October 23, 2021

TTbar deformation and new integrable models

Yunfeng Jiang

1 Abstract

TTbar and other solvable irrelevant deformations have received considerable attention in the past few years. In this lecture, I will first explain the importance of TTbar deformation for the understanding of several fundamental issues in quantum field theories. These include the S-matrix bootstrap program and UV completeness of quantum field theories. After that, the main features of TTbar deformation will be discussed. Finally I will show how these deformations can be defined in a much broader family of integrable models including lattice models such as integrable spin chains and cold atom systems. The physical properties of the deformed models will also be discussed.

2 Motivations

What is a QFT ?

1. Lagrangian/Hamiltonian description (perturbative)
2. Axiomatic QFT
   (a) Wightman axioms (Lorentzian)
   (b) Osterwalder-Schrader axioms (Euclidean)
   (c) Rigorous, but hard to do computations
3. Bootstrap: conformal bootstrap (for CFT), S-matrix bootstrap.

S-matrix bootstrap

- Proposed by W. Heisenberg in 1950's to replace QFT;
- A dominating theory in 50s'-70s' (Mandelstam, Chew) to understand strong interaction;
- Construct S-matrix by self-consistency relations;
- Leads to some interesting results such as Regge theory, Veneziano amplitude, etc;
- Becomes extremely complicated. Replaced by QCD;
- A revival in recent years (inspired by conformal bootstrap)

2D and integrability

- Simpler kinematics.
- Higher conserved charges $\mathcal{Q}_s$;
- In $d > 2$, S-matrix is trivial (Coleman-Mandula theorem);
- In $d = 2$ (Coleman-Mandula does not apply), but
  1. S-matrix factorizes into 2 to 2 S-matrices;
  2. Purely elastic;

Bootstrap axioms

Kinematics in $2d$

$$E^2 - p^2 = m^2, \qquad E(\theta) = m\cosh\theta, \quad p(\theta) = m\sinh\theta. \tag{2.1}$$

The S-matrix satisfies the following axioms (Zamolodchikov-Zamolodchikov 1979)

- **Unitarity**

$$S_{ij}^{kl}(\theta_1-\theta_2)S_{lk}^{nm}(\theta_2-\theta_1) = \delta_i^m\delta_j^n. \tag{2.2}$$

- **Crossing symmetry**

$$S_{ij}^{kl}(\theta) = S_{\bar{l}i}^{\bar{j}k}(i\pi-\theta) \tag{2.3}$$

- **Yang-Baxter equation**

$$S_{ij}^{pr}(\theta_1-\theta_2)S_{pk}^{lq}(\theta_1-\theta_3)S_{rq}^{mn}(\theta_2-\theta_3) = S_{jk}^{pr}(\theta_2-\theta_3)S_{ir}^{qn}(\theta_1-\theta_3)S_{qp}^{lm}(\theta_1-\theta_2). \tag{2.4}$$

- Can be solved in a number of important cases;

The CDD factors

- Functions satisfy axioms

$$\Phi(\theta)\Phi(-\theta) = 1, \qquad \Phi(i\pi-\theta) = \Phi(\theta). \tag{2.5}$$

  Castillejo-Dalitz-Dyson (CDD) factors (1956)

- A family of functions

$$\Phi(\theta) = \exp\left(i\sum_s \alpha_s \sinh(s\theta)\right), \qquad s \quad \text{is odd} \tag{2.6}$$

- Another family

$$\Phi(\theta) = \prod_p^N \frac{B_p - i\sinh\theta}{B_p + i\sinh\theta} \tag{2.7}$$

  where B_p are real negative, or conjugate pairs with real negative part.

Physics Question

- Multiply $S_{ij}^{kl}(\theta)$ by a CDD factor $\Phi(\theta)$
  1. Integrability preserved → Integrable deformation;
  2. IR physics not changed → Irrelevant deformation;
- What does it do at the Lagrangian level ? What is the property of the new theory ?
- TTbar and higher deformations !

3 Review of TTbar deformation

Definitions

- Lagrangian formalism

$$\partial_\lambda \mathcal{L}_\lambda = T\bar{T}, \qquad T\bar{T} = \det T_{ij} = T_{11}T_{22} - T_{12}T_{21} \tag{3.1}$$

- S-matrix formalism

$$S_{ij}^{kl}(\theta_1, \theta_2) \mapsto e^{-i\lambda(p_1 E_2 - p_2 E_1)} S_{ij}^{kl}(\theta_1, \theta_2) \tag{3.2}$$

- Hamiltonian formalism (more detail later)

$$\partial_\lambda H = \int T\bar{T}(x)\mathrm{d}x \tag{3.3}$$

Classical aspects

- Free massless boson

$$\mathcal{L}_0 = \partial\phi\bar{\partial}\phi, \qquad \partial = \frac{1}{2}(\partial_x - i\partial_y), \qquad \bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y). \tag{3.4}$$

  Deformed Lagrangian

$$\mathcal{L}_0 \mapsto \mathcal{L}_\lambda = \frac{1}{2\lambda}\left(\sqrt{4\lambda\partial\phi\bar{\partial}\phi + 1} - 1\right) = -\frac{1}{2\lambda} + \mathcal{L}_{\mathrm{NG}} \tag{3.5}$$

  where

$$\mathcal{L}_{\mathrm{NG}} = \frac{1}{2\lambda}\sqrt{\det(\partial_\alpha X \cdot \partial_\beta X)}, \qquad X^1 = x, \quad X^2 = y, \quad X^3 = \sqrt{\lambda}\phi/2. \tag{3.6}$$

  Some comments

  1. Relation to string theory (non-locality);
  2. Quantization (effective QCD string);

- Relation to 2D gravity

  1. $T\bar{T}$ deformation of a QFT $\leftrightarrow$ Couple the QFT to 2D gravity

2. Deformed action

$$S_0 \mapsto S_\lambda = S_{\text{grav}}[g_{\mu\nu}] + S_0[g_{\mu\nu}, \phi]. \tag{3.7}$$

where $S_0[g_{\mu\nu}, \phi]$ is action on metric $g_{\mu\nu}$. First order formalism

$$g_{\mu\nu} = \delta_{ab} e^a_\mu e^b_\nu \tag{3.8}$$

and

$$S_{\text{grav}}[g_{\mu\nu}] = \frac{1}{2\lambda} \int \mathrm{d}^2 x\, \varepsilon^{\mu\nu} \varepsilon_{ab} (\delta^a_\mu - e^a_\mu)(\delta^b_\nu - e^b_\nu). \tag{3.9}$$

We have

$$Z_\lambda[\phi] \sim \int \mathcal{D}e\, e^{\frac{1}{2\lambda} \int \mathrm{d}^2 x\, \varepsilon^{\mu\nu} \varepsilon_{ab} (\delta^a_\mu - e^a_\mu)(\delta^b_\nu - e^b_\nu)} Z_0[e^a_\mu; \phi] \tag{3.10}$$

Integrate out e^a_μ, at classical level, we obtain $S_\lambda[\phi]$.

Finite volume spectrum

- Factorization formula (Zamolodchikov 2004)

$$\langle n | T\bar{T} | n \rangle = \langle n | T_{xx} | n \rangle \langle n | T_{yy} | n \rangle - \langle n | T_{xy} | n \rangle \langle n | T_{yx} | n \rangle \tag{3.11}$$

  Proof based on

  1. Translational invariance;
  2. Conservation law $\partial_\mu T^{\mu\nu} = 0$.

- Expectation value of stress tensor

$$\mathcal{E}_n(R, \lambda) = -R \langle n | T_{yy} | n \rangle, \qquad \partial_R \mathcal{E}_n(R, \lambda) = -\langle n | T_{xx} | n \rangle \tag{3.12}$$

  and (for relativistic QFT)

$$P_n = -iR \langle n | T_{xy} | n \rangle = -iR \langle n | T_{yx} | n \rangle. \tag{3.13}$$

- Flow equation
  From definition of $T\bar{T}$ deformation

$$\partial_\lambda \mathcal{E}(R, \lambda) = -R \langle n | T\bar{T} | n \rangle. \tag{3.14}$$

  We obtain flow equation

$$\partial_\lambda \mathcal{E}_n(R, \lambda) = \mathcal{E}_n(R, \lambda) \partial_R \mathcal{E}_n(R, \lambda) + \frac{1}{R} P_n^2 \tag{3.15}$$

  For $P_n = 0$, inviscid Burgers' equation

$$\mathcal{E}_n(R, \lambda) = \mathcal{E}_n(R + \lambda \mathcal{E}_n, 0) \tag{3.16}$$

- Undeformed CFT spectrum

$$\mathcal{E}_n^{\mathrm{CFT}}(R,0) = E_n(R) = \frac{1}{R}\left(n+\bar{n}-\frac{c}{12}\right), \qquad P_n(R) = \frac{1}{R}(n-\bar{n}). \tag{3.17}$$

Deformed spectrum

$$\mathcal{E}_n(R,\lambda) = \frac{R}{2\lambda}\left(\sqrt{1+\frac{4\lambda E_n}{R}+\frac{4\lambda^2 P_n^2}{R^2}}-1\right). \tag{3.18}$$

Comments

1. Different signs of λ;
2. For other cases, solve PDE numerically.

Torus partition function

- CFT torus partition function

$$Z(\tau,\bar{\tau}) = \sum_n e^{2\pi i R\tau_1 RP_n - 2\pi\tau_2 RE_n}, \qquad \tau = \tau_1 + i\tau_2, \qquad \bar{\tau} = \tau_1 - i\tau_2. \tag{3.19}$$

The $T\bar{T}$ deformed partition function

$$Z_{T\bar{T}}(\tau,\bar{\tau}|\lambda) = \sum_n e^{2\pi i R\tau_1 RP_n - 2\pi\tau_2 R\mathcal{E}_n(\lambda)} \tag{3.20}$$

- Modular invariance

$$Z\left(\frac{a\tau+b}{c\tau+d},\frac{a\bar{\tau}+b}{c\bar{\tau}+d}\right) = Z(\tau,\bar{\tau}), \qquad a,b,c,d\in\mathbb{Z}, \quad ad-bc=1. \tag{3.21}$$

Under $T\bar{T}$ deformation, not conformal invariant. However,

$$Z_{T\bar{T}}\left(\frac{a\tau+b}{c\tau+d},\frac{a\bar{\tau}+b}{c\bar{\tau}+d}\middle|\,\lambda\right) = Z_{T\bar{T}}\left(\tau,\bar{\tau}|\,\lambda\right) \tag{3.22}$$

- Uniqueness.
  Consider trivially solvable deformation

$$H \mapsto \mathcal{H}(H,P,\tilde{\lambda}), \qquad P\mapsto P \tag{3.23}$$

$\tilde{\lambda}$ is dimensionless parameter. Trivially diagonalized

$$\mathcal{H}(H,P,\tilde{\lambda})|n\rangle = \mathcal{H}(E_n,P_n,\tilde{\lambda})|n\rangle, \qquad P|n\rangle = P_n|n\rangle. \tag{3.24}$$

Define

$$\mathcal{E}_n(\tilde{\lambda}) = \mathcal{H}(H, P, \tilde{\lambda}) \tag{3.25}$$

and

$$Z_{\text{def}}(\tau, \bar{\tau}|\tilde{\lambda}) = \sum_n e^{2\pi i R\tau_1 RP_n - 2\pi\tau_2 R\mathcal{E}_n(\tilde{\lambda})}. \tag{3.26}$$

Require modular invariance

$$Z_{\text{def}}\left(\frac{a\tau+b}{c\tau+d}, \frac{a\bar{\tau}+b}{c\bar{\tau}+d}\middle| \frac{\tilde{\lambda}}{|c\tau+d|^2}\right) = Z_{\text{def}}(\tau, \bar{\tau}|\tilde{\lambda}) \tag{3.27}$$

We can fix $\mathcal{H}(H, P, \tilde{\lambda})$ uniquely to be

$$\mathcal{E}_n(\tilde{\lambda}) = \frac{1}{\tilde{\lambda}\pi R}\left(\sqrt{1 + 2\pi\tilde{\lambda}RE_n + \tilde{\lambda}^2\pi^2R^2P_n^2} - 1\right) \tag{3.28}$$

where $\tilde{\lambda} = 2\lambda/(\pi R^2)$.

- Density of states

$$\rho(E) = \frac{2\pi R(2c/3\pi)^{1/4}}{[E(E\lambda + 4R)]^{3/4}} \times \exp\left[\sqrt{\frac{2\pi cRE}{3}\left(1 + \frac{E\lambda}{4R}\right)}\right] \tag{3.29}$$

1. In the IR limit $Et \ll R$,

$$\rho_{\text{IR}}(E) \approx \mathcal{N}_C\, E^{-3/4} \exp\left(\sqrt{\frac{2c\pi RE}{3}}\right), \qquad \text{Cardy behavior} \tag{3.30}$$

2. In the UV limit $Et \gg R$,

$$\rho_{\text{UV}}(E) \approx \mathcal{N}_H\, E^{-3/2} \exp\left(\sqrt{\frac{\pi c\lambda}{6}}\, E\right), \qquad \text{Hagedorn behavior.} \tag{3.31}$$

Comment

1. 2d local QFT exhibit Cardy behavior
2. Hagedorn behavior is typical for strings.
3. Hagedorn temperature $\beta_H = \sqrt{\pi c\lambda/6}$. Theoretical temperature upper bound.

4 Higher deformations

Higher conserved currents

- Integrable QFT;
- Conserved currents

$$\partial_{\bar{z}} T_{s+1}(z,\bar{z}) = \partial_z \Theta_{s-1}(z,\bar{z}), \qquad \partial_z \bar{T}_{s+1} = \partial_{\bar{z}} \bar{\Theta}_{s-1}(z,\bar{z}) \tag{4.1}$$

- Irrelevant operators

$$X_s(z,\bar{z}) = \lim_{(z,\bar{z})\to(z',\bar{z}')} \big(T_{s+1}(z,\bar{z})\bar{T}_{s+1}(z',\bar{z}') - \Theta_{s-1}(z,\bar{z})\bar{\Theta}(z',\bar{z}')\big) + \text{derivative terms} \tag{4.2}$$

where $s = 1, 3, \cdots$. Here $s = 1$ is the $T\bar{T}$ operator.

- Deformation

$$\frac{\partial}{\partial\lambda} S_\lambda = \int X_s(z,\bar{z})\, \mathrm{d}^2 z. \tag{4.3}$$

- Factorization/solvability

$$\langle n|X_s|n\rangle = \langle n|T_{s+1}|n\rangle\langle n|\bar{T}_{s+1}|n\rangle - \langle n|\Theta_{s-1}|n\rangle\langle n|\bar{\Theta}_{s-1}|n\rangle \tag{4.4}$$

- CDD factors

$$\frac{\partial \mathcal{L}_\lambda}{\partial \lambda} = X_s \quad \leftrightarrow \quad S_{ij}^{kl}(\theta) \mapsto S_{ij}^{kl}(\theta) e^{i\lambda \sinh(s\theta)} \tag{4.5}$$

where $\lambda = g_s\, m^{2s}$.

5 Hamiltonian formalism

Bilinear deformation

- Two conserved currents

$$\partial_\mu \mathcal{J}_1^\mu = 0, \qquad \partial_\mu \mathcal{J}_2^\mu = 0. \tag{5.1}$$

with

$$\mathcal{J}_1^\mu = (\hat{q}_1, J_1), \qquad \mathcal{J}_2^\mu = (\hat{q}_2, J_2). \tag{5.2}$$

Conserved charges

$$Q_1 = \int \hat{q}_1(x)\mathrm{d}x, \qquad Q_2 = \int \hat{q}_2(x)\mathrm{d}x. \tag{5.3}$$

- Bilinear deformation

$$\frac{\mathrm{d}H_\lambda}{\mathrm{d}\lambda} = \int \mathcal{O}_{JJ}(x)\mathrm{d}x, \qquad \mathcal{O}_{JJ} = -\varepsilon_{\mu\nu}\mathcal{J}_1^\mu \mathcal{J}_2^\nu. \tag{5.4}$$

- Taking two conserved currents

$$\mathcal{J}_1^\mu = \mathcal{J}_{\mathcal{H}}^\mu = (\mathcal{H}, J_{\mathcal{H}}), \qquad \mathcal{J}_2^\mu = \mathcal{J}_{\mathcal{P}}^\mu = (\mathcal{P}, J_{\mathcal{P}}) \tag{5.5}$$

we have $\mathcal{O}_{JJ} = T\bar{T}$.

Bilocal deformation

- Define

$$\frac{\mathrm{d}H_\lambda}{\mathrm{d}\lambda} = [X, H_\lambda] \tag{5.6}$$

- Algebra preserving deformation.

$$[Q_a, Q_b] = f_{abc}\, Q_c, \qquad \frac{\mathrm{d}}{\mathrm{d}\lambda} Q_a(\lambda) = [X, Q_a(\lambda)]. \tag{5.7}$$

Easy to prove that

$$[Q_a(\lambda), Q_b(\lambda)] = f_{abc}\, Q_c(\lambda). \tag{5.8}$$

- Integrable models : $[Q_a, Q_b] = 0$

$$[Q_a(\lambda), Q_b(\lambda)] = 0. \tag{5.9}$$

Equivalence of two deformations

- Take X to be bilocal operator

$$X_{JJ} = i\int_{x_1<x_2} \hat{q}_1(x_1)\hat{q}_2(x_2)\, \mathrm{d}x_1\mathrm{d}x_2. \tag{5.10}$$

Using $\partial_t\hat{q}_1 = i[H, \hat{q}_1]$ and $\partial_t\hat{q}_2 = i[H, \hat{q}_2]$, we find

$$[X_{JJ}, H] = \int_{s_{\mathrm{L}}}^{s_{\mathrm{R}}} \mathcal{O}_{JJ}(x)\mathrm{d}x - J_1(s_{\mathrm{L}})Q_2 + J_2(s_{\mathrm{R}})Q_1. \tag{5.11}$$

- Boundary conditions

1. Infinite line

$$[X_{JJ}, H] = \int_{-\infty}^{\infty} \mathcal{O}_{JJ}(x)\mathrm{d}x, \tag{5.12}$$

   Two deformations are equivalent.

2. Periodic boundary condition

$$[X_{JJ}, H] = \int_0^R \mathcal{O}_{JJ}(x)\mathrm{d}x - J_1(0)Q_2 + J_2(0)Q_1. \tag{5.13}$$

- Flow equation
  Take $\mathcal{J}_1^\mu = \mathcal{J}_\mathcal{H}^\mu$ and $\mathcal{J}_2^\mu = \mathcal{J}_\mathcal{P}^\mu$,

$$\langle n|H|n\rangle = E_n, \quad \langle n|P|n\rangle = P_n, \quad \langle n|J_\mathcal{H}|n\rangle = P_n/R, \quad \langle n|J_\mathcal{P}|n\rangle = -\partial_R E_n. \tag{5.14}$$

  and

$$\langle n|[X_{JJ}, H]|n\rangle = 0. \tag{5.15}$$

  We find that

$$\partial_\lambda E_n = E_n \partial_R E_n + \frac{P_n^2}{R} \tag{5.16}$$

- Comments

  1. First proposed for deforming spin chains;
  2. Realized it is related to TTbar in 2019;
  3. Applied to Bose-gas;

Deformed S-matrix

- An infinite system (QFT, non-relativistic QFT, spin chain)

- Asymptotic two-particle state

$$|u, u'\rangle = a(u, u')|u < u'\rangle + a(u', u)|u' < u\rangle + \text{local contributions} \tag{5.17}$$

  where the S-matrix is given by

$$S(u, u') = \frac{a(u', u)}{a(u, u')}. \tag{5.18}$$

- Deformed asymptotic state

$$|u,u'\rangle_\lambda \approx a_\lambda(u,u')|u<u'\rangle + a_\lambda(u',u)|u'<u\rangle + \cdots \tag{5.19}$$

From eigenvalue equation

$$H_\lambda|u,u'\rangle_\lambda = [h(u)+h(u')]\,|u,u'\rangle_\lambda \tag{5.20}$$

- Taking derivatives with respect to λ

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(H_\lambda|u,u'\rangle_\lambda\right) = [h(u)+h(u')]\,\frac{\mathrm{d}}{\mathrm{d}\lambda}|u,u'\rangle_\lambda \tag{5.21}$$

- We obtain

$$X_{JJ}|u,u'\rangle_\lambda = \frac{\mathrm{d}a_\lambda(u,u')}{\mathrm{d}\lambda}|u<u'\rangle + \frac{\mathrm{d}a_\lambda(u,u')}{\mathrm{d}\lambda}|u'<u\rangle \tag{5.22}$$

Using

$$\begin{aligned} X_{JJ}|u<u'\rangle &= [h_1(u)h_2(u') + f_12(u) + f_{12}(u')]|u<u'\rangle, \\ X_{JJ}|u'<u\rangle &= [h_1(u')h_2(u) + f_12(u') + f_{12}(u)]|u'<u\rangle, \end{aligned} \tag{5.23}$$

we obtain

$$S_\lambda(u,u') = e^{-i\lambda[h_1(u)h_2(u')-h_1(u')h_2(u)]}S(u,u'). \tag{5.24}$$

where

$$Q_1 = \sum_{j=1}^N h_1(u_j), \qquad Q_2 = \sum_{j=1}^N h_2(u_j). \tag{5.25}$$

Comments

1. The CDD factors;
2. Derivation is universal (QFT, spin chain)

6 Deformed cold atom system

Lieb-Liniger model

- The Lieb-Liniger model (first quantized form)

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2c \sum_{i<j}^{N} \delta(x_i - x_j). \tag{6.1}$$

Alternatively as a non-relativistic QFT with Lagrangian (second quantized form)

$$\mathcal{L} = \frac{i}{2}\left(\phi^\dagger \partial_t \phi - \phi \partial_t \phi^\dagger\right) - \partial_x \phi \partial_x \phi^\dagger - c\, \phi\phi\phi^\dagger\phi^\dagger. \tag{6.2}$$

The Hamiltonian and momentum are given by

$$\begin{aligned} H &= \int \mathrm{d}x \left[\partial_x \phi(x) \partial_x \phi^\dagger(x) + c\, \phi^\dagger(x)\phi^\dagger(x)\phi(x)\phi(x)\right], \\ P &= -\frac{i}{2}\int \left(\phi^\dagger(x)\partial_x\phi(x) - \partial_x\phi^\dagger(x)\phi(x)\right). \end{aligned} \tag{6.3}$$

- Spectrum (Solved by Bethe ansatz $|\mathbf{u}_N\rangle$)

$$e^{ip(u_j)L} \prod_{k\neq j}^{N} S(u_j, u_k) = 1, \qquad e(u) = u^2, \quad p(u) = u \tag{6.4}$$

and

$$S(u,v) = \frac{u - v - ic}{u - v + ic}. \tag{6.5}$$

We find

$$E_N = \sum_{j=1}^{N} e(u_j), \qquad Q_a|\mathbf{u}_N\rangle = \sum_{j=1}^{N} h_a(u_j)|\mathbf{u}_N\rangle \tag{6.6}$$

Bilinear deformation

- Bilinear/bilocal deformation with $\mathcal{J}_a^\mu = (\hat{q}_a, J_a)$, $a = 1, 2$.

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} H_\lambda = i \int \left(\hat{q}_1(x) J_2(x) - \hat{q}_2(x) J_1(x)\right) \mathrm{d}x \tag{6.7}$$

- Deformed S-matrix

$$S(u,v) \mapsto S_\lambda(u,v) = \frac{u - v - ic}{u - v + ic}\, e^{-i\lambda[h_1(u)h_2(v) - h_1(v)h_2(u)]} \tag{6.8}$$

- Conserved charges

$$\{Q_a\} = \{Q_0, Q_1, Q_2, \cdots\} \tag{6.9}$$

where

$$Q_0 = \hat{N}, \qquad Q_1 = \hat{P}, \qquad Q_2 = \hat{H}. \tag{6.10}$$

- $[Q_1 Q_2] \leftrightarrow T\bar{T}$, The simplest case $[Q_0 Q_1]$ (hard rod deformation)

$$h_0(u) = 1, \qquad h_1(u) = u, \qquad h_2(u) = u^2. \tag{6.11}$$

  phase shift

$$\theta(u,v) = -i \log S(u,v), \qquad \lim_{c\to 0} \theta(u,v) = -\pi \mathrm{sgn}(u-v) \tag{6.12}$$

  Deformed S-matrix

$$\theta_\lambda(u,v) = -\pi\,\mathrm{sgn}(u-v) + \lambda\,(u-v). \tag{6.13}$$

  The hard rod model

$$H_{\mathrm{HR}} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \sum_{i<j}^{N} v(x_i - x_j), \qquad v(x) = \begin{cases} \infty, & |x| < a; \\ 0, & |x| > a. \end{cases} \tag{6.14}$$

  This deforms point particles to hard rods of length $|\lambda|$.

- Deformed spectrum

$$\partial_\lambda E_N = N \partial_R E_N - P_N \langle \mathbf{u}_N | J_{\hat{N}} | \mathbf{u}_N \rangle \tag{6.15}$$

  For $P_N = 0$, we have $\partial_\lambda E_N = N \partial_R E_N$.
  Comment

  1. Different signs of λ,
  2. Discuss for free boson;

- $T\bar{T}$ deformation

$$\partial_\lambda E_N = E_N \partial_R E_N - P_N \langle \mathbf{u}_N | J_H | \mathbf{u}_N \rangle \tag{6.16}$$

- Deformed classical Lagrangian can be found
- Interpreted in terms of Newton-Cartan geometry

7 Deformed spin chain

Spin chain current

- The spin chain Hamiltonian

$$H = \sum_x \hat{h}(x) \tag{7.1}$$

The Hamiltonian density $\hat{h}(x)$ is well-defined. Current density $J_q(x)$ can be found by

$$\partial_t \hat{q}(x) = i[H, \hat{q}(x)] = -\partial_x J_q(x). \tag{7.2}$$

- No momentum density operator. $P = \log U$.
- No $T\bar{T}$ and hard rod deformation.

Hard rod deformation

- What is a hard rod deformation for spin chain ?
- Constrained XXZ spin chain

$$H_t = -\frac{1}{2}\sum_{i=1}^{L} \mathrm{P}_t \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z\right) \mathrm{P}_t \tag{7.3}$$

where

$$\mathrm{P}_t = \prod_i \left[\frac{1}{2}(1-\sigma_i^z) + \frac{1}{2}(1+\sigma_i^z)\prod_{l=1}^{t}(1-\sigma_{i+l}^z)\right] \tag{7.4}$$

is a projection.

- No two down spins have distance smaller than t.
- Folded XXZ spin chain

$$H_2 = -\frac{1}{4}\sum_{j=1}^{L}(1+\sigma_j^z \sigma_{j+3}^z)(\sigma_{j+1}^+ \sigma_{j+2}^- + \sigma_{j+1}^- \sigma_{j+1}^+) \tag{7.5}$$

The S-matrix is given by

$$S(p_1, p_2) = -e^{-i(p_1 - p_2)}. \tag{7.6}$$

A spin chain hard rod of length 2.

- Interesting also from thermalization.

8 Future directions

- CDD factors and local QFTs;
- Correlation functions (form factors);
- Deformed algebra (Virasoro, Yangian);