Heavy-to-Light Semileptonic Decays through N^3LO in QCD

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USTC Seminar · 3 September 2026

Based on work with Xiang Chen, Long Chen, Yan-Qing Ma
2309.01937, 2602.11537, 2602.11879.

Cabibbo-Kobayashi-Maskawa (CKM) mechanism

$$L_{\mathrm{SM}} \ni \overline{u}_{Li}\, \gamma^{\mu} V_{ij} d_{Lj} W_{\mu} + \mathrm{h.c.}$$

$$V_{\mathrm{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

Quark flavor mixing is characterized by a unitary matrix V

Explain many decay experiments

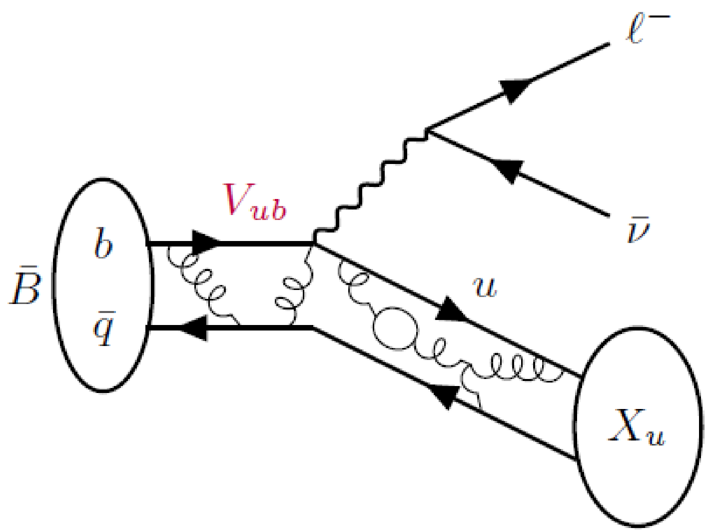
Irreducible phase

CP violation source

One of the most stringent precision tests of the Standard Model

Different experiments determine different entries

Only four independent real parameters



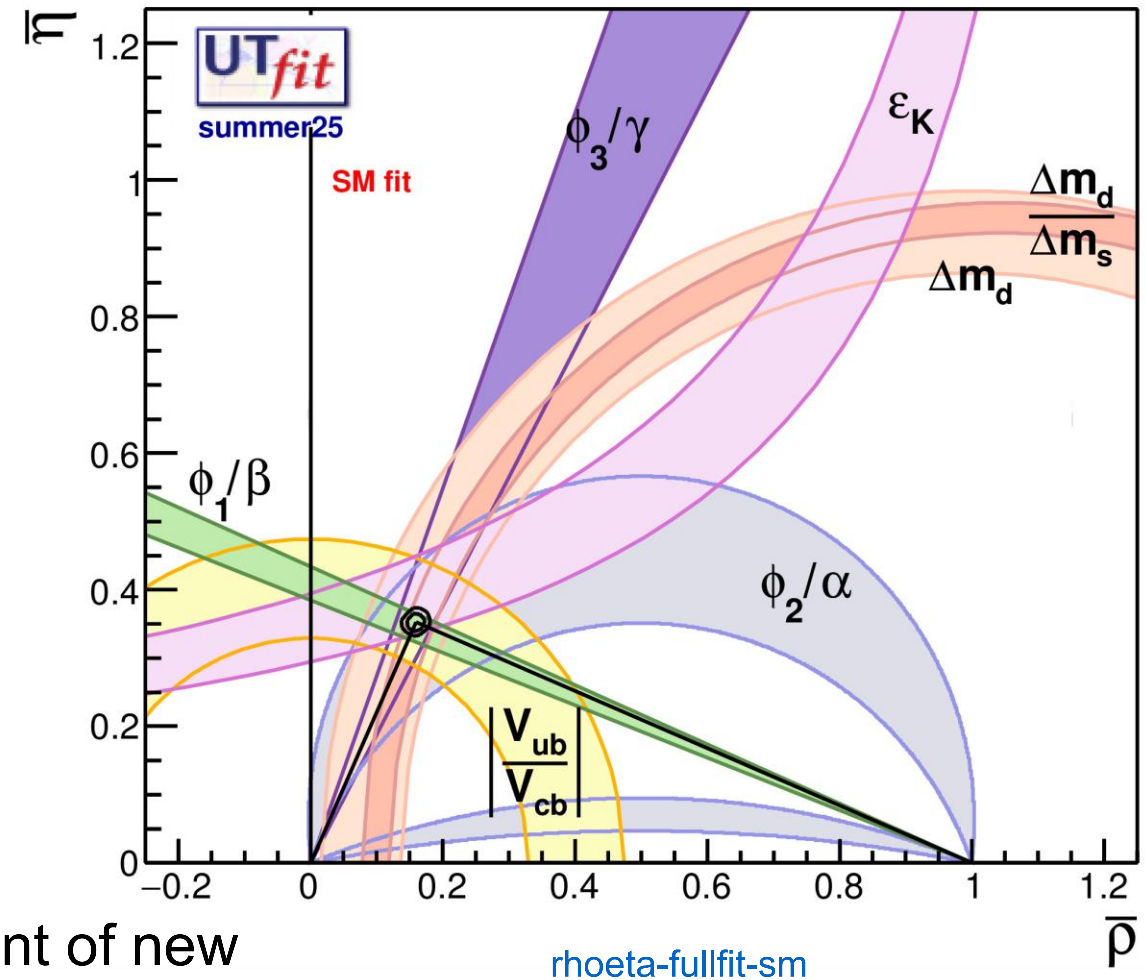
CKM unitarity triangle

$$V^{\dagger}V = 1$$

$$\Rightarrow \frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*} + 1 + \frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*} = 0$$

$$-(\bar{\rho} + i\,\bar{\eta})$$

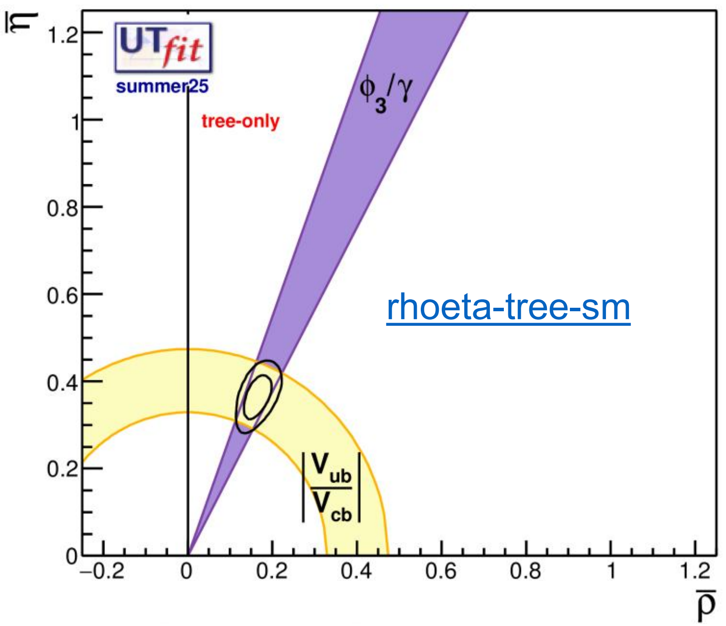
Measurements overlap in a narrow region → quark flavor mixing is consistent with the Standard Model.



rhoeta-fullfit-sm

Any significant inconsistency could be a hint of new physics.

V_{ub}

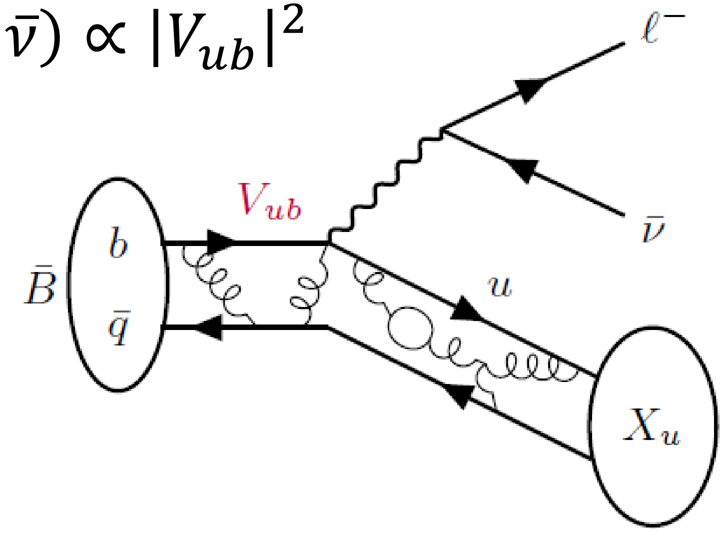


V_{ub} plays an important role

Determine the length of one side

Limit the precision tests.

$$\Gamma(b \to u\, l\, \bar{\nu}) \propto |V_{ub}|^2$$



Two determinations are consistent:

Inclusive: $B \to X_u l\, \bar{\nu}$ rpp2025-rev-vcb-vub.pdf

$$|V_{ub}|_{\mathrm{incl}} \sim 4.1 \times 10^{-3}, \pm(4-7)\%$$

Exclusive: $B \to \pi\, l\, \bar{\nu}$

$$|V_{ub}|_{\mathrm{excl}} \sim 3.7 \times 10^{-3}, \pm 5\%$$

Inclusive semileptonic B decay

$|V_{ub}|_{\text{incl}}^{\text{GGOU}} \sim 4.1 \times 10^{-3}$, $\sim 4\%$ uncertainty rpp2025-rev-vcb-vub.pdf

GGOU: A framework to treat non-perturbative effects (shape function)

Higher-order QCD alone is comparable to the projected Belle II experimental precision

Gambino, Giordano, Ossola, Uraltsev, 0707.2493

- Current higher-order QCD: ~ 1.5%
- Current experiment: ~3%
- Belle II future experiment: ~1.7%

BelII, 1808.10567

To exploit Belle II, the perturbative QCD uncertainty must be reduced.

Fully differential distribution

1. Kinematic selections require fully differential distributions

To suppress $B \to X_c\, l\, \nu$ background (50x larger)

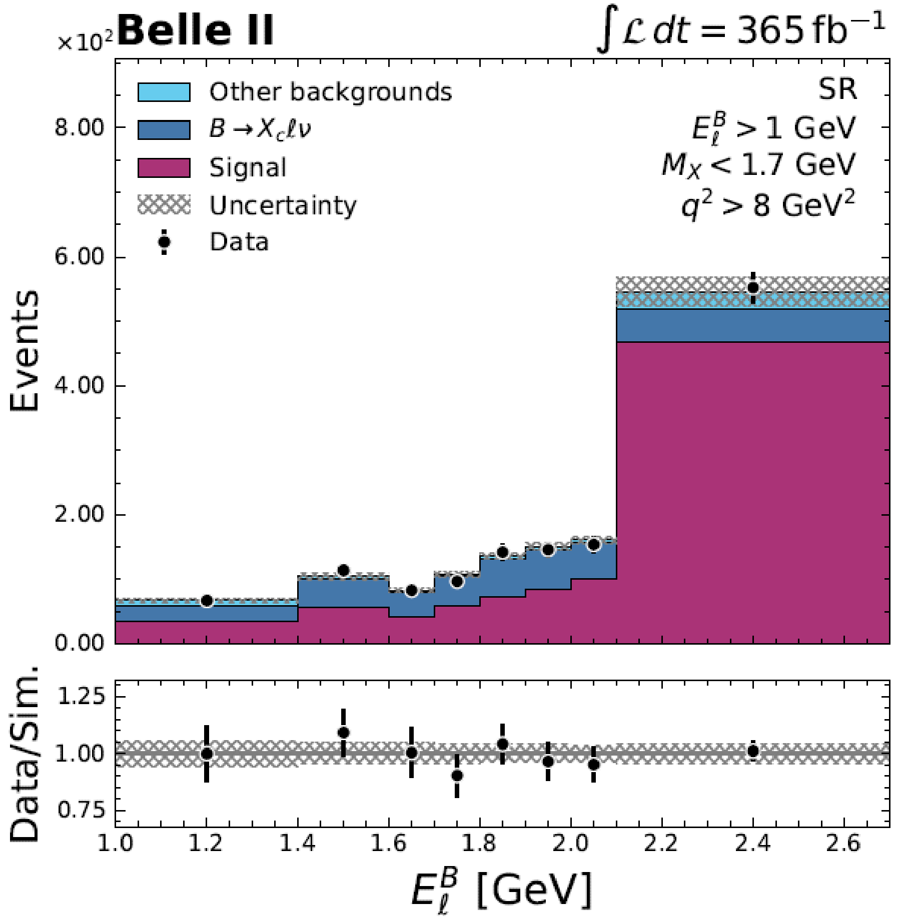
$M_X < 1.7\ \text{GeV},\ E_l^B > 1\ \text{GeV},$
$q^2 = (p_l + p_\nu)^2 > 8\ \text{GeV}^2$

2. NNLO distribution is still incomplete

- β_0 enhanced (BLM) contributions (Used in the V_{incl}) Gambino, Gardi, Ridolfi, 0610140
- Virtual corrections
- Double real-radiation corrections

Bonciani, Ferroglia, 0809.4687; Asatrian, Greub, Pecjak, 0810.0987
Beneke, Huber, Li, 0810.1230; Bell, 0810.5698

Bonciani, Broggio, Cieri et al., 1807.01681



Why fully differential N3LO?

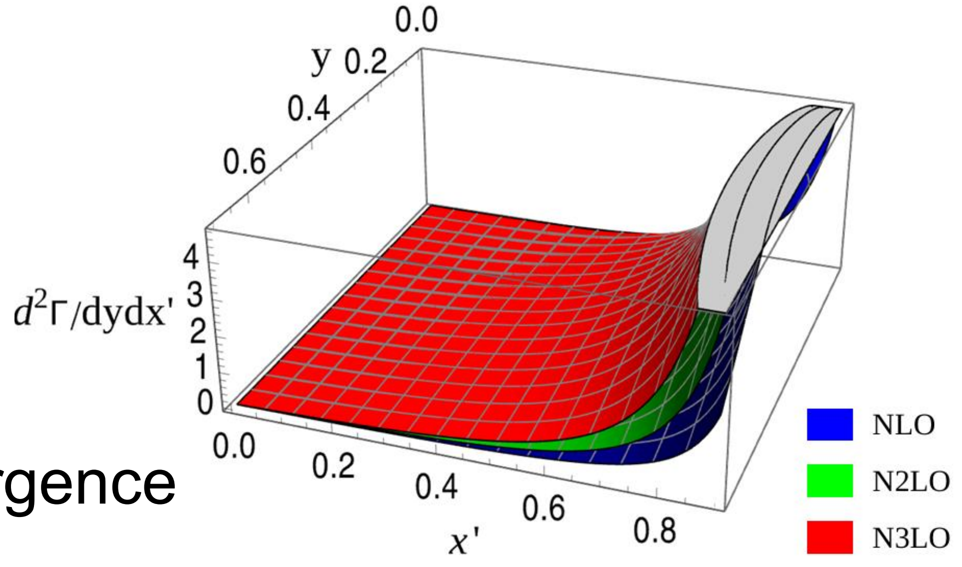
Complete NNLO differential distributions

For completeness

Complete N3LO differential distributions

The first real test of perturbative convergence and the residual scale dependence!

Chen, Chen, XG et al., 2602.11879
Chen, Chen, XG et al., 2602.11537



*Note

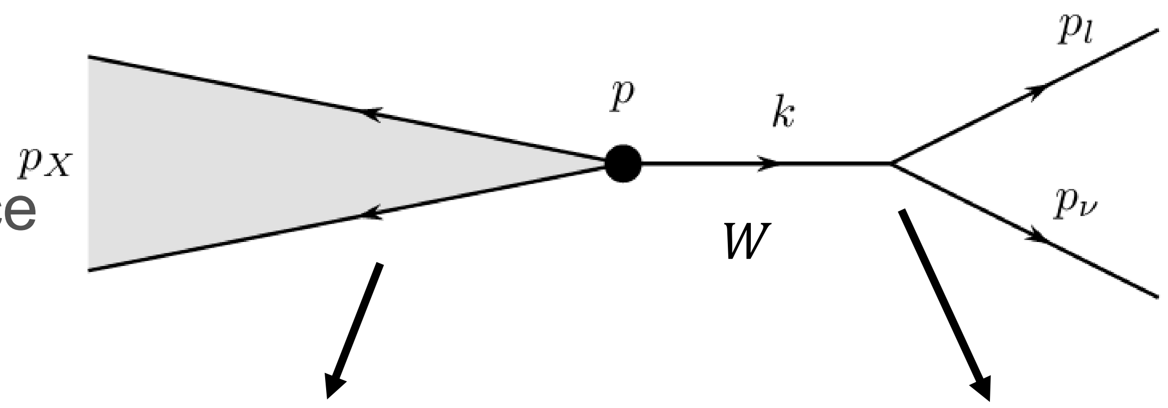
An independent NNLO calculation based on phase space slicing + SCET

Broggio, Capdevila, Ferroglia et al., 2601.15447

Decay kinematics

$Q \to W + X_u$ **encodes all QCD dynamics**

Integrate out the hadronic phase space



Leptonic phase space

Fixed external momentum

QCD corrections to $Q \to W + X_u$

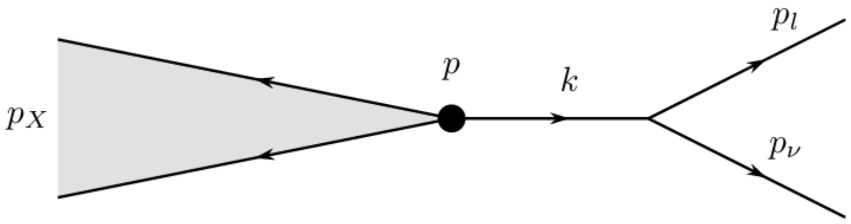
Tree level

Three-dimensional phase space: $\mathrm{d\,PS_{L3}} \sim \mathrm{d}\,E_l\ \mathrm{d}E_\nu\ \mathrm{d}\cos(\theta_{l\nu})$

A convenient parametrization: $\mathrm{d\,PS_{L3}} \sim \mathrm{d}m_w^2\ \mathrm{d}E_w\ \mathrm{d}E_l$

$\boldsymbol{E_l}$ **dependence is a simple two-body decay.**
The nontrivial part is the two-dimension distribution ($\mathrm{d}m_w^2\ \mathrm{d}E_w$)

Five structure functions



$$\mathrm{d\,PS_{L3}} \sim \mathrm{d}m_w^2\, \mathrm{d}E_w\, \mathrm{d}E_l$$

The two-dimension distribution is described by hadronic tensors $W_{Qq}^{\mu\nu}(p,k)$

Lorentz symmetry decomposition

$$\begin{aligned}\mathcal{W}_{Qq}^{\mu\nu}(p,k) &= W_1\left(-g^{\mu\nu}p^2\right) + W_2\,k^\mu k^\nu + W_3\,p^\mu p^\nu \\ &\quad + W_4\left(p^\mu k^\nu + k^\mu p^\nu\right) - W_5\,i\epsilon^{\mu\nu\rho\sigma}p_\rho k_\sigma\,,\end{aligned}$$

W_i are hadronic structure functions.
W_i depends on ϵ, m_w^2 and E_w

$p^2 = m_Q^2 = 1, k^2 = m_w^2,$
$k \cdot p = E_w, d = 4 - 2\epsilon$

hadronic structure functions describe the semileptonic decay

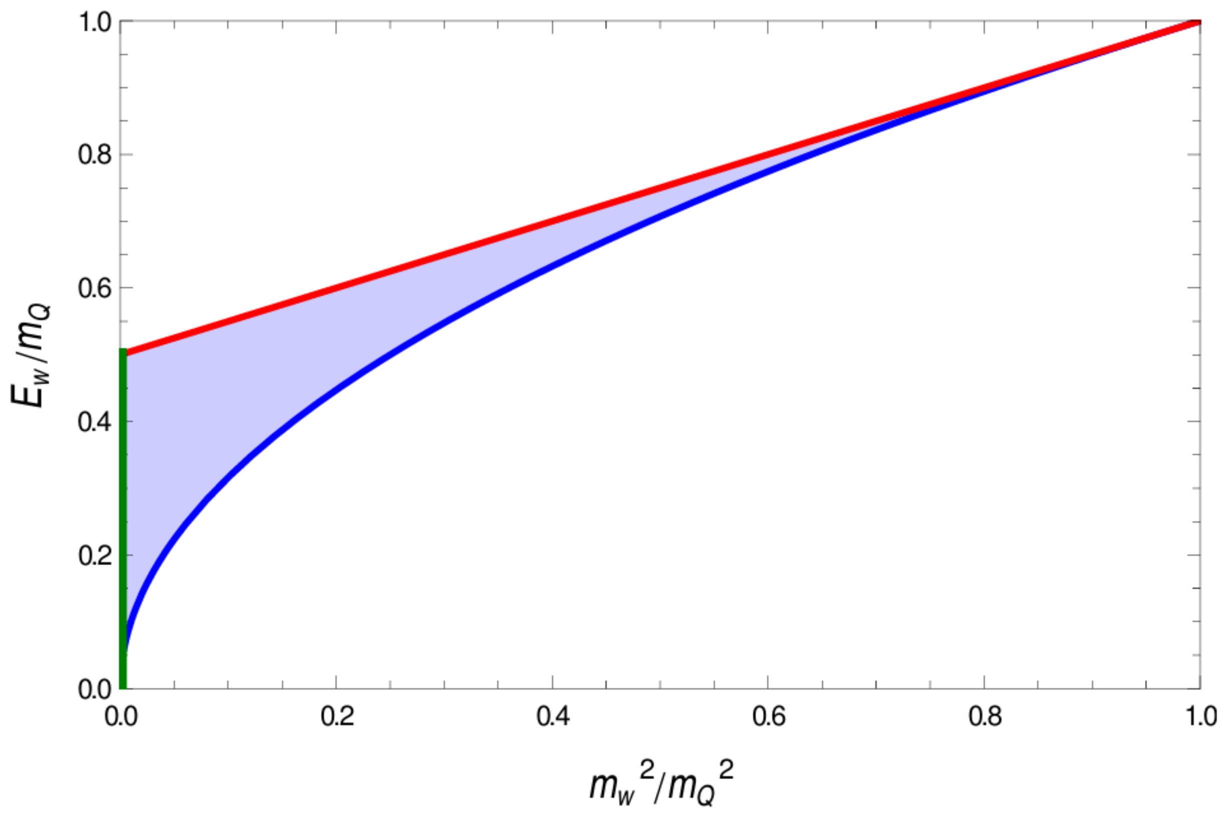
Phase space

$$\mathrm{d\,PS_{L3}} \sim \mathrm{d}m_w^2\,\mathrm{d}E_w\,\mathrm{d}E_l$$

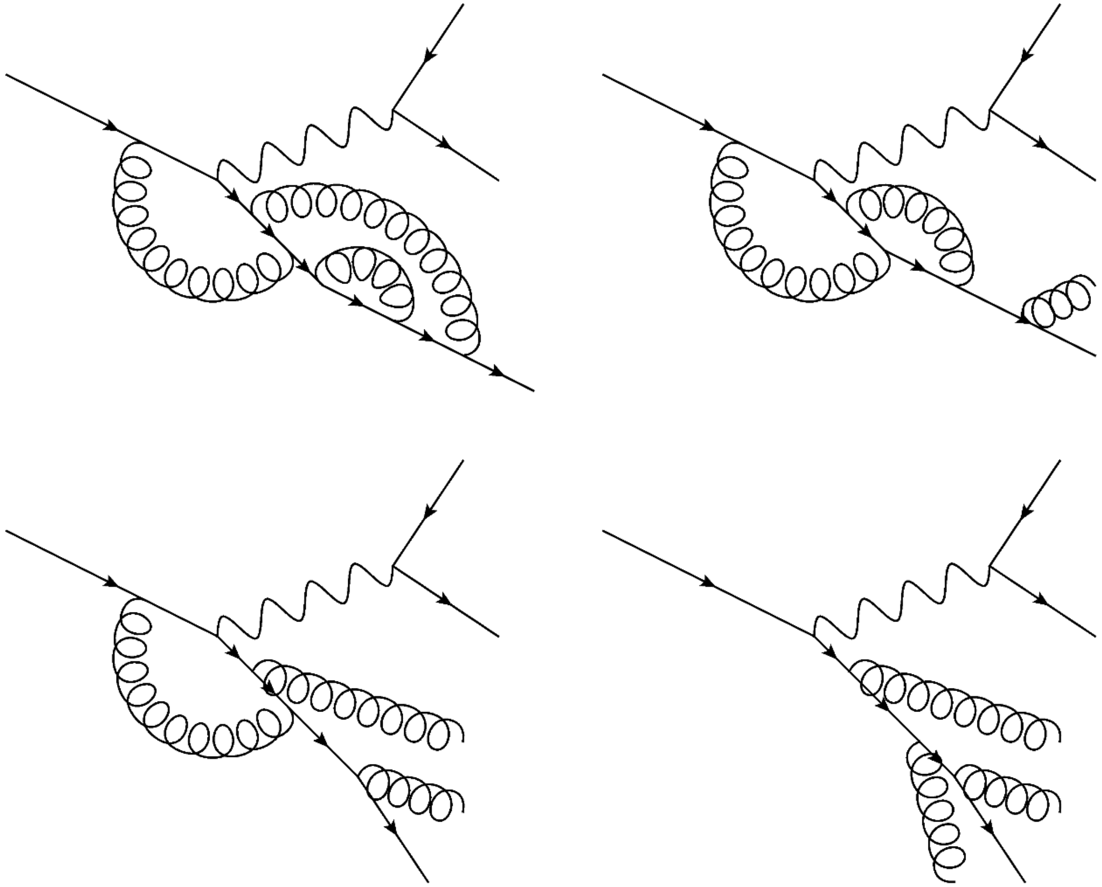
RED $E_w = (1 + m_W^2)/2$
Born / virtual edge; radiation is soft or collinear

BLUE $E_w = m_W$
the W boson is produced at rest

GREEN $m_W = 0$
the two massless leptons are collinear



Feynman diagrams



Virtual + Real:

From three-loop virtual corrections to tree-level triple-real radiation.

Each contribution is separately infrared divergent; only their inclusive sum is physical.

Interfered with tree amplitudes

Qgraf: Nogueira et al., Comput. Phys. 1993

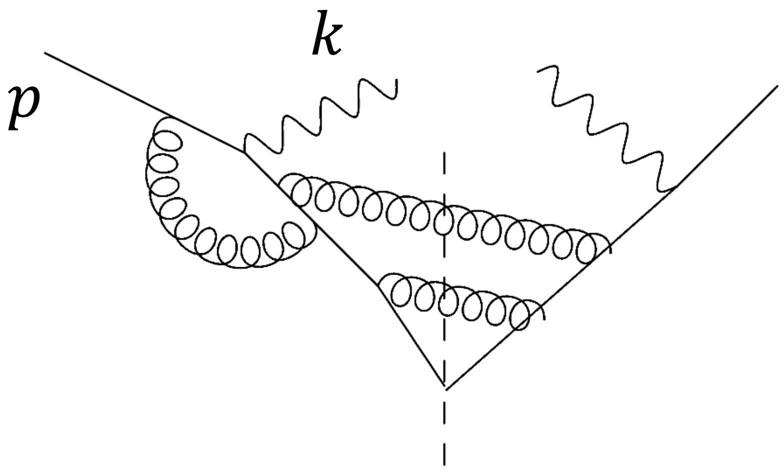
DiaGen, Czakon, private

Feynman integral topologies

Reverse unitarity for phase-space integrals

$$\delta(p_i^2) \sim \frac{1}{p_i^2 + i0} - \frac{1}{p_i^2 - i0}$$

Anastasiou, Melnikov, 0207004



Make use of loop-calculation techniques

Explore symmetries and classify Feynman integrals into topologies

12 indices: 10 propagators and 2 irreducible scalar products

$$\int \frac{\mathrm{d}^d l_1 \mathrm{d}^d p_1 \mathrm{d}^d p_2 [(p+p_2)^2]^{-a_{11}} [(k+p_2)^2]^{-a_{12}}}{[(k+p_1)^2 - m_b^2]^{a_1} [(k-p+p_1)^2]^{a_2} [l_1^2]^{a_3} [(l_1+p\ \ )^2 - m_b^2]^{a_4} [(l_1+p_1)^2]^{a_5} [(l_1+p-p_2)^2 - m_b^2]^{a_6} [(k-l_1-p+p_2)^2]^{a_7} [(k-p+p_1+p_2)^2]^{a_8} [(p_1)^2]^{a_9} [(p_2)^2}$$

CalcLoop: Ma et al., https://gitlab.com/multiloop-pku/calcloop

FORM, Vermaseren, 0010025

Integration-by-part (IBP) reduction

$$0 = \int \prod_{i}^{L} \frac{\mathrm{d}^d l_i}{i\,\pi^{d/2}} \frac{\partial}{\partial l^\mu} \frac{v^\mu}{D_1^{a_1} \ldots D_{15}^{a_{15}}}$$

Chetyrkin et al., NPB(1981)
K.G. Chetyrkinand A.L. Kataev,1981

Laporta, 0102033

$$\Rightarrow c_1\, I_1 + c_2\, I_2 + \cdots + c_n\, I_n = 0$$

c_i are rational function coefficients

100k integrals $\xrightarrow{\text{After using relations}}$ ~3000 master integrals

Our tools

B **Blade**

Link: https://gitee.com/multiloop-pku/blade

项目信息

Block-triangular form improved Feynman integral decomposition .

Blade: XG, Liu, Ma, Wu. 2405.14621

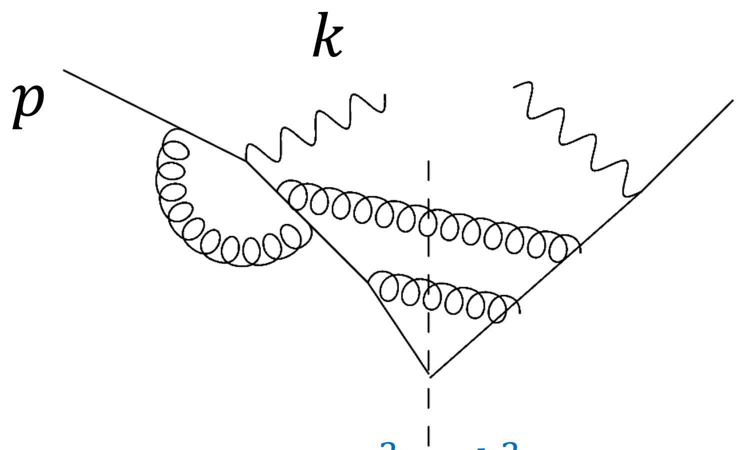
Differential equations for master integrals

Kotikov, PLB(1991); Remiddi, 9711188; Gehrmann–Remiddi, 9912329

Use IBP to construct closed differential equations (DEs) for master integrals

$$\frac{\partial I[1,1,1,1]}{\partial E_w} = -\frac{m_w^2 - E_w^2}{E_w} I[1,2,1,1] + \cdots$$

IBP $$= \frac{1}{E_w} I[1,1,1,1] + \cdots$$



$m_w^2 = k^2$
$E_w = k \cdot p$
$d = 4 - 2\epsilon$

Two-variable system

$$\frac{\partial}{\partial E_w} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix} = B_{E_w}(m_w^2, E_w, \epsilon) \begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix}$$

Same for $B_{m_W^2}$ **Solve DEs analytically or numerically (series expansion)**

Numerical differential equation

Series expansion

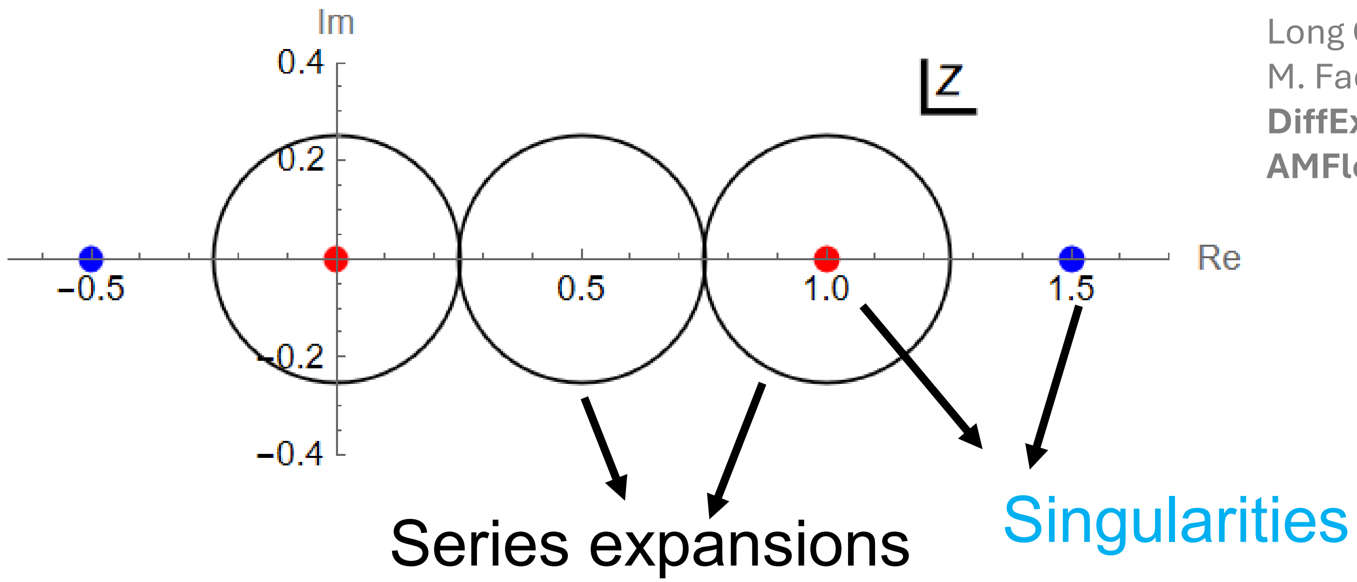
Systematic and efficient

Univariate problem is well understood

Matching and transportation

$$I(\epsilon, z) = \sum_{\mu,k,n} c_{\mu,k,n}(\epsilon) z^{\mu(\epsilon)} \log^k z^n$$

R. N. Lee, A. V. Smirnov, V. A. Smirnov, 1709.07525
R. Bonciani, G. Degrassi, P. P. Giardino, R. Gröber, 1812.02698
H. Frellesvig, M. Hidding, L. Maestri, F. Moriello, G. Salvatori, 1911.06308
Long Chen, M. Czakon, M. Niggetiedt, 2109.01917
M. Fael, F. Lange, K. Schönwald, M. Steinhauser, 2202.05276
DiffExp, M. Hidding, 2006.05510
AMFlow, X. Liu, Y.-Q. Ma, 2201.11669



Boundary condition from AMFlow

X. Liu, Y.-Q. Ma, 2201.11669

Tricks for solving series expansions

Two practical tricks to reduce symbolic complexity

Numerical ϵ

Evaluate at $\epsilon = 10^{-3} + n \times 10^{-4}, n = 0,1.$

Do not reconstruct deep Laurent series for divergent intermediates.

Extrapolate $\epsilon \to 0$ only for the final finite observable.

Liu, Ma, 2201.11637
Liu, Ma, 2201.11669

Pseudo-singularities

Many singularities are spurious or non-physical, which affects numerical stability

Increase working precision to bypass their effects.

Chen, XG, He, Li et al., 2209.14953

The bottleneck: a two-variable system

Hybrid solution: solve one direction, sample the other

1. Sample m_W^2 — Stratified Gauss–Kronrod nodes

2. Solve E_w for fixed m_W^2 — Piecewise functions in terms of series expansions

3. Interpolate in m_W^2 — Arbitrary points require only cheap series evaluation plus local interpolation

A m_W^2-differential equation is the fallback

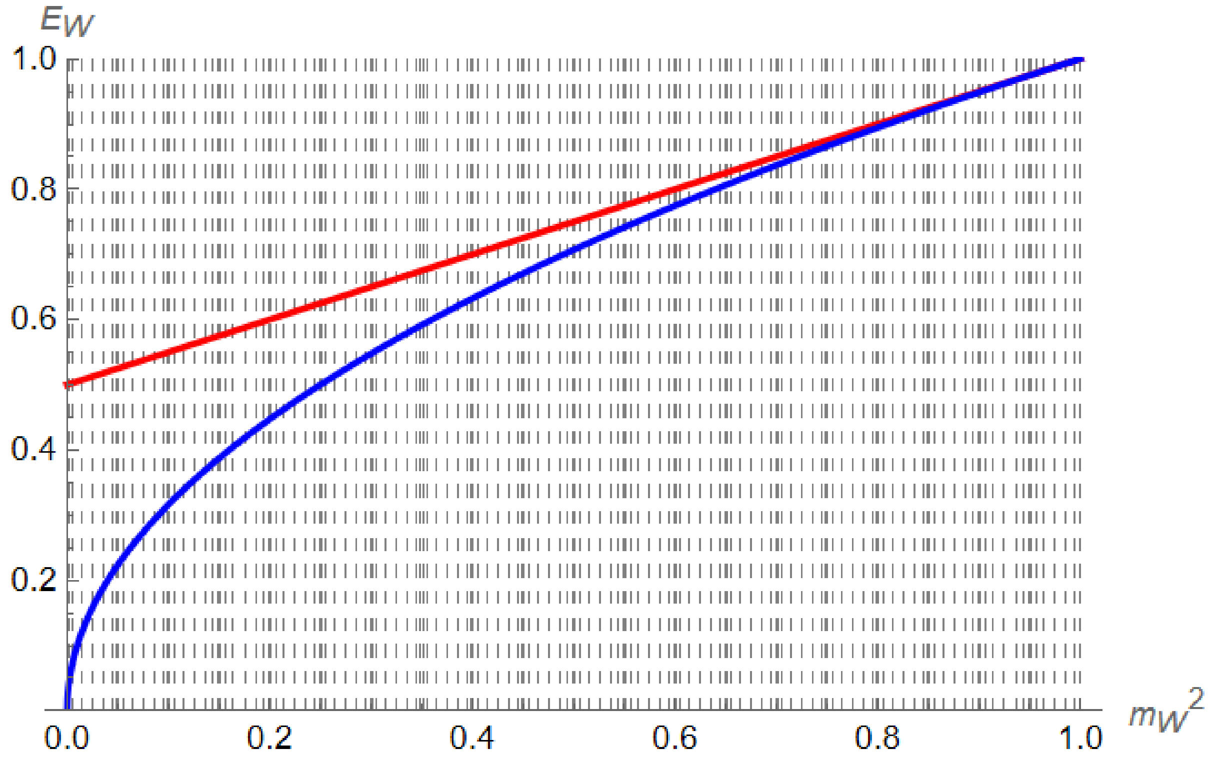
Gauss-Kronrod sampling

Why use G-K sampling?

- Support accurate integration and stable interpolation
  - Uniform 7 points: ~ 5 digits
  - GK 7 points: ~ 8 digits
- Good error estimation

How many samples?

- 20 segments × 7 GK nodes = 140 samples



Master integral grids

140 piecewise functions

- Piecewise function in E_W, consists of series expansions.
- Truncated to 200 terms

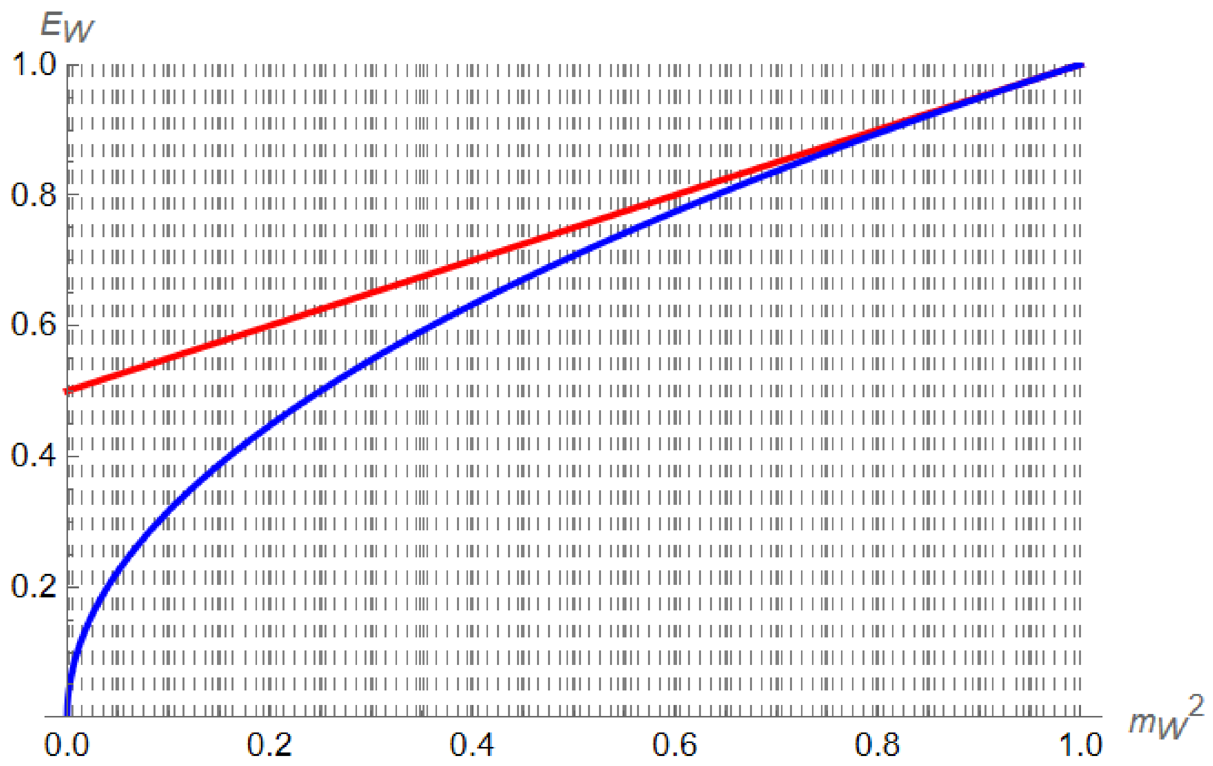
Divergences are regularized by ϵ

e.g. $E_W^{-1-2\epsilon}$

Integration gives $1/\epsilon$ IR divergence

Arbitrary phase point

1. Evaluate adjacent piecewise functions at the same E_W.
2. Interpolate m_W^2
3. m_W^2-differential equation is the fallback



From master integrals to physical structure functions

Integrand → Integral reduction → Master integrals → Bare structure functions → UV renormalization → IR cancellation

virtual + real corrections

Finite results in the bulk phase space

m_Q in the on-shell scheme;
α_s in the $\overline{\mathrm{MS}}$ scheme

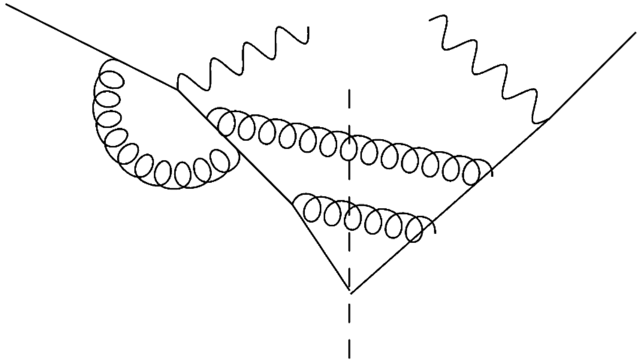
The structure functions are saved in terms of 140 piecewise functions consist of series expansions.

Valid in the whole two-dimension phase space

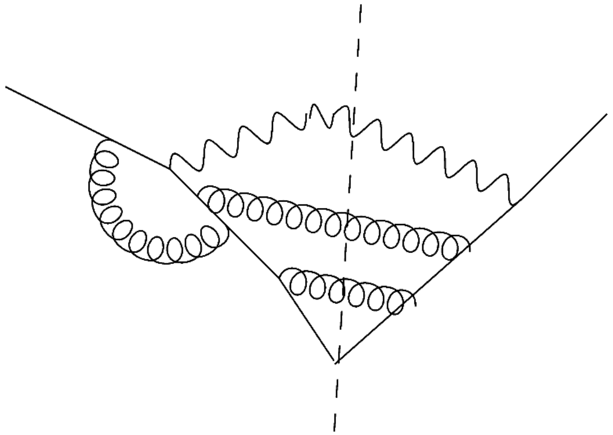
How do we know such a complicated calculation is correct?

Validation

Differential integration agrees with an independent inclusive calculation.



Differential



Inclusive

Fix m_W, complete the integration of the differential decay width in E_W.

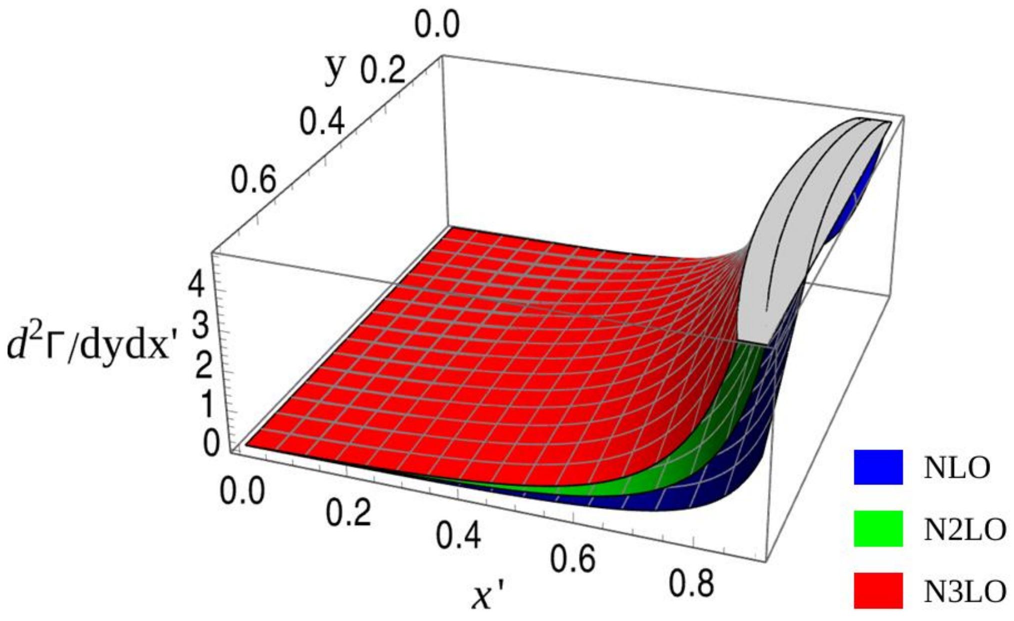
1. Integrate W-boson momentum in the integrand.
2. Use reverse unitarity to handle $\delta(k^2 - m_W^2)$
3. New Feynman integrals, new computations

Chen, Chen, XG et al., 2309.01937
Fael, Usovitsch, 2310.03685

Double differential results

$$\frac{\mathrm{d}^3\,\Gamma^{\mathrm{sl}}}{\mathcal{N}\,\mathrm{d}m_w^2\,\mathrm{d}E_w\,\mathrm{d}E_l} = W_1\,m_Q^2\,m_w^2 \;+\; W_3\,m_Q^2\left(2\,E_w\,E_l - 2\,E_l^2 - m_w^2/2\right) \;+\; W_5\,m_Q\,m_w^2\left(E_w - 2\,E_l\right)$$

- Born/virtual edge excluded: the displayed rate starts at O(a_S).
- Corrections grow for larger lepton pair invariant mass (m_W^2) and larger E_W (smaller hadronic invariant mass).
- Near the endpoint, fixed order need to be matched to resummation and non-perturbative physics.



$x = E_W$
$y = m_W^2$
$x' = \dfrac{2(E_W - m_W)}{(1 - m_W)^2}$

Regular region:
$0 < y < 0.7$
$\sqrt{y} < x < 0.9\,x_{\max}$

Why does the inclusive series converge so poorly?

Heavy Quark Expansion

Bigi, Shifman, Uraltsev et al., 9304225
Bigi, Uraltsev, Vainshtein, 9207214
Uraltsev, 9905520

Non-perturbative parameters

$$\Gamma(B \to X_u l\, \bar{\nu}_l) = \Gamma_0(m_b)\left[C_p\left(1 - \frac{\mu_\pi^2 - \mu_G^2}{2m_b^2}\right) - 2\frac{\mu_G^2}{m_b^2} + O\left(\frac{\Lambda_{QCD}^3}{m_b^3}\right)\right]$$

Perturbative QCD corrections to free b-quark decay

$$\Gamma_0(m_b) = \frac{G_F^2 |V_{ub}|^2 m_b^5 A_{ew}}{192\pi^3}$$

$$C_p = 1 - 2.41307\frac{\alpha_S}{\pi} - 21.2955\left(\frac{\alpha_S}{\pi}\right)^2 - 269.788\left(\frac{\alpha_S}{\pi}\right)^3 + O(\alpha_S^4) \qquad \mu = m_b$$

The perturbative coefficients grow rapidly.

The pole mass carries infrared sensitivity

$$\Gamma \propto m_b^5\, C_p$$

Pole mass carries a $O(\Lambda_{\mathrm{QCD}})$ infrared renormalon ambiguity.

$$\frac{\delta\Gamma}{\Gamma} \cong 5\,\frac{\delta m_b}{m_b} \sim O\left(\frac{\Lambda_{QCD}}{m_b}\right)$$

Amplified

But the HQE has no $(1/m_b)$ correction

⇨ The ambiguity must cancel against C_p

In the pole scheme this cancellation leads to poor perturbative convergence

Solution: re-express the decay width in a short-distance mass.

Short-distance masses reorganize the expansion

$\overline{\text{MS}}$

High-scale short-distance mass

- Renormalon-free
- Universal and precisely determined

Kinetic

Low-scale short-distance mass

- Renormalon-free
- Subtracts low-momentum contributions below $\mu_{\text{kin}} \sim 1$ GeV
- Tailored to inclusive heavy-hadron decays

Convert and re-expand both the coefficients and the m_b^5 prefactor consistently.

Does this distinction actually matter numerically? Yes—dramatically.

The kinetic scheme stabilizes the perturbative series

$\overline{\text{MS}}$

$$\Gamma_0(m_b)\, C_p = \Gamma_0(\overline{m}_b)(1 + 0.3036 + 0.1367 + 0.0687)$$

$\mu = \overline{m}_b(\overline{m}_b)$ $\overline{m}_b(\overline{m}_b) = 4.18\ \text{GeV}$

- Corrections are reduced from tens of percent to a few percent.
- δ_{scale}~ [+6%, -8%] $\mu \in [1.4, 6]$ GeV

Kinetic

$$\Gamma_0(m_b)\, C_p = \Gamma_0\big(m_{b,\text{kin}}[1]\big)(1 - 0.0271 + 0.0251 + 0.0218)$$

$\mu = m_{b,\text{kin}}\,[1]/2$ $m_{b,\text{kin}}[1] =$ 4.554 ± 0.018 GeV

HFLAV, 1909.12524

- Increase the width by 2% at NNLO and N^3LO
- δ_{scale}~ 2% $\mu \in \left[m_{b,\text{kin}}\,[1]/2\,, 2m_{b,\text{kin}}[1]\right]$

Mass scheme has a major impact on how efficiently the perturbative series is organized!

Inclusive charmless B decay width

State of the art predictions

Combine the perturbative corrections (kinetic) and non-perturbative parameters

$$\Gamma(B \to X_u \ell \bar{\nu}_\ell) = \frac{|V_{ub}|^2}{|3.82 \times 10^{-3}|^2} \left(6.53 \underbrace{\pm 0.12}_{\delta m_{b,\text{kin}}} \underbrace{\pm 0.13}_{\delta_{\text{scale}}} \underbrace{\pm 0.03}_{\delta_{\text{HQE}-\text{parameters}}}\right) \times 10^{-16}\,\text{GeV}$$

The N3LO result delivers a percent level perturbative normalization.

A stable inclusive prediction does not mean that the perturbative correction is small everywhere in phase space.

A subtlety: the phase-space boundary also moves

The conversion for inclusive quantity is straightforward.

At $\mathrm{N^3LO}$, changing the quark-mass scheme of a differential distribution develops a new subtlety. $m = m_0 + \delta_m$

$$\int_0^m f(\alpha_S, m; x)\, dx = \int_0^{m_0} f(\alpha_S, m_0 + \delta_m; x)\, dx + \int_{m_0}^{m} f(\alpha_S, m_0 + \delta_m; x) dx$$

$$= \int_0^{m_0} \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n f(\alpha_S, m; x)}{\partial m^n}\bigg|_{m=m_0} (\delta m)^n\, dx$$

$$+ \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \frac{\partial^k f(\alpha_S, m_0 + \delta_m; x)}{\partial x^k}\bigg|_{m=m_0} (\delta m)^{k+1}$$

Boundary effect terms become non-vanishing at $\mathrm{N^3LO}$!

Can only be incorporated in histogrammed spectrum (the last bin)

The inclusive width hides large local corrections

- Good convergence behavior

- Low-q^2 and high-q^2 corrections partially cancel in the inclusive width.

  Explains the small corrections to the total width at NNLO.

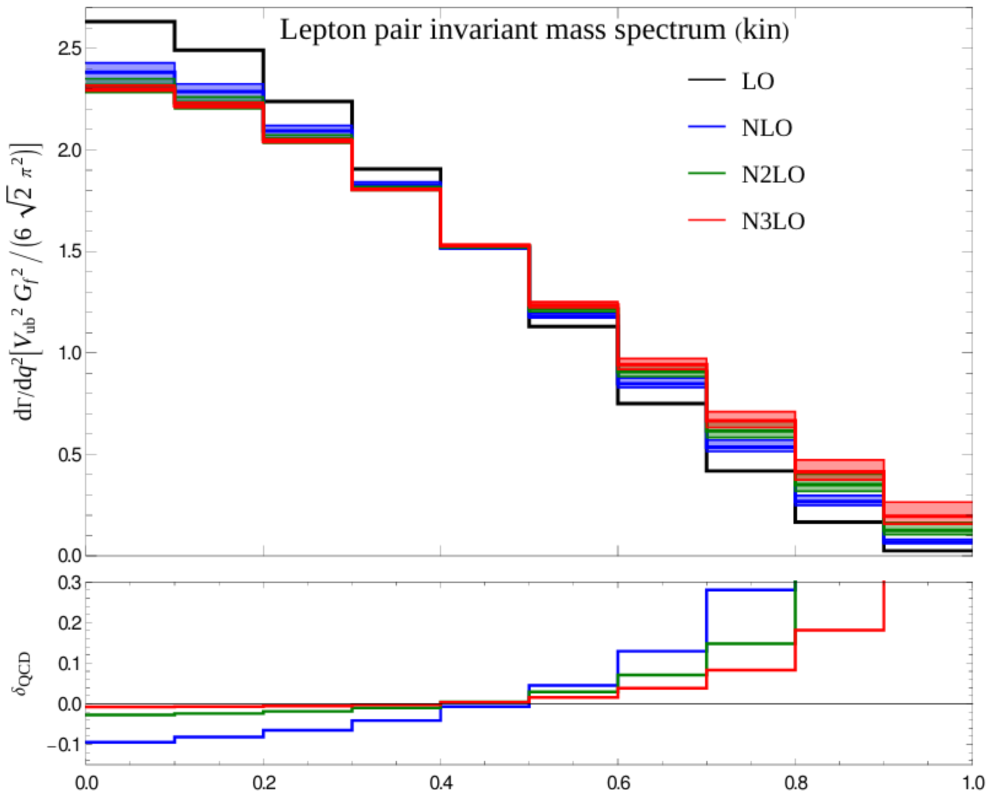
- N3LO becomes sizeable at large q^2.

  Last five bins: about +9.5%, with a −4% to +6% scale range.

  The large-q^2 region had played an important role in the inclusive determination of V_{ub}.

- It would be interesting to investigate the potential impact of our findings

Lepton pair invariant mass spectrum (kin)



q^2/m_b^2 $\quad$ $q^2 = m_W^2$

Beyond B decays: $c \to q\, l\, \bar{\nu}_l$

The same structure functions find applications in c-quark physics

Inclusive electron-energy moments (EEM) of $D_s \to X\, l\, \bar{\nu}_l$ ($c \to q\, l\, \bar{\nu}_l$) are important observables sensitive to V_{cq}.

$q = d, s$

Percent level precision at BESIII

BESIII, 2104.07311
Santis et al., 2504.06063

$$\langle E_e^0 \rangle = 8.9466 - 1.1871\varepsilon - 0.4440\varepsilon^2 - 0.0609\varepsilon^3$$
$$\langle E_e^1 \rangle = 4.1602 - 0.4994\varepsilon - 0.1307\varepsilon^2 + 0.0861\varepsilon^3$$
$$\langle E_e^2 \rangle = 2.1494 - 0.2311\varepsilon - 0.02498\varepsilon^2 + 0.1107\varepsilon^3$$

- Even at the charm scale, inclusive moments remain controlled through N3LO in the 1S scheme.
- The above results up to O(ε^2) have been employed to improve the precision of the inclusive determination of $|V_{cs}|$, $V_{cd}|$ and non-perturbative parameters.

Shao, Huang, Qin, 2502.05901
Shao, Feng, Liu et al., 2509.11404

Beyond B decays: $t \to Wb$ at $\mathrm{N}^3\mathrm{LO}$

$m_t \gg \Lambda_{QCD}$, perturbative corrections alone is reliable

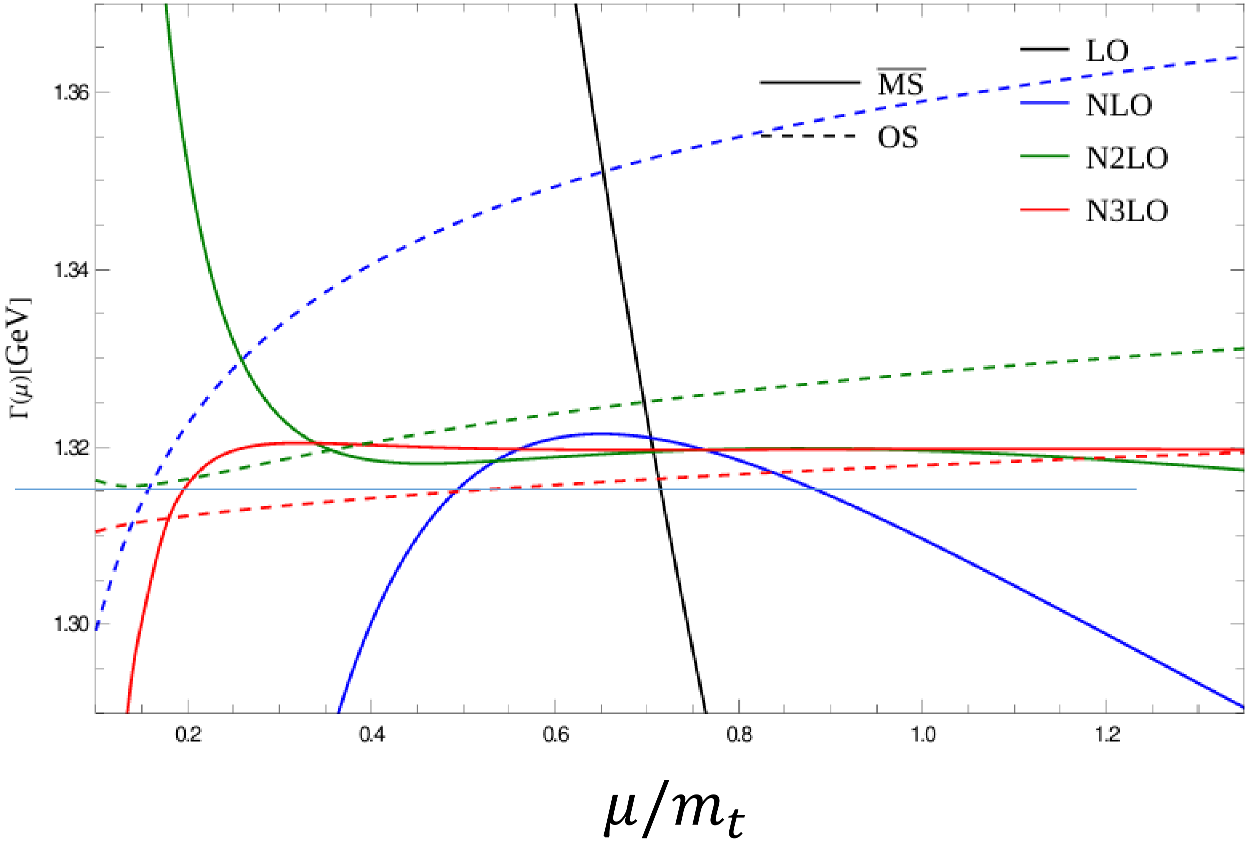
$\Gamma_t = 1.4864 - 0.1275 - 0.0307 - 0.0104$
$= 1.3178$ GeV

N^3LO shift ~ 10 MeV, larger than NNLO scale variation.

Short-distance mass schemes significantly improve perturbative stability.

Meets future experiment requirements

CEPC, 2207.12177



Conclusion

1. Fully differential N3LO. Complete structure functions enable precision predictions with realistic kinematic cuts.

2. Hybrid strategy (interpolation + series expansion) for solving the bivariate master integrals.

3. A suitable mass scheme stabilizes the perturbative series, while sizeable local corrections can be hidden by cancellations.

Outlook: combine fixed-order results with resummation and exploit them for precision extractions of CKM matrix elements and nonperturbative parameters.

Thank you!