

Dynamical Generation of the VY Superpotential in 4D $\mathcal{N} = 1$ SYM

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Outline

Motivation for the question

Towards a semiclassical understanding of VY superpotential

Conclusion

Motivation: IR Dynamics of Yang–Mills

$$\beta(g) < 0 \quad \Longrightarrow \quad g_{\text{IR}} \gg 1$$

Strong coupling in the IR:

confinement, mass gap, vacuum structure

- ▶ Gauge field only in UV
- ▶ Asymptotically free

$\mathcal{N} = 1$ **Pure** $SU(N)$ **SYM**

Field content in UV:

$$A_\mu + \lambda_\alpha$$

with λ_α an adjoint Majorana gluino.

Dynamical Generation of the VY Superpotential

For $4d \mathcal{N} = 1$ pure $SU(N)$ SYM, the IR dynamics is encoded by the glueball superfield

$$S \sim \text{Tr } W^\alpha W_\alpha.$$

What is known

Symmetry, holomorphy, and anomaly matching strongly constrain the effective superpotential (Veneziano–Yankielowicz 1982) to be

$$W_{\text{VY}}(S) = NS \left(\log \frac{S}{\Lambda^3} - 1 \right).$$

What are still unknown

The formula itself is highly constrained, but the underlying dynamical mechanism is not manifest and is therefore often treated as a strong-coupling black box.

Goal: understand the dynamics reason behind the VY description.

Why Quantum Corrections to Semiclassical Objects Matter

Semiclassical configurations are not just classical saddles.

They carry quantum information:

instanton / monopole / ... + fluctuation determinant + zero modes



nonperturbative amplitude $\sim e^{-S_{\text{cl}}} \times (\text{quantum corrections})$

These amplitudes generate terms in the IR effective theory:

$$\Delta\mathcal{L}_{\text{IR}}, \quad W_{\text{eff}}, \quad V_{\text{eff}}.$$

Main point

Quantum corrections by semiclassical objects determine which IR effects are generated: mass gap, confinement, vacuum structure...

Why Non-Perturbative Physics Matters

- Semiclassical expansion around classical saddles:

$$Z \sim e^{-S_{\text{cl}}/\hbar} \sum_n a_n \hbar^n$$

misses non-perturbative sectors

$$\sim e^{-S_{\text{inst}}/\hbar}.$$

System	Non-perturbative effect	Physical consequence
Double well QM	Instanton tunneling	Energy splitting
3D compact QED	Monopole instantons	Confinement and mass gap
4D Yang–Mills	Instanton sectors	θ -vacuum, chiral anomaly
JT gravity	Wormholes/topology sum	Non-perturbative completion

Key message

Non-perturbative sectors can qualitatively change the semiclassical picture.

Instantons: What They Suggest and What They Miss

Both YM and $\mathcal{N} = 1$ SYM have UV instanton sectors:

$$Q = 1, \quad S_{\text{inst}} = \frac{8\pi^2}{g^2}.$$

Instantons naturally generate nonperturbative factors:

$$e^{-8\pi^2/g^2 + i\theta} \sim \Lambda^{b_0}.$$

Pure YM

Instantons suggest nonperturbative physics, but the instanton measure has large-size sensitivity:

$$\int d\rho \rho^{-5} (\rho\Lambda)^{\frac{11}{3}N}.$$

The integral is dominated by large instantons:

$$\rho \sim \Lambda^{-1},$$

where semiclassics breaks down. So we could not easily organize the contribution of instantons in pure YM.

Zero-Mode Criterion for Generating a Superpotential

Before talking about 4d N=1 SYM, let us recall some basic facts. A nonperturbative saddle contributes to the superpotential only if its collective coordinates match the superspace F-term measure:

$$\int d^4x d^2\theta W_{\text{eff}}.$$

Therefore the saddle should have exactly

4 bosonic zero modes

for its position in spacetime, and exactly

2 fermionic zero modes.

Criterion

$$n_{\text{bosonic}} = 4, \quad n_{\text{fermionic}} = 2$$

Only then can the instanton amplitude directly generate a local superpotential term.

Why the 4d SYM Instanton Does Not Directly Generate W

For a one-instanton in pure $SU(N)$ $\mathcal{N} = 1$ SYM:

$$n_{\text{fermionic}} = 2N.$$

But a superpotential contribution needs only

$$d^2\theta \iff 2 \text{ fermionic zero modes.}$$

The extra fermionic zero modes must be absorbed by operator insertions.

Thus the instanton generates a higher fermion correlator:

$$\langle (\lambda\lambda)^N \rangle \sim \Lambda^{3N},$$

rather than directly generating

$$W_{\text{VY}}(S).$$

Message

The 4d instanton knows about the correct nonperturbative scale and chiral selection rule, but it has too many zero modes to be a direct superpotential-generating saddle.

A Long-Standing Tension

Historically, many techniques were developed to make instanton physics semiclassically controllable:

- ▶ Higgsing the gauge group
- ▶ Adding matter multiplets
- ▶ Compactification on $\mathbb{R}^3 \times S^1$
- ▶ String/D-brane embeddings
- ▶ ...

These approaches successfully resolve issues such as

large instantons, fermion zero modes, IR divergences.

However

Most of these constructions modify the original IR dynamics: spectrum, vacua, symmetry realization.

Thus the relation to the undeformed strongly coupled $4d$ theory can become indirect.

Can one obtain controlled instanton dynamics without changing the original IR physics?

Toward a Direct Semiclassical Description

Goal:

keep the original IR fixed

while still obtaining

controlled semiclassical saddles

Recent proposal by myself:

UV quantum dynamics $\implies$ emergent IR structure.

Instead of modifying the IR theory, the proposal is to reorganize the nonperturbative dynamics so that

the IR strong-coupling system itself
admits a semiclassical UV description.

A Useful Analogy: GLSM $\rightarrow$ NLSM $\rightarrow$ $\widetilde{W}_{\text{eff}}$

My proposal was inspired from my research experience in 2d GLSM (Witten at 1979 and 1993)

A familiar story in 2d supersymmetric gauge theory:

$$\text{UV GLSM} \longrightarrow \text{IR NLSM} \longrightarrow \text{Far-IR } \widetilde{W}_{\text{eff}}.$$

Main lesson

The UV theory may contain extra massive fields, but after integrating them out, their quantum effects generate an IR effective superpotential.

Schematically,

$$\text{massive UV charged fields} \Rightarrow \text{quantum correction} \Rightarrow \widetilde{W}_{\text{eff}}.$$

The IR superpotential is not put in by hand; it is generated by UV dynamics.

Example: GLSM for $\mathbb{CP}^{N-1}$

Start with a $2d$ $\mathcal{N} = (2, 2)$ $U(1)$ gauge theory.

Matter content:

$$N \text{ chiral fields } \Phi_i, \quad Q_i = +1.$$

There is also a FI-theta parameter

$$t = r - i\theta.$$

Semi-classically, the D-term equation is

$$\sum_{i=1}^N |\phi_i|^2 = r,$$

modulo the $U(1)$ gauge symmetry.

Therefore, for $r > 0$,

$$\frac{\{\phi_i \neq 0, \sum_i |\phi_i|^2 = r\}}{U(1)} = \mathbb{CP}^{N-1}.$$

So the UV GLSM flows to the IR (intermediate scale) nonlinear sigma model on $\mathbb{CP}^{N-1}$.

Quantum Generation of the Twisted Superpotential

The vector multiplet contains a twisted chiral field

$$\Sigma.$$

When $\Sigma \neq 0$, the charged fields become massive:

$$m_i \sim \Sigma.$$

Integrating out the N massive charged chirals generates

$$\widetilde{W}_{\text{eff}}(\Sigma) = -t\Sigma + N\Sigma(\log \Sigma - 1).$$

The vacuum equation is

$$\frac{\partial \widetilde{W}_{\text{eff}}}{\partial \Sigma} = 0 \quad \Longrightarrow \quad \Sigma^N = e^{-t} \equiv \Lambda^N.$$

Hence

$$\langle \Sigma \rangle = \Lambda e^{2\pi i k/N}, \quad k = 0, \dots, N-1.$$

Role of Matter Fields: Activating Domain-Wall Sectors

In the GLSM description, the charged matter fields are not merely spectators.

Integrating out these massive fields generates quantum corrections to the effective twisted superpotential:

$$\widetilde{W}_{\text{eff}}(\Sigma).$$

Physical interpretation

The matter fields probe different branches of the effective theory and interpolate between distinct vacua (Witten at 79).

Definition

A domain wall is a codimension-one configuration connecting two vacua:

$$\lim_{x \rightarrow -\infty} \Sigma(x) = \Lambda e^{2\pi i k/N}, \quad \lim_{x \rightarrow +\infty} \Sigma(x) = \Lambda e^{2\pi i(k+\ell)/N}.$$

Their quantum effects effectively activate the domain-wall sectors, which are BPS objects.

Analogy for 4d $\mathcal{N} = 1$ SYM

In pure SYM, the IR theory has

N vacua

labeled by

$$\langle S \rangle = \Lambda^3 e^{2\pi i k / N}.$$

Therefore domain walls are intrinsic nonperturbative objects connecting different vacua.

My proposal relies that suitable UV fields to activate the domain wall sectors, analogous to how charged matter fields generate

$$\widetilde{W}_{\text{eff}}(\Sigma)$$

in the GLSM.

Main idea

IR effective superpotentials emerge from quantum corrections associated with objects interpolating between different vacua.

Domain Walls in 4d $\mathcal{N} = 1$ SYM

Pure $SU(N)$ $\mathcal{N} = 1$ SYM has

$$N$$

supersymmetric vacua:

$$\langle S \rangle = \Lambda^3 e^{2\pi i k / N}, \quad k = 0, \dots, N-1.$$

Therefore the theory admits BPS domain walls interpolating between different vacua.

Again A domain wall is a codimension-one configuration connecting two vacua:

$$\lim_{x \rightarrow -\infty} S(x) = \Lambda^3 e^{2\pi i k / N}, \quad \lim_{x \rightarrow +\infty} S(x) = \Lambda^3 e^{2\pi i (k+\ell) / N}.$$

References:

- ▶ Dvali–Shifman (1997)
- ▶ Acharya–Vafa (2001)

A New Proposal: Fields Exciting Domain Walls

Ordinary $4d$ instantons are not sufficient to directly generate the VY superpotential.

The proposal introduces auxiliary UV fields whose excitations couple to the domain-wall sectors of the theory.

Schematically,

$$\text{UV fields} \quad \Longleftrightarrow \quad \text{domain-wall excitations} \quad \Longrightarrow \quad W_{\text{VY}}.$$

This is analogous to the GLSM story:

$$\text{massive charged matter} \quad \Longrightarrow \quad \widetilde{W}_{\text{eff}}.$$

Main conceptual new idea in my proposal

The VY superpotential is interpreted not as a purely formal anomaly constraint, but as an emergent effective superpotential generated by quantum fluctuations associated with domain-wall sectors.

Field Content in the New Proposal

To organize the dynamics of domain walls, introduce a higher-form sector:

$$B_2 \quad \text{and} \quad A_3.$$

B_2 : two-form “matter” field, A_3 : three-form gauge field.

They enter through a Stückelberg/Higgs-type combination

$$A_3 - dB_2.$$

In superspace:

$B_2 \longrightarrow$ real linear multiplet L ,

$A_3 \longrightarrow$ three-form multiplet.

Physical meaning

The higher-form fields provide elementary variables that can excite domain-wall sectors in $4d$, analogous to charged matter fields in the $2d$ GLSM.

Why 4d Needs Higher-Form Fields

The dimensionality of the wall changes the required field content.

theory	domain wall	natural field
$2d$	worldline	ordinary matter
$4d$	three-dimensional wall	two-/three-form fields

In $2d$,

charged particle worldline

can represent a kink/domain wall.

In $4d$, the wall is an extended object, so ordinary particle fields are not enough.

domain wall couples naturally to A_3 :

$$\int_{\text{wall}} A_3.$$

So instead of considering only the SYM action, we introduce matter fields to excite the domain walls:

$$\mathcal{L}_{\text{SYM}} + \Delta\mathcal{L}(B_{\mu\nu}, A_3, \dots).$$

As we did in 2d, in principle, integrating out the matter fields may generate the superpotential we want. However, it involves a loop computation that is complicated in 4d. To make this process easier, let us recall the mirror symmetry in 2d.

Mirror Perspective on the GLSM Story

For the $2d \mathcal{N} = (2, 2)$ GLSM for $\mathbb{CP}^{N-1}$,

$$U(1) \quad + \quad N \text{ charged chirals.}$$

The quantum GLSM dynamically generates

$$\widetilde{W}_{\text{eff}}(\Sigma).$$

Mirror theory is a Landau–Ginzburg model with the superpotential

$$W_{\text{mirror}} = \Sigma \left(\sum_{i=1}^N Y_i - t \right) + \sum_{i=1}^N e^{-Y_i}.$$

Why the Mirror Description Is Powerful

The mirror superpotential is holomorphic and perturbative RG protected.

Key simplification

Quantum effects of GLSM become classical operations in the mirror theory.

$$\frac{\partial W}{\partial Y_i} = 0 \quad \Longrightarrow \quad e^{-Y_i} = \Sigma,$$

$$\frac{\partial W}{\partial \Sigma} = 0 \quad \Longrightarrow \quad e^{-\sum_i Y_i} = e^{-t},$$

which reproduces

$$\Sigma^N = e^{-t}.$$

Complicated quantum dynamics $\Longrightarrow$
classical algebraic equations.

Integrating Out Mirror Fields Gives $\widetilde{W}_{\text{eff}}$

$$W_{\text{mirror}} = \Sigma \left(\sum_{i=1}^N Y_i - t \right) + \sum_{i=1}^N e^{-Y_i}.$$

The Y_i equations of motion are

$$\frac{\partial W_{\text{mirror}}}{\partial Y_i} = 0 \quad \Rightarrow \quad e^{-Y_i} = \Sigma.$$

Substituting back gives

$$\widetilde{W}_{\text{eff}}(\Sigma) = -t\Sigma - N\Sigma(\log \Sigma - 1).$$

Then

$$\frac{\partial \widetilde{W}_{\text{eff}}}{\partial \Sigma} = 0 \quad \Rightarrow \quad \Sigma^N = e^{-t} \equiv \Lambda^N.$$

So the exact twisted effective superpotential follows from a classical elimination in the mirror.

Why This Mirror Description Is Useful

The nonperturbative terms

$$e^{-Y_i}$$

come from vortex-instanton effects in the original GLSM.

But in the mirror description they appear as ordinary classical superpotential terms.

Key advantage

The difficult quantum dynamics of the GLSM is converted into classical algebraic operations:

$$\frac{\partial W_{\text{mirror}}}{\partial Y_i} = 0.$$

Thus

quantum RG dynamics $\implies$ classical mirror superpotential geometry.

How to get the mirror: Abelian Duality in the GLSM

For each charged chiral field

$$\Phi_i = |\phi_i| e^{i\varphi_i},$$

the phase φ_i has a $U(1)$ shift symmetry.

Abelian duality replaces this phase by a periodic twisted chiral field

$$Y_i \sim Y_i + 2\pi i.$$

The real part of Y_i is related to the radial variable:

$$\text{Re } Y_i \sim |\phi_i|^2.$$

The GLSM gauge coupling becomes a linear coupling

$$\Sigma \left(\sum_i Y_i - t \right),$$

where Σ is the twisted chiral field strength.

2d GLSM Vortex Equations

For a $U(1)$ GLSM with chiral fields ϕ_i of charge Q_i :

$$S \supset \int d^2x \left[|D_\mu \phi_i|^2 + \frac{1}{2e^2} F_{12}^2 + \frac{e^2}{2} \left(\sum_i Q_i |\phi_i|^2 - r \right)^2 \right]$$

The (BPS-like) vortex equations are

$$D_{\bar{z}} \phi_i = 0$$

$$F_{12} + e^2 \left(\sum_i Q_i |\phi_i|^2 - r \right) = 0$$

with vortex number

$$k = \frac{1}{2\pi} \int F_{12} \in \mathbb{Z}$$

and instanton weight $e^{-S_k} \sim e^{-kt}$, $t = r - i\theta$.

Interpretation:

- ▶ $D_{\bar{z}} \phi_i = 0$: holomorphic matter fields
- ▶ Magnetic flux balanced by D-term / FI parameter
- ▶ Vortices generate nonperturbative twisted superpotential corrections

Vortex Instantons in Dual Variables

In the dual description, vortex sectors are encoded in, or reorganized into, the exponential terms:

$$\Delta \widetilde{W} = \sum_{i=1}^N e^{-Y_i}.$$

Thus the exact mirror twisted superpotential is

$$W_{\text{mirror}} = \Sigma \left(\sum_{i=1}^N Y_i - t \right) + \sum_{i=1}^N e^{-Y_i}.$$

Hori–Vafa lesson

Vortex-generated quantum effects in the abelian GLSM become ordinary classical superpotential terms in the mirror LG model. (For the mirrors of nonabelian GLSMs, it was solved by Gu and Sharpe at 18).

From the 2d Mirror Story to 4d SYM

In the 2d GLSM:

charged matter $\Phi_i \longrightarrow$ dual twisted chiral Y_i .

Vortex sectors are encoded in, or reorganized into, the exponential terms:

$$e^{-Y_i},$$

and classically integrating out Y_i gives

$$\widetilde{W}_{\text{eff}}(\Sigma).$$

4d analogue

Embed the higher-form matter, B_2 , into the linear super-field and take the linear-chiral dual:

$$B_2 \longrightarrow Z.$$

where Z is a 4d chiral super-field.

Domain-wall instantons generate

$$e^{-Z}.$$

Classically integrating out Z gives the VY superpotential:

$$W_{\text{VY}}(S).$$

2d GLSM Vortex vs. 4d $\mathcal{N} = 1$ VY

2d GLSM vortex	4d higher-form analogue
A_1	A_3
$F_{12} + e^2 \left(\sum_i Q_i \phi_i ^2 - r \right) = 0$	$F_{1234} = e^2 (N \rho^2 - \xi)$
$D_{\bar{z}} \phi_i = 0$	$dC = *_E \mathcal{H}_3$
$k = \frac{1}{2\pi} \int_{\mathbb{R}^2} F_2$	$n_4 = -\frac{1}{2\pi} \int_{M_4} F_4$
e^{-Y}	$W_{\text{np}} \sim \Lambda^3 e^{-Z}$
$\widetilde{W}_{\text{eff}}(\Sigma)$	$W_{\text{VY}}(S) = -NS \left(\log \frac{S}{\Lambda^3} - 1 \right)$

4d VY as a higher-form analogue of 2d vortex dynamics

4d Mechanism: From Domain-Wall Variables to VY

Introduce a higher-form/domain-wall sector and dualize it to a chiral field

$$Z.$$

The holomorphic effective superpotential is

$$W_{\text{eff}}(S, Z) = \frac{1}{2} \left[\left(NZ - \tilde{\tau} + 3N \log \frac{\Lambda}{\mu} \right) S + N\mu^3 e^{-(Z-\tilde{\tau}/N)} \right].$$

Here

$$S = -\frac{1}{32\pi^2} \text{Tr } W^\alpha W_\alpha$$

is the glueball superfield.

Interpretation

The exponential term is the higher-form analogue of the 2d vortex-instanton term:

$$e^{-Y} \longrightarrow e^{-(Z-\tilde{\tau}/N)}.$$

Integrating Out Z Gives the VY Superpotential

The Z -equation gives

$$NS = N\mu^3 e^{-(Z-\tilde{\tau}/N)}.$$

Therefore

$$Z - \frac{\tilde{\tau}}{N} = -\log \frac{S}{\mu^3}.$$

Substituting back into $W_{\text{eff}}(S, Z)$ gives

$$W_{\text{eff}}(S) = -\frac{1}{2}NS \left(\log \frac{S}{\Lambda^3} - 1 \right).$$

With the convention

$$S_{\text{F-term}} = -\frac{1}{2} \left[\int d^2\theta \, W(S) + \text{h.c.} \right],$$

the holomorphic superpotential is

$$W_{\text{VY}}(S) = NS \left(\log \frac{S}{\Lambda^3} - 1 \right).$$

Conclusion

- ▶ Pure YM and $\mathcal{N} = 1$ SYM contain YM instanton sectors, but ordinary $4d$ YM instantons alone do not directly explain the IR strong-coupling dynamics.
- ▶ The $2d$ GLSM/mirror story provides a prototype:

UV quantum dynamics $\implies$ IR effective superpotential.

- ▶ In the mirror description, vortex instanton effects become classical exponential terms:

$$e^{-Y_i}.$$

Integrating out dual variables then gives the exact

$$\widetilde{W}_{\text{eff}}.$$

- ▶ My proposal imports this logic into $4d$ SYM using higher-form/domain-wall variables.
- ▶ The higher form matter fields and gauge fields have the instantons which have the correct zero modes and generate holomorphic exponential terms, and integrating out the dual variables reproduces the Veneziano–Yankielowicz superpotential: $W_{\text{VY}}(S)$.

Thank You!