

Isomorphism of Schur Sectors of Argyres-Douglas Theories

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7th Fields and Strings

Work in progress

Dualities

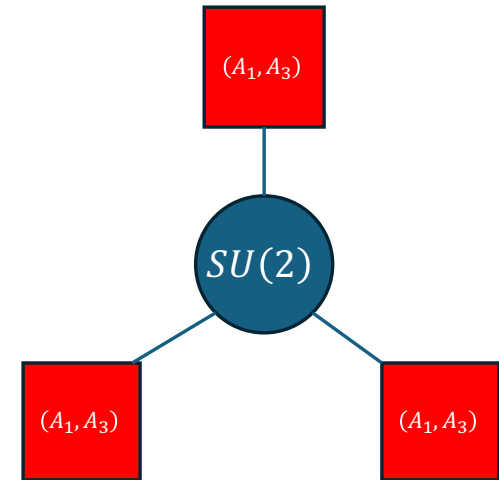
- **Duality** is a powerful tool in physics
 - Two seemingly different theory is actually the same: Hilbert space, correlators, dynamics, and etc.
- Famous examples:
 - **Strong/weak duality**: EM duality, Seiberg duality, etc
 - **Gauge/gravity duality**: AdS/CFT in various dimensions
- Today: **less** than duality

Isomorphism

- Two theories are not **dual**, but share the same subsector of Hilbert space.
- Subspaces of Hilbert spaces of different theories are **isomorphic** as **graded vector spaces / graded modules of flavor symmetry**.
- Example: [Buican-Nishinaka 2020]
 - Schur sectors of $SU(2)$ $N = 4$ SYM and $(3,2)$ theories are **isomorphic** as **graded vector space**.

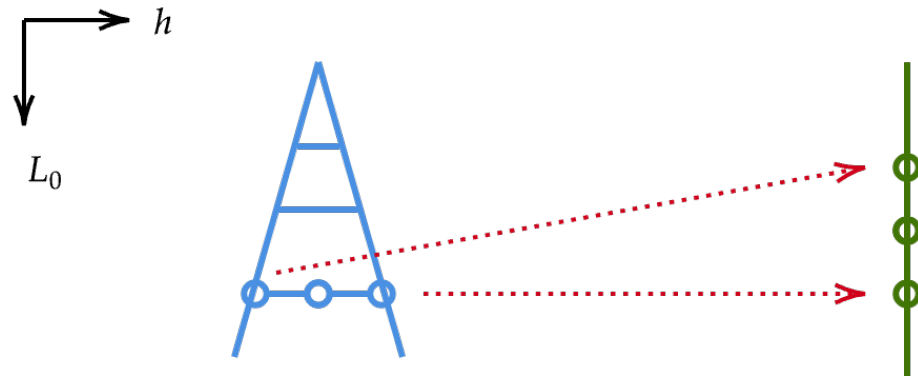
Example of isomorphism

- Theory 1: $SU(2) \ N = 4 \ SYM$, $a = c = 3/4$, treating as $N = 2$, $SU(2)_F$ symmetry
- Theory 2: $(3,2)$ theory, 3 copies of (A_1, A_3) Argyres-Douglas (AD) theories with $SU(2)$ gauged, $a = c = 2$
 - $\beta = 0$
- Schur sector
 - BPS operators:
 - $[Q_-^1, O(0)] = [\tilde{Q}_{2-}, O(0)] = [S_1^-, O(0)] = [\tilde{S}^{2+}, O(0)] = 0$
 - Index: $I_{(3,2)}(q) = I_{SYM}(q^{\frac{1}{2}}; q^3)$
 - **Isomorphism**: $Schur_{SYM} \xrightarrow{\sim} Schur_{(3,2)}$



More on isomorphism

- Cartan of superconformal algebra: E, j_1, j_2, R, r
 - SYM: $I_{SYM}(x; q) = tr_S(-1)^F x^h q^{E-R}$, h : Cartan of flavor $su(2)_F$
 - $(3,2) : I_{(3,2)}(q) = tr_S(-1)^F q^{E-R}$
- $I_{(3,2)}(q) = I_{SYM}\left(q^{\frac{1}{2}}; q^3\right)$: **different** $U(1)$ grading
 - SYM: $3(E - R) + h$
 - $(3,2) : E - R$

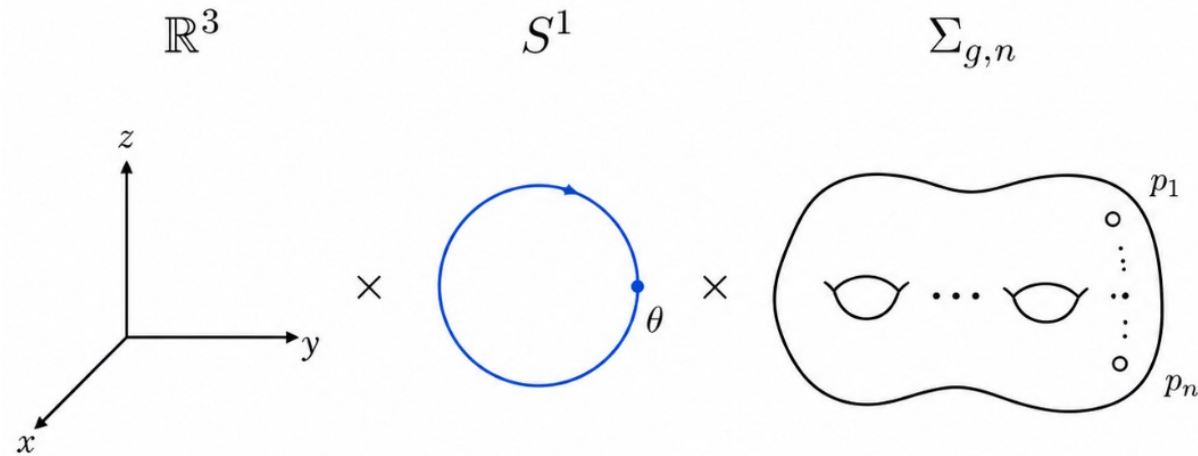


Implication on VOA

- 4d SCFT / VOA correspondence [BLLPRvR]
 - $\text{SYM} \Rightarrow$ small $N = 4$ superconformal algebra (SCA)
 - $(3,2)$ theory $\Rightarrow A(6)$ algebra
- Isomorphism between Schur sector $\Rightarrow$ **Isomorphism** between two VOAs
- **Question**: is this phenomenon general?
 - [Adamovic-Moller 25]: **generalization** to affine VOA (affine Kac-Moody) $L_{-2+p/q}(\mathfrak{sl}_2)$ and rational Virasoro $(p, 3q)$ VOA
 - When $p = 2$, **both sides** have 4d uplift as AD theories (A_1, D_{2n+1}) , (A_1, A_{6n})
- More **isomorphisms** in general AD theories

Class-S theories

- Class-S construction [Gaiotto, GMN]:
 - 4d $N = 2$ SCFTs from 6d (2,0) SCFTs compactified on a **Riemann surface with punctures** with **topological twist**
- 6d (2,0) SCFTs on $\mathbb{R}^3 \times S^1 \times \Sigma$

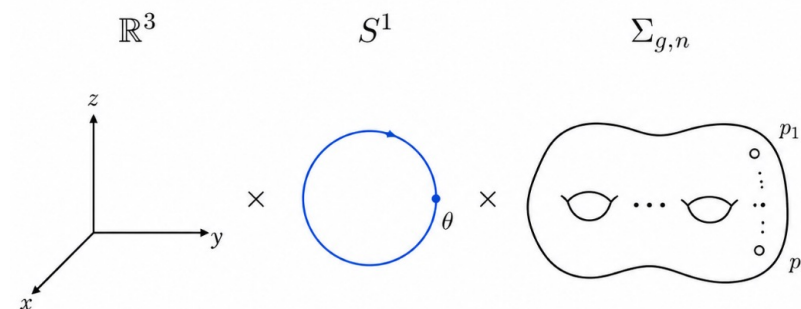


6d (2,0) SCFT of type ADE on $\mathbb{R}^3 \times S^1 \times \Sigma$

1. **Compactify** on S^1 (radius R): 5d $N = 2$ SYM on $\mathbb{R}^3 \times \Sigma$
2. Compactify on Σ with suitable **topological twist**
3. To get non-simplylaced group, extra outer automorphism twist

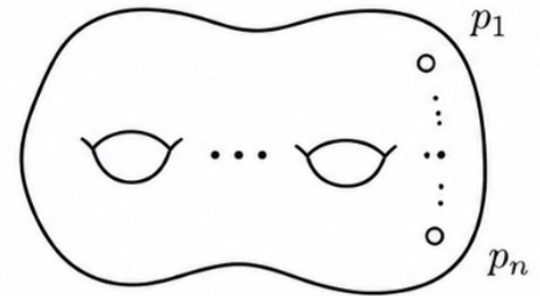
- Adjoint scalars: $\phi = Y^1 + iY^2, Y^3, Y^4, Y^5$
 - After twisted compactification ϕ : (1,0)-form on Σ
 - Gauge fields along Σ direction: $A = A_z dz + A_{\bar{z}} d\bar{z}$
 - BPS equations => **Hitchin equation** for ϕ

$$\begin{aligned} F(A) + R^2 [\phi, \bar{\phi}] &= 0, \\ \partial_{\bar{z}} \phi + [A_{\bar{z}}, \phi] &= 0, \\ \partial_z \bar{\phi} + [A_z, \bar{\phi}] &= 0. \end{aligned}$$



Boundary condition

- Different boundary condition at punctures
 - **Regular** puncture:
 - $\phi(z) \sim \frac{f}{z} + \dots$, f : nilpotent element in $\mathfrak{g}$
 - **Irregular** puncture:
 - $\phi(z) \sim \frac{s}{z^n} + \dots$, s : semisimple element in $\mathfrak{g}$, $n > 1$
- Determines physical properties
 - Flavor symmetry
 - Coulomb branch operator spectrum
 - Dimension of Higgs branch



Generalized Argyres-Douglas theories

- Generalized AD theories: **fractional** dim Coulomb branch operator, non-Lagrangian
- Class-S construction: $\Sigma = \mathbb{P}^1 \simeq S^2$ [GMN, Xie]
 - One irregular puncture at 0: $\phi(z) \sim \frac{s}{z^{1+\frac{u}{b}}} + \dots, \quad u > 0$
 - One regular puncture at ∞ : $\phi(w) \sim \frac{f}{w} + \dots$
- **Label**: $(\mathfrak{g}, u, b, f)$
 - Possible value of b depends on $\mathfrak{g}$
 - $u \geq 0, \gcd(u, b) = 1$
 - f is a nilpotent element of $\mathfrak{g}$, nilpotent Higgsing from $f = 0$
- For our purpose: fix $b = h^\vee$ **dual Coxeter number**
 - e.g. $\mathfrak{g} = \mathfrak{sl}_n, h^\vee = n$

Associated VOA

- 4d SCFT / VOA correspondence [BLLPRvR]
 - 4d $N = 2$ SCFT $T \Rightarrow$ VOA V_T
 - $c_{2d} = -12c_{4d}$
 - $\chi_0(z; q) = I_{Schur}(z; q)$
- AD theories $T = (\mathfrak{g}, u, h^\vee, f)$ [CS, SXY, XY, etc]
 - $f = 0$, $V_T = L_{-h^\vee + \frac{h^\vee}{u}}(\mathfrak{g})$, affine VOA (affine KM) at boundary admissible level $-h^\vee + \frac{h^\vee}{u}$
 - General f , $V_T = W_{-h^\vee + \frac{h^\vee}{u}}(\mathfrak{g}, f)$, affine W-algebra at boundary admissible level $-h^\vee + \frac{h^\vee}{u}$

Schur operators [GRRY]

- Schur operators

- **BPS condition:** $[Q_-^1, O(0)] = [\tilde{Q}_{2\dot{-}}, O(0)] = [S_1^-, O(0)] = [\tilde{S}^{2\dot{-}}, O(0)] = 0$

- Quantum number: (E, j_1, j_2, R, r) eigenvalues of Cartans of $SU(2,2|2)$

- **Constraint:**

$$\frac{1}{2}(E - j_1 - j_2) - R = 0, \quad r + (j_1 - j_2) = 0$$

- **Schur index**

- Generalized **Witten index:** RG invariant, counts operators and relations

- $I_S = \text{tr}_S(-1)^F x^h q^{E-R}$

Schur index of generalized AD theories

- AD theories $T = (\mathfrak{g}, u, h^\vee, f)$
 - (e, f, x) : $su(2)$ triple in $\mathfrak{g}$ generated by f
 - $\mathfrak{g}_F$: **centralizer** of f in $\mathfrak{g}$, Lie algebra of **flavor symmetry**
 - $su(2) \times \mathfrak{g}_F \in \mathfrak{g}$.
 - Decompose $\text{adj}_{\mathfrak{g}} = \sum_j V_j \otimes R_j$, V_j : spin j rep of $su(2)$, R_j : rep of $\mathfrak{g}_F$
- Schur index of AD theories $T = (\mathfrak{g}, u, h^\vee, f)$ [SXY, KW, XY]
 - $I_S = \text{PE} \left[\sum_j \frac{q^{1+j}}{1-q} \chi_{R_j}(\mathbf{y}) - \frac{q^u}{1-q^u} \sum_j \chi_{V_j}(q) \chi_{R_j}(\mathbf{y}) \right]$, y : fugacities of $\mathfrak{g}_F$
 - $\text{PE}[f(a, b, \dots)] = \exp \sum_{n=1} \frac{1}{n} f(a^n, b^n, \dots)$

Examples:

- Full puncture: $f = 0$

- $I_S(z; q) = \text{PE} \left[\frac{q}{1-q} \chi_{adj}(\mathbf{z}) - \frac{q^u}{1-q^u} \chi_{adj}(\mathbf{z}) \right] = \text{PE} \left[\frac{q-q^u}{(1-q)(1-q^u)} \chi_{adj}(\mathbf{z}) \right]$

- Fully closed puncture: $f \in O_{principal}$

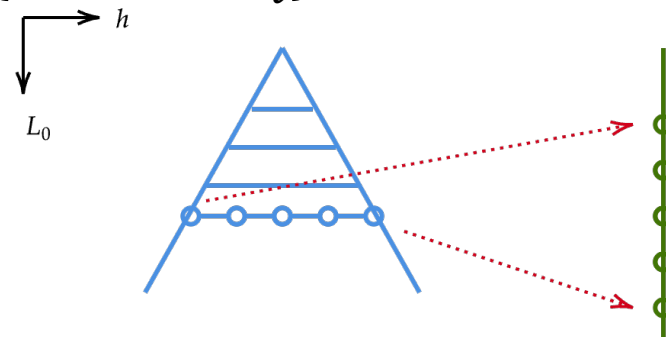
- $I_S(q) = \text{PE} \left[\frac{\sum_i q^{d_i} - \sum_i q^{u+1-d_i}}{(1-q)(1-q^u)} \right], \quad \{d_i\}: \text{degrees of Casimir of } \mathfrak{g}$

Graded isomorphism

- Hint from [BN]: **specialize** flavor fugacity to $U(1)$ grading
- AD theory: $T_1 = (\mathfrak{g}, u, h^\vee, 0)$
 - $\chi_{adj}(q^{-\rho}) = q^{-\frac{h}{2}-1} \chi_{\frac{h}{2}}(q) (\sum_i q^{d_i}), \quad h: \text{Coxeter number}$
 - $\chi_{\frac{h}{2}}(q) = q^{-\frac{h}{2}} (1 + q + \cdots + q^h)$
 - $I_1(q^{-\rho}; q^{h+1}) = \text{PE} \left[\frac{q^{h+1} - q^{(h+1)u}}{(1-q^{h+1})(1-q^{(h+1)u})} q^{-h-1} (1 + q + \cdots + q^h) (\sum_i q^{d_i}) \right]$
- AD theory: $T_2 = (\mathfrak{g}, (h+1)u, h^\vee, f_{prin})$
 - $I_2(q) = \text{PE} \left[\frac{\sum_i q^{d_i} - \sum_i q^{(h+1)u+1-d_i}}{(1-q)(1-q^{(h+1)u})} \right]$

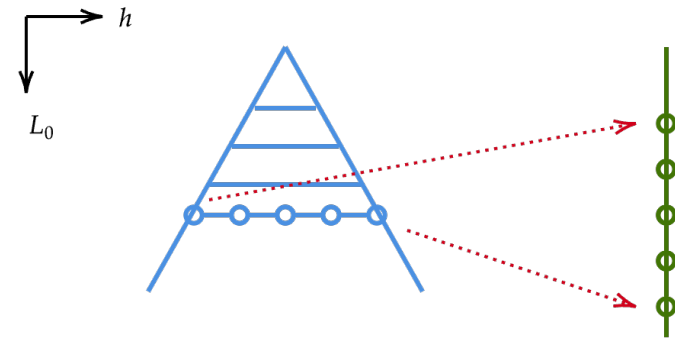
Isomorphism

- AD theory: $T_1 = (\mathfrak{g}, u, h^\vee, 0)$
 - $I_1(q^{-\rho}; q^{h+1}) = \text{PE} \left[\frac{q^{h+1} - q^{(h+1)u}}{(1-q^{h+1})(1-q^{(h+1)u})} q^{-h-1} (1 + q + \dots + q^h) (\sum_i q^{d_i}) \right]$
- AD theory: $T_2 = (\mathfrak{g}, (h+1)u, h^\vee, f_{prin})$
 - $I_2(q) = \text{PE} \left[\frac{\sum_i q^{d_i} - \sum_i q^{(h+1)u+1-d_i}}{(1-q)(1-q^{(h+1)u})} \right]$
- $I_1(q^{-\rho}; q^{h+1}) = I_2(q)$, use the fact: $\{d_i\} = \{h+2-d_i\}$
 - Isomorphism as graded vector space
 - $x + (h+1)(E-R)$ vs $E-R$



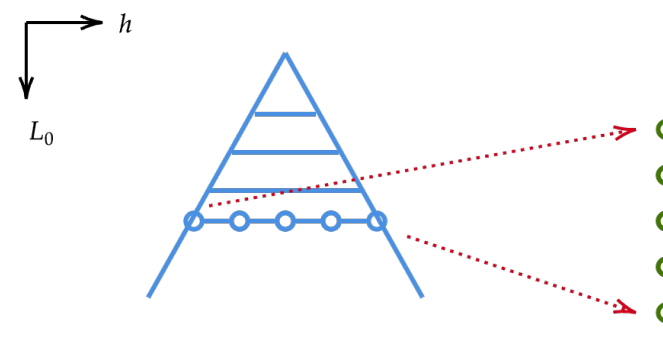
Implication on VOA

- 4d / VOA: Isomorphism of Schur sector $\Rightarrow$ Isomorphism of VOAs as graded vector space
 - $\chi_L(q^{-\rho}; q^{(h+1)}) = \chi_W(q)$
 - $L_{-h^\vee + \frac{h^\vee}{u}}(\mathfrak{g}) \xrightarrow{\sim} W_{-h^\vee + \frac{h^\vee}{(h+1)u}}(\mathfrak{g}, f_{prin}), \quad u > h^\vee$
- To prove this one need construct the map using strong generators and singular vectors



Isomorphism for other f

- AD theory: $T_1 = (\mathfrak{g}, u, h^\vee, 0)$
 - $I_S(z; q^m) = \text{PE} \left[\frac{q^m}{1-q^m} \chi_{adj}(z) - \frac{q^{mu}}{1-q^{mu}} \chi_{adj}(z) \right]$
- AD theories $T_2 = (\mathfrak{g}, mu, h^\vee, f)$
 - $I_S = \text{PE} \left[\sum_j \frac{q^{1+j}}{1-q} \chi_{R_j}(y) - \frac{q^{mu}}{1-q^{mu}} \sum_j \chi_{V_j}(q) \chi_{R_j}(y) \right]$
- Necessary condition for isomorphism
 - $\chi_{adj}(q^{-x}y) = V_{\frac{m-1}{2}}(q)(\dots)$
 - $m \mid \dim adj_{\mathfrak{g}}, \gcd(m, h^\vee) = 1$



Type A

- $SU(N)$: $\dim = (N - 1)(N + 1)$
 - $h + 1 = N + 1$, already discussed
 - $m = N - 1, f \in O_{[N-1,1]}$ subregular nilpotent orbit, $\mathfrak{g}_F = U(1)$
 - $I_1(c q^{-x}, q^{N-1}) = I_2(c, q)$
 - $L_{-N+\frac{N}{u}}(\mathfrak{su}(N))$ is isomorphic to $W_{-N+\frac{N}{(N-1)u}}(\mathfrak{su}(N), [N - 1, 1])$ as graded $U(1)$ modules
 - $m = 3, \gcd(m, N) = 1, f \in O_{[3^l, 2]}$, $\mathfrak{g}_F = U(l)$
 - $L_{-N+\frac{N}{u}}(\mathfrak{su}(N))$ is isomorphic to $W_{-N+\frac{N}{3u}}(\mathfrak{su}(N), f)$ as graded $U(l)$ modules
 - $N = ml + 1, f \in O_{[m^l, 1]}$, $\mathfrak{g}_F = U(l)$
 - $L_{-N+\frac{N}{u}}(\mathfrak{su}(N))$ is isomorphic to $W_{-N+\frac{N}{mu}}(\mathfrak{su}(N), f)$ as graded $U(l)$ modules
- etc

Other types

- Type B, $SO(2N + 1)$, $\dim = (2N + 1)N$
 - $m = 2N + 1, f \in \text{principal orbit discussed before}$
 - $N \text{ odd}, m = N, f \in O_{[N^2, 1]}$
- Type C, $Sp(2N)$, $\dim = (2N + 1)N$
 - $m = 2N + 1, f \in \text{principal orbit discussed before}$
 - $N \text{ odd}, m = N, f \in O_{[N^2]}$
- Type D, $SO(2N)$, $\dim = (2N - 1)N$
 - $m = 2N - 1, f \in \text{principal orbit discussed before}$
 - $N \text{ odd}, m = N, f \in O_{[N^2]}$
- Conjecture: $L_{-h^\vee + \frac{h^\vee}{u}}(\mathfrak{g})$ is isomorphic to $W_{-h^\vee + \frac{h^\vee}{mu}}(\mathfrak{g}, f)$ as graded $\mathfrak{g}_F$ -modules

Extend isomorphisms to modules

- $L_{-h^\vee + \frac{h^\vee}{u}}(\mathfrak{g})$ is isomorphic to $W_{-h^\vee + \frac{h^\vee}{mu}}(\mathfrak{g}, f)$ as graded $\mathfrak{g}_F$ -modules for combinations $\mathfrak{g}, m, f$
- **Injection:**
 - π : **admissible modules** of $L_{-h^\vee + \frac{h^\vee}{u}}(\mathfrak{g}) \rightarrow$ **simple modules** $W_{-h^\vee + \frac{h^\vee}{mu}}(\mathfrak{g}, f)$
 - Usually **not** bijection, more simple modules for W-algebra
 - $\chi_M(y q^{-x}; q^m) = \chi_{\pi(M)}(y; q^m)$
 - Conjecture: $M \simeq \pi(M)$ as graded $\mathfrak{g}_F$ -modules
- Implication on 4d: **equivalence** of correlators of line operators

Summary

- For suitable combinations $\mathfrak{g}, m, f$, we have
 - **Isomorphisms** between Schur sector of AD theories $(\mathfrak{g}, u, h^\vee, 0)$ and AD theories $(\mathfrak{g}, mu, h^\vee, f)$ as graded $\mathfrak{g}_F$ -modules
 - Although these two theories are **not dual** to each other
 - 4d/VOA: $L_{-h^\vee + \frac{h^\vee}{u}}(\mathfrak{g})$ is **isomorphic** to $W_{-h^\vee + \frac{h^\vee}{mu}}(\mathfrak{g}, f)$ as graded $\mathfrak{g}_F$ -modules for combinations $\mathfrak{g}, m, f$
 - These **isomorphism** also extend to simple modules of VOA

Outlook

- Physics
 - **Physical mechanism** behind these isomorphisms? More fundamental way to obtain these isomorphisms?
 - Understand relations between line operators?
 - **Gravity dual**? Different gravity background, same modes?
- VOA
 - **Generalization** to admissible level? No 4d counterpart
 - **Implications** on modular properties of characters? Fusion rings?

Thank you very much!