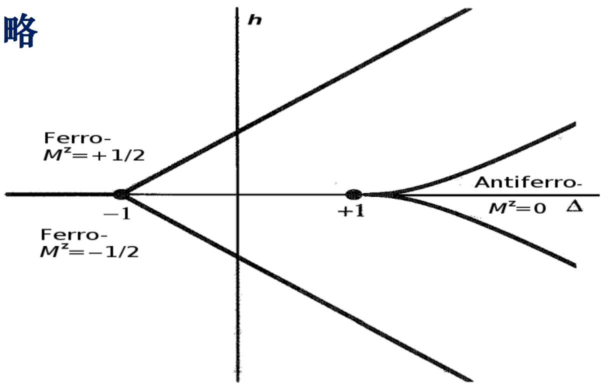
$$\varepsilon_n(\varphi) = \varepsilon_n^0(\varphi) + \sum_m \hat{A}_{nm} * \varepsilon_m^-(\varphi)$$

$$\varepsilon_n^0(\varphi) = -2\pi J \sinh\xi a_{n/2}(x) + nh,$$

低能时可以忽略

$$\varepsilon^-(\varphi) = -T\ln\left(1 + \mathrm{e}^{-\varepsilon(\varphi)/T}\right)$$

$$f = \sum_j \int \mathrm{d}\varphi s_j a(\varphi, \kappa_j)\varepsilon_j^-(\varphi) + \frac{\Delta J}{4} - \frac{h}{2}.$$

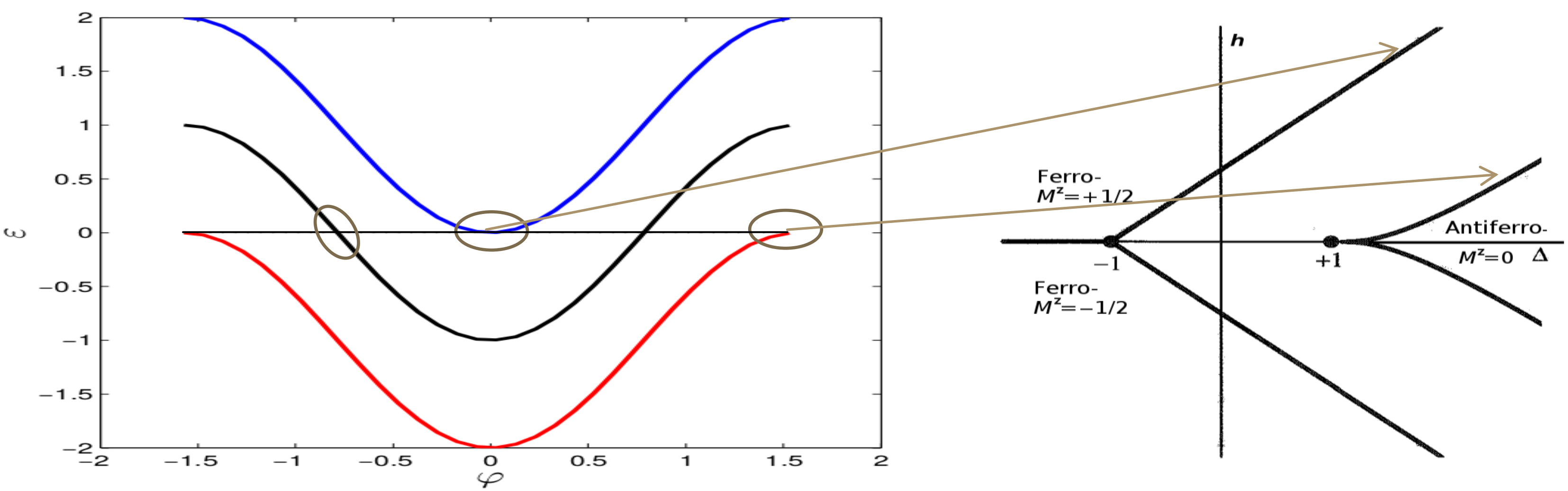


热力学

- **热力学基态的TBAE**

$$\varepsilon(\varphi) = h - 2\pi J \sinh\xi a_{1/2}(\varphi) + \int_{-Q}^{Q} a_1(\varphi - \varphi')\varepsilon(\varphi')\mathrm{d}\varphi'$$

不同的相区，积分有不同的截断



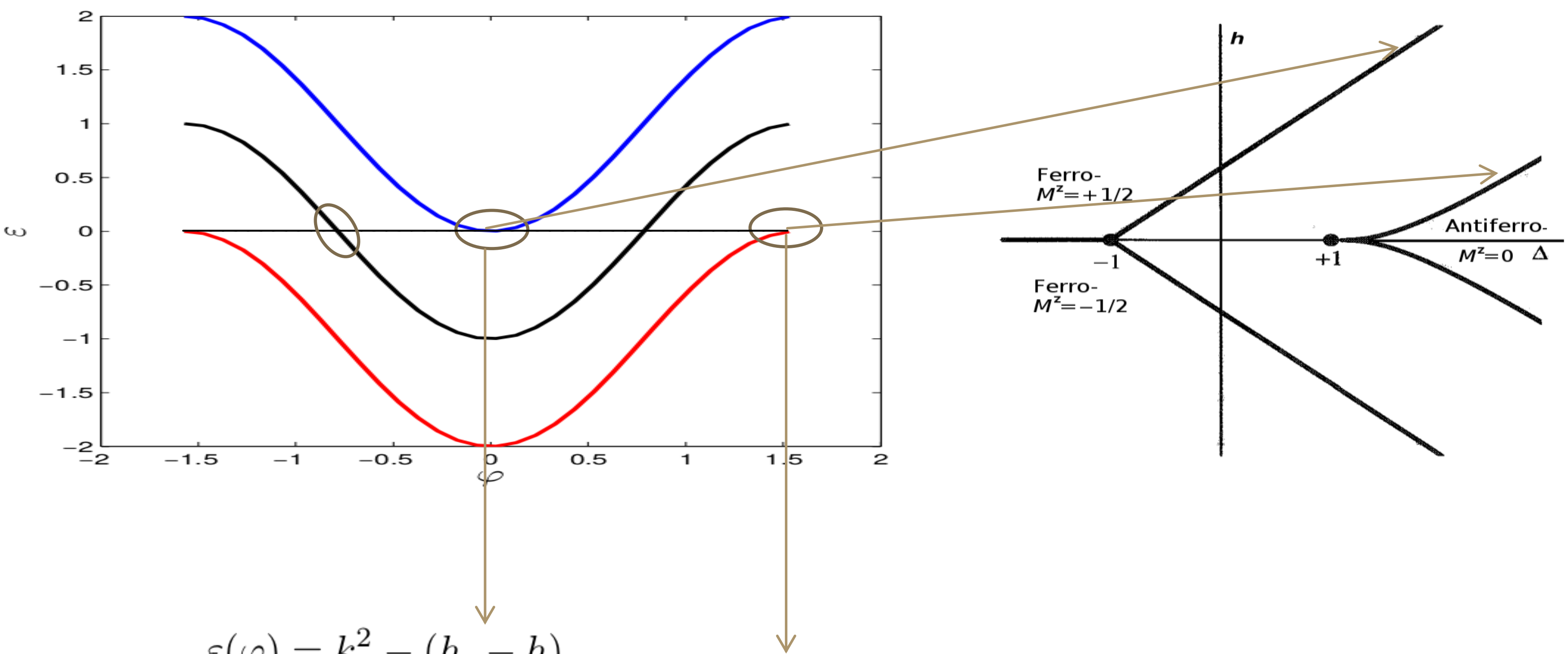
顺磁(TLL) $Q \in (0, \pi/2)$ **铁磁** $Q = 0$ **反铁磁** $Q = \pi$

热力学

- **低温TBAE**

$$\varepsilon(\varphi) = h - 2\pi J \sinh\xi a_{1/2}(\varphi) + \int_{-\infty}^{\infty} a_1(\varphi - \varphi')\varepsilon^-(\varphi')\mathrm{d}\varphi'$$

$$\varepsilon^-(\varphi) = -T\ln\left(1 + \mathrm{e}^{-\varepsilon(\varphi)/T}\right)$$



$$\varepsilon(\varphi) = k^2 - (h_s - h)$$

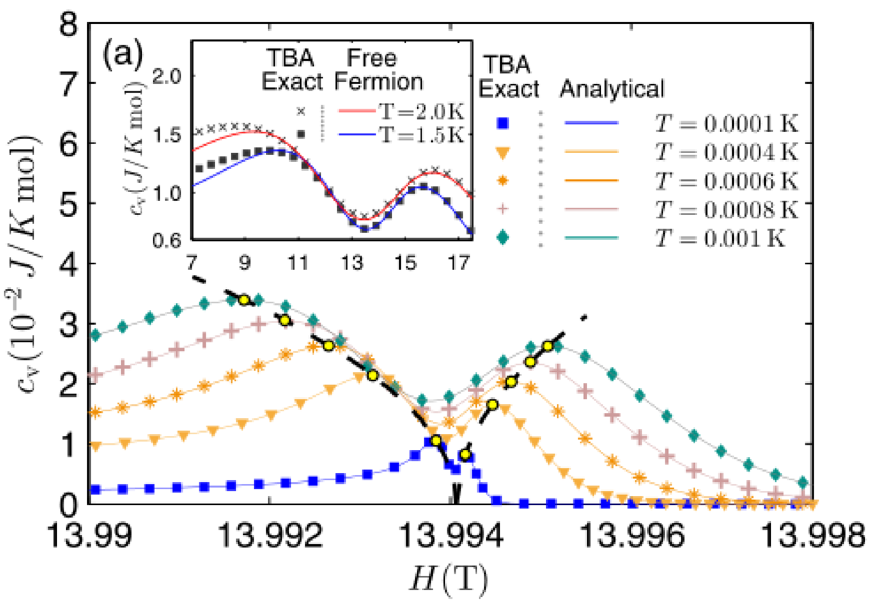
$$\varepsilon(\varphi) = -k^2 - (h_c - h) + ** = -\left(k^2 - \frac{1}{2}(h - h_c)\right)$$

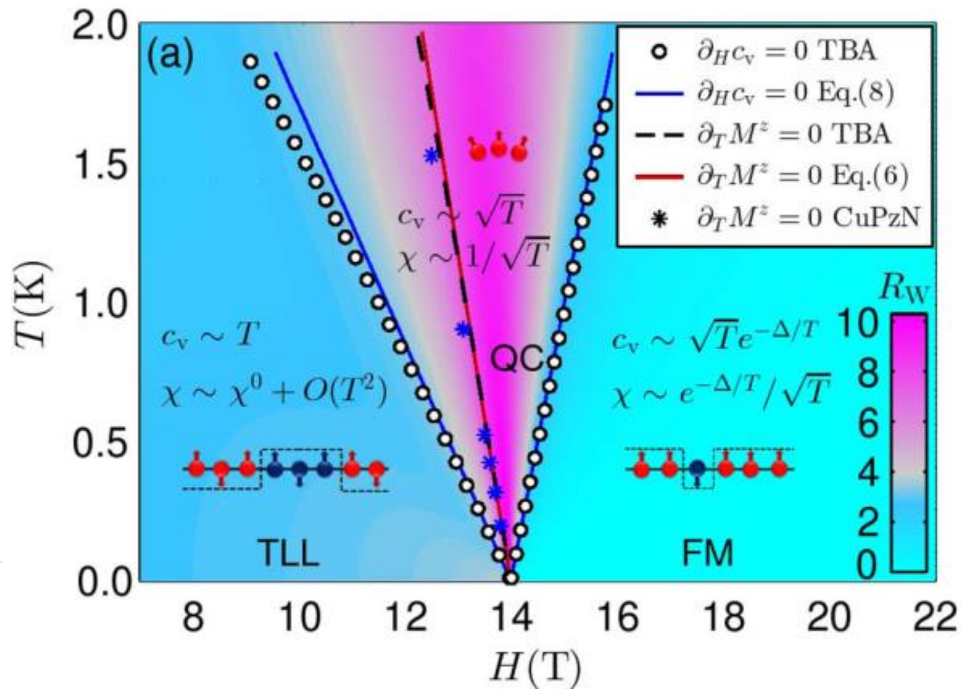
量子临界性

- 铁磁-顺磁相变的量子临界性

$$f \approx \frac{\sqrt{\pi}T^{\frac{3}{2}}}{4\sqrt{J}} \mathrm{Li}_{\frac{3}{2}}\left(-e^{\frac{A}{T}}\right) \qquad A = 4J - H$$

$\Delta = 1$





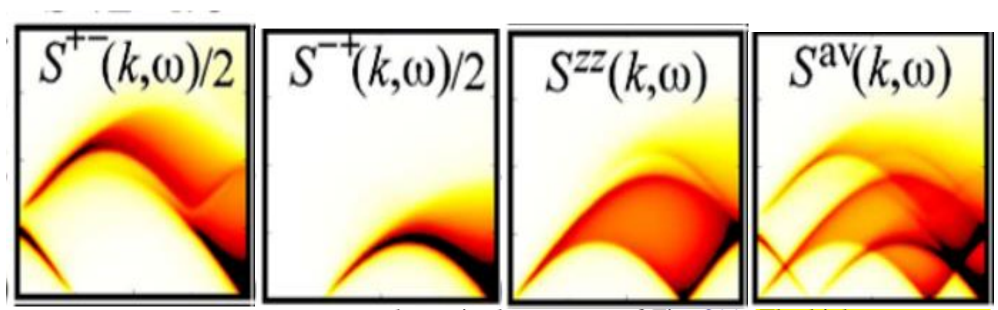
Phys Rev B 96, 220401 (R) (2017)

结论与展望

1, 比热双峰结构可以很好的描写TLL到量子临界区过渡

2, 在量子临界区，多体的相互作用不仅仅影响“自由费米子”的有效质量也会影响到化学势

3, 如何利用可观测关联函数（如：动力学结构因子）来分析热力学普适规律、量子临界性



shown in the top row of Fig. 3(a). The high-energy continuum ($\omega/J \gtrsim 2$), which is mainly due to $S2$, goes up to higher energies as the magnetic field increases, separated

Masanori Kohno
PRL 102, 037203 (2009)

Thanks!