Continuous Global Symmetries

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Based on
2503.04546, 2509.13170, 2510.14722 and 2601.$pqrsm/n$
with Qiang Jia, Ran Luo, Yi-Nan Wang and Yi Zhang

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What Questions Are We Facing

Some Records of History

- Moore-Seiberg: Bringing fusion categories into physics, and non-invertible lines, and more
- Wen et al.: Beyond Landau paradigm, notably armed with highly abstract math
- GKSW: Revolutionzing (or rewriting?) the concept of symmetry
- Schäfer-Nameki et al.: Stringy realization and (maybe) more field-theoretical examples
- * Vafa et al.: Swampland program says a lot about symmetry

Some Questions, Math and Physics

- People tend to study finite symmetries, why not continuous ones? And why not non-abelian ones?
- It is known that rearranging defects detects anomaly for finite symmetry, how about continuous symmetries?
- It is claimed that symmetry is topological order in one-higher dimension, can we see the extra dimension pops up?
- Topological order in one-higher dimension is usually described by a category, can it be extended to continuous symmetries?
- Since topological field theories are close related to categories, can we formulate TQFTs in other interesting manner?

Ultimately, we ask ...

Why do we care about

- Continuous global symmetry? (Sure we do.)
- Category theory? (We are not sure.)

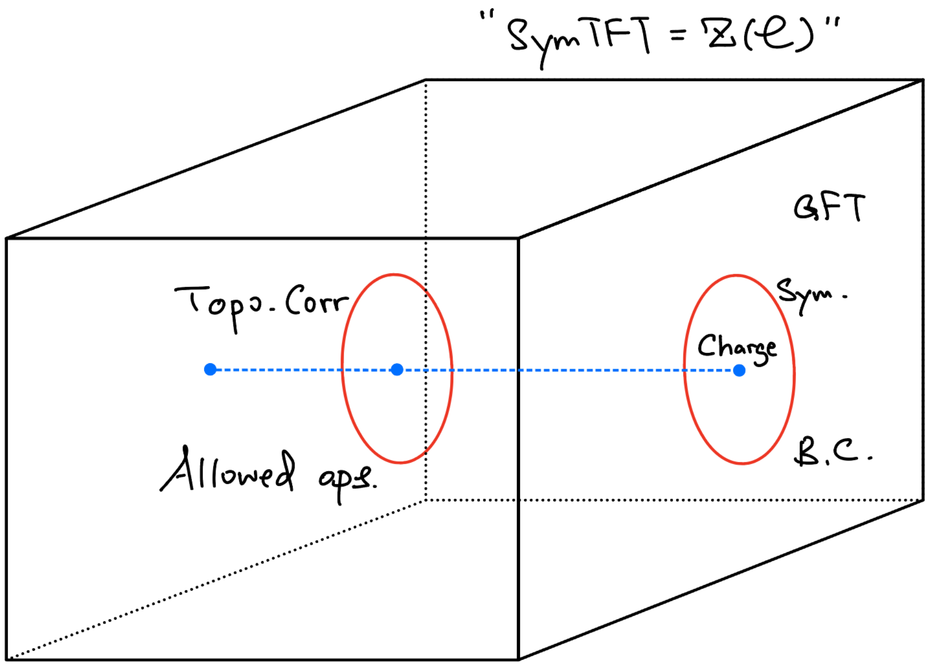
The Theoretical Minimum

Symmetry in Mordern Language

- Symmetry is a topological operator
- Inserting discrete symmetry operator $\Leftrightarrow$ Coupling to flat gauge field
- SymTFT governs the topological correlation
- B.C of SymTFT determines the physical theory
- * Symmetry is a category

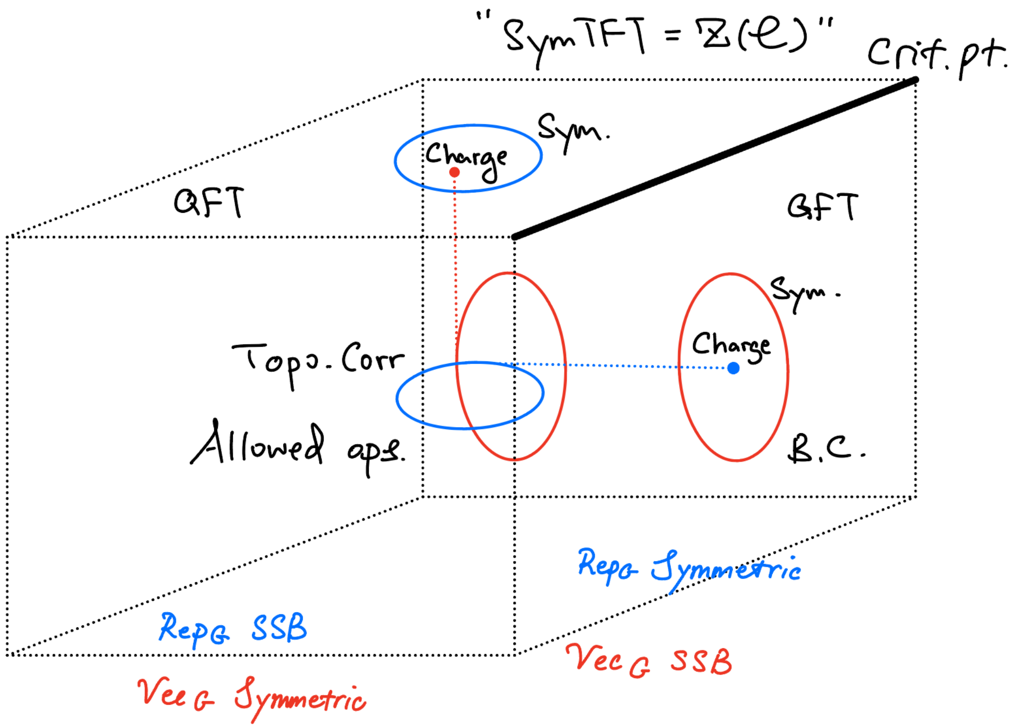
"Treasure Map"

I try to summarize things in previous slides in one "Treasure map" as:



Treasure Map ver 2

Or, more categorical:



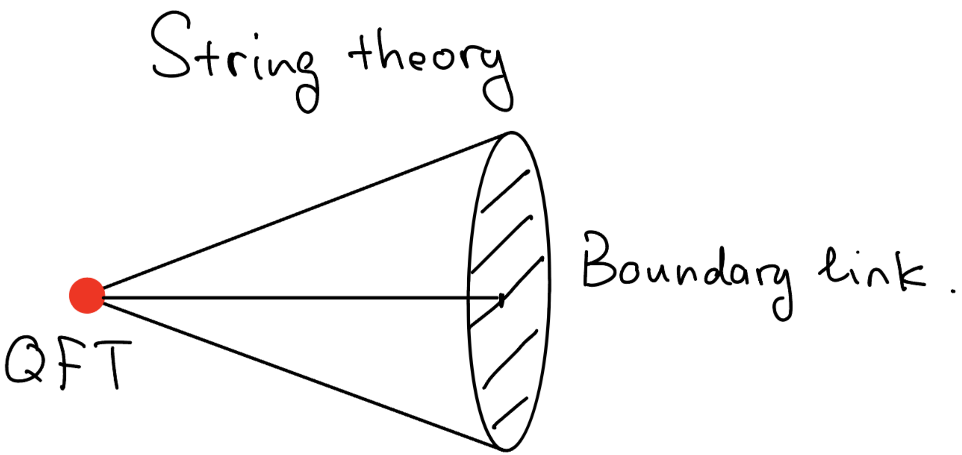
Why Is It Hard?

- How to "guess" a SymTFT for continuous symmetries?
- How to reconcile higher-codimension support with non-commutativity (homotopical constraint)?
- How to generalize defect webs to continuous symmetries?
- How to understand the appearance of an extra dimension?
- How to construct a ~~fusion~~ category for continuous symmetry?
- * Can we put it on firm math grounds?

The TFT That Governs Continuous Global Symmetries

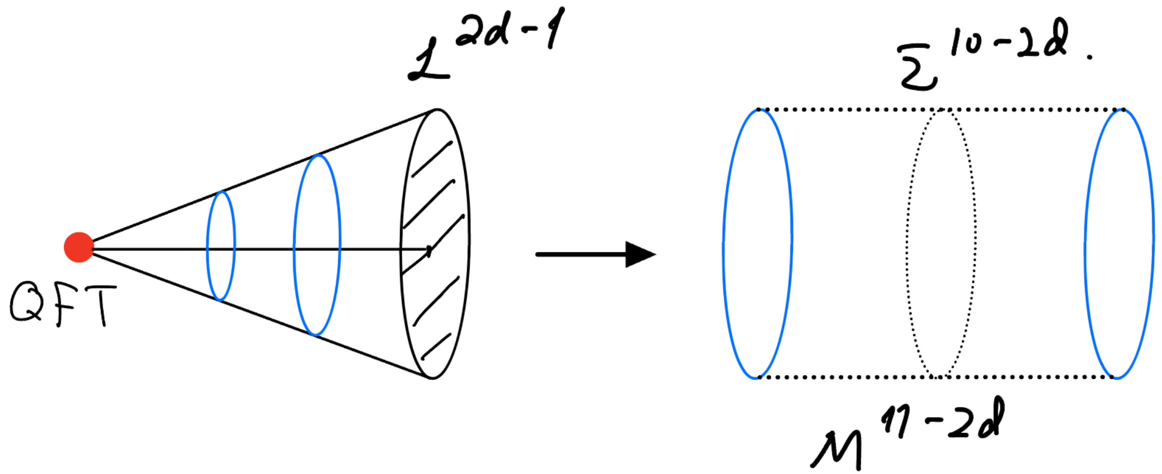
SymTFT

- Quick recap:
  - SymTFT action governs the topological correlations.
  - SymTFT B.C. governs various gaugings.
  - SymTFT lives in one-higher dimension and can be constructed as a "sandwich".



A Different View: Quantization

Define a theory on M^{11-2d}, get Hilbert space on Σ^{10-2d}.



Questions:

- What theory do we need?
- What is its Hilbert space?

A Different View: Quantization

- The QFT we face:

$$Z[\mathcal{W}] = \int \mathcal{D}\phi \; e^{iS[\phi] + \int_{\mathcal{W}} * J}$$

- Poincaré dual:

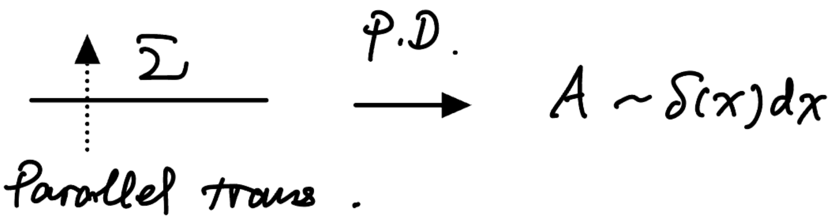
$$\int_{\Sigma} \star J = \int A \star J \;\Rightarrow\; \overset{A \sim PD(\Sigma)}{\overset{\uparrow}{\rule{4cm}{0.4pt}}}{}_{\Sigma} \;\Rightarrow\; \text{Active} = \text{Transport by } e^{i\int_{\ell} A}$$

- Consistency of active view:

$$e^{i\int_{\ell} A} = e^{i\int_{\ell'} A} \;\Rightarrow\; \text{Flat } A$$

A Different View: Quantization

An example of PD:



Choose an A that reproduces the correct parallel transport:

$$A_\Sigma = i\alpha\delta(x)dx = e^{-i\alpha H(x)}de^{i\alpha H(x)}$$

Leading to:

$$Z[A] = \int \mathcal{D}\phi \; e^{iS[\phi]+\int A*J}$$

A Different View: Quantization

- The QFT we face now:

$$Z[A] = \int \mathcal{D}\phi \; e^{iS[\phi]+\int A*J}$$

is a function of $A \in \widetilde{\mathcal{A}}_0$.

- So what is the space $\mathrm{Fun}(\widetilde{\mathcal{A}}_0)$?

If there is a theory $\mathcal{S}$ in one-higher dimension and the QFT lives on a "slice", we would expect:

$\mathrm{Fun}(\widetilde{\mathcal{A}}_0)$ is the "Hilbert space" of $\mathcal{S}$.

A Different View: Quantization

- Lessons from known examples:

$$S = \int \underbrace{b \cup \delta c}_{\text{“Kinetic”}} + \underbrace{c^3}_{\text{Twist}}$$

So we are in favor of having:

> “BF-like” action whose quantization leads to Fun$(\widetilde{\mathcal{A}}_0)$.

- A fact by [Baez]:

> State space of BF-theory is Fun$(\widetilde{\mathcal{A}}_0)$.

A Different View: Quantization

Conclusion I

SymTFT of global G symmetry is BF-theory with G-gauge symmetry.

But there are still problems:

- Any concrete way to see the Hilbert space?
- What are the topological operators?
- What are the topological correlations?

The Hilbert Space (Unpublished)

- Hilbert space first.

Consider a (free) field theory model:

Free fermions on torus $\frac{\theta}{2\pi} + \frac{it}{2\pi}$ with twisted boundary condition.

Partition function is:

$$Z^{R}_{[a,b]}(\tau) = \frac{\vartheta\left[\begin{smallmatrix} a-\frac{1}{2} \\ \frac{1}{2}-b \end{smallmatrix}\right](0,\tau)}{\eta(\tau)} = \frac{\vartheta\left[\begin{smallmatrix} \frac{\alpha}{2\pi}-\frac{1}{2} \\ \frac{1}{2}-\frac{\alpha\tau_1}{2\pi} \end{smallmatrix}\right](0,\tau)}{\eta(\tau)}, \; A = \frac{\alpha}{2\pi} d\theta$$

Take $\tau \to i0_+$, we have:

Untwisted torus + KK leads to

$$\lim_{\tau\to i0_+} Z^{R}_{[a,b]}(\tau) \propto Z_0[h] \in L^2(Cl(G))$$

Topological Operators (Published)

- Topological operator next.

Typically one may expect $U_\alpha(\Sigma) \sim e^{i\alpha}$, hence:

$$U_\alpha(\Sigma)U_\beta(\Sigma) \neq U_\beta(\Sigma)U_\alpha(\Sigma)\,.$$

But this CANNOT be true for any codim > 1 operators since higher homotopy commute.

So we define (note that it depends only on $[g]$, not on g):

Definition

$$\widetilde{U}_{[g]}(\Sigma) = \int dh\ \exp\left(i \int_\Sigma (\mathsf{ad}_{h_{\gamma_{xp}}} \mathsf{ad}_h \alpha_0(p), B)\right)$$

Topological Correlators (Unpublished)

- Finally, topological correlators.

We would like to calculate:

Correlator

$$\int \mathcal{D}A\mathcal{D}B \; e^{iS_{BF}} \; \widetilde{U}_\alpha(\Sigma) W_\lambda(\ell)$$

The key observation comes from [Witten, Beasley]:

Definition

$$W_\lambda(\ell) := \lim_{\eta\to\infty} \int \mathcal{D}U \; e^{i\int_{M_{d+1}} \delta_\ell \wedge (L_\sigma(\eta,U)+C_\lambda(U))}$$

And the result is [Cordova-Ohmori-Rudelius, Jia-Luo-JT-Wang-Zhang]:

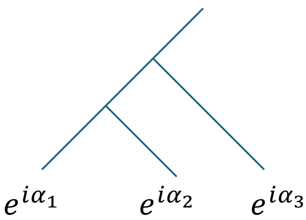
Quantum

$$\langle \widetilde{U}_\alpha(\Sigma) W_\lambda(\ell)\rangle \propto \frac{\chi_{r_\lambda}(e^{i\alpha})}{\dim r_\lambda} := \text{Normalized character}\,.$$

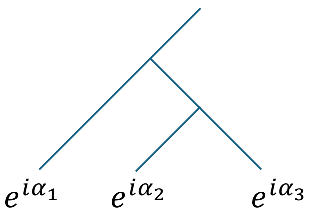
Anomaly, F-move, and Extra Dimension

F-move

• Starting point: F ( )-move (and we will focus on 2D from now on) [Moore-Seiberg, Bhardwaj-Tachikawa, Chang-Lin-Shao-Wang-Yin]



$$= \quad e^{-i\alpha_1([\alpha_2+\alpha_3]-\alpha_2-\alpha_3)} \times$$



Question:

- Can we do this for continuous symmetry?
- What can we learn if we could?

Some Preparation

- A claim yet to be concretized:

Claim [Bhardwaj et al.]

Rearrangements of topological defects ⇔ Gauge transformations

- Let us do what should have been done:

Suppose:

$$A_{\curlywedge_2} = \Lambda^{-1} A_{\curlywedge_1} \Lambda + \Lambda^{-1} d\Lambda$$

We have:

$$\langle \curlywedge_1 \rangle = \int \mathcal{D}\psi \; e^{iS[\psi]} \curlywedge_1 = \int \mathcal{D}\psi \; e^{iS[\psi] + i\,\mathrm{tr}\int_{M_2} A_{\curlywedge_1} \star J[\psi]},$$

$$\langle \curlywedge_2 \rangle = \int \mathcal{D}\psi \; e^{iS[\psi]} \curlywedge_2 = \int \mathcal{D}\psi \; e^{iS[\psi] + \int_{M_d} A_{\curlywedge_2} \wedge \star J} = e^{i\mathcal{A}[A_{\curlywedge_2};\Lambda]} \langle \mathcal{W} \rangle$$

What We Shall Compute

- Aim is to compute [Yonekura-Witten]:

$$\mathcal{A}[A_{\curlywedge};\Lambda] = \mathrm{tr}\int_{M_2\times\gamma} cs(A)$$

Via a topological magic:

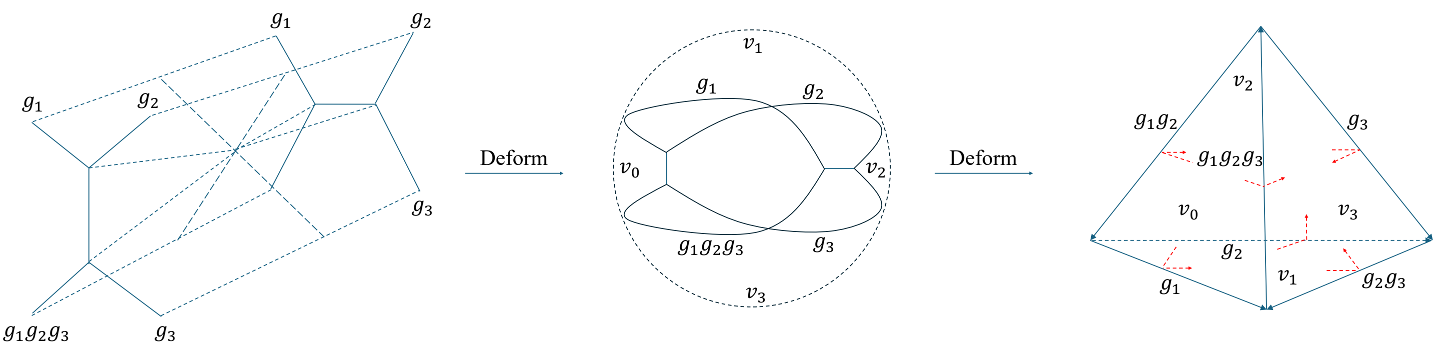
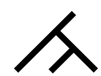


Figure 1: The deformations of the combined  and  configuration. The group multiplications in this figure are all taken from the right.

The Result

- The upshot is:

(Integrated) 't Hooft anomaly:

$$H^3(\mathfrak{g},\mathbb{Z}) \cong H^3_{\mathsf{dR}}(G,\mathbb{Z}) \cong H^3_{\mathsf{singular}}(G,\mathbb{Z}) \cong H^4(BG,\mathbb{Z})$$

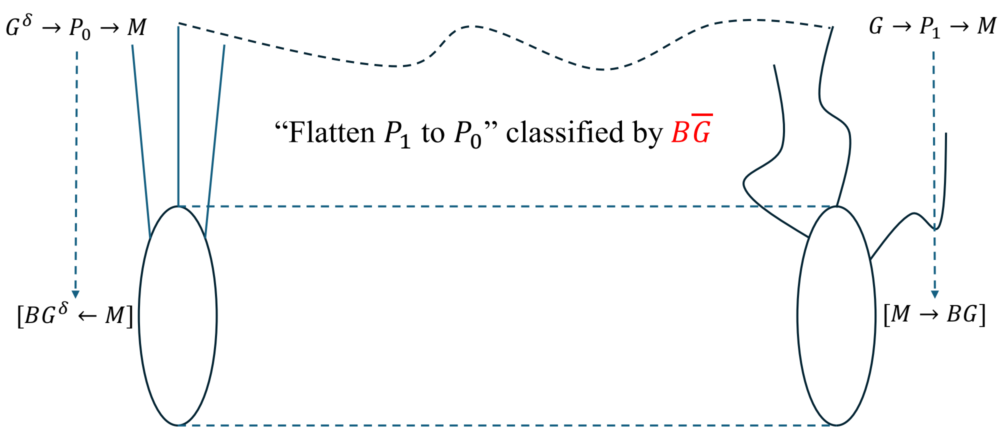
Questions:

- Why this weird $H^4(BG,\mathbb{Z})$? (Spoiler: This plays a key role in the symmetry category.)
- What else can we learn from the above structures?

For the latter there is one more piece of information:
There is a homomorphism $w : H^3(\mathfrak{g},\mathbb{R}) \to H^3(B\overline{G},\mathbb{R})$ where $B\overline{G}$ is the classifying space of flat G-trivial bundles.

An Extra Dimension

- Some mathematics [Stasheff, Morita, Rudolph-Schmidt]:
  - $w: H^*(\mathfrak{g}) \to H^*(B\overline{G})$, $B\overline{G}$ classifies flat G-bundle with trivialization.
  - $BG^\delta \xrightarrow{B\overline{G}} BG$. In human language: $B\overline{G}$ extrapolates between BG^δ and BG.
- Visualizing the abstract non-sense:



The Categorification

Issues To Be Understood (by us)

- Symmetry is topological order in one-higher dimension.
- Symmetry is described by a category $\mathscr{C}$.
- SymTFT = Drinfeld center of $\mathscr{C}$
- Different $\mathcal{Z}(\mathscr{C})$ correspond to different set of topological operators.
- Different Lagrangian algebra objects of $\mathcal{Z}(\mathscr{C})$ correspond to different gaugings/phases.

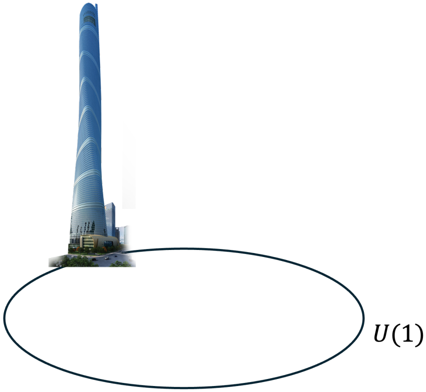
Categorical $U(1)$ Symmetry

This was developed in [Freed-Hopkins-Lurie-Teleman].

Definition

Sky$^\tau(U(1))$ is the category of skyscraper sheaves of finite dimensional vector spaces with finite support on $U(1)$, with convolution tensor product twisted by $\tau \in H^4(BU(1), \mathbb{Z})$.

So it looks like:



Categorical $U(1)$ Symmetry

- Convolution tensor product given by:

A pair of hermitian line bundles and isometry $(K \to U(1) \times U(1), \theta)$, giving the tensor product

$$\mathbb{C}_x * \mathbb{C}_y = K_{x,y} \otimes \mathbb{C}_{xy}$$

with

$$\theta_{x,y,z} : K_{y,z} \otimes K^{-1}_{xy,z} \otimes K_{x,yz} \otimes K^{-1}_{x,y} \to \mathbb{C}$$

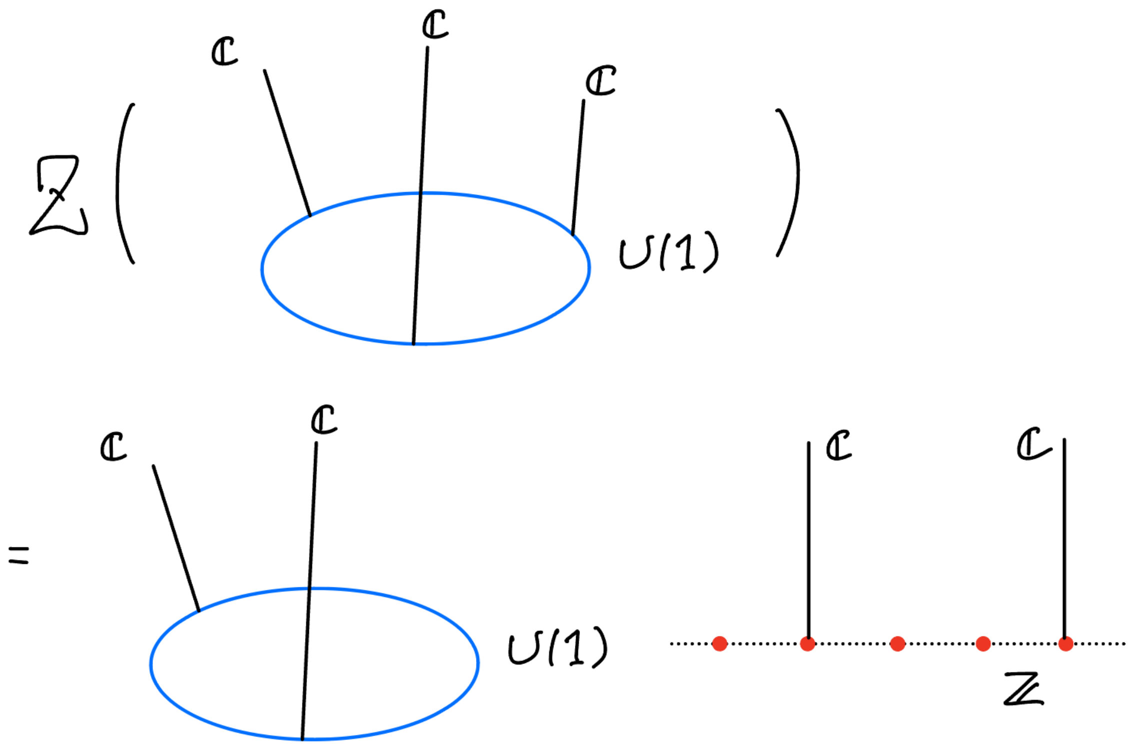
- The Drinfeld center is described as:

A simple object in Sky(C) is labeled by

$$(x, z)\ ,\ \ x \in U(1)\ ,\ \ z \in \Lambda \cong \mathrm{IrRep}(U(1)) \cong \mathbb{Z}\ .$$

Categorical $U(1)$ Symmetry

- So it looks like:



Issue of Finiteness

- We will need to generalize $\mathsf{Sky}^{\tau}(U(1))$ to something like $\mathscr{C}^{\tau}(G)$, where $\tau \in H^4(BG, \mathbb{Z})$ is the anomaly.

- First issue: Sky is NOT enough because the center is not big enough.

This is because we expect that the center is labeled by conjugacy classes of G, this is incompatible with the requirement that the sheaf is finitely supported. This is also pointed out in [Henriques].

- Have to relax finite support condition.

- We do not know how to rigorously do this yet. But we have some guess.

The Convolution Tensor Product

We proceed with the insufficiency of Sky kept in mind.

• Convolution tensor product is given by:

$$X * Y = m_*(p_1^*(X) \otimes p_2^*(Y) \otimes K)$$

for a line bundle $K \to G \times G$ (more precisely a multiplicative bundle gerbe, [Weis]).

• Fact: A multiplicative bundle gerbe on compact G is characterized by $\tau \in H^4(BG, \mathbb{Z})$.

• This mathematical fact partially explains the motivation we introduced the weird $H^4(BG, \mathbb{Z})$ earlier.

Gluing Pieces Together

- We have:

The Symmetry Category

The symmetry category of continuous global G-symmetry with 't Hooft anomaly $H^4(BG, \mathbb{Z})$ is $\mathscr{C}^\tau(G)$, where $\mathscr{C}$ is the category of sheaves of finite dimensional vector spaces, with the constraint of finite support relaxed.

- Issues to be resolved:
  - Relaxing finiteness means we go out of the rose garden of fusion category. Namely we drop nice properties like semi-simpleness, etc. This requires the development of a whole new field.
  - One may ask: What sheaves? Coherent? Quasi-coherent? Locally free? The answer to this is tricky.

What's More?

Gaugable Subsymmetries (with Jia)

Certainly we care about gaugings of global symmetries, but how?

It is said it is about the Lagrangian algebra objects of the Cat.
Key properties: commutative + "mid-dimension".

Fact: Given $f : H \subset G$,

$$f_*\mathcal{O}_H \text{ is commutative in } (\mathbf{QCoh}(G), *)$$ [1]

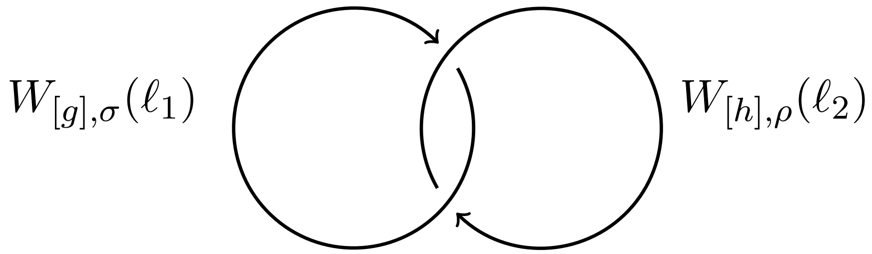
and we will need to generalize the above fact to $\mathbf{QCoh}_G(G)$ (and surely to smooth Lie groups)

$\Rightarrow$ Upshot 1: Closed subgroup of G can be gauged.

[1] $\mathbf{QCoh}(X)$: Cat. of finite rank quasi-coherent $\mathcal{O}_X$-modules for algebraic X

Gaugable Subsymmetries (with Jia)

- S and T matrices:



$$S^{(k)}_{([g],\sigma),([h],\rho)} = \int \mathcal{D}B\mathcal{D}A \; e^{i\int_{S^3} BF + \frac{k}{4\pi}\mathsf{CS}} \; W_{[g],\sigma}(\ell_1)\; W_{[h],\rho}(\ell_2)$$
$$\propto \sum_{w\in W} \; e^{-\frac{ik}{2\pi}\,\mathrm{tr}(\alpha w(\beta))} \; \chi^*_\rho(e^{iw^{-1}(\alpha)}) \; \chi^*_\sigma(e^{iw(\beta)})$$

and similarly

$$T^{(k)}_{([g],\sigma),([h],\rho)} = \delta_{[g],[h]} \; \delta_{\sigma,\rho} \; e^{\frac{ik}{4\pi}\,\mathrm{tr}\;\;\alpha^2} \; \frac{\chi_\sigma(g)}{d_\sigma} \quad \rightarrow \quad \text{Normalized character}$$

Going Smooth

- $\mathbf{QCoh}(G)$ is not good enough.

Proposal

Perf(G) is the right stuff.

Motivated by:

- It is "tensorial".
- It has "Wilson-'t Hooft lines" in it.
- It admits natural equivariant quotient.

Summary

What's done?

- Wrote down the SymTFT governing continuous symmetries in [2503.04546]
- (Re)formulated anomaly via defects for continuous symmetries in [2510.14722]
- Proposed the symmetry category for continuous symmetries in [2509.13170]

What to be done?

- To classify various gaugings of categorical symmetry, to appear
- To achive a different formulation of ETQFT, to appear
- To understand why we need category at all, long term, but will appear