Calabi-Yau period motives in quantum field theory and general relativity

Peng Huanwu Center for Fundamental Theory, USTC Hefei

Albrecht Klemm, BCTP/HCM Bonn University

April 1 2024

Based on work with

Kilian Bönisch, Claude Duhr, Fabian Fischbach, Florian Loebbert, Christoph Nega, Jan Plefka, Franzika Porkert, Reza Safari, Benjamin Sauer, Lorenzo Tancredi

[1]=arXiv:1912.06201v2, [2]=arXiv:2008.10574v1,
[3]=arXiv:2108.05310, in JHEP
[4]=arXiv:2209.05291 in PRL and [5]=arXiv: 2212.09550 in JHEP,
[6]= arXiv:2310.08625 acc. JHEP, [7]= arXiv:2402.xxxxx,
[8]= arXiv:2401.07899 sub. PRL

Overview

The Period Motive of Calabi-Yau manifolds that we know in string theory especially the B-model approach to mirror symmetry and the B-type topological string has new applications to

Overview

The Period Motive of Calabi-Yau manifolds that we know in string theory especially the B-model approach to mirror symmetry and the B-type topological string has new applications to

I. Evaluation of higher loop corrections to Quantum Field Theory, for the new precision test of the Standard Model at future collider experiments CERN . . ..

Overview

The Period Motive of Calabi-Yau manifolds that we know in string theory especially the B-model approach to mirror symmetry and the B-type topological string has new applications to

I. Evaluation of higher loop corrections to Quantum Field Theory, for the new precision test of the Standard Model at future collider experiments CERN . . ..

II. Amplitude evaluations in systems with Yangian integrable symmetries, like 4d N=4 Super-Yang-Mills theory and Fishnet Theories.

Overview

The Period Motive of Calabi-Yau manifolds that we know in string theory especially the B-model approach to mirror symmetry and the B-type topological string has new applications to

I. Evaluation of higher loop corrections to Quantum Field Theory, for the new precision test of the Standard Model at future collider experiments CERN . . ..
II. Amplitude evaluations in systems with Yangian integrable symmetries, like 4d N=4 Super-Yang-Mills theory and Fishnet Theories.
III. Post Minkowskian (PM) Worldline Quantum Field Theory approximation to General Relativity to predict the gravitational wave forms in black hole scattering/mergers detected by LIGO,. . ..

Introduction perturbative QFT

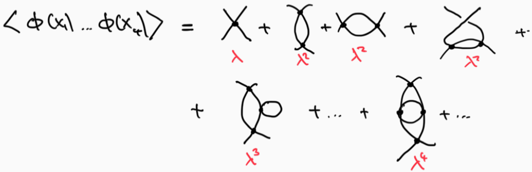
$$Z[J] = \int \mathcal{D}\phi \exp\left[\frac{i}{\hbar}\int \mathrm{d}^D x(\mathcal{L} + J\phi)\right] \ .$$

E.g. with $\mathcal{L} = \int \mathrm{d}^D x \left[\frac{1}{2}(\partial_u \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4\right]$.

All physical correlators are of the form

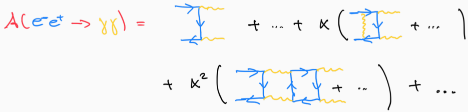
$$\langle\phi(x_1)..\phi(x_n)\rangle = Z[J]^{-1}\left(\frac{\delta}{\delta J(x_1)}\right)..\left(\frac{\delta}{\delta J(x_n)}\right) Z[J]\Big|_{J=0}$$

In interacting theories $\lambda \neq 0$ this is expanded asymptotically in Feynman graphs

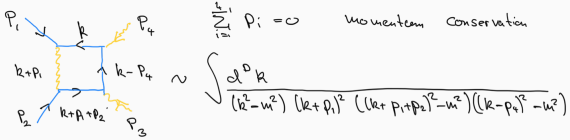


Introduction perturbative QFT

Realistic theories: Probability for e^- e^+ to annihilate to two photons $P(e^- e^+ \to \gamma\gamma) \sim |\mathcal{A}(e^- e^+ \to \gamma\gamma)|^2$, $\alpha \sim \frac{1}{137}$



Scalar part e.g. for e.g. the box integral I: Propagators $\frac{1}{q^2 - m^2 + i \cdot 0}$



$D = D_{cr} - 2\epsilon$, $I = \sum_{k=-n}^{\infty} I_k \epsilon^n$ with I_k functions of masses and Lorentz invariant products of the external momenta that we need to know!

Emerging relation Feyman Integrals and Periods

Feynman integrals $\Leftrightarrow$ Periods of algebraic varieties

Planar Feynman graph	Max. Cut Integrals	Period - Geometry
1-loop	rational functions	Pts in Fano 1-fold

Emerging relation Feyman Integrals and Periods

$$\text{Feynman integrals} \Leftrightarrow \text{Periods of algebraic varieties}$$

Planar Feynman graph	Max. Cut Integrals	Period - Geometry
1-loop	rational functions	Pts in Fano 1-fold
2-loop	elliptic functions	families of elliptic curve

Emerging relation Feyman Integrals and Periods

Feynman integrals $\Leftrightarrow$ Periods of algebraic varieties

Planar Feynman graph	Max. Cut Integrals	Period - Geometry
1-loop	rational functions	Pts in Fano 1-fold
2-loop	elliptic functions	families of elliptic curve
3-loop	fullfil 3 ord. hom diff eqs.	families of K3
4-loop	fullfil 4 ord. hom diff eqs.	families of CY-3-fold
⋮	⋮	⋮

Emerging relation Feyman Integrals and Periods

Feynman integrals $\Leftrightarrow$ Periods of algebraic varieties

Planar Feynman graph	Max. Cut Integrals	Period - Geometry
1-loop	rational functions	Pts in Fano 1-fold
2-loop	elliptic functions	families of elliptic curve
3-loop	fullfil 3 ord. hom diff eqs.	families of K3
4-loop	fullfil 4 ord. hom diff eqs.	families of CY-3-fold
⋮	⋮	⋮

The full Feynman integral has boundaries: Periods integrals are replaced by chain integrals; hom. diff. eqs. are replaced by inhom. diff. eqs.

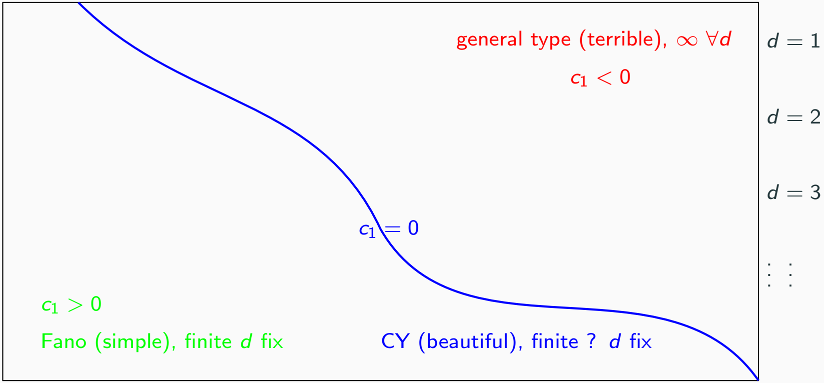
Emerging relation Feyman Integrals and Periods

Feynman integrals $\Leftrightarrow$ Periods of algebraic varieties

Planar Feynman graph	Max. Cut Integrals	Period - Geometry
1-loop	rational functions	Pts in Fano 1-fold
2-loop	elliptic functions	families of elliptic curve
3-loop	fullfil 3 ord. hom diff eqs.	families of K3
4-loop	fullfil 4 ord. hom diff eqs.	families of CY-3-fold
$\vdots$	$\vdots$	$\vdots$

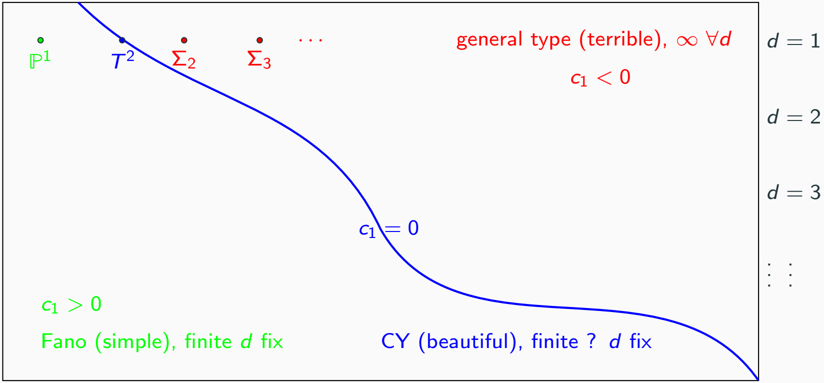
The full Feynman integral has boundaries: Periods integrals are replaced by chain integrals; hom. diff. eqs. are replaced by inhom. diff. eqs. I. Gel'fand, S. Bloch, P. Vanhove, M.Kerr, C. Duran, S. Weinzierl, F. Brown, O. Schnetz, J. Bourjaily, A. Mc Leod, M. Hippel, M. Wilhelm, J. Broedel, L Trancredi, S. Müller-Stach, ... + 248 cits. in [3]

Kodaira map of algebraic varieties

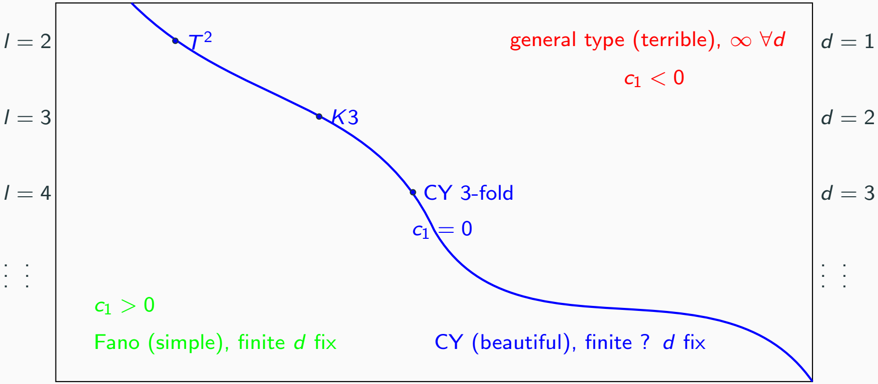


Kodaira map of algebraic varieties

$$\begin{array}{ccccc} l=0 & l=1 & l=2 & l=3 & \cdots \\ g=0 & g=1 & g=2 & g=3 & \cdots \end{array}$$



Kodaira map of algebraic varieties



Dictionary Feynman graphs/amplitudes and geometry

Perturbative QFT	Geometry X	Differential eq.	Arithmetic Geometry
maximal cut Feynman integral	Period integral $\underline{\Pi}$ (ϵ-deformed)	Homogeneous Gauss Manin $(d-A(z))\underline{\Pi}=0$	Motive defined by l-adic coh $H^k_{et}(\overline{X},\mathbb{Q}_l)$
	$\circlearrowleft$ Monodromy group $\in \Gamma(\mathbb{Z})$; irreducible ?		$\circlearrowleft$ Galois group $\mathrm{Gal}(\overline{K}/K)$ irreducible ?
actual Feynman integral	Chain integral ϵ-deformed	Inhomogeneous Gauss Manin connection $(d-A(z))\underline{\Pi}=B(z)$	Extended motive

Gauss Manin connection and sub sectors

One way to get the Gauss-Manin connection and the inhomogeneous term is to use the integration by by parts relations IBP relation between so called master integrals. Consider l-loop Feynman integrals in general dimensions $D \in \mathbb{R}_+$ of the form

$$I_{\underline{\nu}}(\underline{x}, D) := \int \prod_{r=1}^{l} \frac{\mathrm{d}^D k_r}{i\pi^{\frac{D}{2}}} \prod_{j=1}^{p} \frac{1}{D_j^{\nu_j}} \qquad (1)$$

$D_j = q_j^2 - m_j^2 + i \cdot 0$ for $j = 1, \ldots, p$ are the propagators, q_j is the j^{th} momenta through D_j, $m_j^2 \in \mathbb{R}_+$ are masses, $i \cdot 0$ indicates the choice of contour/branchcut in $\mathbb{C}$. Subject to momentum conservation the q_j are linear in the external momenta $p_1, \ldots, p_E$, $\sum_{i=j}^{E} p_j = 0$ and the loop momenta k_r. We defined $\epsilon := \frac{D_{cr} - D}{2}$.

Master Integrals and integration by parts relations

The Feynman integral depends besides D (ϵ) on dot products of p_i and the masses m_j^2, written compactly in a vector $\underline{w} = (w_1, \ldots, w_N) = (p_{i_1} \cdot p_{i_2}, m_j^2)$ and dimensional analysis of $I_{\underline{\nu}}$ shows that it depends only on the ratios of two parameters x_i, we chose

$$x_k := \frac{w_k}{w_N} \qquad \text{for } 1 \leq k < N$$

and label now the parameters of the integrals $I_{\underline{\nu}}$ by the dimensionless parameters $\underline{x}$.

Master Integrals and integration by parts relations

The propagator exponents and $D \in \mathbb{Z}$ span a lattice $(\underline{\nu}, D) \in \mathbb{Z}^{p+1}$. The $I_{\underline{\nu}}(\underline{x}, D)$ are called master integrals.

Master Integrals and integration by parts relations

The propagator exponents and $D \in \mathbb{Z}$ span a lattice $(\underline{\nu}, D) \in \mathbb{Z}^{p+1}$. The $I_{\underline{\nu}}(\underline{x}, D)$ are called master integrals.

The integration by parts (IBP) identities

$$\int \prod_{r=1}^{l} \frac{\mathrm{d}^D k_r}{i\pi^{\frac{D}{2}}} \frac{\partial}{\partial k_k^{\mu}} \left(q_l^{\mu} \prod_{j=1}^{p} \frac{1}{D_j^{\nu_j}} \right) = 0 \ .$$

relate the master integrals with different exponents $\underline{\nu}$.

Master Integrals and integration by parts relations

The propagator exponents and $D \in \mathbb{Z}$ span a lattice $(\underline{\nu}, D) \in \mathbb{Z}^{p+1}$. The $I_{\underline{\nu}}(\underline{x}, D)$ are called master integrals.

The integration by parts (IBP) identities

$$\int \prod_{r=1}^{l} \frac{\mathrm{d}^D k_r}{i\pi^{\frac{D}{2}}} \frac{\partial}{\partial k_k^\mu} \left(q_l^\mu \prod_{j=1}^{p} \frac{1}{D_j^{\nu_j}} \right) = 0 \ .$$

relate the master integrals with different exponents $\underline{\nu}$.

There is a finite region in the lattice that contains all non-vanishing master integrals. In a basis of master integrals one can express derivatives w.r.t. the z_k as a linear combination rational coefficients by the IBP relations.

Master Integrals and integration by parts relations

The basis of master integrals (graph cohomology) corresponds to the basis of the cohomology $H^{l-1}(M_l, \mathbb{Z})$.

.

Master Integrals and integration by parts relations

The basis of master integrals (graph cohomology) corresponds to the basis of the cohomology $H^{l-1}(M_l, \mathbb{Z})$.

The integration by parts relations correspond to the Griffith reduction formula.

.

Master Integrals and integration by parts relations

The basis of master integrals (graph cohomology) corresponds to the basis of the cohomology $H^{l-1}(M_l, \mathbb{Z})$.

The integration by parts relations correspond to the Griffith reduction formula.

A complete set of IBP relations corresponds to the complete Picard Fuchs ideal of Gauss-Manin connection for the period integrals.

Master Integrals and integration by parts relations

- The basis of master integrals (graph cohomology) corresponds to the basis of the cohomology $H^{l-1}(M_l, \mathbb{Z})$.
- The integration by parts relations correspond to the Griffith reduction formula.
- A complete set of IBP relations corresponds to the complete Picard Fuchs ideal of Gauss-Manin connection for the period integrals.

Among the elements in the lattice $\mathbb{Z}^p$ and, in particular, for the master integrals one can define sectors and a semi-ordering on the latter by defining a map

$$\underline{\nu} \mapsto \underline{\vartheta}(\underline{\nu}) =: (\theta(\nu_j))_{1 \leq j \leq p} \ .$$

where θ is the Heaviside step function. The semi-ordering is then defined by $\underline{\vartheta}(\underline{\nu}) \leq \underline{\vartheta}(\underline{\tilde{\nu}})$, iff $\theta(\nu_j) \leq \theta(\tilde{\nu}_j)$, $\forall j$. This defines an inclusive order on subgraphs with less propagators and therefore simpler topology.

IBP relation summary:

The IBP relations characterise a suitable finite set of master integrals

$$I_{\underline{\nu}}(\underline{x}, D) := \int \prod_{r=1}^{l} \frac{\mathrm{d}^D k_r}{i\pi^{\frac{D}{2}}} \prod_{j=1}^{p} \frac{1}{D_j^{\nu_j}} ,$$

with $D_j = q_j^2 - m_j^2 + i \cdot 0$ for $j = 1, \ldots, p$ propagators and $(\underline{\nu}, D)$ in a finite region in $\mathbb{Z}^{p+1}$, by a first order Gauss Manin connection

$$\boxed{d\underline{I}(\underline{x}, \epsilon) = \mathbf{A}(\underline{x}, \epsilon)\underline{I}(\underline{x}, \epsilon)}$$

$\epsilon = (D_{cr} - D)/2.$

Master Integral Basis Change possibly to canonical form

$$\underline{I}(\underline{x}, \epsilon) \to \underline{I}^{better}(\underline{z}(x); \epsilon) = R_0(\underline{z}(x); \epsilon)\underline{I}(\underline{z}(x); \epsilon)$$

$$\mathbf{A}(\underline{z}; \epsilon)^{better} = [R_0(\underline{z}; \epsilon)\mathbf{A} + dR_0(\underline{z}; \epsilon)]R_0(\underline{z}; \epsilon)^{-1}$$

Master Integral Basis Change possibly to canonical form

$$\underline{I}(\underline{x},\epsilon) \to \underline{I}^{better}(\underline{z}(x);\epsilon) = R_0(\underline{z}(x);\epsilon)\underline{I}(\underline{z}(x);\epsilon)$$

$$\mathbf{A}(\underline{z};\epsilon)^{better} = [R_0(\underline{z};\epsilon)\mathbf{A} + dR_0(\underline{z};\epsilon)]R_0(\underline{z};\epsilon)^{-1}$$

$$\mathbf{A}(\underline{z};\epsilon)^{best} = [R_n(\underline{z};\epsilon)\mathbf{A}^{n-1} + dR_n(\underline{z};\epsilon)]R_n(\underline{z};\epsilon)^{-1} = \epsilon A(\underline{z})$$

Master Integral Basis Change possibly to canonical form

$$\underline{I}(\underline{x},\epsilon) \to \underline{I}^{better}(\underline{z}(x);\epsilon) = R_0(\underline{z}(x);\epsilon)\underline{I}(\underline{z}(x);\epsilon)$$
$$\mathbf{A}(\underline{z};\epsilon)^{better} = [R_0(\underline{z};\epsilon)\mathbf{A} + dR_0(\underline{z};\epsilon)]R_0(\underline{z};\epsilon)^{-1}$$

$$\mathbf{A}(\underline{z};\epsilon)^{best} = [R_n(\underline{z};\epsilon)\mathbf{A}^{n-1} + dR_n(\underline{z};\epsilon)]R_n(\underline{z};\epsilon)^{-1} = \epsilon A(\underline{z})$$

$$\left[d_z - \epsilon \begin{pmatrix} & \mathbf{0} & & & \mathbf{0} & & & \mathbf{0} & \\ * & \dots & * & A^1_{11} & \cdots & A^1_{1r_1} & & & \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & & \mathbf{0} & \\ * & \dots & * & A^1_{r_1} & \cdots & A^1_{r_1 r_1} & & & \\ & \vdots & & & \ddots & & & & \\ * & \dots & * & & & & A^n_{11} & \cdots & A^n_{1r_1} \\ \vdots & \ddots & \vdots & & \mathbf{0} & & \vdots & \ddots & \vdots \\ * & \dots & * & & & & A^n_{r_2} & \cdots & A^n_{r_n r_n} \end{pmatrix} \right] \begin{pmatrix} I^{sub} \\ \Pi^1_1 \\ \vdots \\ \Pi^1_{r_1} \\ \vdots \\ \Pi_{n_1} \\ \vdots \\ \Pi_{n_r} \end{pmatrix}^{best} = 0$$

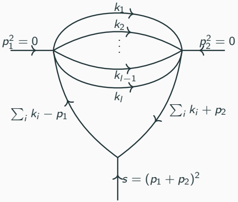
The blocks

Here $A_{ij}^k(z)$ are $d\log(\mathrm{alg}(z))$ and the $*$ are rational functions in z and we typically have a situation, where the l-loop block in this improved IBP first order flat connection above is described by period integrals in the sense of Kontsevich and Zagier fullfilling the Gauss-Manin flat connection of a geometry X, which is typically a (non-smooth) Calabi-Yau manifold.

The blocks

Here $A^k_{ij}(z)$ are $d\log(\mathrm{alg}(z))$ and the $*$ are rational functions in z and we typically have a situation, where the l-loop block in this improved IBP first order flat connection above is described by period integrals in the sense of Kontsevich and Zagier fullfilling the Gauss-Manin flat connection of a geometry X, which is typically a (non-smooth) Calabi-Yau manifold.

Example: From the (l+1)-loop ice-cone graph



it is clear that it contains l-loop banana graph as block(s).

Dictionary for the blocks

	$l = (n+1)$-**loop in block integrals in** D_{cr} **dimensions**	**Calabi-Yau (CY) geometry**
1	Maximal cut integrals in D_{cr} dimensions	$(n, 0)$-form periods of CY manifolds or CY motives
2	Dimensionless ratios $z_i = {m_i}^2/p^2$	Unobstructed compl. moduli of M_n, or equi'ly Kähler moduli of the mirror W_n
3	Integration-by-parts (IBP) reduction	Griffiths reduction method
4	Integrand-basis for maximal cuts of of master integrals in D_{cr}	Middle (hyper) cohomology $H^n(M_n)$ M_n
5	Complete set of differential operators annihilating a given maximal cut in D_{cr} dimensions	Homogeneous Picard-Fuchs differential ideal (PFI) / Gauss-Manin (GM) connection

Definition of compact Calabi-Yau (CY) n - folds

A Calabi -Yau n-fold M is a compact *complex manifold* of complex dimension n that

Definition of compact Calabi-Yau (CY) n - folds

A Calabi -Yau n-fold M is a compact *complex manifold* of complex dimension n that

ω) is a Kähler manifold, i.e. has a Kähler $(1,1)$-form ω,

Definition of compact Calabi-Yau (CY) n - folds

A Calabi -Yau n-fold M is a compact *complex manifold* of complex dimension n that

ω) is a Kähler manifold, i.e. has a Kähler $(1,1)$-form ω,

Ω) and has unique no-where vanishing holomorphic $(n,0)$-form Ω.

Definition of compact Calabi-Yau (CY) n - folds

A Calabi -Yau n-fold M is a compact *complex manifold* of complex dimension n that

ω) is a Kähler manifold, i.e. has a Kähler $(1,1)$-form ω,

Ω) and has unique no-where vanishing holomorphic $(n,0)$-form Ω.

The latter condition is equivalent to

1) the canonical class is trivial $K_M = c_1(T_M) = 0$,
2) given a Kähler class, $\exists$ metric g with $R_{i\bar{\jmath}}(g) = 0$,
3) the holonomy of the metric g is $(\subset SU(n))$ $SU(n)$,

Definition of compact Calabi-Yau (CY) n - folds

A Calabi -Yau n-fold M is a compact *complex manifold* of complex dimension n that

ω) is a Kähler manifold, i.e. has a Kähler $(1,1)$-form ω,

Ω) and has unique no-where vanishing holomorphic $(n,0)$-form Ω.

The latter condition is equivalent to

1) the canonical class is trivial $K_M = c_1(T_M) = 0$,
2) given a Kähler class, $\exists$ metric g with $R_{i\bar{\jmath}}(g) = 0$,
3) the holonomy of the metric g is $(\subset SU(n))$ $SU(n)$,

Remarks: CY n-fold are generalisations of elliptic curves

- CY 1-fold is an elliptic curve, say $y^2 = x(x-1)(x-z)$ with Ω given by $\frac{dx}{y}$ and $\omega = \frac{dx}{y} \wedge \frac{d\bar{x}}{\bar{y}}$ is its volume form.

Definition of compact Calabi-Yau (CY) n - folds

A Calabi -Yau n-fold M is a compact *complex manifold* of complex dimension n that

ω) is a Kähler manifold, i.e. has a Kähler $(1,1)$-form ω,

Ω) and has unique no-where vanishing holomorphic $(n,0)$-form Ω.

The latter condition is equivalent to

1) the canonical class is trivial $K_M = c_1(T_M) = 0$,
2) given a Kähler class, $\exists$ metric g with $R_{i\bar{\jmath}}(g) = 0$,
3) the holonomy of the metric g is $(\subset SU(n))$ $SU(n)$,

Remarks: CY n-fold are generalisations of elliptic curves

- CY 1-fold is an elliptic curve, say $y^2 = x(x-1)(x-z)$ with Ω given by $\frac{dx}{y}$ and $\omega = \frac{dx}{y} \wedge \frac{d\bar{x}}{\bar{y}}$ is its volume form.
- We use $SU(n)$ rather then $\subset SU(n)$ to avoid trivial products of lower CY n-folds in the generalisation.

Construction of Calabi-Yau n-folds hypersurface in projective spaces

Let M be a degree $\mathcal{N} = dH$ embedding of M into $H \subset \mathbb{P}^{n+1}$.
Then the splitting of the exact sequence

$$0 \to TM \to T\mathbb{P}^{n+1} \to \mathcal{N} \to 0$$

Construction of Calabi-Yau n-folds hypersurface in projective spaces

Let M be a degree $\mathcal{N} = dH$ embedding of M into $H \subset \mathbb{P}^{n+1}$.
Then the splitting of the exact sequence

$$0 \to TM \to T\mathbb{P}^{n+1} \to \mathcal{N} \to 0$$

at $T\mathbb{P}^{n+1}$ implies with $c_1(T\mathbb{P}^{n+1}) = (1+H)^{n+2}$ and $c_1(\mathcal{N}) = (1+dH)$ that $ch(TM)$ equals

$$\frac{(1+H)^{n+2}}{1+dH} = 1 + \underbrace{[(n+2)-d]H}_{c_1(TM)=0!} + \underbrace{[(1-d)^2 + \tfrac{1}{2}n(n+3-2d)]H^2}_{c_2(TM)=c_2 H^2} + \ldots$$

Construction of Calabi-Yau n-folds hypersurface in projective spaces

Let M be a degree $\mathcal{N} = dH$ embedding of M into $H \subset \mathbb{P}^{n+1}$.
Then the splitting of the exact sequence

$$0 \to TM \to T\mathbb{P}^{n+1} \to \mathcal{N} \to 0$$

at $T\mathbb{P}^{n+1}$ implies with $c_1(T\mathbb{P}^{n+1}) = (1+H)^{n+2}$ and $c_1(\mathcal{N}) = (1+dH)$ that $ch(TM)$ equals

$$\frac{(1+H)^{n+2}}{1+dH} = 1 + \underbrace{[(n+2)-d]H}_{c_1(TM)=0!} + \underbrace{[(1-d)^2 + \tfrac{1}{2}n(n+3-2d)]H^2}_{c_2(TM)=c_2H^2} + \ldots$$

$\Rightarrow$ 1) cubic $d = 3$ in $\mathbb{P}^2$ $wy^2 = 4x^3 - g_2(z)xw^2 - g_3(z)w^3$ is complex family of elliptic curves with complex modulus z.

Construction of Calabi-Yau n-folds hypersurface in projective spaces

Let M be a degree $\mathcal{N} = dH$ embedding of M into $H \subset \mathbb{P}^{n+1}$.
Then the splitting of the exact sequence

$$0 \to TM \to T\mathbb{P}^{n+1} \to \mathcal{N} \to 0$$

at $T\mathbb{P}^{n+1}$ implies with $c_1(T\mathbb{P}^{n+1}) = (1+H)^{n+2}$ and $c_1(\mathcal{N}) = (1+dH)$ that $ch(TM)$ equals

$$\frac{(1+H)^{n+2}}{1+dH} = 1 + \underbrace{[(n+2)-d]H}_{c_1(TM)=0!} + \underbrace{[(1-d)^2 + \frac{1}{2}n(n+3-2d)]H^2}_{c_2(TM)=c_2 H^2} + \ldots$$

$\Rightarrow$ 1) cubic $d = 3$ in $\mathbb{P}^2$ $wy^2 = 4x^3 - g_2(z)xw^2 - g_3(z)w^3$ is complex family of elliptic curves with complex modulus z.

$\Rightarrow$ 2) quartic in $\mathbb{P}^3$ is a CY 2-fold with 19 complex moduli.

Construction of Calabi-Yau n-folds hypersurface in projective spaces

Let M be a degree $\mathcal{N} = dH$ embedding of M into $H \subset \mathbb{P}^{n+1}$.
Then the splitting of the exact sequence

$$0 \to TM \to T\mathbb{P}^{n+1} \to \mathcal{N} \to 0$$

at $T\mathbb{P}^{n+1}$ implies with $c_1(T\mathbb{P}^{n+1}) = (1+H)^{n+2}$ and $c_1(\mathcal{N}) = (1+dH)$ that $ch(TM)$ equals

$$\frac{(1+H)^{n+2}}{1+dH} = 1 + \underbrace{[(n+2)-d]H}_{c_1(TM)=0!} + \underbrace{[(1-d)^2 + \tfrac{1}{2}n(n+3-2d)]H^2}_{c_2(TM)=c_2H^2} + \ldots$$

$\Rightarrow$ 1) cubic $d = 3$ in $\mathbb{P}^2$ $wy^2 = 4x^3 - g_2(z)xw^2 - g_3(z)w^3$ is complex family of elliptic curves with complex modulus z.

$\Rightarrow$ 2) quartic in $\mathbb{P}^3$ is a CY 2-fold with 19 complex moduli.

$\Rightarrow$ 3) quintic in $\mathbb{P}^4$ is a CY 3-fold with 101 complex moduli.

More on constructions of Calabi-Yau n-folds

Number of complex moduli $\#mon - |Aut(\mathbb{P}^*)|$:

1) $(x_i^3; 3, x_i^2 x_j; 6, \prod x_i; 1)$: 10-9=1,

More on constructions of Calabi-Yau n-folds

Number of complex moduli $\#mon - |Aut(\mathbb{P}^*)|$:

1) $(x_i^3; 3, x_i^2 x_j; 6, \prod x_i; 1)$: 10-9=1,
2) $(x_i^4; 4, x_i^3 x_j; 12, x_i^2 x_j^2; 6, x_i^2 x_j x_k; 12, \prod x_i; 1)$: 35-16=19,
3) Likewise 126-25=101.

More on constructions of Calabi-Yau n-folds

Number of complex moduli $\#mon - |Aut(\mathbb{P}^*)|$:

1) $(x_i^3; 3, x_i^2 x_j; 6, \prod x_i; 1)$: 10-9=1,
2) $(x_i^4; 4, x_i^3 x_j; 12, x_i^2 x_j^2; 6, x_i^2 x_j x_k; 12, \prod x_i; 1)$: 35-16=19,
3) Likewise 126-25=101.

Euler number (Gauss Bonnet): $\chi = \int_{M_n} c_n(TM) = c_n d$,

1) $\chi = 0$, $\chi = 2g - 2 \Rightarrow g = 1$ one topological type E.

More on constructions of Calabi-Yau n-folds

Number of complex moduli $\#mon - |Aut(\mathbb{P}^*)|$:

1) $(x_i^3; 3, x_i^2 x_j; 6, \prod x_i; 1)$: 10-9=1,
2) $(x_i^4; 4, x_i^3 x_j; 12, x_i^2 x_j^2; 6, x_i^2 x_j x_k; 12, \prod x_i; 1)$: 35-16=19,
3) Likewise 126-25=101.

Euler number (Gauss Bonnet): $\chi = \int_{M_n} c_n(TM) = c_n d$,

1) $\chi = 0$, $\chi = 2g - 2 \Rightarrow g = 1$ one topological type E.
2) By $c_2(TM) = 6H^2 \Rightarrow \chi = 24$. HRR for arithmetic genus of surface $\chi_0 = \sum_{i=0}^{2}(-1)^i h^{0,i} = \frac{1}{12}\int_{M_2}(c_1^2 + c_2)$. Now by definition $h^{00} = h^{02} = 1$, $h^{01} = 0$ because of $SU(2)$ hol, i.e. $\chi_0(M_2) = 2$ and since $c_1 = 0 \Rightarrow \chi(M_2) = 24$ and we have only one topological type the $K3$ surface
3) By $c_3(TM) = -40H^3 \Rightarrow \chi = -200$. Hirzebruch Riemann Roch $\chi_0 = \frac{1}{24}\int_{M_3} c_1 c_2 = 1 - 0 + 0 - 1\checkmark$, $\chi_1 = -h^{11} + h^{21} = \frac{1}{24}\int_{M_3} c_1 c_2 - 12 c_3 \Rightarrow \chi = 2(h^{11} - h^{21})\checkmark$

More on constructions of Calabi-Yau n-folds

Theorem (C.T.C Wall): The topological type of a Calabi-Yau 3-fold M is fixed by their Hodge numbers (h_{21}, h_{11}), their triple intersection $D_i \cap D_j \cap D_k \in \mathbb{N}$ and $c_2(TM) \cdots D_k$, $D_k \in H_4(M, \mathbb{Z})$.

More on constructions of Calabi-Yau n-folds

Theorem (C.T.C Wall): The topological type of a Calabi-Yau 3-fold M is fixed by their Hodge numbers (h_{21}, h_{11}), their triple intersection $D_i \cap D_j \cap D_k \in \mathbb{N}$ and $c_2(TM) \cdots D_k$, $D_k \in H_4(M, \mathbb{Z})$. The topological classification Calabi-Yau n-folds for $n \geq 3$ is an open problem, but it is relatively easy to find classes of constructions:

More on constructions of Calabi-Yau n-folds

Theorem (C.T.C Wall): The topological type of a Calabi-Yau 3-fold M is fixed by their Hodge numbers (h_{21}, h_{11}), their triple intersection $D_i \cap D_j \cap D_k \in \mathbb{N}$ and $c_2(TM) \cdots D_k$, $D_k \in H_4(M, \mathbb{Z})$. The topological classification Calabi-Yau n-folds for $n \geq 3$ is an open problem, but it is relatively easy to find classes of constructions:

CICYs: Complete intersections: The vanishing locus of r polynomials $P_k = 0$, $k = 1, \ldots, r$ in $\mathbb{P} = \otimes_{l=1}^{m} \mathbb{P}_l^{n_l}$ define a CY $(\sum_{l=1}^{m} n_l - r)$ -fold if $\sum_{k=1}^{r} d_{kl} = n_l + 1$, $\forall l = 1, \ldots, m$, with d_{kl} are degrees of the k'th polynomial in the l'th factor: 2d n-1 loop bananas.

More on constructions of Calabi-Yau n-folds

Theorem (C.T.C Wall): The topological type of a Calabi-Yau 3-fold M is fixed by their Hodge numbers (h_{21}, h_{11}), their triple intersection $D_i \cap D_j \cap D_k \in \mathbb{N}$ and $c_2(TM) \cdots D_k$, $D_k \in H_4(M, \mathbb{Z})$. The topological classification Calabi-Yau n-folds for $n \geq 3$ is an open problem, but it is relatively easy to find classes of constructions:

CICYs: Complete intersections: The vanishing locus of r polynomials $P_k = 0$, $k = 1, \ldots, r$ in $\mathbb{P} = \otimes_{l=1}^{m} \mathbb{P}_l^{n_l}$ define a CY $\left(\sum_{l=1}^{m} n_l - r\right)$ -fold if $\sum_{k=1}^{r} d_{kl} = n_l + 1$, $\forall l = 1, \ldots, m$, with d_{kl} are degrees of the k'th polynomial in the l'th factor: 2d n-1 loop bananas.

BCs: Branched covers: Let $\mathbb{P}$ be a n-dimensional Fano variety with positive canonical class $K(\mathbb{P}) = c_1(\mathbb{P}) > 0$ then a d-fold cover that is branched at $\frac{d}{d-1} K(\mathbb{P})$ is a non necessarily smooth CY n-fold: $\mathbb{P} = (\mathbb{P}^1)^n$ and $d = 2, 3$ are relevant for 2d n-loop fishnets .

General properties of Calabi-Yau n-fold fold families

Theorem Tian/Todorov: The complex moduli space $\mathcal{M}_{cs}(M)$ of a CY n-fold M is parametrized for by $h^{n-1,1} = \dim_{\mathbb{C}}(H^{n-1,1}(M))$ globally unobstructed complex deformation parameters z, i.e. is a manifold of complex dimension $h^{n-1,1} =: r$ (E and K_3 are special).

General properties of Calabi-Yau n-fold fold families

Theorem Tian/Todorov: The complex moduli space $\mathcal{M}_{cs}(M)$ of a CY n-fold M is parametrized for by $h^{n-1,1} = \dim_{\mathbb{C}}(H^{n-1,1}(M))$ globally unobstructed complex deformation parameters z, i.e. is a manifold of complex dimension $h^{n-1,1} =: r$ (E and K_3 are special).

Example: We counted $101 = h^{2,1}$ complex deformation parameters for the quintic in $\mathbb{P}^4$ and by the Lefshetz hyperplane theorem $h^{1,1} = 1$ (inherited from $\mathbb{P}^4$), hence $\chi = 2(h^{1,1} - h^{21}) = -200$ in accordance with Gauss-Bonnet.

General properties of Calabi-Yau n-fold fold families

Theorem Tian/Todorov: The complex moduli space $\mathcal{M}_{cs}(M)$ of a CY n-fold M is parametrized for by $h^{n-1,1} = \dim_{\mathbb{C}}(H^{n-1,1}(M))$ globally unobstructed complex deformation parameters z, i.e. is a manifold of complex dimension $h^{n-1,1} =: r$ (E and K_3 are special).

Example: We counted $101 = h^{2,1}$ complex deformation parameters for the quintic in $\mathbb{P}^4$ and by the Lefshetz hyperplane theorem $h^{1,1} = 1$ (inherited from $\mathbb{P}^4$), hence $\chi = 2(h^{1,1} - h^{21}) = -200$ in accordance with Gauss-Bonnet.

Application: The complex moduli dependent period integrals on CY n-fold families generalize elliptic functions. They are identified for important examples with the maximal cut Feynman higher-loop integrals, where the complex moduli z are identified with the scale invariant physical parameters e.g. $z_i = p^2/m_i^2, \ldots$.

Periods integrals

$$\Pi_{ij}(\underline{z}) = \int_{\Gamma_i} \gamma^j(\underline{z})$$

define a non-degenerate pairing between between (middle) homology and (middle) cohomology well defined by the theorem of Stokes:

$$\Pi : H_n(M_n, \mathbb{K}) \times H^n(M_n, \mathbb{C}) \to \mathbb{C}\,.$$

It is possible and natural to have $\mathbb{K}$ to be $\mathbb{Z}$. There is an intersection pairing

$$\Sigma : H_n(M_n, \mathbb{K}) \times H_n(M_n, \mathbb{K}) \to \mathbb{K},$$

that can be made in particular integral. If n is odd Σ is antisymmetric and can be made symplectic. If n is even Σ is a symmetric on the even self dual lattice $H_n(M_n, \mathbb{K})$. E.g. for $K3$ $b_2 = 22$ and $\sigma = b_2^+ - b_2^- = \frac{1}{3}\int_{M_2} c_1^2 - 2c_2 = -16$ hence b_2 has signature $(3, 19)$ and is $E_8(-1)^{\oplus 2} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus 3}$.

If n is odd we fix can integral symplectic basis $\underline{\Gamma} = \{A_I, B^I\}$, $I = 0, \ldots, r$ with $\mathrm{Span}_{\mathbb{Z}}(\underline{\Gamma}) = H_n(W, \mathbb{Z})$ and

$$A_I \cap A_J = B^I \cap B^J = 0, \quad A_I \cap B^J = -B^J \cap A_I = \delta_I^J .$$

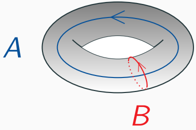
It is clearly defined up only to an $\mathrm{Sp}(b_n(M), \mathbb{Z})$ choice.

If n is odd we fix can integral symplectic basis $\underline{\Gamma} = \{A_I, B^I\}$, $I = 0, \ldots, r$ with $\mathrm{Span}_{\mathbb{Z}}(\underline{\Gamma}) = H_n(W, \mathbb{Z})$ and

$$A_I \cap A_J = B^I \cap B^J = 0, \quad A_I \cap B^J = -B^J \cap A_I = \delta_I^J.$$

It is clearly defined up only to an $\mathrm{Sp}(b_n(M), \mathbb{Z})$ choice.

Exp: Calabi-Yau 1-fold: $p_3 = wy^2 - x(x-w)(x-wz) = 0 \subset \mathbb{P}^2$

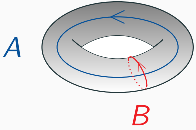


If n is odd we fix can integral symplectic basis $\underline{\Gamma} = \{A_I, B^I\}$, $I = 0, \ldots, r$ with $\mathrm{Span}_{\mathbb{Z}}(\underline{\Gamma}) = H_n(W, \mathbb{Z})$ and

$$A_I \cap A_J = B^I \cap B^J = 0, \quad A_I \cap B^J = -B^J \cap A_I = \delta_I^J .$$

It is clearly defined up only to an $\mathrm{Sp}(b_n(M), \mathbb{Z})$ choice.

Exp: Calabi-Yau 1-fold: $p_3 = wy^2 - x(x-w)(x-wz) = 0 \subset \mathbb{P}^2$



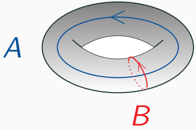
$\Omega(z) = \oint \frac{2dx \wedge dy}{p_3} = \frac{dx}{y}$, $\partial_z \Omega(z) \sim \frac{xdx}{y}$

If n is odd we fix can integral symplectic basis $\underline{\Gamma} = \{A_I, B^I\}$, $I = 0, \ldots, r$ with $\mathrm{Span}_{\mathbb{Z}}(\underline{\Gamma}) = H_n(W, \mathbb{Z})$ and

$$A_I \cap A_J = B^I \cap B^J = 0, \quad A_I \cap B^J = -B^J \cap A_I = \delta_I^J.$$

It is clearly defined up only to an $\mathrm{Sp}(b_n(M), \mathbb{Z})$ choice.

Exp: Calabi-Yau 1-fold: $p_3 = wy^2 - x(x-w)(x-wz) = 0 \subset \mathbb{P}^2$



$\Omega(z) = \oint \frac{2dx\wedge dy}{p_3} = \frac{dx}{y}$, $\partial_z \Omega(z) \sim \frac{xdx}{y}$

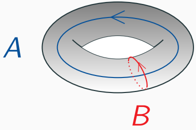
$E_1(z) = \oint_A \Omega$, $E_2(z) = \oint_B \Omega$ Elliptic integrals.

If n is odd we fix can integral symplectic basis $\underline{\Gamma} = \{A_I, B^I\}$, $I = 0, \ldots, r$ with $\mathrm{Span}_{\mathbb{Z}}(\underline{\Gamma}) = H_n(W, \mathbb{Z})$ and

$$A_I \cap A_J = B^I \cap B^J = 0, \quad A_I \cap B^J = -B^J \cap A_I = \delta_I^J.$$

It is clearly defined up only to an $\mathrm{Sp}(b_n(M), \mathbb{Z})$ choice.

Exp: Calabi-Yau 1-fold: $p_3 = wy^2 - x(x-w)(x-wz) = 0 \subset \mathbb{P}^2$



$$\Omega(z) = \oint \tfrac{2dx \wedge dy}{p_3} = \tfrac{dx}{y},\ \partial_z \Omega(z) \sim \tfrac{xdx}{y}$$

$E_1(z) = \oint_A \Omega,\ E_2(z) = \oint_B \Omega$ Elliptic integrals.

Well studied in part because they solve Keplers problem

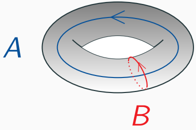
Periods annihilated by Picard-Fuchs (1881) 2cd order linear operator $L^{(2)}$.

If n is odd we fix can integral symplectic basis $\underline{\Gamma} = \{A_I, B^I\}$, $I = 0, \ldots, r$ with $\mathrm{Span}_{\mathbb{Z}}(\underline{\Gamma}) = H_n(W, \mathbb{Z})$ and

$$A_I \cap A_J = B^I \cap B^J = 0, \quad A_I \cap B^J = -B^J \cap A_I = \delta_I^J .$$

It is clearly defined up only to an $\mathrm{Sp}(b_n(M), \mathbb{Z})$ choice.

Exp: Calabi-Yau 1-fold: $p_3 = wy^2 - x(x-w)(x-wz) = 0 \subset \mathbb{P}^2$



$$\Omega(z) = \oint \frac{2dx \wedge dy}{p_3} = \frac{dx}{y},\ \partial_z \Omega(z) \sim \frac{xdx}{y}$$

$E_1(z) = \oint_A \Omega,\ E_2(z) = \oint_B \Omega$ Elliptic integrals.

Well studied in part because they solve Keplers problem

Periods annihilated by Picard-Fuchs (1881) 2cd order linear operator $L^{(2)}$.

$$\mathcal{L} \int_\Gamma \Omega = \left[(1-z)\partial_z^2 + (1-2z)\partial_z - \frac{1}{4}\right] \int_\Gamma \Omega = 0 \ .$$

Period geometry on CY n-fold

The main constraints which govern the period geometry of CY-folds are the Riemann bilinear relations

$$e^{-K} = i^{n^2} \int_{M_n} \Omega \wedge \bar{\Omega} > 0 \qquad (2)$$

defining the real positive exponential of the Kähler potential $K(z)$ for the Weil-Peterssen metric $G_{i\bar{\jmath}} = \partial_{z_i} \bar{\partial}_{\bar{z}_{\bar{\jmath}}} K(z)$ on $\mathcal{M}_{cs}(M_n)$.

Period geometry on CY n-fold

The main constraints which govern the period geometry of CY-folds are the Riemann bilinear relations

$$e^{-K} = i^{n^2} \int_{M_n} \Omega \wedge \bar{\Omega} > 0 \tag{2}$$

defining the real positive exponential of the Kähler potential $K(z)$ for the Weil-Peterssen metric $G_{i\bar{\jmath}} = \partial_{z_i} \bar{\partial}_{\bar{z}_{\bar{\jmath}}} K(z)$ on $\mathcal{M}_{cs}(M_n)$. As well as from relations on holomorphic bilinears and their derivatives that follow from Griffiths transversality

$$\int_{M_n} \Omega \wedge \underline{\partial}^k_{I_k} \Omega = \begin{cases} 0 & \text{if } k < n \\ C_{I_n}(z) & \text{if } k = n\ . \end{cases} \tag{3}$$

Here $\underline{\partial}^k_{I_k} \Omega = \partial_{z_{I_1}} \ldots \partial_{z_{I_k}} \Omega \in F^{n-k} := \bigoplus_{p=0}^k H^{n-p,p}(W)$ are arbitrary combinations of derivatives w.r.t. to the z_i, $i = 1, \ldots, r$.

Period geometry on CY n-fold

The $C_{I_n}(z)$ are rational functions labelled by I_n up to permutations. The differential ideals $\mathcal{L}\vec{\Pi} = 0$ also determine the $C_{I_n}(z)$ up to a multiplicative normalisation

Period geometry on CY n-fold

The $C_{I_n}(z)$ are rational functions labelled by I_n up to permutations. The differential ideals $\mathcal{L}\vec{\Pi} = 0$ also determine the $C_{I_n}(z)$ up to a multiplicative normalisation

Remark 1:W.r.t the Hodge decomposition the pairings (2) and (3) have the property that if $\alpha_{m,n} \in H^{m,n}(M_n)$ and $\beta_{p,q} \in H^{r,s}(M_n)$ then $\int_W \alpha_{m,n} \wedge \beta_{p,q} = 0$ unless $m + p = n + q = 3$.

Period geometry on CY n-fold

The $C_{I_n}(z)$ are rational functions labelled by I_n up to permutations. The differential ideals $\mathcal{L}\vec{\Pi} = 0$ also determine the $C_{I_n}(z)$ up to a multiplicative normalisation

Remark 1: W.r.t the Hodge decomposition the pairings (2) and (3) have the property that if $\alpha_{m,n} \in H^{m,n}(M_n)$ and $\beta_{p,q} \in H^{r,s}(M_n)$ then $\int_W \alpha_{m,n} \wedge \beta_{p,q} = 0$ unless $m + p = n + q = 3$.

Remark 2: In terms of a basis of periods compatible with Σ they can be written as

$$\int_{M_n} \Omega \wedge \bar{\Omega} = \vec{\Pi}^\dagger \Sigma \vec{\Pi}, \qquad \int_{M_n} \Omega \wedge \underline{\partial}^k_{I_k} \Omega = -\vec{\Pi}^T \Sigma \underline{\partial}^k_{I_k} \vec{\Pi} \ ,$$

(Relative) Calabi-Yau periods via Symanzik representation

The GM identification would be of limited use if there would not be direct ways to associate the block with a geometry X. E.g. in the Symanzik representation the contribution of an l-loop graph yields an integral with a rational integrand defined by the graph polynomials $\mathcal{U}(\underline{x})$ and $\mathcal{F}(\underline{x}, \underline{p}, \underline{m})$, $\underline{p}$ independent momenta, $\underline{m}$ masses

$$I_{\sigma_{n-1}}(\underline{p}, \underline{m}) = \int_{\sigma_{n-1}} \prod_i x_i^{\nu_i - 1} \frac{\mathcal{U}^{\omega - \frac{D}{2}}}{\mathcal{F}^{\omega}} \mu_{n-1}$$

(Relative) Calabi-Yau periods via Symanzik representation

The GM identification would be of limited use if there would not be direct ways to associate the block with a geometry X. E.g. in the Symanzik representation the contribution of an l-loop graph yields an integral with a rational integrand defined by the graph polynomials $\mathcal{U}(\underline{x})$ and $\mathcal{F}(\underline{x},\underline{p},\underline{m})$, $\underline{p}$ independent momenta, $\underline{m}$ masses

$\omega = \sum_{i=1}^{n} \nu_i - lD/2$, l # of loops

n # of edges, ν_i their multiplicity D space time dim

$$I_{\sigma_{n-1}}(\underline{p},\underline{m}) = \int_{\sigma_{n-1}} \prod_i x_i^{\nu_i - 1} \frac{\mathcal{U}^{\omega - \frac{D}{2}}}{\mathcal{F}^{\omega}} \mu_{n-1}$$

μ_{n-1} measure on $\mathbb{P}^{n-1}$

$\sigma_{n-1} = \{[x_1 : \ldots : x_n] \in \mathbb{P}^{n-1} | x_i \in \mathbb{R}_{\geq 0} \,\forall\, 1 \leq i \leq n\}$ an open domain.

(Relative) Calabi-Yau periods via Symanzik representation

The GM identification would be of limited use if there would not be direct ways to associate the block with a geometry X. E.g. in the Symanzik representation the contribution of an l-loop graph yields an integral with a rational integrand defined by the graph polynomials $\mathcal{U}(\underline{x})$ and $\mathcal{F}(\underline{x}, \underline{p}, \underline{m})$, $\underline{p}$ independent momenta, $\underline{m}$ masses

$\omega = \sum_{i=1}^{n} \nu_i - lD/2$, l # of loops

n # of edges, ν_i their multiplicity, D space time dim

$$I_{\sigma_{n-1}}(\underline{p}, \underline{m}) = \int_{\sigma_{n-1}} \prod_i x_i^{\nu_i - 1} \frac{\mathcal{U}^{\omega - \frac{D}{2}}}{\mathcal{F}^{\omega}} \mu_{n-1}$$

μ_{n-1} measure on $\mathbb{P}^{n-1}$

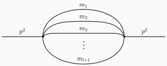
$\sigma_{n-1} = \{[x_1 : \ldots : x_n] \in \mathbb{P}^{n-1} | x_i \in \mathbb{R}_{\geq 0} \, \forall \, 1 \leq i \leq n\}$ an open domain.

$D = D_{cr} - 2\epsilon$, $I = \sum_{k=-n}^{\infty} I_k \epsilon^n$ with I_k functions of masses and Lorentz invariant products of the external momenta.

E.g. for the banana banana graph integrals in critical dimension $D_{cr} = 2$:

Feyman graphs and (relative) Calabi-Yau periods

E.g. for the banana banana graph integrals in critical dimension $D_{cr} = 2$:

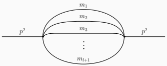


the Symanzik parametrisation leads directly to period integrals in a Calabi-Yau geometry footnotesize(with $t = \frac{p^2}{\mu^2}$, $\xi_i = \frac{m_i}{\mu}$ $(z_i = \frac{m_i^2}{p^2})$).

$$I_{\sigma_l} = \int_{\sigma_l} \frac{\mu_l}{\mathcal{F}(t, \xi_i; x)} = \int_{\sigma_l} \frac{\mu_l}{\left(t - \left(\sum_{i=1}^{l+1} \xi_i^2 x_i\right)\left(\sum_{i=1}^{l+1} x_i^{-1}\right)\right) \prod_{i=1}^{l+1} x_i}$$

Feyman graphs and (relative) Calabi-Yau periods

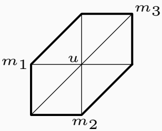
E.g. for the banana banana graph integrals in critical dimension $D_{cr} = 2$:

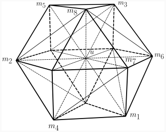


the Symanzik parametrisation leads directly to period integrals in a Calabi-Yau geometry footnotesize(with $t = \frac{p^2}{\mu^2}$, $\xi_i = \frac{m_i}{\mu}$ $(z_i = \frac{m_i^2}{p^2})$).

$$I_{\sigma_l} = \int_{\sigma_l} \frac{\mu_l}{\mathcal{F}(t, \xi_i; x)} = \int_{\sigma_l} \frac{\mu_l}{\left(t - \left(\sum_{i=1}^{l+1} \xi_i^2 x_i\right)\left(\sum_{i=1}^{l+1} x_i^{-1}\right)\right)\prod_{i=1}^{l+1} x_i}$$

as the Newton polytopes of $\mathcal{F}$ is reflexive, hence $\mathcal{F} = 0$ defines a Calabi-Yau manifold.





For a better representation of the geometry

Consider the complete intersection of two polynomials of degree $(1, \ldots, 1)$ in the cartesian product of $(\mathbb{P}^1)'s$

$$\mathbb{P}_{l+1} := \otimes_{i=1}^{l+1} \mathbb{P}^1_{(i)} \,.$$

Such a complete intersection manifold in a product of manifolds is denoted in short as

$$M_{l-1}^{\mathrm{CI}} = \left(\begin{array}{c} \mathbb{P}^1_{(1)} \\ \vdots \\ \mathbb{P}^1_{(l+1)} \end{array} \middle\| \begin{array}{cc} 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \end{array} \right) \subset \left(\begin{array}{c} \mathbb{P}^1_{(1)} \\ \vdots \\ \mathbb{P}^1_{(l+1)} \end{array} \middle\| \begin{array}{c} 1 \\ \vdots \\ 1 \end{array} \right) =: F_l \subset \mathbb{P}_{l+1} \,.$$

GKZ system for the complete intersection geometry

$$
\begin{aligned}
P_1 &= a_0\, w_0^{(1)} + \sum_{m=1}^{l+1} a_{2m-1}\, w_m^{(1)} = a_0 \prod_{k=1}^{l+1} x_1^{(k)} + \sum_{m=1}^{l+1} a_{2m-1}\; x_2^{(m)} \prod_{k\neq m}^{l+1} x_1^{(k)} \\
P_2 &= \tilde{a}_0\, w_0^{(2)} + \sum_{m=1}^{l+1} a_{2m}\, w_m^{(2)} = \tilde{a}_0 \prod_{k=1}^{l+1} x_2^{(k)} + \sum_{m=1}^{l+1} a_{2m}\; x_1^{(m)} \prod_{k\neq m}^{l+1} x_2^{(k)} \; .
\end{aligned}
$$

On these parameters $a_i, \tilde{a}_i$ in the canonical representation

$$
\int_\Gamma \Omega(\underline{z}) = \int_\Gamma \frac{1}{(2\pi i)^r} \oint_{S_1^1} \oint_{S_2^1} \frac{\wedge_{i=1}^m \mu_{n_i}}{P_1 P_2} \; ,
$$

of the periods integrals, the $(\mathbb{C}^*)^{l+1}$-scaling symmetries

$$
\begin{aligned}
\ell^{(1)} &= \quad (-1,-1;1,1,0,0,\cdots,0,0,0,0) \\
\ell^{(2)} &= \quad (-1,-1;0,0,1,1,\cdots,0,0,0,0) \\
&\vdots \\
\ell^{(l)} &= \quad (-1,-1;0,0,0,0,\cdots,1,1,0,0) \\
\ell^{(l+1)} &= \quad (-1,-1;0,0,0,0,\cdots,0,0,1,1)
\end{aligned}
$$

Advantages of the geometric representation

act and yield $(l+1)$ second order GKZ operators in the Batyrev large radius coordinates $z_k = \prod_{i=1}^{2(l+2)} a_i^{\ell_i^{(k)}}/(a_0\tilde{a}_0)$, $k=1,\ldots,l+1$.

Advantages of the geometric representation as(complete intersection) Calabi-Yau manifold

1.) The GKZ system in the yields immediately all period integrals $\underline{\Pi}$ and near the point of maximal unipotent monodromy $z_i = 0$ a canonical integral basis w.r.t. to the global monodromy $\mathcal{O}(\Sigma, \mathbb{Z})$. In particular one identifies the physical period and its analytic properties.
2.) Once the analytic continuation of $\underline{\Pi}$ to the other critical divisors in the discriminate locus is known they can be calculated to very high precision everywhere in the physical parameter space in extremely short time.

Further Advantages:

3.) Griffith-transversality (3) implies

a.) The Inverse of the Wronskian is up rational factors linear in the periods $W^{-1} = \Sigma W^T Z^{-1}$

$$Z^{-1} = \frac{(2\pi i)^3}{C} \begin{pmatrix} 0 & \frac{C''}{C} - 2\frac{C'}{C} + \frac{c_2}{c_4} & -\frac{C'}{C} & 1 \\ 2\frac{C'}{C} - \frac{C''}{C} - \frac{c_2}{c_4} & 0 & -1 & 0 \\ \frac{C'}{C} & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

b.) The Gauss-Manin connection can be brought into a canonical form

$$\partial_{t^i_*} \begin{pmatrix} \mathcal{V}_0 \\ \mathcal{V}_j \\ \mathcal{V}^j \\ \mathcal{V}^0 \end{pmatrix} = \begin{pmatrix} 0 & \delta_{ik} & 0 & 0 \\ 0 & 0 & C_{ijk} & 0 \\ 0 & 0 & 0 & \delta^j_i \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{V}_0 \\ \mathcal{V}_k \\ \mathcal{V}^k \\ \mathcal{V}^0 \end{pmatrix} .$$

4.) a.) Implies that that in the "variation of constant" procedure the inhomogeneous solution is an iterated integral of the periods $\partial_n^k \Pi$ modulo rational functions. b.) implies that the higher terms in ϵ can be similar written as iterated integrals.

Calabi-Yau motives:

The periods of Calabi-Yau varieties and their extensions (inhomogeneous solutions) evaluate Feynman integrals. Different Calabi-Yau varieties have the same periods structures.

Calabi-Yau motives:

The periods of Calabi-Yau varieties and their extensions (inhomogeneous solutions) evaluate Feynman integrals. Different Calabi-Yau varieties have the same periods structures. Therefore we abstract the essential data of the period $\Pi(z)$ into the notion of a Calabi-Yau period motive with the properties

(a) $\Pi(z)$ is restricted by the real Griffiths bilinear relations defining positivity of volumes and the holomorphic Griffiths transversality conditions.

(b) $\vec{\Pi}(z)$ is a flat section of the Hodge bundle over the moduli space and fulfils a first-order homogeneous differential equation $\nabla_{GM}\Pi = (\partial_z - N(z))\Pi = 0$, or equivalently a set of higher order homogenous differential equations $\mathcal{L}^{(k)}\vec{\Pi} = 0$. The higher-order operators $\mathcal{L}^{(k)}(z, \partial_z)$ generate the Picard-Fuchs differential ideal.

Calabi-Yau motives:

(c) There is a $\mathbb{Z}[\alpha]$-integer intersection form Σ with entries $\Sigma_{ab} = \Gamma_a \cap \Gamma_b$, which is anti-symmetric and symplectic for n odd with signature $\left(\frac{b_n}{2}, \frac{b_n}{2}\right)$, and for n even it is symmetric of a signature $(m, b_n - m)$ determined by the Hirzebruch signature index.

(d) Flat sections of the Hodge bundle are determined by their monodromies M_γ for loops γ around special divisors of $\mathcal{M}_{cs}(M)$, that for a choice of basis $\Gamma_a \in H_n(M, \mathbb{Z}[\alpha])$ generate the monodromy group $\Gamma_M \subset \mathrm{Sp}(b_n, \mathbb{Z}[\alpha])$ for n odd and $\Gamma_M \subset O(\Sigma, \mathbb{Z}[\alpha])$ for n even. In particular, $\vec{\Pi}(z)$ defines a representation of Γ_M.

N=4 Super-Yang-Mills and integrablity

Driving question: Which symmetries allow to solve n.t. QFT's.

N=4 Super-Yang-Mills and integrablity

Driving question: Which symmetries allow to solve n.t. QFT's.

Original Stage: $\mathcal{N} = 4$ SYM in the $\text{AdS}_5/\text{SCFT}_4$ correspondence.

N=4 Super-Yang-Mills and integrablity

Driving question: Which symmetries allow to solve n.t. QFT's.

Original Stage: $\mathcal{N} = 4$ SYM in the $AdS_5/SCFT_4$ correspondence.

Symmetry: Drinfeld Yangian symmetry, as first identified as symmetry of the dilatation op. Dolan, Nappi & Witten: (03,04) and later found to allow to find solution of certain scattering amplitudes, Wilson loops observables and S-matrix elements. Dummond, Henn, Plefka, Zarembo (09,13) Beisert, Garus, Rosso (17),...

N=4 Super-Yang-Mills and integrablity

Driving question: Which symmetries allow to solve n.t. QFT's.
Original Stage: $\mathcal{N} = 4$ SYM in the $AdS_5/SCFT_4$ correspondence.
Symmetry: Drinfeld Yangian symmetry, as first identified as symmetry of the dilatation op. Dolan, Nappi & Witten: (03,04) and later found to allow to find solution of certain scattering amplitudes, Wilson loops observables and S-matrix elements. Dummond, Henn, Plefka, Zarembo (09,13) Beisert, Garus, Rosso (17),...
Integrable Deformations: Marginal β deformations Leigh, Strassler (95) Maldacena Luni (05). Here most relevant the supersymmetry breaking γ_i, $i = 1, 2, 3$ deformations in the double scaling limit $g \to 0$, $\gamma_3 \to i\infty$ with $\xi^2 = g^2 N_c e^{-i\gamma_3}$ fixed Gürdoğan, Kazakov (16), with Caetano (18) and the bi-scalar model χFT Kazakov, Olivucci (18) leading to holographic dual pairs of integrable fishnet and fishchain theories in D dimensions.

Orginal Fishnet Lagrangians

These bi-“scalar” fishnet theories in D dimensions have a Lagrangian with quartic interaction $V = 4$

$$\mathcal{L}^{\omega D}_{\mathrm{quad}} = N_{\mathrm{c}}\mathrm{tr}[-X(-\partial_\mu\partial^\mu)^\omega \bar{X} - Z(-\partial_\mu\partial^\mu)^{\frac{D}{2}-\omega}\bar{Z} + \xi^2 XZ\bar{X}\bar{Z}]\,.$$

ω determines the propagator power in the Feynman graphs. E.g. $D = 4$, $\omega = 1$ and $D = 2$, $\omega = 1/2$ are conformal choices.

Orginal Fishnet Lagrangians

These bi-"scalar" fishnet theories in D dimensions have a Lagrangian with quartic interaction $V = 4$

$$\mathcal{L}_{\text{quad}}^{\omega D} = N_{\text{c}}\text{tr}[-X(-\partial_\mu\partial^\mu)^\omega \bar{X} - Z(-\partial_\mu\partial^\mu)^{\frac{D}{2}-\omega}\bar{Z} + \xi^2 XZ\bar{X}\bar{Z}] \,.$$

ω determines the propagator power in the Feynman graphs. E.g. $D = 4$, $\omega = 1$ and $D = 2$, $\omega = 1/2$ are conformal choices.
Most importantly this theory exhibit as symmetry the Yangian extension of the bosonic conformal symmetry.

Hexagonal Fishnets Lagrangian

A generalization with analogous symmetry properties are Fishnet theories with cubic interaction $V = 3$ Kazakov, Olivucci (23) and Lagrangian

$$\mathcal{L}^{D}_{\mathrm{cub}} = N_{\mathrm{c}} \mathrm{tr}\big[-X(-\partial_\mu\partial^\mu)^{\omega_1}\bar{X} - Y(-\partial_\mu\partial^\mu)^{\omega_2}\bar{Y} - Z(-\partial_\mu\partial^\mu)^{\omega_3}\bar{Z} + \xi_1^2 \bar{X} Y Z + \xi_2^2 X \bar{Y}\bar{Z}\big]\,,$$

with $\sum_{i=1}^{V} \omega_i = D$ at vertex, e.g. $D = 2$ and $\omega_1 = \omega_2 = \omega_3 = 2/3$.

Hexagonal Fishnets Lagrangian

A generalization with analogous symmetry properties are Fishnet theories with cubic interaction $V=3$ Kazakov, Olivucci (23) and Lagrangian

$$\begin{aligned}\mathcal{L}^{D}_{\text{cub}} = \quad & N_{\text{c}}\text{tr}\big[-X(-\partial_\mu\partial^\mu)^{\omega_1}\bar{X} - Y(-\partial_\mu\partial^\mu)^{\omega_2}\bar{Y} - Z(-\partial_\mu\partial^\mu)^{\omega_3}\bar{Z} \\ & +\xi_1^2\bar{X}YZ + \xi_2^2 X\bar{Y}\bar{Z}\big]\,,\end{aligned}$$

with $\sum_{i=1}^{V}\omega_i = D$ at vertex, e.g. $D=2$ and $\omega_1=\omega_2=\omega_3=2/3$. Scalar field have conformal dimension $\Delta_\phi = (D-2)/2$ and conformal interactions have to have valency $V=2D/(D-2)$, i.e. $D=6,4,3$ enforce $V=3,4,6$ respectively.

Hexagonal Fishnets Lagrangian

A generalization with analogous symmetry properties are Fishnet theories with cubic interaction $V = 3$ Kazakov, Olivucci (23) and Lagrangian

$$\mathcal{L}^D_{\rm cub} = \quad N_{\rm c}{\rm tr}\big[-X(-\partial_\mu\partial^\mu)^{\omega_1}\bar{X} - Y(-\partial_\mu\partial^\mu)^{\omega_2}\bar{Y} - Z(-\partial_\mu\partial^\mu)^{\omega_3}\bar{Z}$$
$$+\xi_1^2\bar{X}YZ + \xi_2^2 X\bar{Y}\bar{Z}\big]\,,$$

with $\sum_{i=1}^{V}\omega_i = D$ at vertex, e.g. $D = 2$ and $\omega_1 = \omega_2 = \omega_3 = 2/3$. Scalar field have conformal dimension $\Delta_\phi = (D-2)/2$ and conformal interactions have to have valency $V = 2D/(D-2)$, i.e. $D = 6, 4, 3$ enforce $V = 3, 4, 6$ respectively.
Then the (planar) fishnet graphs can be cut by a closed oriented curve from the three regular tilings of the plane:

Regular tilings and Calabi-Yau motives

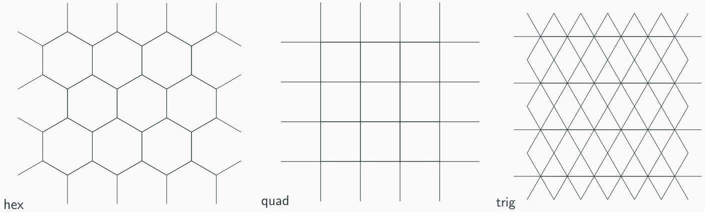


Figure 1: The three regular tilings of the plan with vertices of valence $\nu = 3, 4, 6$ respectively.

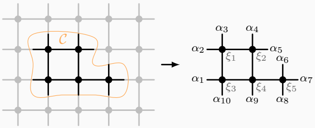


Figure 2: Ten-point five-loop fishnet integral cut out of a square tiling of the plane.

Regular tilings and Calabi-Yau motives

To obtain a graph G consider a convex closed oriented curve $\mathcal{C}$ that cuts edges of the tiling and does not pass to vertices. To each vertex inside the curve $\mathcal{C}$ we associate a $\mathbb{P}^1$ with homogeneous coordinates $[x_i : u_i]$, $i = 1, \ldots, l$ over which we want to integrate with the measure

$$\mathrm{d}\mu_i = u_i \mathrm{d}x_i - x_i \mathrm{d}u_i \ . \tag{4}$$

To the end point of each cut edge outside $\mathcal{C}$ we associate a parameter $a_j \in \mathbb{C}$, $j = 1, \ldots, r$. The graph is constructed by the l vertices with propagators

$$P^I_{ij} = \frac{1}{(x_i - x_j)^{w_{ij}}}, \qquad P^E_{ij} = \frac{1}{(x_i - a_j)^{w_{ij}}} \ . \tag{5}$$

To be conformal in D dimension the weights of propagators incident to each vertex V_i has to fullfill

$$\sum w_{ij} = D \tag{6}$$

Regular tilings and Calabi-Yau motives

We deal mainly with $D = 2$ and choose the propagator weights all equal $w_{ij} = w = 2/\nu(V)$, where $\nu(V)$ is the valence of the vertices, i.e. for the hexagonal tiling we have $w = \frac{2}{3}$, for the quartic tiling $w = \frac{1}{4}$ amd for the trigonal tiling $w = \frac{1}{3}$.

To the hexagonal and the quartic lattice we can associate an in general singular l-dimensional Calabi-Yau variety M_l as the $d = 3$ or $d = 2$ fold cover

$$W = \frac{y^d}{d} - P([\underline{x} : \underline{u}]; \underline{a}) = 0 \qquad (7)$$

over the base $B = (\mathbb{P}^1)^l$ branched at

$$P([\underline{x} : \underline{w}]; \underline{a}) = \prod_{ij} (u_j x_i - x_j u_i) \prod_{ij} (x_i - a_j u_i) = 0 \;, \qquad (8)$$

respectively. The orders of the covering automorpishm exchanging the sheets will play a crucial role in the following geometric analysis.

Regular tilings and Calabi-Yau motives

Note that (**??**) defines a Calabi-Yau manifold, because the canonical class of the base is with H_i the hyperplane class of the i'th $\mathbb{P}^1$ given by

$$K_B = 2\bigoplus_{i=1} H_i, \tag{9}$$

and the Calabi-Yau condition ensuring $K_{M_l} = 0$

$$\frac{d}{d-1} K_B = [P([\underline{x} : \underline{u}]; \underline{a})] = \nu \bigoplus_{i=1} H_i \tag{10}$$

is true with $d = 3, 2$ as $\nu = 3, 4$ for graphs from the hexagonal and the quartic tiling, respectively.

Regular tilings and Calabi-Yau motives

Another way of stating this is that the periods over the unique holomorphic (ℓ,0)-form, given by the Griffiths residuum form Ω

$$\Pi_G = \int_C \Omega = \int_C \frac{1}{2\pi i} \oint_\gamma \frac{dy \prod_{i=1}^{l} d\mu_i}{W} = \int_C \frac{\prod_{i=1}^{l} d\mu_i}{\partial_y W} = \int_C \frac{\prod_{i=1}^{l} d\mu_i}{P^{\frac{d-1}{d}}} = \int_C \prod_{ij} P_{ij}^{I} \prod_{ij} P_{ij}^{E} \prod_{i=1}^{l} d\mu_i \,, \tag{11}$$

are well defined. The significance for the application is that these period integrals over cycles $C \in H_l(M_l, \mathbb{Z})$ are building blocks for the amplitudes.

$$I_G = \int_C \Omega = \int \sqrt{\left| \prod_{ij} P_{ij}^{I} \prod_{ij} P_{ij}^{E} \right|^2} \prod_{i=1}^{l} d\mu_i \wedge d\bar{\mu}_i \,, \tag{12}$$

Regular tilings and Calabi-Yau motives

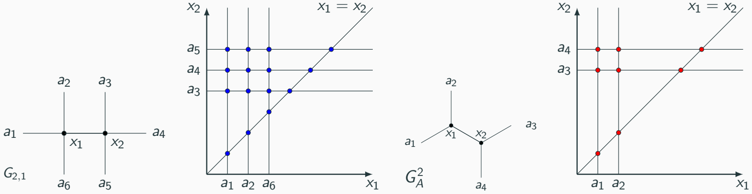


Figure 3: Singularities of the $K3$ denoted for the valence 4 graph $M_{G_{1,2}}$ and the valence 3 graph $M_{G_A^2}$. Note that 3 of the a_i can be set to $0, 1, \infty$ by a diagonal $PSL(2, \mathbb{C})$ acting on the projective plane in which the a_i lie

Claim 1: To each graph G we can associate a Calabi-Yau variety X whose periods determine I.

Regular tilings and Calabi-Yau motives

Claim 1: To each graph G we can associate a Calabi-Yau variety X whose periods determine I.

Claim 2: Each I gives rise to a Calabi-Yau motive with integer symmetry (l even) or antisymmetric (l odd) intersection form Σ, a point of maximal unipotent monodromy and a period vector $\Pi(\underline{z}) = \int_{\Gamma_i} \Omega$ with $\Gamma_i \in H_l(W^{(m,n)}, \mathbb{Z})$. The Feynman amplitude is given near the Mum points by the quantum volume of the mirror

$$I = i^{l^2} \Pi^\dagger \Sigma \Pi = e^{-K(\underline{z},\underline{\bar{z}})} = Vol_q(M^{(m,n)})$$

and globally by analytic continuation of the periods. Here $M^{(m,n)}$ is the mirror of $W^{(m,n)}$.

Regular tilings and Calabi-Yau motives

Claim 3: There exist an integrable conformal fishnet theories (CFNT) developed first (Gürdogan, Kazakov 2015) as deformation of $N = 4$ $SU(N_c)$ SYM theory. Let X, Z be $SU(N_c)$ matrix fields then the Lagrangian is

$$\mathcal{L}_{FN} = N_c \mathrm{tr}\left(-\partial_\mu X \partial^m u \bar{X} - \partial_\mu Z \partial^m u \bar{Z} + \xi^2 X Z \bar{X} \bar{Z}\right)$$

Each $I_{m,n}$ integral is an amplitude in the CFNT, i.e. $I_{m,n}(\underline{z})$ has to be single valued i.e. a Bloch Wigner dilogarithm or in the $D = 2$ case e^{-K}.

Claim 3: There exist an integrable conformal fishnet theories (CFNT) developed first (Gürdogan, Kazakov 2015) as deformation of $N = 4$ $SU(N_c)$ SYM theory. Let X, Z be $SU(N_c)$ matrix fields then the Lagrangian is

$$\mathcal{L}_{FN} = N_c \mathrm{tr}\left(-\partial_\mu X \partial^m u \bar{X} - \partial_\mu Z \partial^m u \bar{Z} + \xi^2 X Z \bar{X} \bar{Z}\right)$$

Each $I_{m,n}$ integral is an amplitude in the CFNT, i.e. $I_{m,n}(\underline{z})$ has to be single valued i.e. a Bloch Wigner dilogarithm or in the $D = 2$ case e^{-K}.

The factorisation of the amplitudes of the integrable system subject to the Yang-Baxter relations imply many non-trivial relations for he periods of the $W^{(m,n)}$. E.g. we the one parameter specialisation the periods of $W^{(n,m)}$ are $(m \times m)$ minors of the periods $W_I^{(1,m+m)}$ etc.

Claim 4: ($Y(SO(3,1)) = Y(Sl(2,\mathbb{R})) \oplus \overline{Y(Sl(2,\mathbb{R}))}$.) The holomorphic Yangian generated by the algebra

$$\begin{aligned} P_j^{\mu} &= -i\partial_{a_j}^{\mu}, & K_j^{\mu} &= -2ia_j^{\mu}(a_j^{\nu}\partial_{a_j,\nu} + \Delta_j) + ia_j^2\partial_{a_j}^{\mu} \\ L_j^{\mu\nu} &= i(a_j^{\mu}\partial_{a_j}^{\nu} - a_j^{\nu}\partial_{a_j}^{\mu}), & D_j &= -i(a_j^{\mu}\partial_{a_j,\mu}), \end{aligned}$$

Claim 4: ($Y(SO(3,1)) = Y(Sl(2,\mathbb{R})) \oplus \overline{Y(Sl(2,\mathbb{R}))}$.) The holomorphic Yangian generated by the algebra

$$\begin{aligned} P_j^\mu &= -i\partial_{a_j}^\mu, & K_j^\mu &= -2ia_j^\mu(a_j^\nu\partial_{a_j,\nu} + \Delta_j) + ia_j^2\partial_{a_j}^\mu \\ L_j^{\mu\nu} &= i(a_j^\mu\partial_{a_j}^\nu - a_j^\nu\partial_{a_j}^\mu), & D_j &= -i(a_j^\mu\partial_{a_j,\mu}), \end{aligned}$$

in differentials w.r.t. to the external position, generates together with the permutation symmetries of the latter a differential ideal that annihilates the $I(\underline{z})$ and is *equivalent* to the Picard-Fuchs differential ideal that describes the variation of the Hodge structure in the middle cohomology of X and annihilated the periods of Ω.

Regular tilings and Calabi-Yau motives

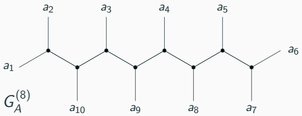


Figure 4: The $G_A^{(8)}$ graph. The A series starts from even dimensional Calabi-Yau spaces

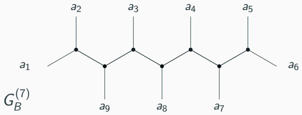


Figure 5: The $G_B^{(7)}$ graph. The B series starts from odd dimensional Calabi-Yau spaces

Regular tilings and Calabi-Yau motives

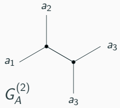


Figure 6: The $G_A^{(2)}$ graph and its transformation to a genus 2 Picard curve

$$y^3 = (x - a_1)(x - a_2)(x - a_3)^2(x - a_4)^2$$

Regular tilings and Calabi-Yau motives

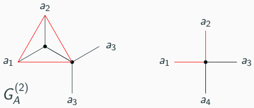


Figure 6: The $G_A^{(2)}$ graph and its transformation to a genus 2 Picard curve

$$y^3 = (x - a_1)(x - a_2)(x - a_3)^2(x - a_4)^2$$

Regular tilings and Calabi-Yau motives

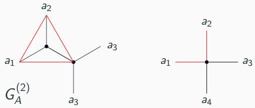


Figure 6: The $G_A^{(2)}$ graph and its transformation to a genus 2 Picard curve

$$y^3 = (x - a_1)(x - a_2)(x - a_3)^2(x - a_4)^2$$

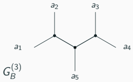


Figure 7: The $G_B^{(3)}$ graph and its transformation to a genus Picard curve

$$y^3 = (x - a_1)(x - a_2)(x - a_3)(x - a_4)(x - a_5)^2$$

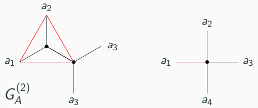


Figure 6: The $G_A^{(2)}$ graph and its transformation to a genus 2 Picard curve

$$y^3 = (x - a_1)(x - a_2)(x - a_3)^2(x - a_4)^2$$

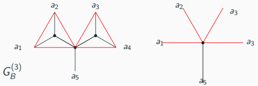
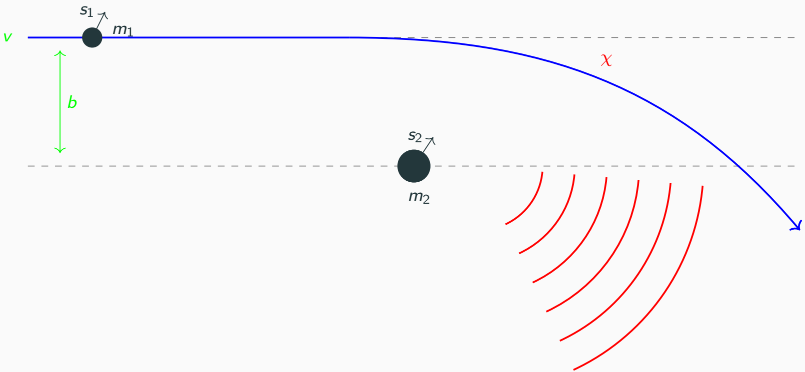


Figure 7: The $G_B^{(3)}$ graph and its transformation to a genus Picard curve

$$y^3 = (x - a_1)(x - a_2)(x - a_3)(x - a_4)(x - a_5)^2$$

Worldline Quantum Field Theory approach to General Relativity

Scattering of two black holes (BH) as starter to the description of BH mergers as the main sources for gravitational waves detected at LIGO, . . .



Worldline Quantum Field Theory approach to General Relativity

The action for the scattering process

$$S = -\sum_{i=1}^{2} m_i \int \mathrm{d}\tau \left[\frac{1}{2} g_{\mu\nu} \dot{x}_i^\mu \dot{x}_i^\nu\right] + S_{\mathrm{EH}}$$

is expanded in Post Minkowskian (PM) approximation in the Worldline Quantum Field Theory (WQFT) approach around the non-interacting background configurations

$$x_i^\mu = b_i^\mu + v_i^\mu \tau + z_i^\mu(\tau)\,, \quad g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{32\pi G}\, h_{\mu\nu}(x)\ .$$

Worldline Quantum Field Theory approach to General Relativity

The goal is to calculate from the initial data: the impact parameter $b^\mu = b_1^\mu - b_2^\mu$ and the incoming velocities v_1, v_2 the physical quantity of interest, which is the radiation induces change in the momentum say $\Delta p_1^\mu = m_1 \int \mathrm{d}\tau \ddot{x}(\tau)$ of the first particle.

In the PM approximation the latter can be expanded in the gravitational coupling G

$$\Delta p_1^\mu = \sum_{n=1}^{\infty} G^n \Delta p^{(n)\,\mu}(x) \ .$$

At each order the contributions $\Delta p^{(n)\,\mu}(x)$ are calculated in the WQFT approach in the Swinger-Keldysh in-in formalism in terms of a Feynman graph expansion with retarded propagators. Here $x = \gamma - \sqrt{\gamma^2 - 1}$ with γ the Lorentz factor of the relative velocities is the only parameter.

Worldline Quantum Field Theory approach to General Relativity

In the 4PM approximation the Feynman integral in the 1SF sector

(K3.0)

involve bilinear of elliptic function which are periods of the $K3$

$$Y^2 = X(X-1)(X-x)Z(Z-1)(Z-1/x)\,.$$

In the 5PM approximation we find in [8] that in the 5PM approximation the following graphs in the 1SF sector

(CY3)

Worldline Quantum Field Theory approach to General Relativity

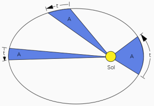
The corresponding smooth CY three-fold one-parameter complex family $x = (2\psi)^{-8}$, can be defined as resolution of four symmetric quadrics

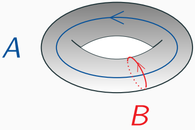
$$x_j^2 + y_j^2 - 2\psi x_{j+1} y_{j+1} = 0,\ j \in \mathbb{Z}/4\mathbb{Z}$$

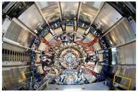
in the homogeneous coordinates x_i, y_j, $j = 0, \ldots, 3$ of $\mathbb{P}^7$. The periods of the above $K3$ and CY threefold determine all special functions that are necessary to solve for $\Delta p^{(5)\,\mu}(x)$ in the 1SF sector.

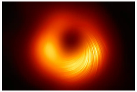
In the 5PM 2SF further different CY and K3 appear.

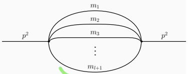
Conclusion and Outlook

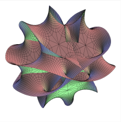






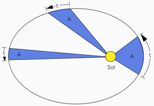


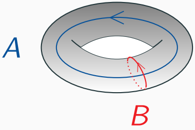




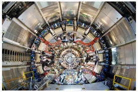
Physics

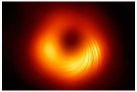
Mathematics



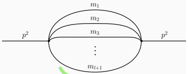


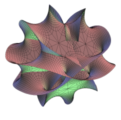
Conclusion and Outlook





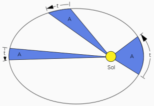
String Theory



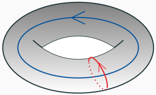


Physics

Mathematics



A



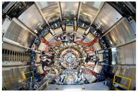
B

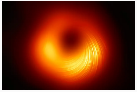
Enumerative Geometry

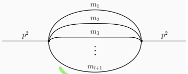
Conclusion and Outlook

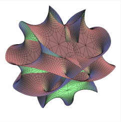
String Theory

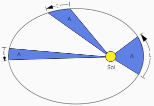
Black Hole Physics





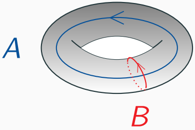






Physics

Mathematics



Arithmetic Geometry

Enumerative Geometry