

An Effective Bethe Ansatz Approach to Non-integrable Models

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arXiv: 2604.04627, 2604.04640

Quantum Many-body System

- Suffer from the curse of dimension

Numerical computations require both memory and time complexity to grow exponentially with the system size $\mathcal{O}(e^L)$.

Analytic approaches are usually difficult. More is different!

Exceptions are free or exactly solvable/integrable systems.

Quantum computer may help in the future!



- Quantum state preparation
- Direct time simulation
- Quantum/Hybrid Machine Learning

Variational Quantum Algorithm
Quantum circuit learning

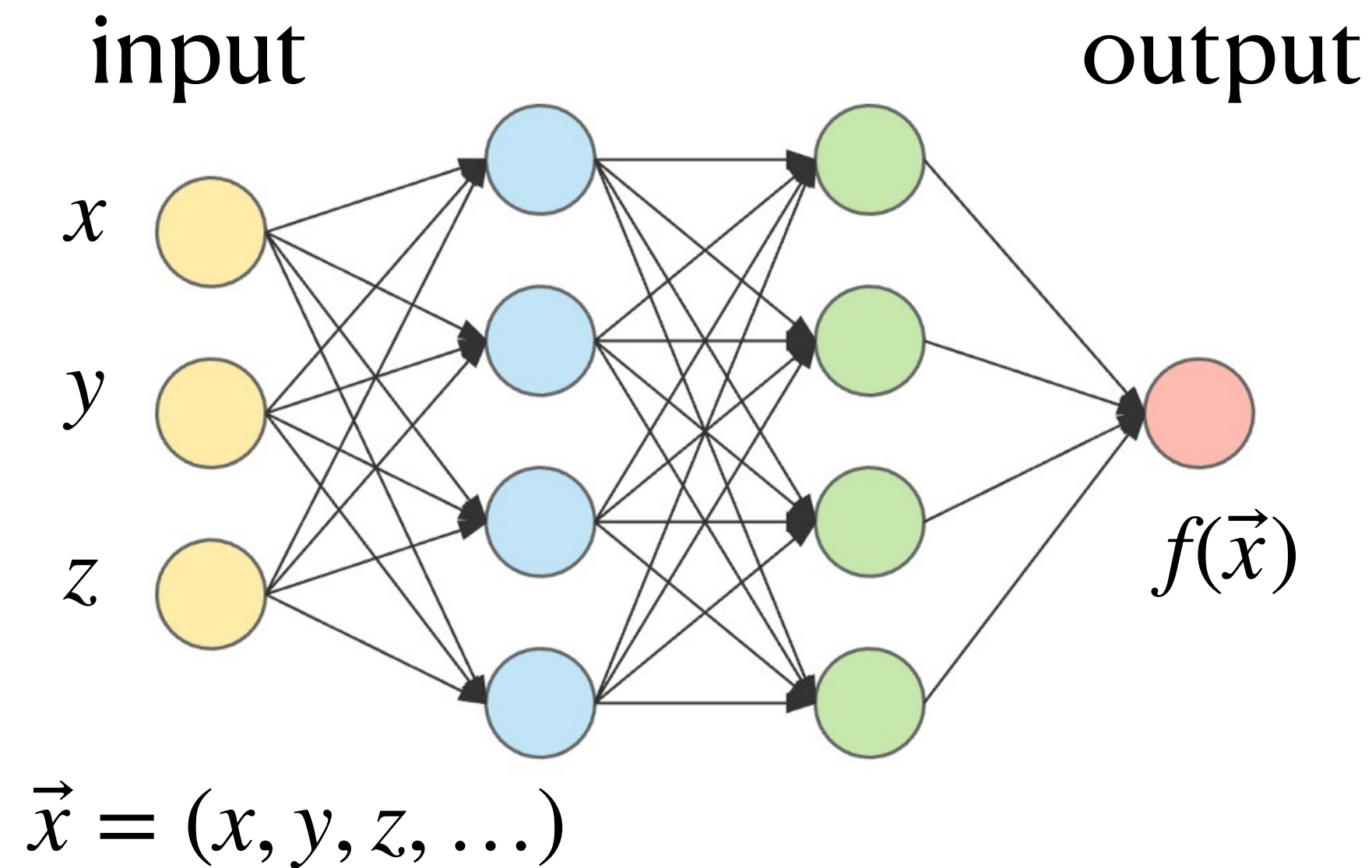


We built a new center!

A brute force approach

Solving a differential equation

$$\mathcal{D}(f(\vec{x})) = 0, \quad \text{bc}(f(\vec{x})) \Big|_{\vec{x}_i} = 0$$

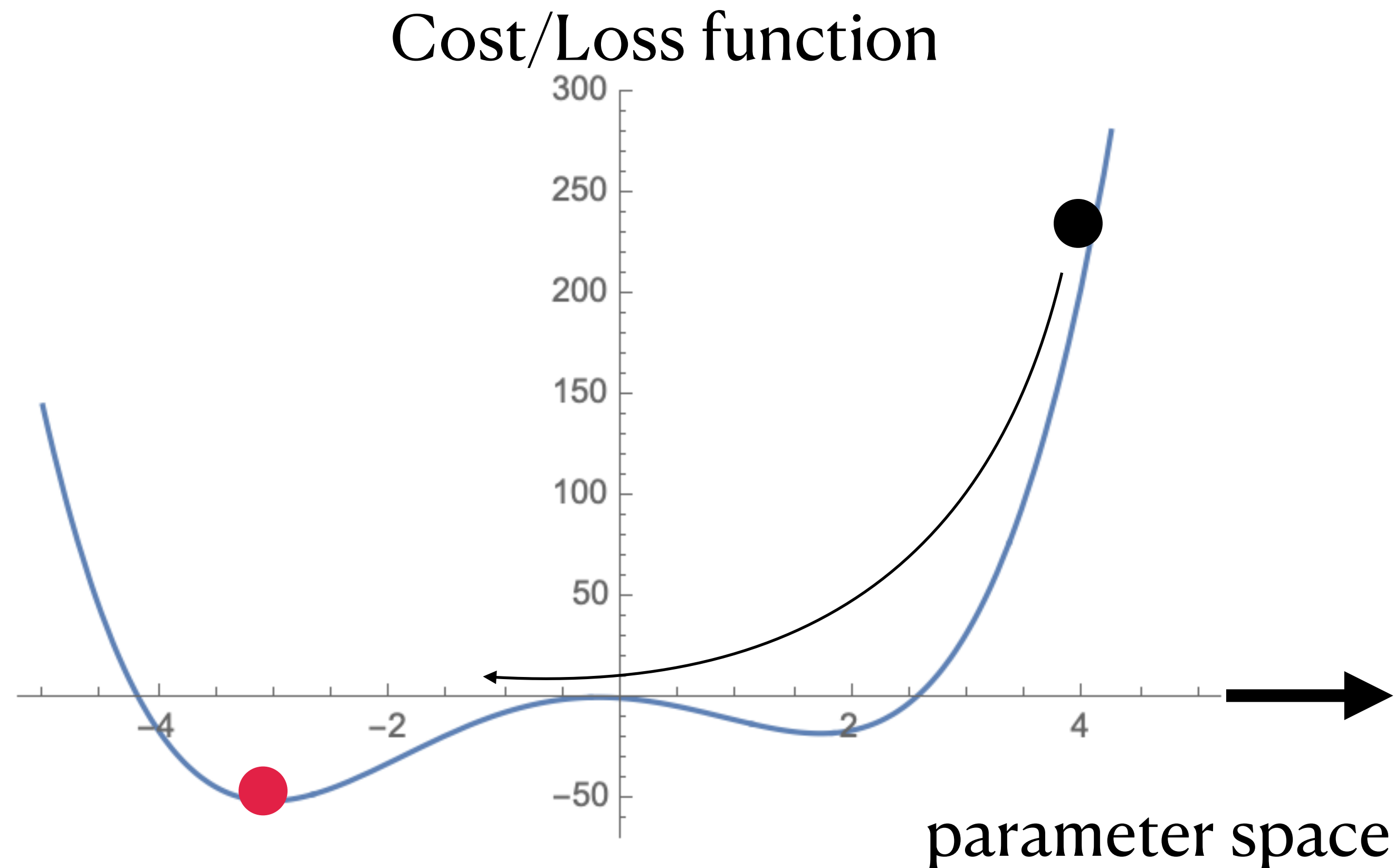


Construct a loss function

$$L = \alpha \sum_j \left| \mathcal{D}(f(\vec{x}^{(j)})) \right|^2 + \beta \sum_i \left(\text{bc}(f(\vec{x})) \Big|_{\vec{x}_i} \right)^2$$

Goal: find the minimum!

Gradient Descent (Optimization/Variation)



We start from a random position in a parameter space, and try to descend to the minimal point.

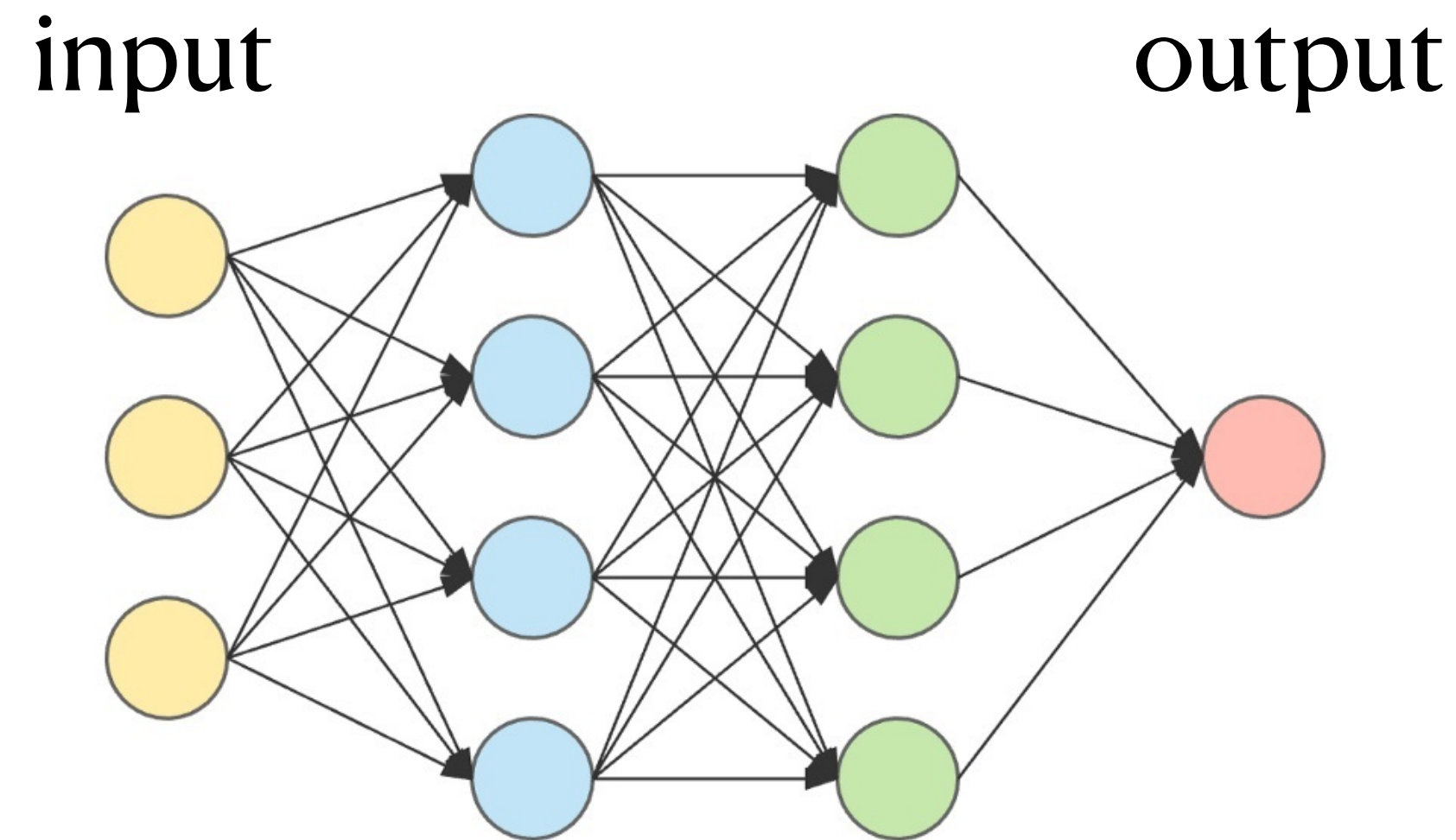
It has many natural applications in physics:

- Find the energy spectrum
- Solve an equation numerically

Two essential problems to solve:

1. Use an ansatz to construct a parameter space
2. Formulate the loss function

Neural network



Cost function computed from back propagation.

(It can be very expensive)

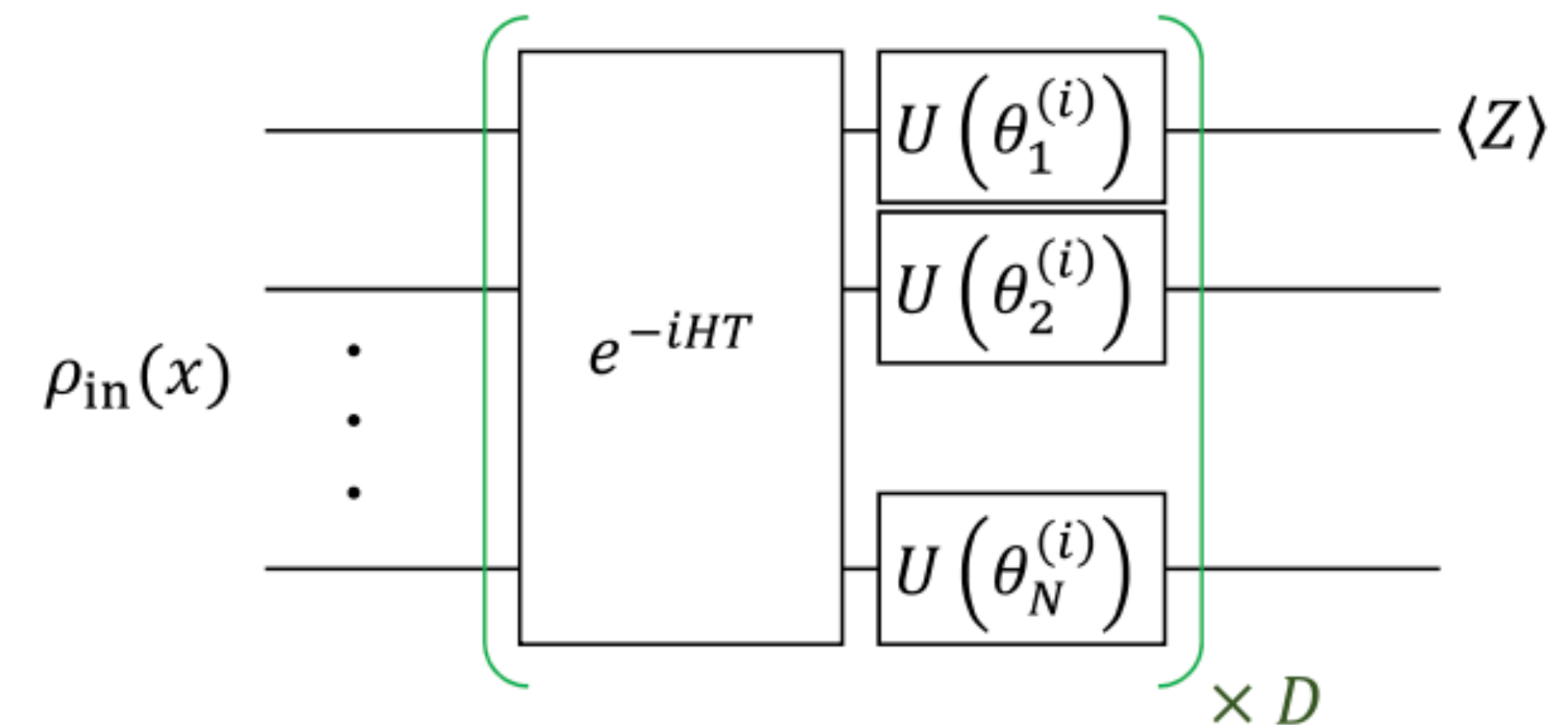
Common goal: to optimize the cost function

Quantum circuit learning

(Quantum machine learning)

c.f. [Mitarai, Negoro, Kitagawa, Fujii (2018)]

Input encoded in the initial state Output from quantum measurement



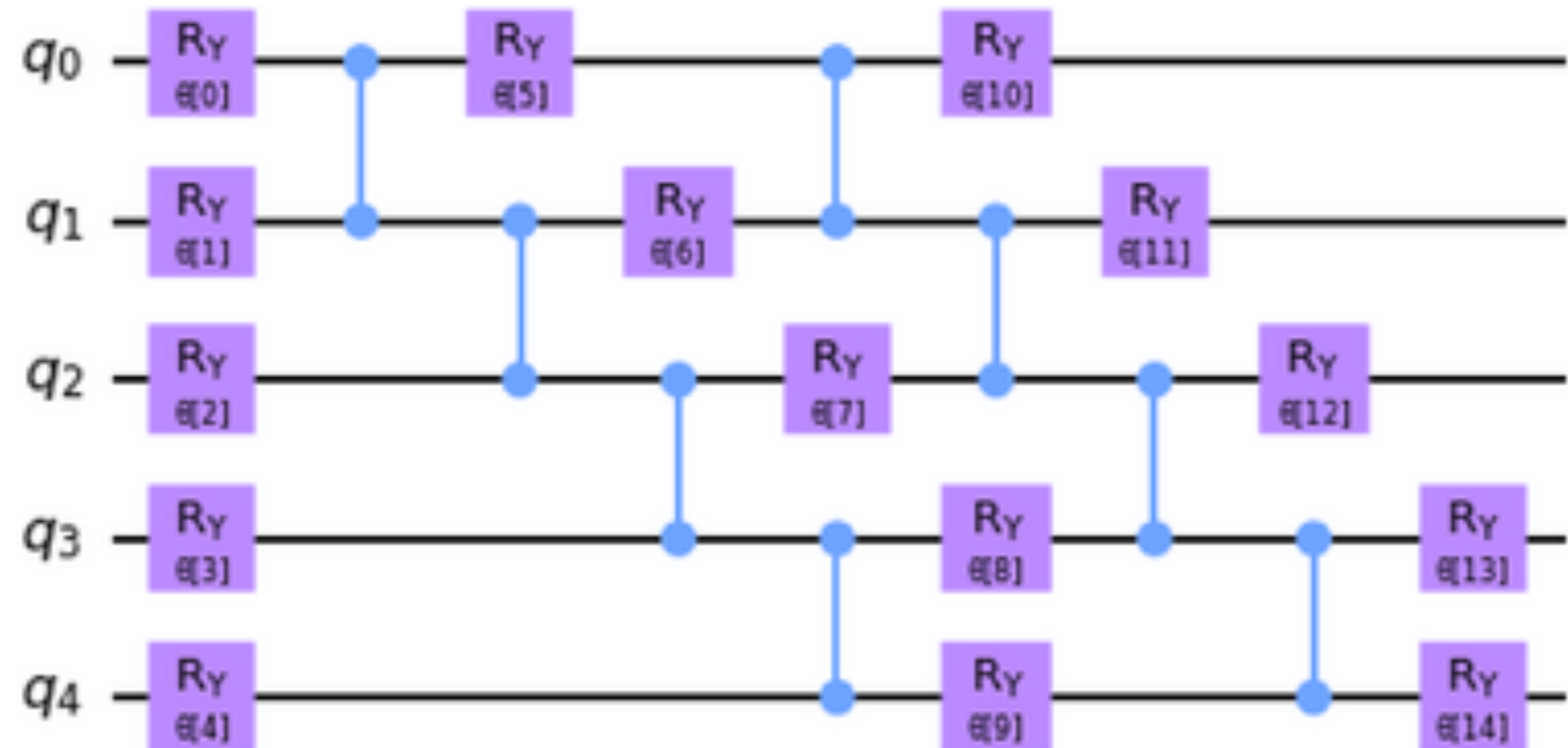
Cost function computed from measurements with $\pm \frac{\pi}{2}$ shift.

$$\frac{\partial \langle B \rangle}{\partial \theta_j} = \frac{1}{2} \text{Tr} \left[B U_{l:j+1} U_j \left(\frac{\pi}{2} \right) \rho_j U_j^\dagger \left(\frac{\pi}{2} \right) U_{l:j+1}^\dagger \right] - \frac{1}{2} \text{Tr} \left[B U_{l:j+1} U_j \left(-\frac{\pi}{2} \right) \rho_j U_j^\dagger \left(-\frac{\pi}{2} \right) U_{l:j+1}^\dagger \right]$$

Variational Quantum Algorithm

For ground state: Variational Quantum Eigensolver (VQE)

For excited states: Variational Quantum Deflation (VQD)



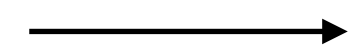
Prepare an Ansatz via the quantum circuit and find the eigenstates by optimizing the parameters.

QCL/VQA more Widely applied in Quantum Chemistry

Simulation of hydrogen molecule H_2

$$H = \sum_{p,q} h_{pq} a_p^\dagger a_q + \sum_{p,q,r,s} h_{pqrs} a_p^\dagger a_q^\dagger a_r a_s$$

Jordan-Wigner



quantum circuit

$$\text{Ansatz } |\psi\rangle = e^{T-T^\dagger} |\psi_0\rangle$$

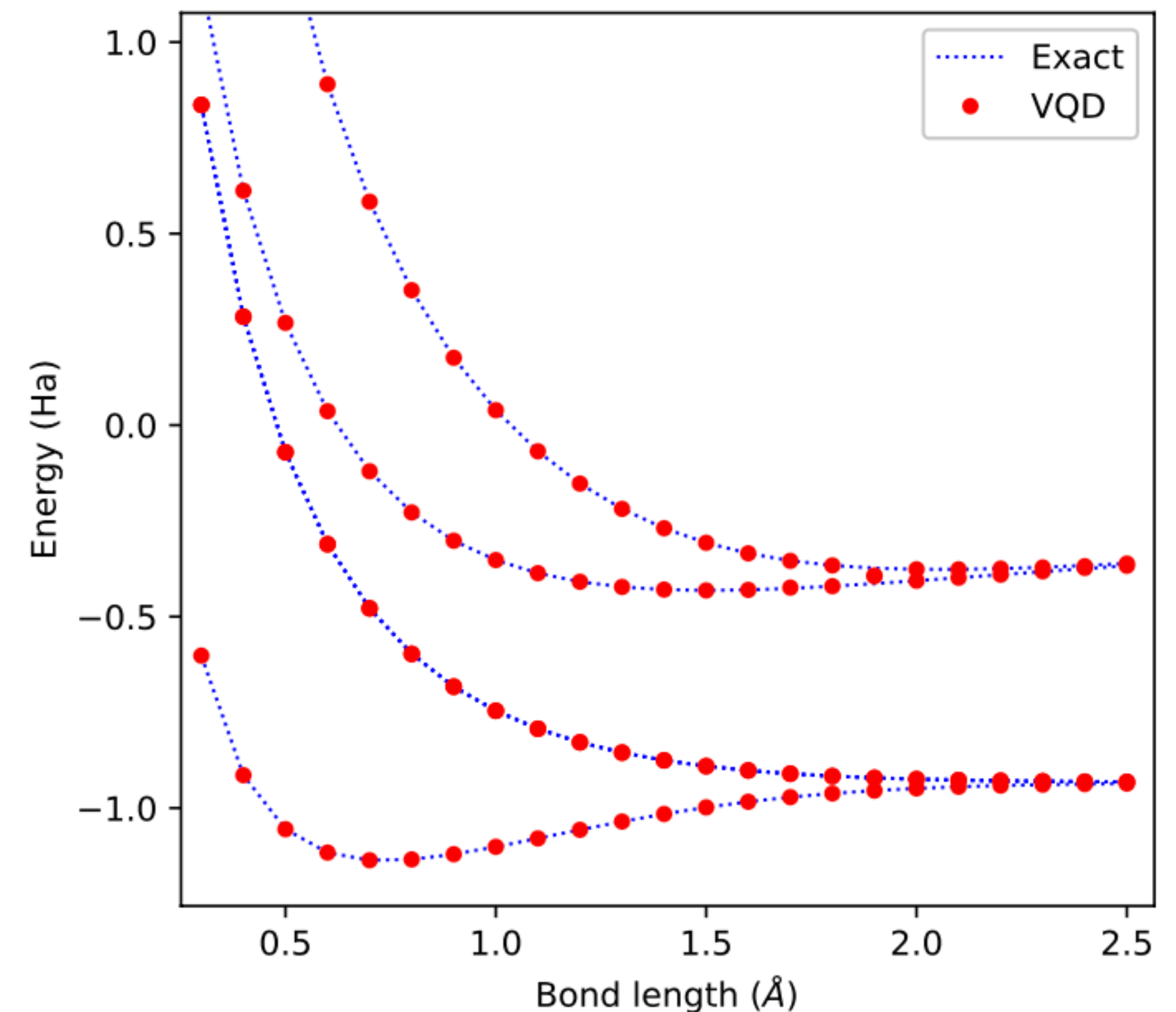
$$T = T_1 + T_2$$

$$T_1 = \sum_{p,q} t_{pq} a_p^\dagger a_q \quad T_2 = \sum_{p,q,r,s} t_{pq}^{rs} a_p^\dagger a_q^\dagger a_r a_s$$

generalised unitary coupled cluster ansatz (UCCGSD)

[J. Lee, W. J. Huggins, M. Head-Gordon, K. B. Whaley (2018)]

[O. Higgott, D. Wang, S. Brierley (2018)]



Idea of Variational Quantum Deflation (VQD)

$$\text{Loss function: } L = \langle \psi(\theta) | H | \psi(\theta) \rangle$$

1. Find the ground state by minimizing L .

$$\rightarrow \text{obtain } |\psi_0\rangle \text{ and } E_0$$

2. Modify the loss function to $L_1 = \langle \psi(\theta) | H | \psi(\theta) \rangle + t | \langle \psi_0 | \psi(\theta) \rangle |$

with t a punishment parameter to force our state orthogonal to $|\psi_0\rangle$.

$$\rightarrow \text{obtain } |\psi_1\rangle \text{ and } E_1$$

3. Repeat it until we have the full spectrum

*We are basically doing all the calculations classically...

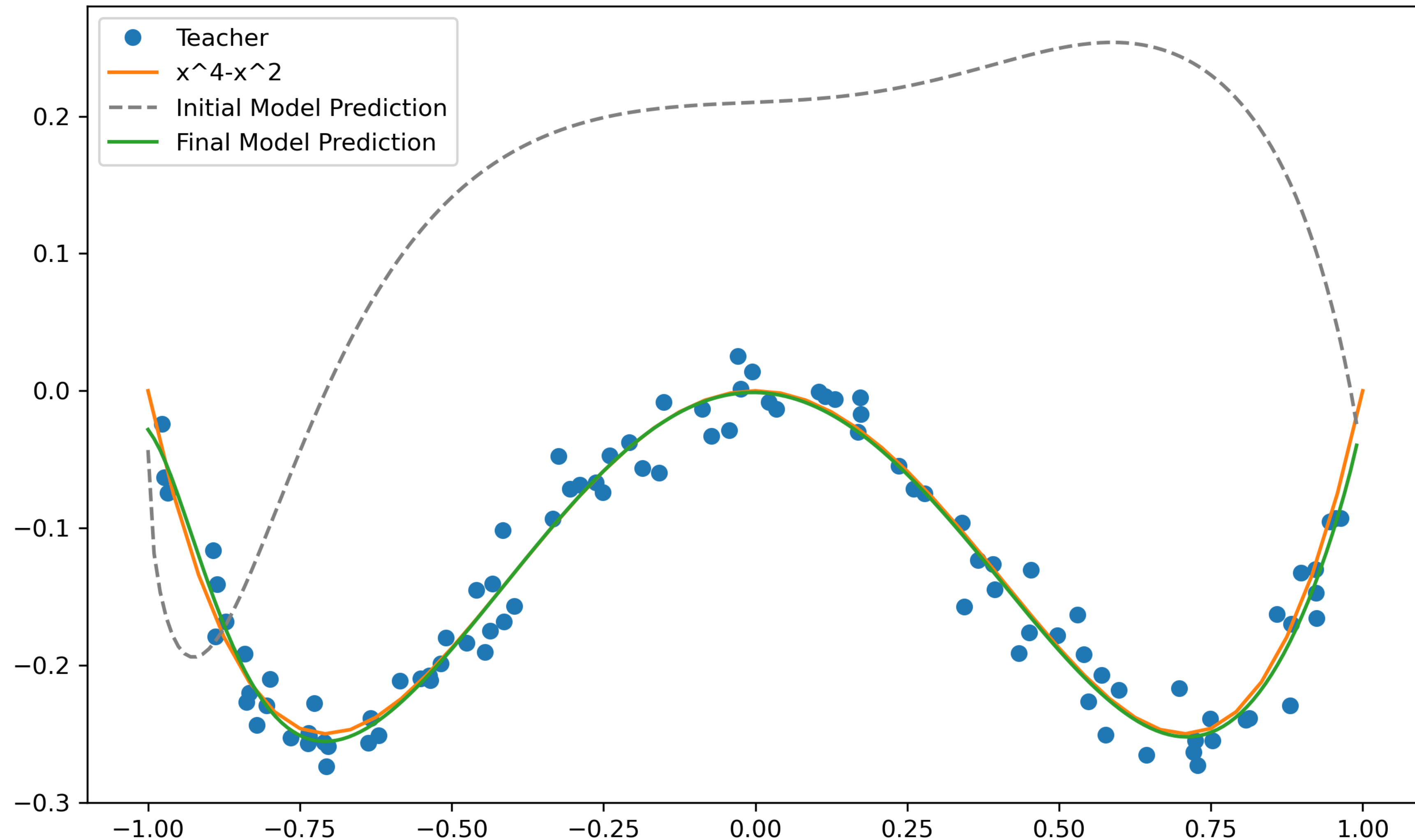
But it can be done in a purely quantum way

(The gradient of the loss function computed from measurements with $\pm \frac{\pi}{2}$ shift).

- Fitting a given function with noise

Using the Z expectation value of the first qubit (multiplied by a free parameter a)

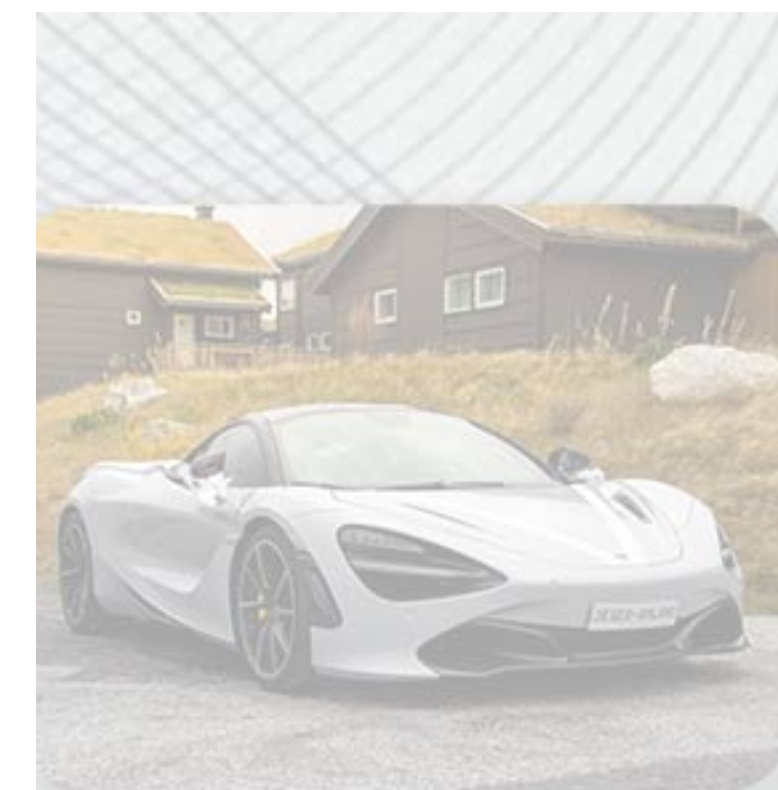
(Optimize the rotation angles with gradient descent method)



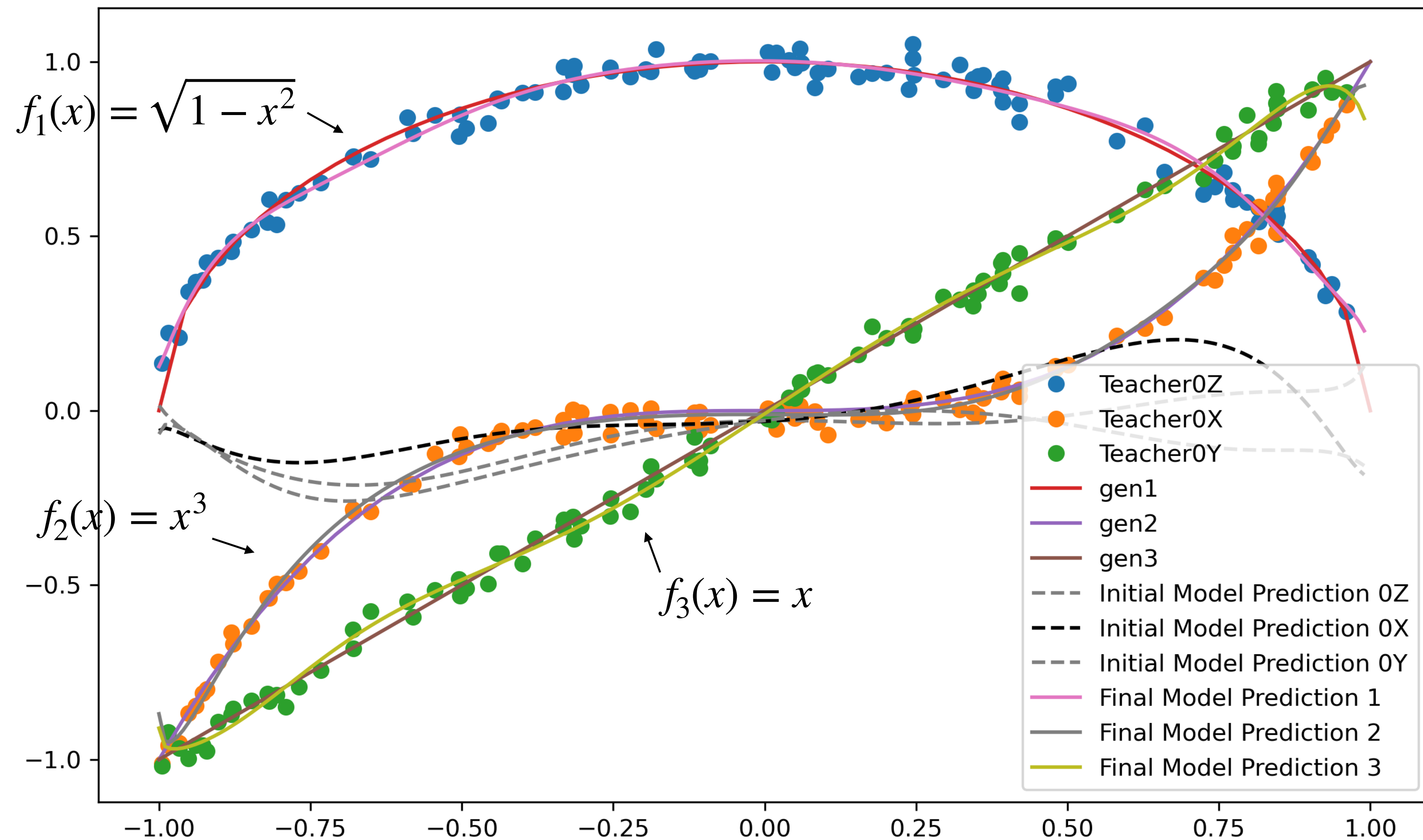
No overfitting with many tunable parameters

Simulation with 6 qubits, $D=6$, $T=10$

One relevant problem:

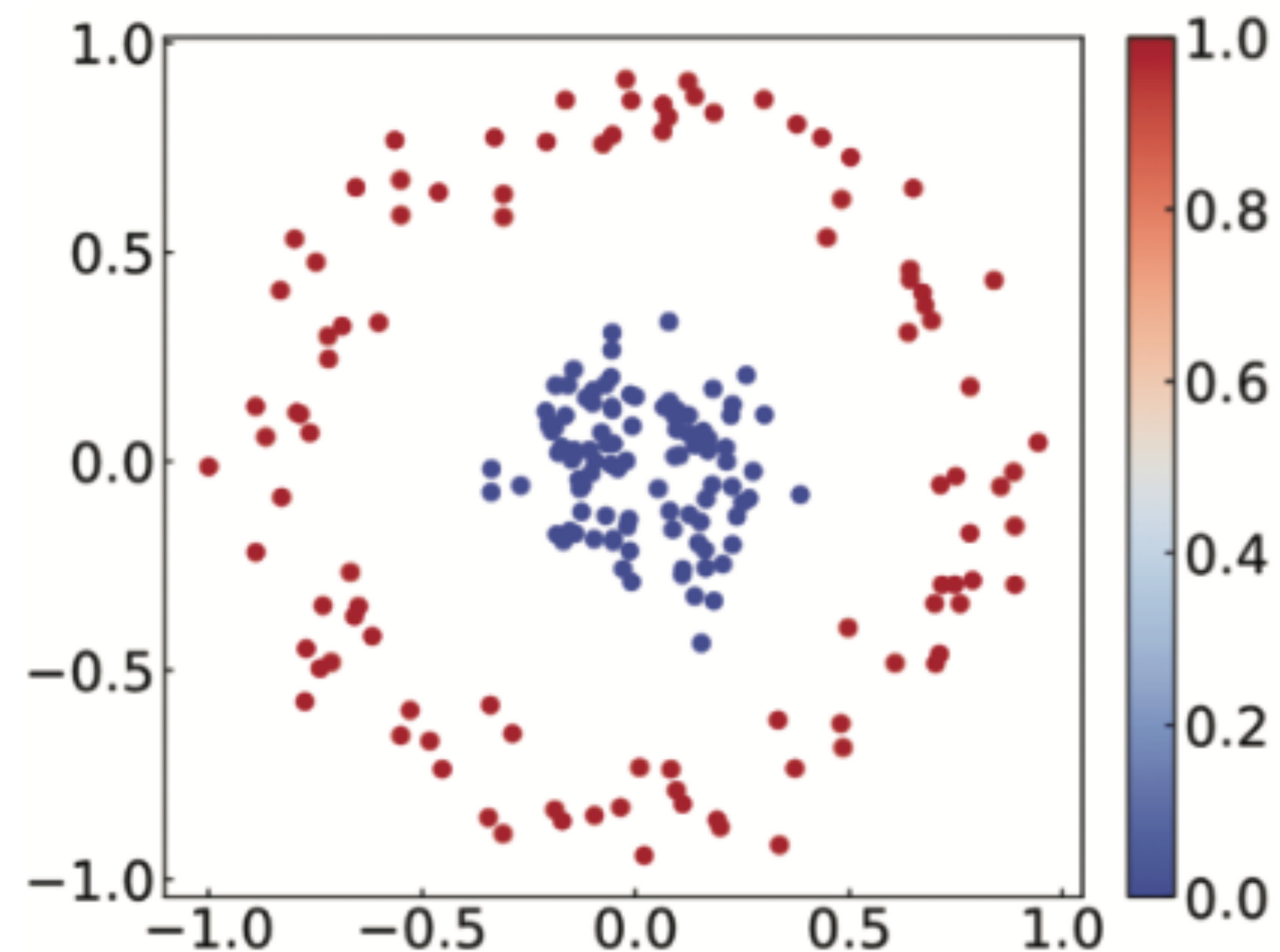


One can fit many functions with one circuit at once. (Multi-outputs)

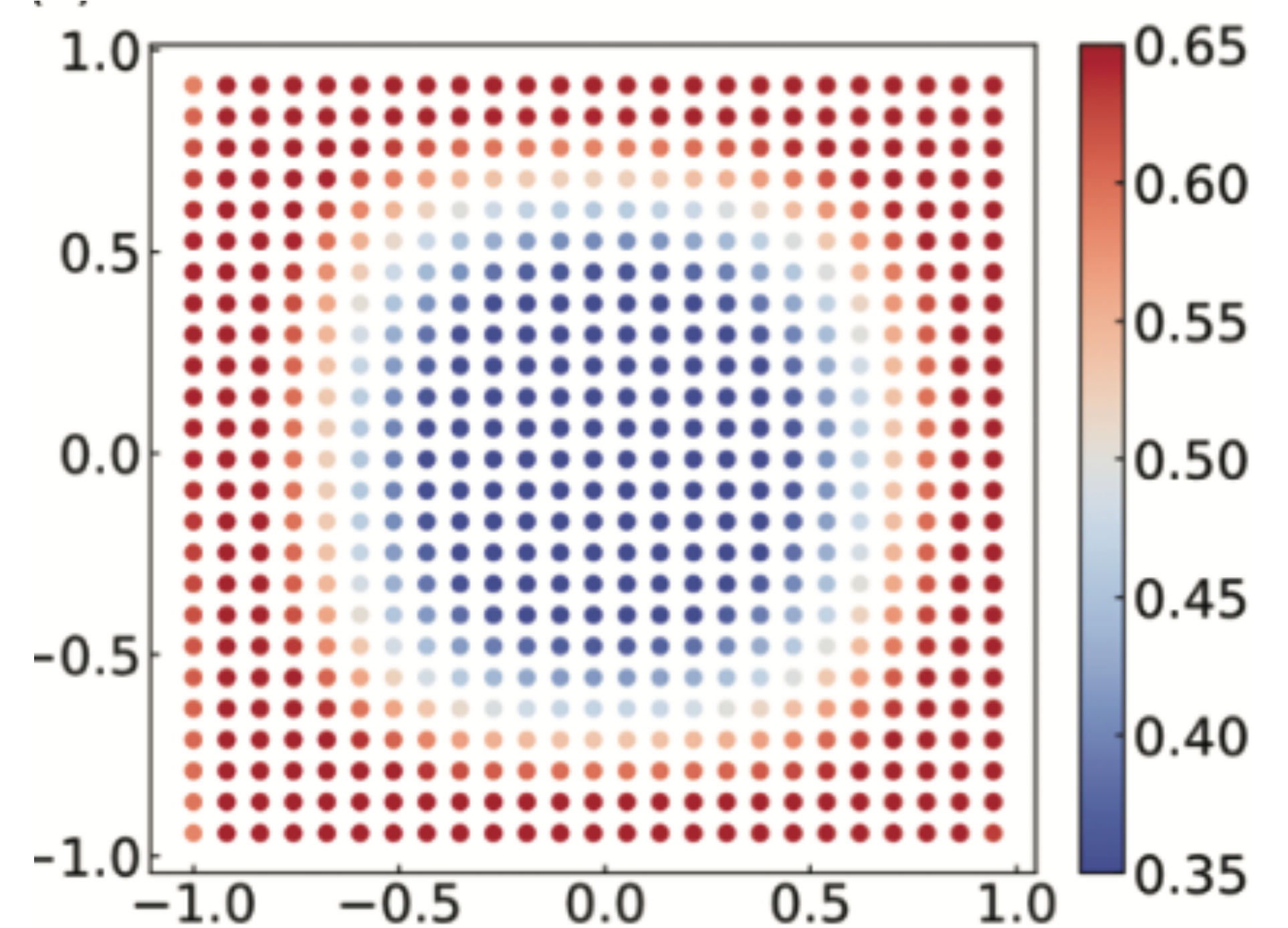


- Classification

Data:



Learned result:



** Good at classification tasks.

[Mitarai, Negoro, Kitagawa, Fujii (2018)]

More advanced applications

- Quantum simulation:

Learn part of the dynamics of 10-spin quantum Ising model (with transverse fields) by using 6-qubit circuit.

Learn some real-time dynamics of 2D QFT.

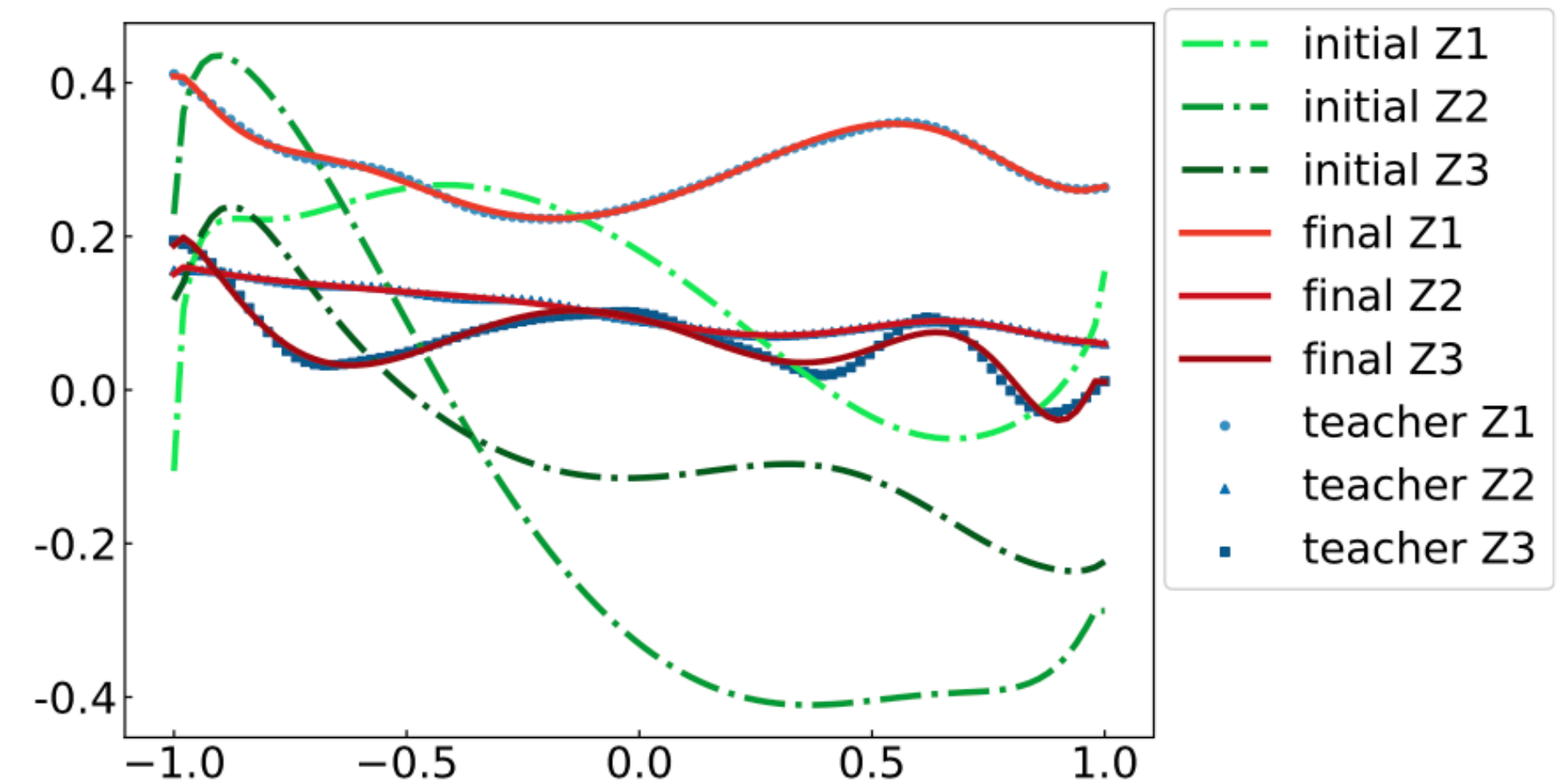
- Heart disease detection

Early prediction of heart disease by quantum deep learning (with quantum circuits)

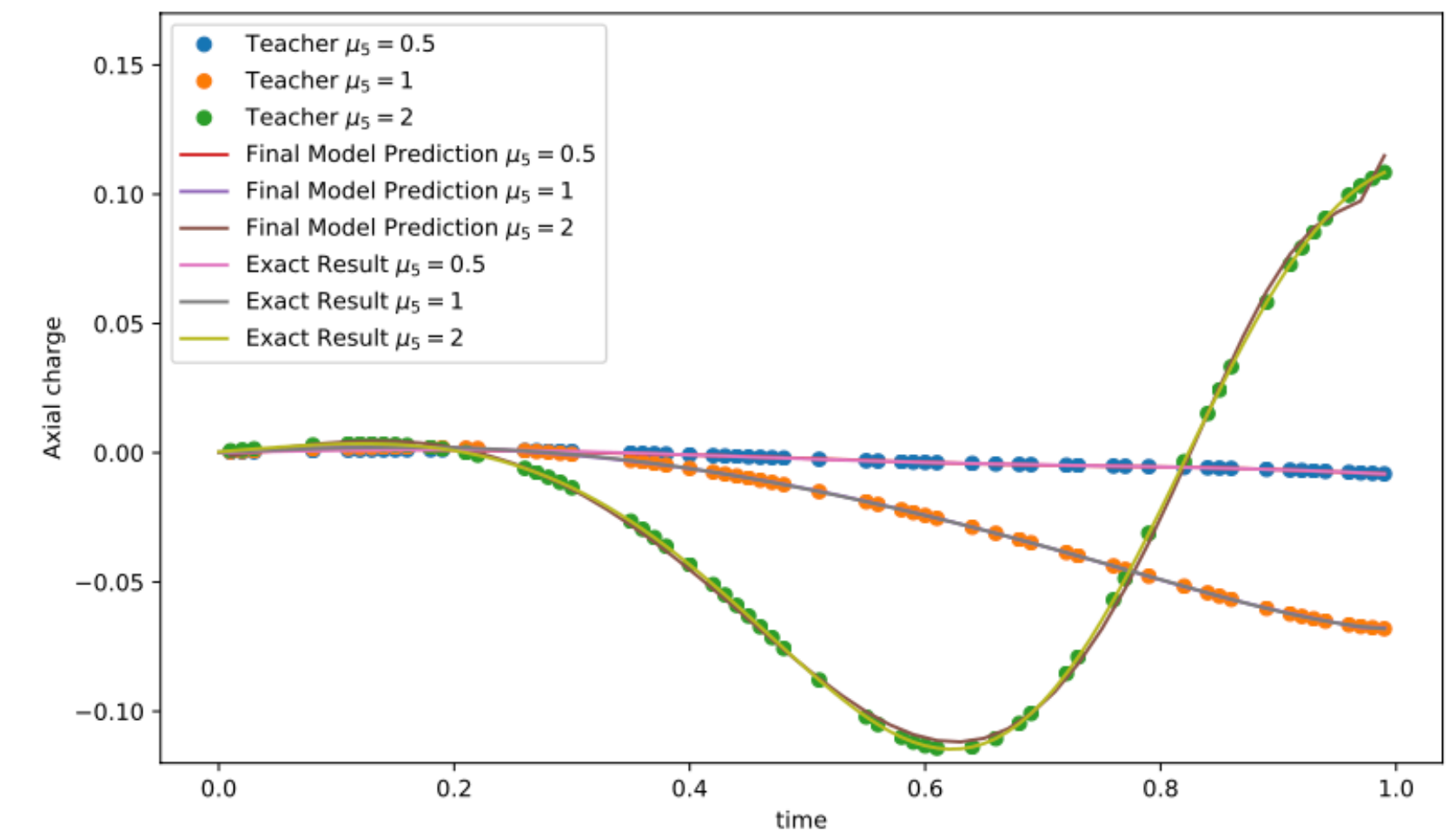
[Alsubai, Alqahtani, Binbusayyis, Sha, Gumaei, Wang (2023)]

Accuracy $\geq 95\%$

better than classical learning approaches



[Mitarai, Negoro, Kitagawa, Fujii (2018)]



[Ikeda (2023)]

Principle behind Quantum Circuit Learning

c.f. [Schuld, Sweke, Meyer (2021)]

[Heimann, Hohenfeld, Schönhoff, Mounzer, Kirchner (2025)]

$$f_{\theta}(x) = \langle 0 | U^{\dagger}(\theta; x) M U(\theta; x) | 0 \rangle$$

Measuring a physical observable M

$$S = \bigotimes_m e^{iH_m}$$

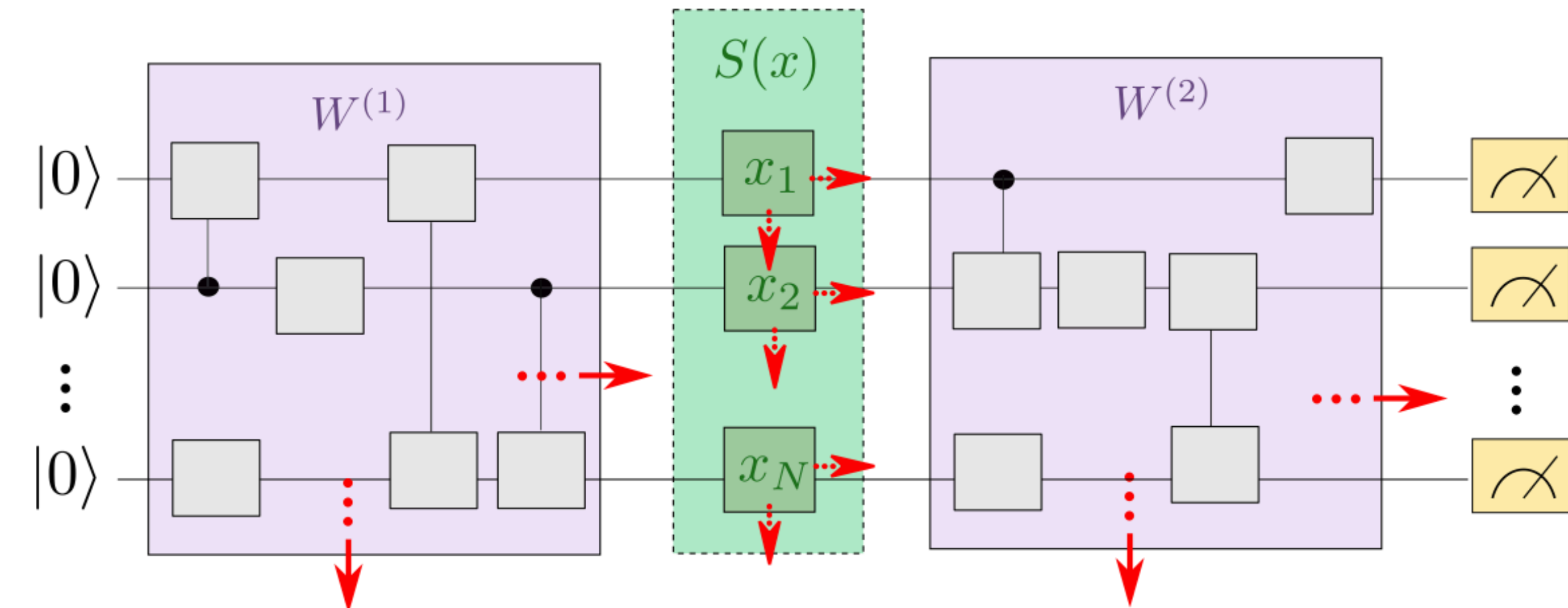
$$\Rightarrow f_{\theta}(x) = \langle \Gamma | S^{\dagger} M S | \Gamma \rangle$$

$$f(x) = \sum_{j,k} \gamma_j^{\star} M_{j,k} \gamma_k e^{ix(\lambda_k - \lambda_j)}$$

Effectively computes a Fourier series of the target function.

Practically we need a large number of layers, and thus a lot of parameters to optimize.

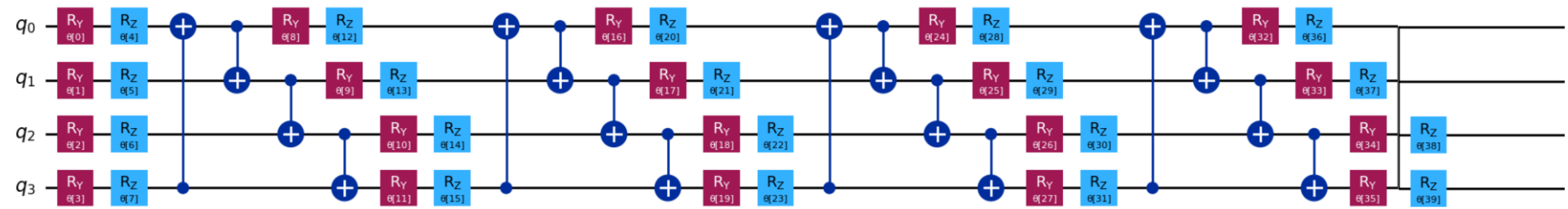
$$U(\theta) = W^{(2)}(\theta^{(2)}) S W^{(1)}(\theta^{(1)})$$



$$\Omega_{\text{par}} = \{ \lambda_{k_1} - \lambda_{j_1} \mid k_1, j_1 \in \{1, \dots, 2^r\} \}$$

Example: VQD for Ising model in quantum simulator

$$H = -J \sum_{i=1}^L Z_i Z_{i+1}$$



$L = 4$

VQD Energy [-10.49924113+0.j -8.11433142+0.j]

Comparison

Level 0:

Exact = -10.503877

VQD 值 = -10.499241+0.000000j

Error = 0.004636

Level 1:

Exact = -8.124311

VQD 值 = -8.114331+0.000000j

Error = 0.009980

$L = 5$

VQD Energy [-10.49924113+0.j -8.11433142+0.j]

Comparison

Level 0:

Exact = -10.503877

VQD 值 = -10.499241+0.000000j

Error = 0.004636

Level 1:

Exact = -8.124311

VQD 值 = -8.114331+0.000000j

Error = 0.009980

Affleck-Kennedy-Lieb-Tasaki (AKLT) model

$$H = \sum_i \vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{3} \sum_i \vec{S}_i \cdot \vec{S}_{i+1}$$

Spin-1
in a symmetry-protected
topological phase



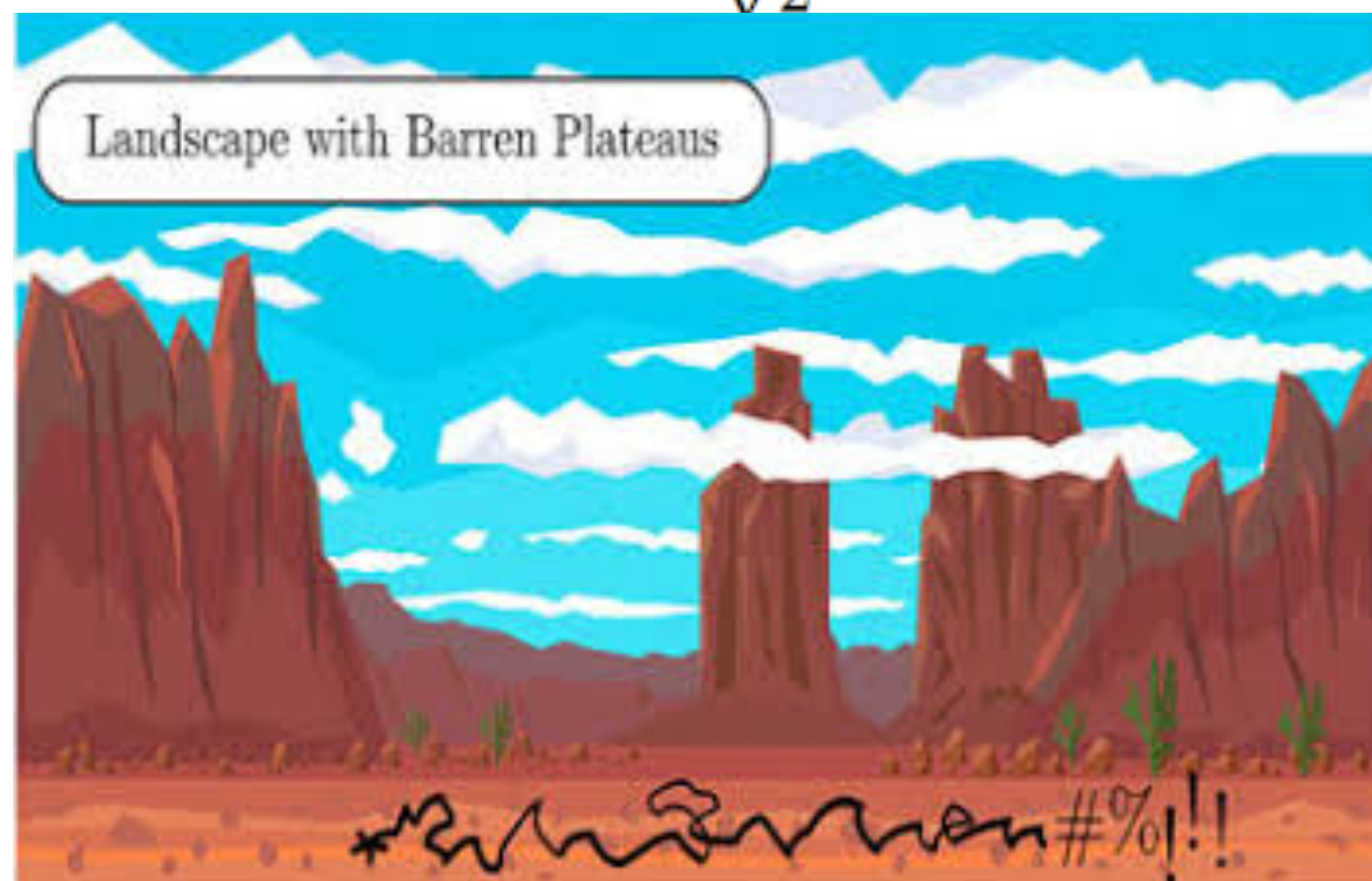
$$\bullet - \bullet = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

$$\bigcirc = |+\rangle\langle\uparrow\uparrow| + |0\rangle\frac{\langle\uparrow\downarrow| + \langle\downarrow\uparrow|}{\sqrt{2}} + |-\rangle\langle\downarrow\downarrow|$$

The ground state can be easily prepared via projectors! And it is straightforward to generalize to 2D.

Barren Plateaus encountered...

Different entanglement layers give rather different optimized results. But all of those we tried are not very good, and need to repeat for many times...

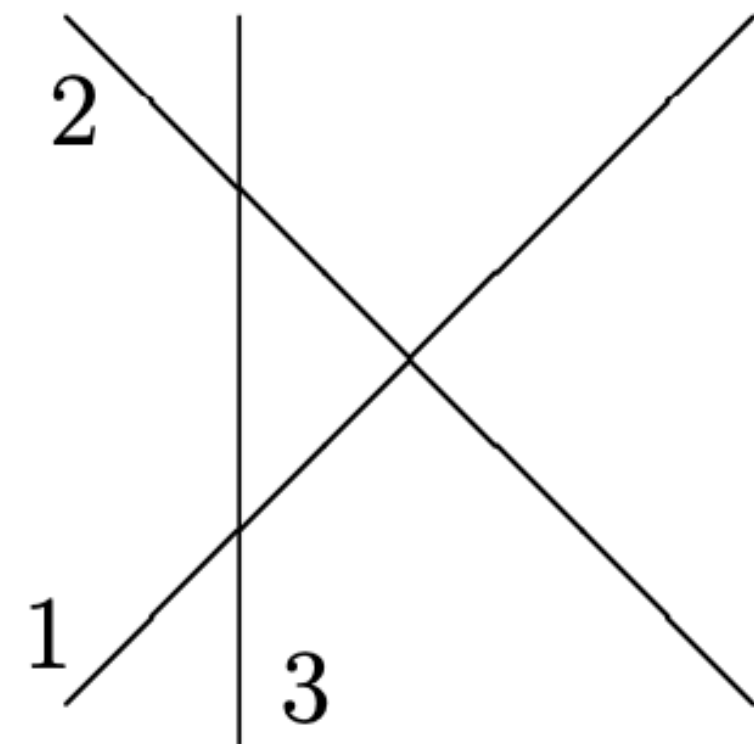


Algebraic Bethe ansatz for integrable models

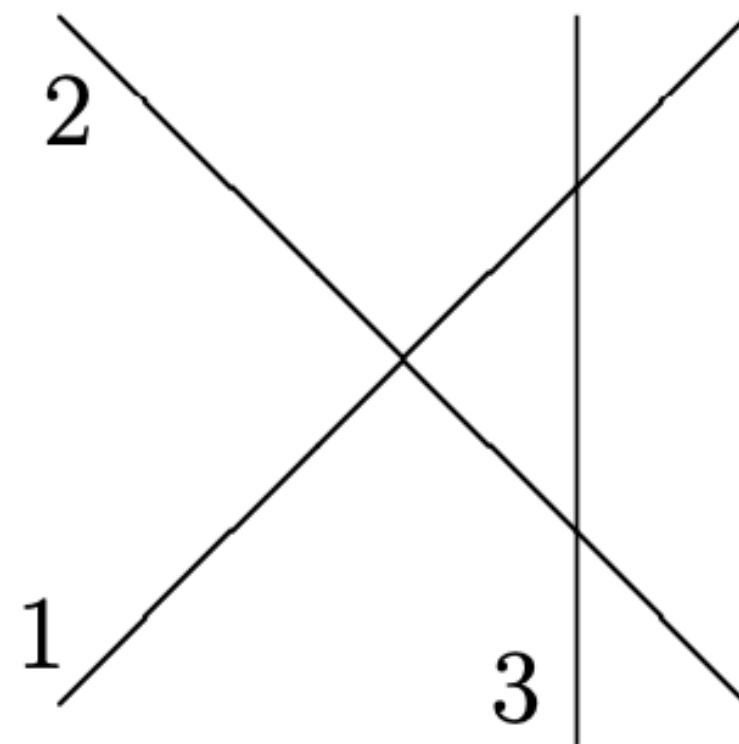
c.f. Faddeev's classical Les Houches lecture

Start from a solution to the Yang-Baxter equation:

$$\mathbf{R}_{12}(u - v)\mathbf{R}_{13}(u)\mathbf{R}_{23}(v) = \mathbf{R}_{23}(v)\mathbf{R}_{13}(u)\mathbf{R}_{12}(u - v)$$



=



~ Many-body scattering of
(quasi-)particles factorized into
two-body ones

(Introduce Lax matrix $\mathbf{L}(u)$ s.t. $\mathbf{R}_{00'}(u - v)\mathbf{L}_{0i}(u)\mathbf{L}_{0'i}(v) = \mathbf{L}_{0'i}(v)\mathbf{L}_{0i}(u)\mathbf{R}_{00'}(u - v)$.)

For simplification, we just take $\mathbf{L}(u) = \mathbf{R}(u)$.)

Define the transfer matrix:

$$t(u) := \text{tr}_0 \left(\mathbf{R}_{0L}(u)\mathbf{R}_{0(L-1)}(u)\dots\mathbf{R}_{01}(u) \right)$$

Conserved charges

Commuting between two transfer matrix at different spectral parameters,

$$[t(u), t(v)] = 0, \quad \forall u, v.$$

Expansion coefficients, $H^{(i)}$'s, in the transfer matrix,

$$t(u) = t(0) \exp \left(\sum_{i=1}^{\infty} H^{(i)} u^i \right),$$

are all commuting with each other.

$$[H^{(i)}, H^{(j)}] = 0, \quad \forall i, j.$$

One can identify one of $H^{(i)}$'s as the Hamiltonian, and the others become the conserved charges in the system.

As many conserved charges as the number of d.o.f.'s in the system. $\Rightarrow$ integrable

Example: Heisenberg (XXX) spin chain

R-matrix given by

$$\mathbf{R}^{(2)}(u) = \begin{pmatrix} u+i & 0 & 0 & 0 \\ 0 & u & i & 0 \\ 0 & i & u & 0 \\ 0 & 0 & 0 & u+i \end{pmatrix} = \begin{pmatrix} u + \frac{i}{2} + iJ_z & iJ_- \\ iJ^+ & u + \frac{i}{2} - iJ_z \end{pmatrix}$$

Hamiltonian:

Spin-1/2:

$$H^{(1)} = \frac{d}{du} \log t(u) \Big|_{u=0} = \frac{1}{2} \sum_{j=1}^L \left(\sigma_1^{(j)} \otimes \sigma_1^{(j+1)} + \sigma_2^{(j)} \otimes \sigma_2^{(j+1)} + \sigma_3^{(j)} \otimes \sigma_3^{(j+1)} \right) + \frac{L}{2}$$

σ_i : Pauli matrices, su(2) generators in spin-1/2

Spin-1:

$$H^{(1)} = \frac{1}{4} \sum_{j=1}^L \left(\vec{S}_j^{(1)} \cdot \vec{S}_{j+1}^{(1)} + (\vec{S}_j^{(1)} \cdot \vec{S}_{j+1}^{(1)})^2 \right) + \text{const.}$$

[Babujian (1983)]

Algebraic Bethe ansatz continued

Rewrite the so-called monodromy matrix into a matrix on the auxiliary space,

$$\mathbf{R}_{0L}(u)\mathbf{R}_{0(L-1)}(u)\dots\mathbf{R}_{01}(u) = \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}(u) \\ \mathcal{C}(u) & \mathcal{D}(u) \end{pmatrix}$$

We wish to diagonalize

$$t(u) = \mathcal{A}(u) + \mathcal{D}(u)$$

with the ansatz for the eigenstate,

$$|\psi(\{u_k\}_{k=1}^M)\rangle = \prod_{k=1}^M \mathcal{B}(u_k) |\Omega\rangle.$$

Action of the transfer matrix:

$$t(u) |\psi(\{u_k\}_{k=1}^M)\rangle = \left[(u+i)^L \prod_{k=1}^M \frac{u_k - u + i}{u_k - u} + u^L \prod_{k=1}^M \frac{u - u_k + i}{u - u_k} \right] |\psi(\{u_k\}_{k=1}^M)\rangle + \dots$$

↓
Unwanted off-diagonal terms

Bethe ansatz equation (BAE)

To kill the unwanted terms/to keep $t(u)$ a polynomial of u

For XXX spin chain (spin-1/2)

$$\frac{(u_j + i)^L}{u_j^L} \prod_{k \neq j} \frac{u_j - u_k - i}{u_j - u_k + i} = 1. \quad \longleftarrow \text{BAE}$$

The periodic boundary condition is used in the above model.

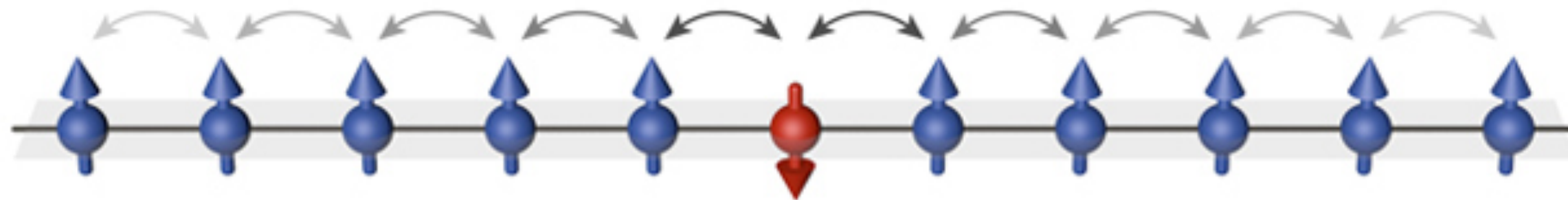
$$J_z^{(L+1)} = J_z^1, \quad J_{\pm}^{(L+1)} = J_{\pm}^{(1)}.$$

The BAEs are highly non-linear, and there are many sophisticated (efficient) methods to (numerically) solve them.

- Wronskian method [Pronko, Stroganov (1998)] [Mukhin, Tarasov, Varchenko (2007)]
- Rational Q-system [Maboe, Volin (2016)] [Gu, Jiang, Sperling (2022)] [Hou, Jiang, RZ (2023)] ...
- Gröbner basis [Jiang, Zhang (2017)]

Integrable Spin Models

Exactly solvable



Conserved charges

(As many as the number of d.o.f.'s)

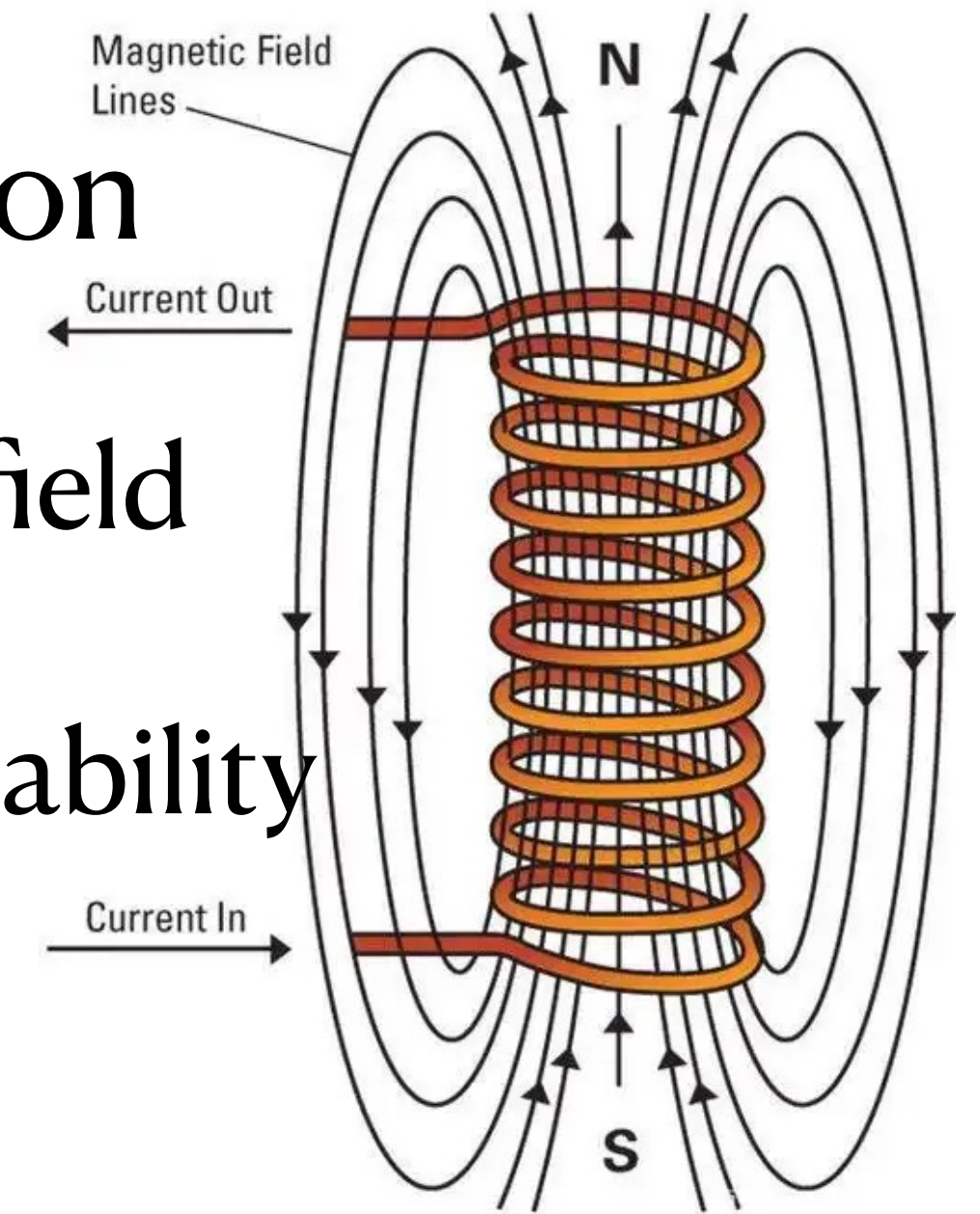
Many physical observables can be computed by solving the Bethe ansatz equation (BAE).

Very efficient (numerically cheap)!

+ deformation

e.g. magnetic field

Breaks the integrability



R-matrix and BAE no longer apply.

How to deal with such systems?

Naive way: perturbation (costly)

Idea:

- To incorporate the physical information via the Bethe ansatz

$$|\psi_{BA}(u_i)\rangle = \prod_{i=1}^M B(u_i) |\Omega\rangle \quad \sim \text{quasi-particle excitations}$$

The number of free parameters to optimize is equal to M (magnon number).

A U(1) symmetry is assumed.

- We do not impose the BAE. Instead we try to find u_i 's from the variational method. (We call it off-shell Bethe ansatz)

$$L = \langle \psi_{BA}(u_i) | H | \psi_{BA}(u_i) \rangle$$

Example: XXX spin chain + perturbation

$$H = - \sum_{i=1}^L (X_i X_{i+1} + Y_i Y_{i+1} + Z_i Z_{i+1}) + \Delta H$$

determined by XXX R-matrix

$$|\psi\rangle = \prod_{i=1}^M \overset{\downarrow}{B(u_i)} |\uparrow \uparrow \dots \uparrow\rangle$$

Example: XXZ spin chain

$$H_{XXZ} = - \sum_{i=1}^L (X_i X_{i+1} + Y_i Y_{i+1} + \Delta Z_i Z_{i+1}) = -H_{XX} - \delta\Delta \sum_i Z_i Z_{i+1} \quad (\Delta = 1 + \delta\Delta)$$

- $L = 4$, $\delta\Delta = 0.1$, energy spectrum

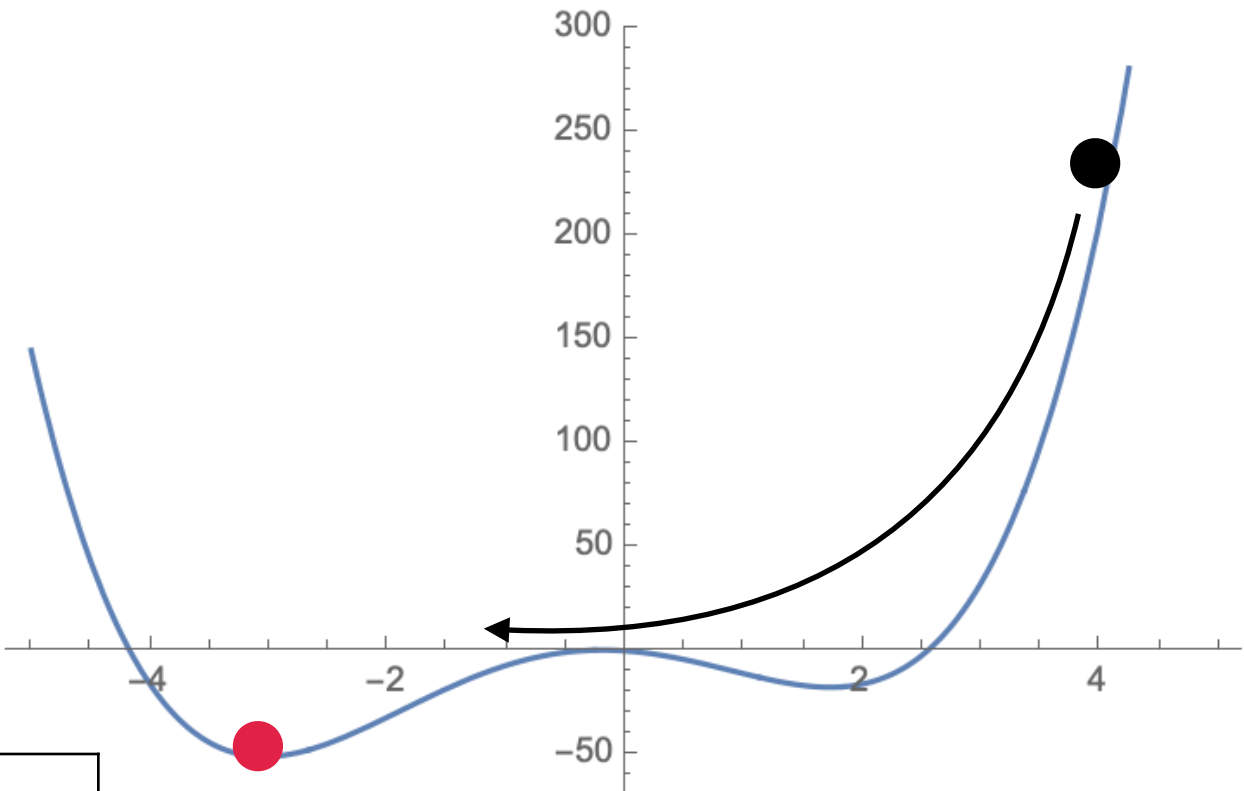
exact diagonalization	-4.4	-4	-3.8696	...	8.2696
off-shell Bethe ansatz	-4.4	-4	-3.867	...	8.267

- $L = 5$, $\delta\Delta = 0.1$, energy spectrum

exact diagonalization	-5.5	-5.1	-4.905	...	7.72298
off-shell Bethe ansatz	-5.5	-5.1	-4.9	...	7.72119

Observations:

- There is a finite (systematic) error remaining, irrelevant to the variational method.
e.g. the XXX Bethe ansatz is not suitable for XXZ model
- The error becomes larger when the perturbation get larger.
- $M = 1$: (almost) precise, $M = 2$: obvious error



$\delta\Delta$	0.1	0.2	0.5	-0.5
L=4	0.0028	0.011	0.06	0.07
L=6	0.005	0.02	0.12	0.12
L=8	0.007	0.028	0.17	0.18

$$\frac{\Delta E}{|E|} \lesssim 1\%$$

$$E \sim \mathcal{O}(2L)$$

$$\text{*Error for antiferro ground state: } \sim 0.001 \times \left(\frac{|\delta\Delta|}{0.1} \right)^2 (L - 2)$$

Example: XXX spin chain with staggered magnetic field (completely breaks the integrability)

$$H = H_0 + hH_z$$

$$H_z = \sum_{n=1}^L (-1)^n S_n^z$$

$$L = 4, \, h = 0.1, \, H_0 = -H_{XXX}$$

Energy level
from exact
diagonalization:

-4.0067
-4.0050
-4
-0.2
0
0.2
...
8.0265

vs

-4.0022
-4.00
-4

-0.200

0.001

0.200

...
8.009

Special

From off-shell
Bethe ansatz

$$\frac{|\psi_{01}\rangle + i|\psi_{02}\rangle}{\sqrt{2}}$$

$$\frac{|\psi_{01}\rangle - i|\psi_{02}\rangle}{\sqrt{2}}$$

$$|\psi_{01}\rangle := |\bar{\psi}_1(0.5 - 0.5i)\rangle$$

$$|\psi_{02}\rangle := |\bar{\psi}_1(-0.5 - 0.5i)\rangle$$

*accuracy can be improved by allowing superposition of Bethe states.

Example: XXX spin chain (with weak breaking of the integrability)

$$H = H_{\text{XXX}} + \lambda H^{(1)} \qquad H^{(1)} = \sum_{n=1}^L \vec{S}_i \cdot \vec{S}_{i+2}$$

[Surace, Motrunich (2023)]

Truncation of a deformed integrable model

[Szász-Schagrin, Pozsgay, Takács (2021)]

$$\frac{\mathrm{d}Q_j(\lambda)}{\mathrm{d}\lambda} = i[X(\lambda), Q_j(\lambda)] \qquad Q_j(\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} Q_j^{(n)}$$

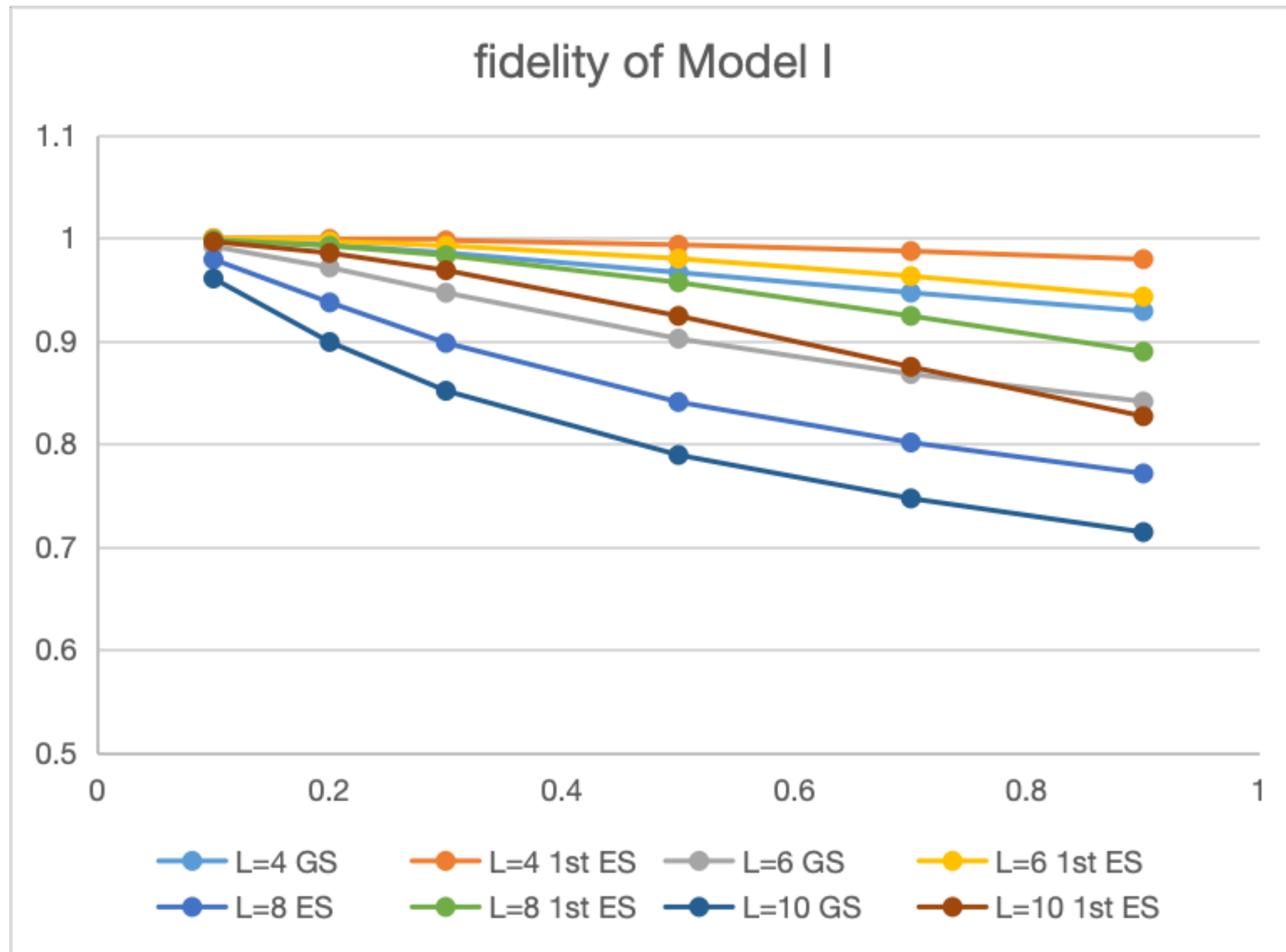
Take $H^{(0)} = H_{\text{XXX}}$, $Q_j^{(1)} = -i[B[Q_3^{(0)}], Q_j^{(0)}]$, $q_j^{(3)} = (\vec{S}_j \times \vec{S}_{j+1}) \cdot \vec{S}_{j+2}$.

Boost operator: $B[Q] = \sum_{n=1}^L n q_n$, $Q = \sum_{n=1}^L q_n$. $\Rightarrow H^{(1)} = \sum_{n=1}^L \vec{S}_i \cdot \vec{S}_{i+2}$

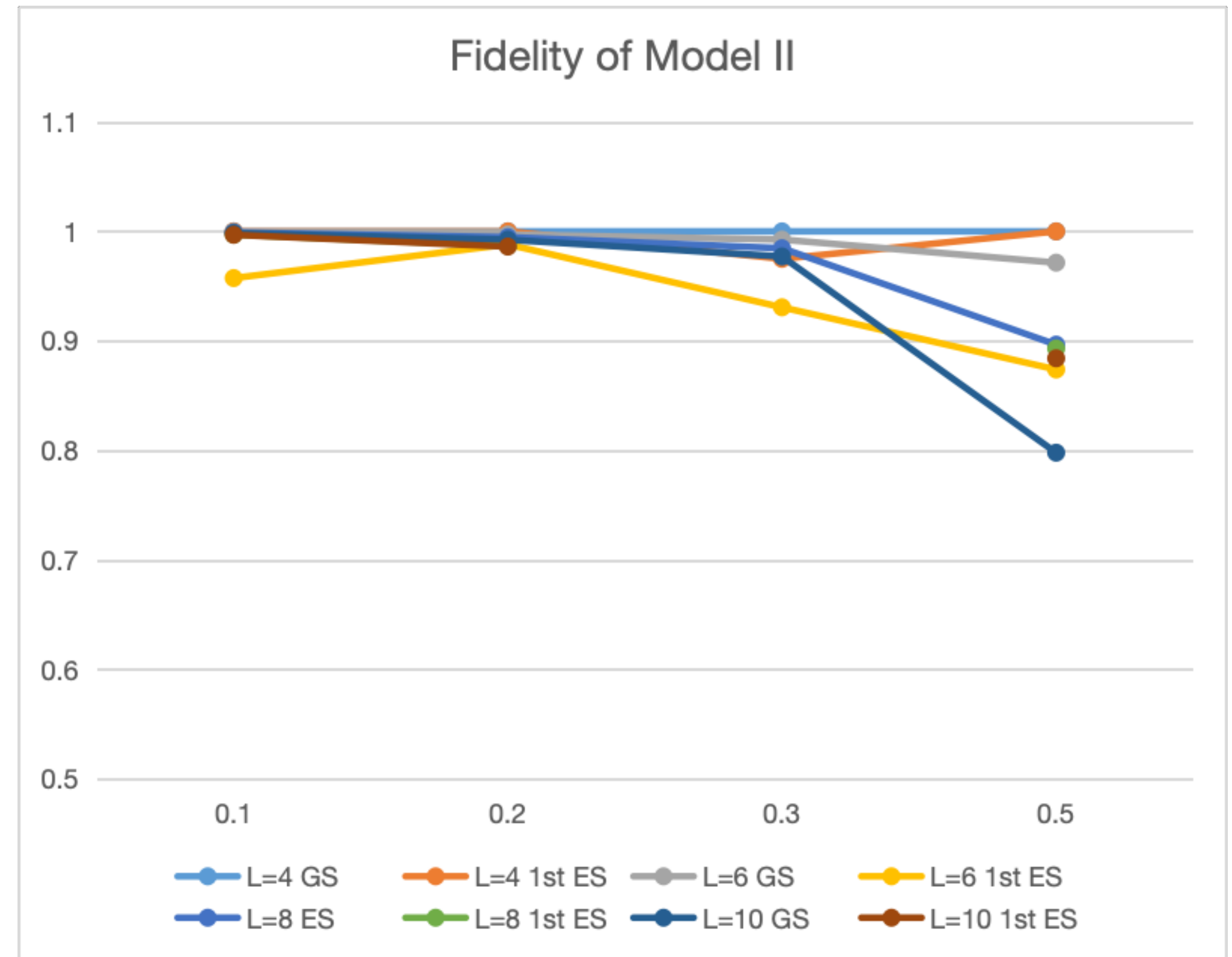
The truncated model is integrable at the order $\mathcal{O}(\lambda^2)$.

Comparison

Model I: $H = H_{XXX} + hH_z$
Strongly integrability breaking

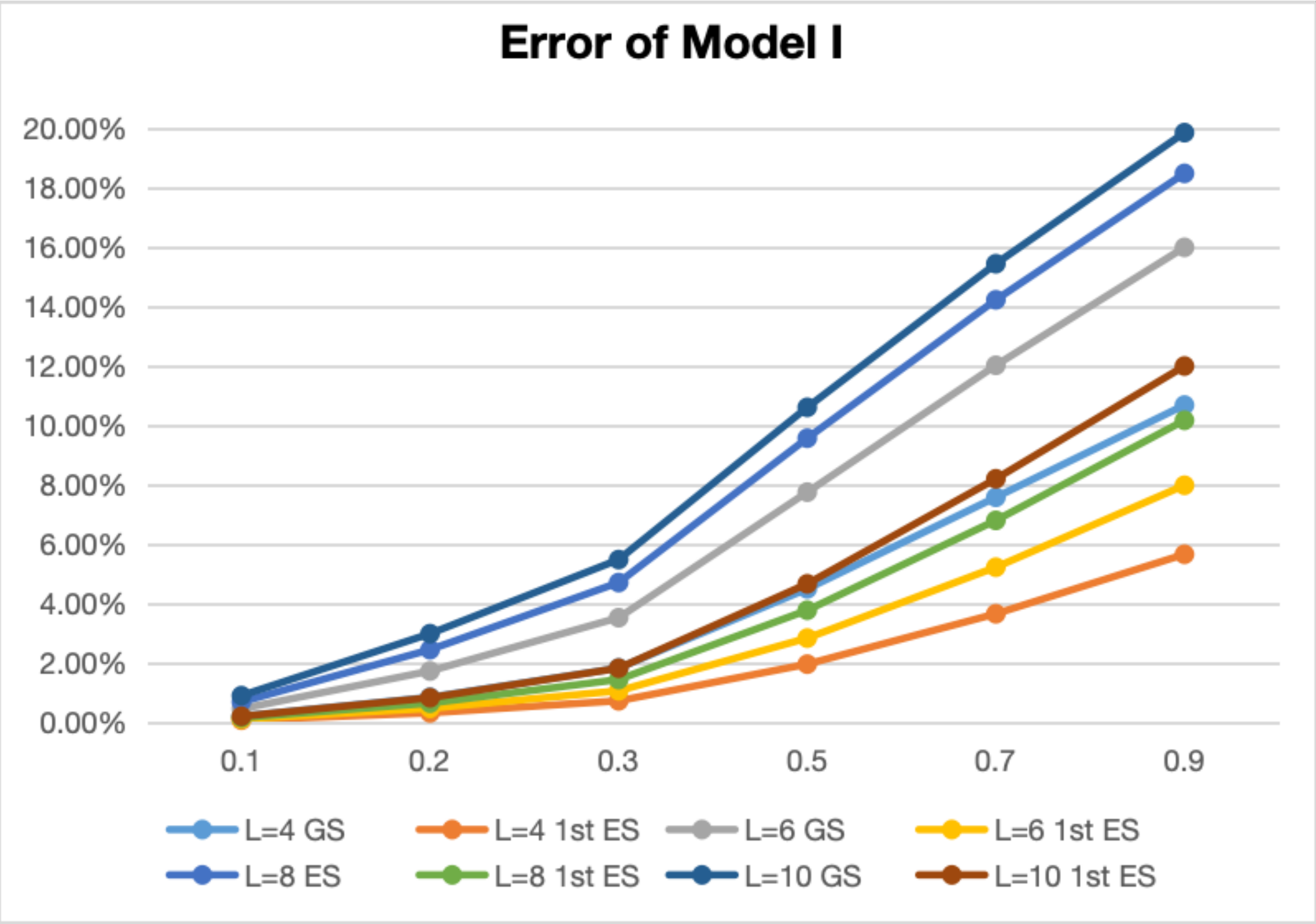


Model II: $H = H_{XXX} + \lambda H^{(1)}$
Weakly integrability breaking

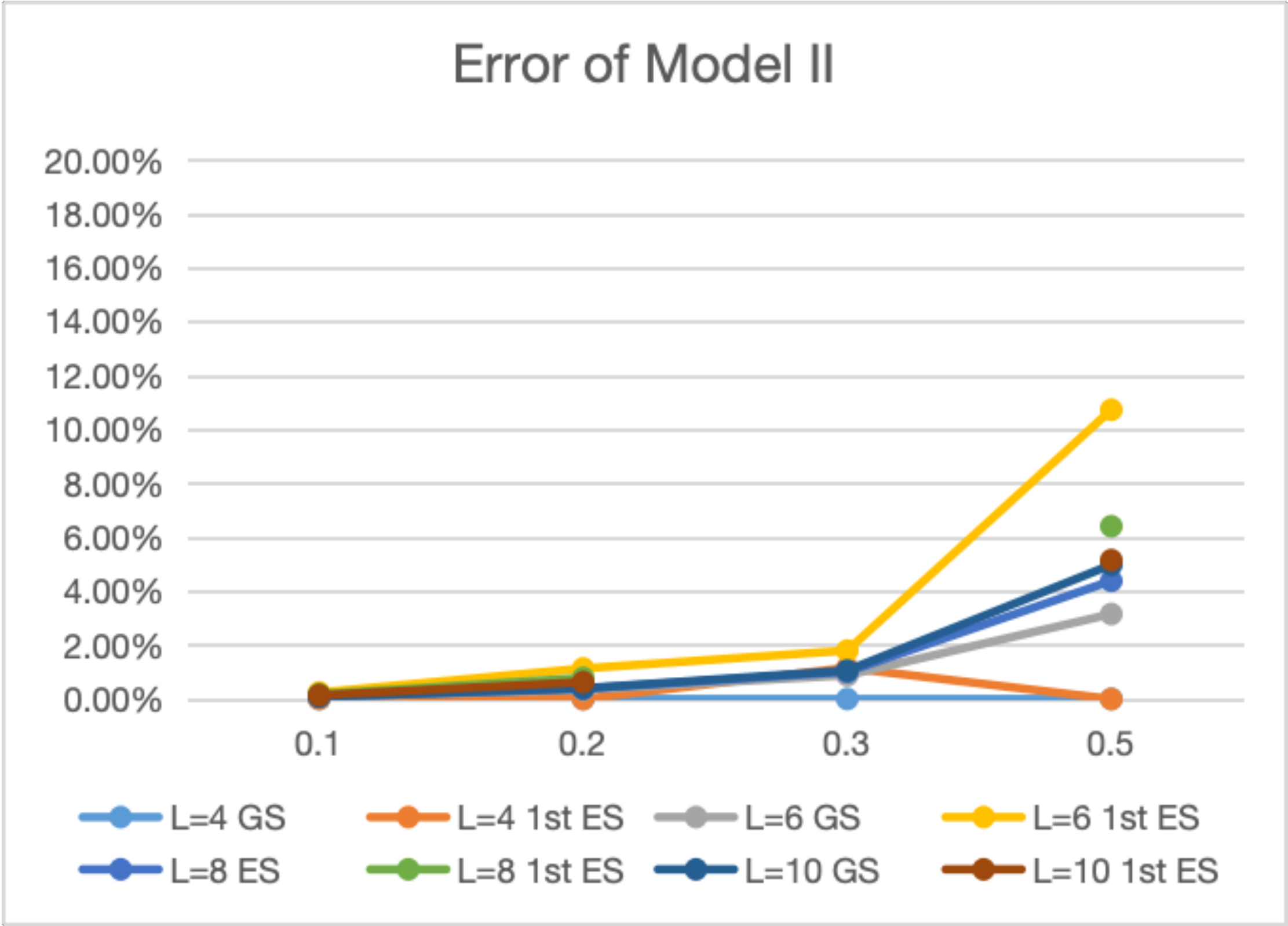


Comparison

Model I: $H = H_{XXX} + hH_z$
Strongly integrability breaking



Model II: $H = H_{XXX} + \lambda H^{(1)}$
Weakly integrability breaking



Observations:

- The optimization can be super short if we start from a solution of BAE even for $L \sim 20$.
One can go beyond ED by saving memory.
- When the perturbation becomes larger, the method gets worse.
- The method works better for Model II (weakly integrability breaking).

Both the error (of energy) and the fidelity of the wave function.

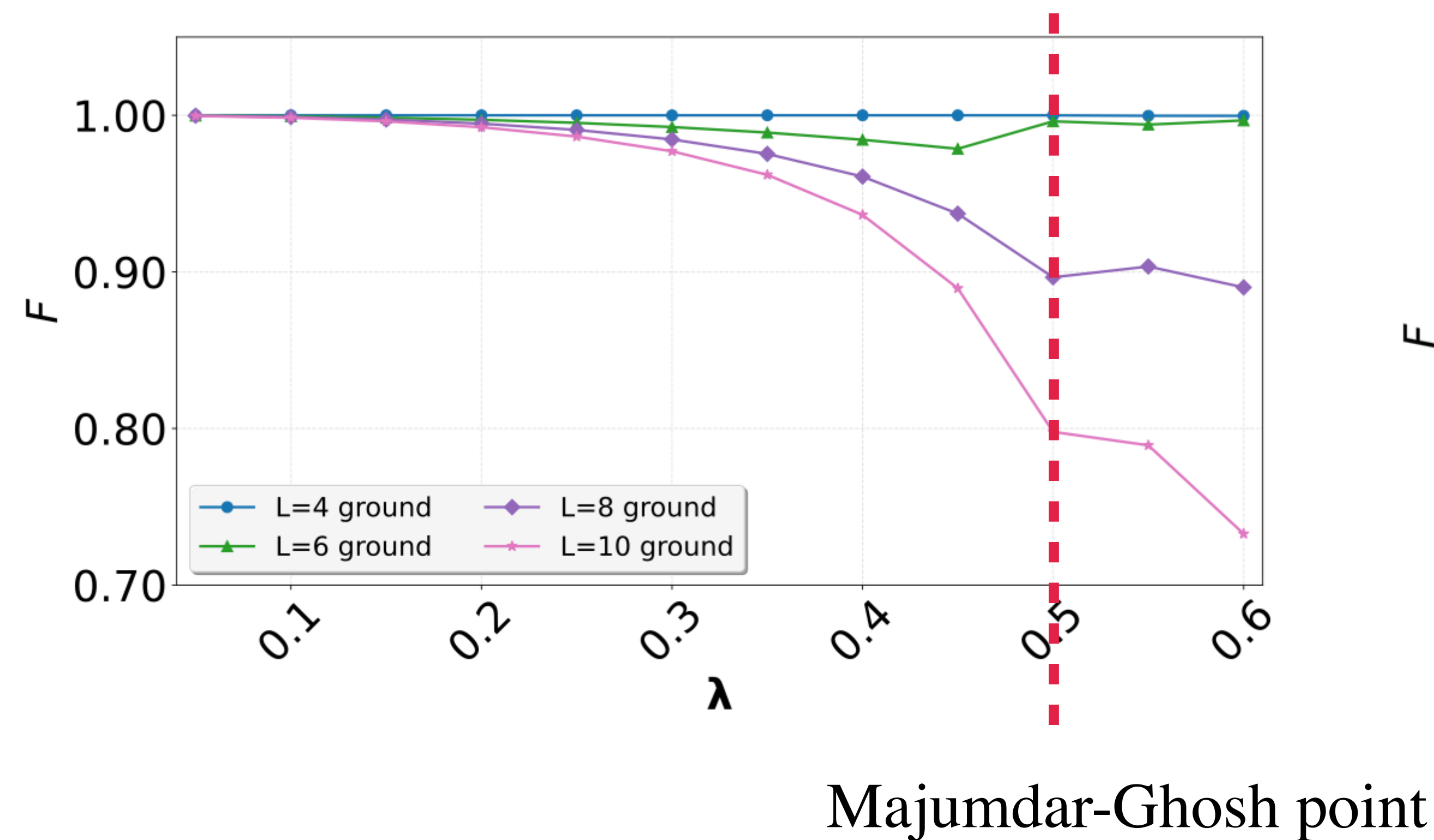
*starting from random initials

Optimization time/sec	state
0.2325	L=4 GS
0.0337	L=4 1st ES
3.5997	L=6 GS
1.8255	L=6 1st ES
26.2177	L=8 ES
13.3114	L=8 1st ES
117.9801	L=10 GS
73.9006	L=10 1st ES

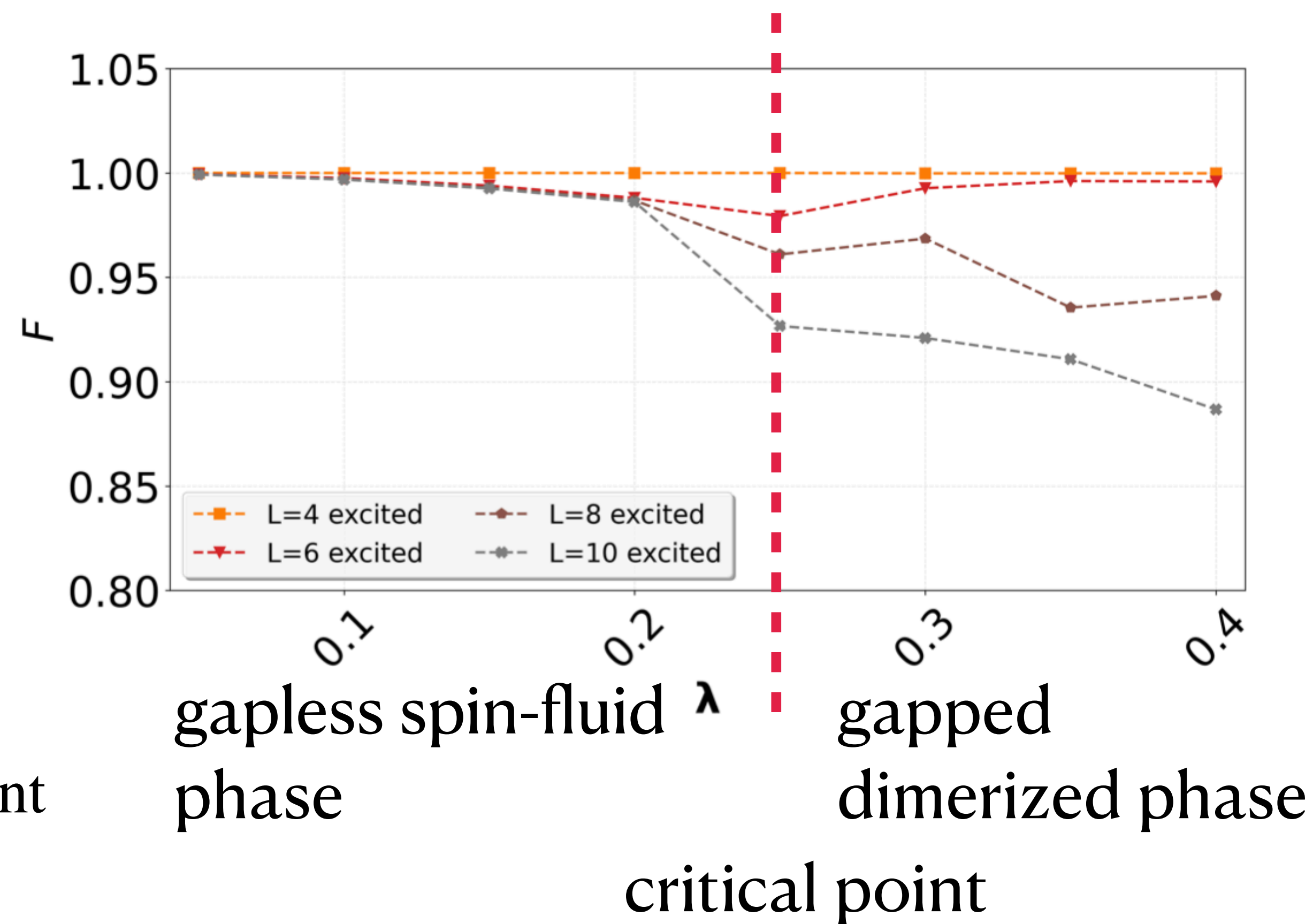
*preparing the ansatz (loss function) requires more time...

EBA probing quantum critical point/level crossing point

Weak integrability-breaking model (J1-J2 model)

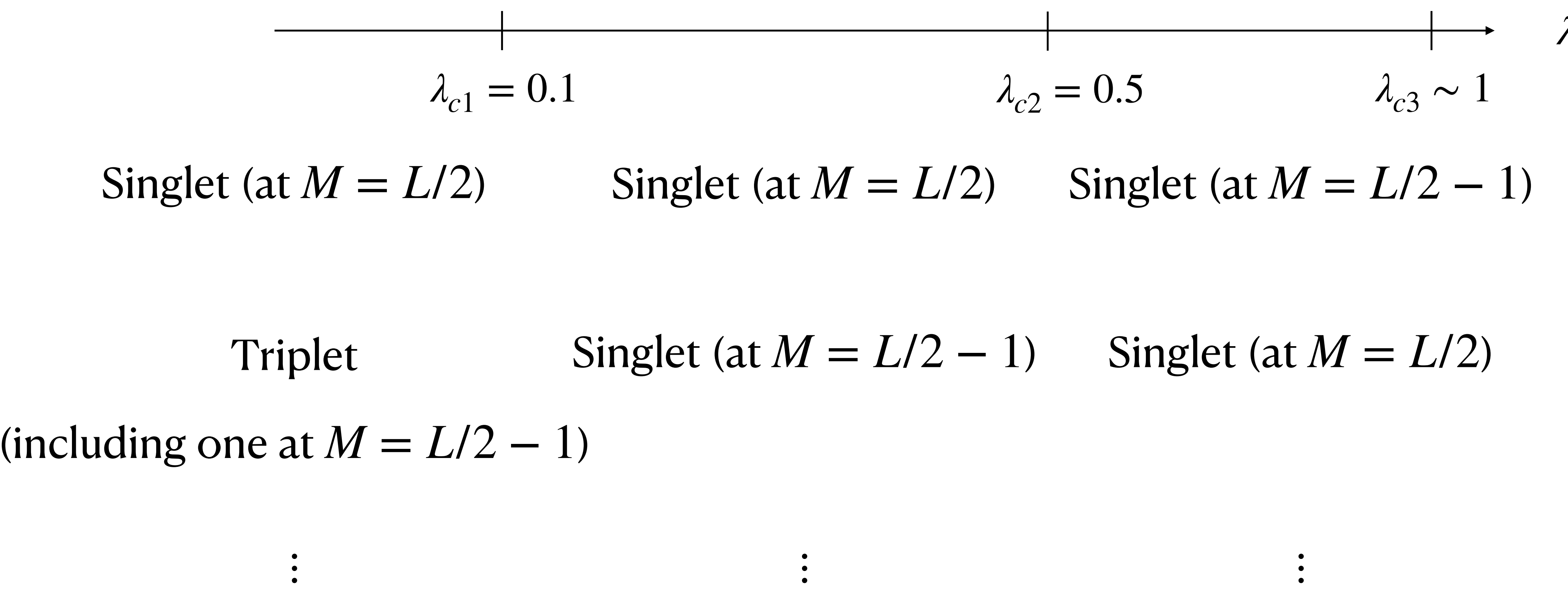


Fidelity for the ground state



Fidelity for the 1st excited state

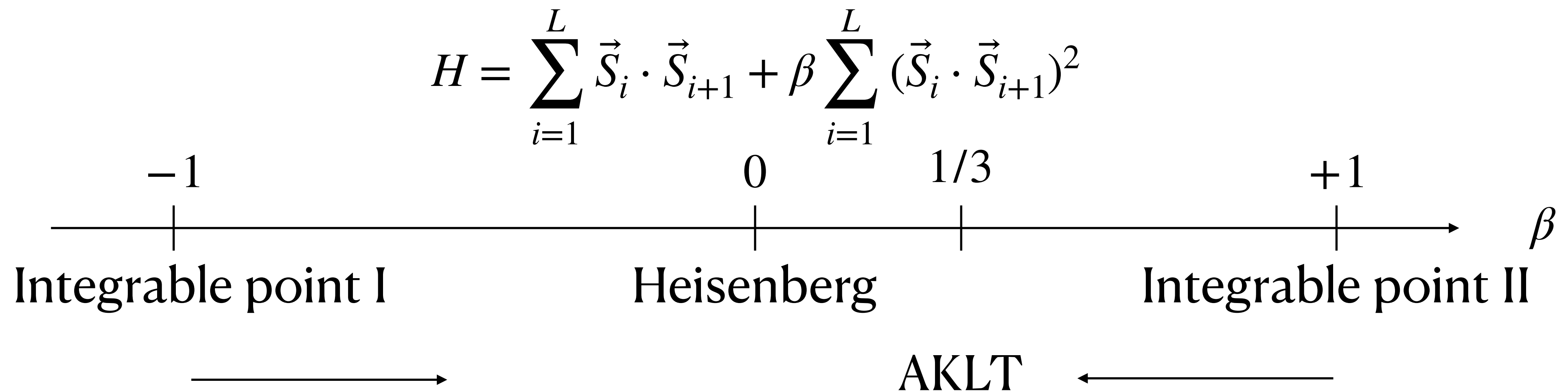
Transition points of Model II (weakly integrability breaking)



Generalizations

We can apply this method to a lot of models with different starting points.

- XXZ model + deformations
- Interpolating between Heisenberg model and AKLT model in spin-1 chain

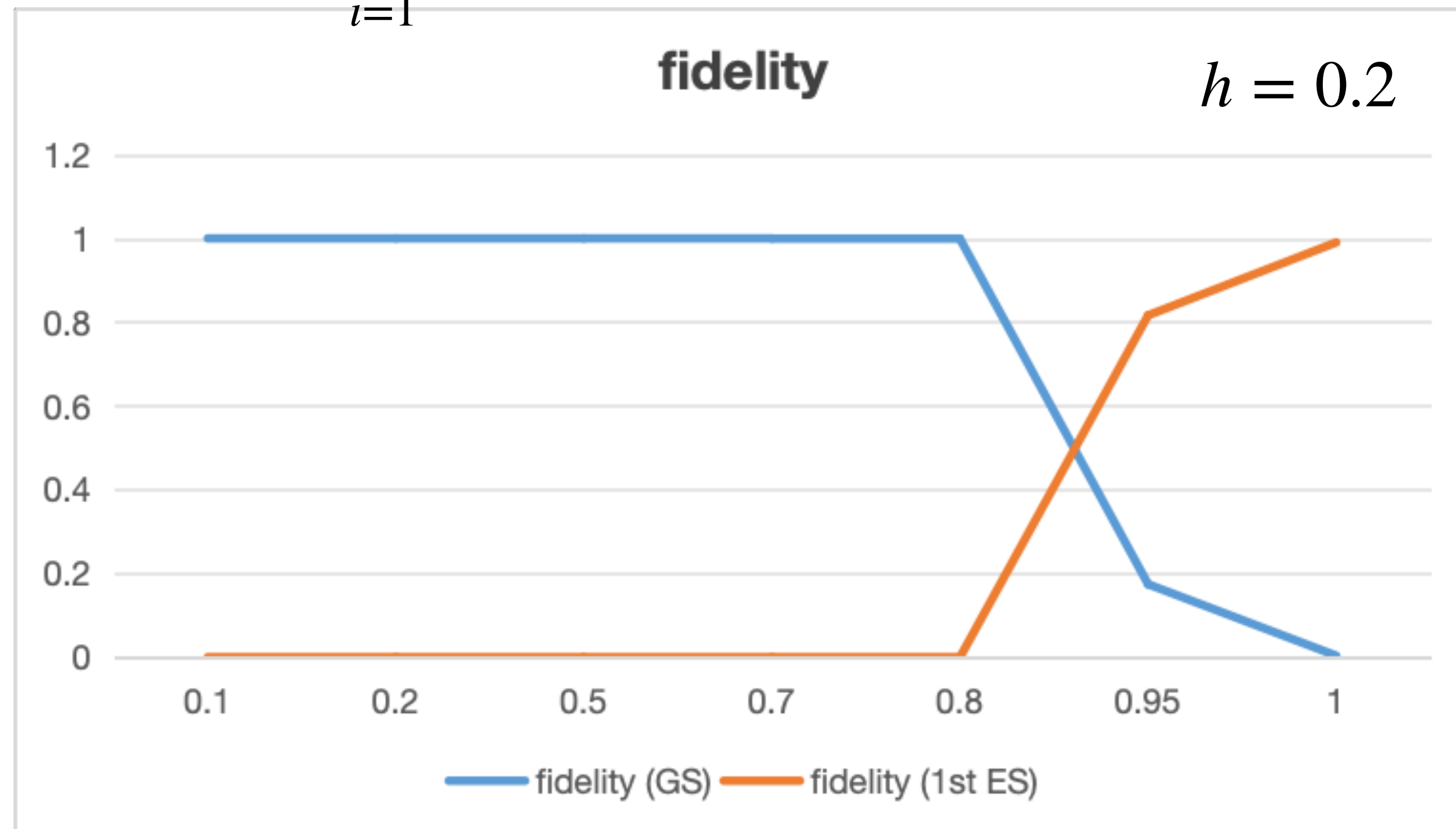


[Wang, RZ (2026)]

Example: interpolation between XXX chain and Transverse-field Ising Model

$$H = (1 - t)H_{xxx} + tH_{TFIM}$$

$$|\psi_{BA}(u_i)\rangle = \prod_{i=1}^{L/2} B(u_i) |\Omega\rangle$$



$$H_{TFIM} = - \sum_i Z_i Z_{i+1} - h \sum_i X_i$$

- transition
- The magnon number is no longer preserved in TFIM.
- Superposition of different magnon sectors $\rightarrow$ excited states?

Improvement I

Adding more parameters into the “R-matrix” improves the accuracy of our method.

e.g. an 8-vertex R-matrix

$$R(\beta, \gamma) = \begin{pmatrix} u + \frac{i(1+\gamma)}{2} & & & \frac{i(1-\beta)}{2} \\ & u + \frac{i(1-\gamma)}{2} & \frac{i}{2} & \\ & \frac{i}{2} & u + \frac{i(1-\gamma)}{2} & \\ \frac{i(1-\beta)}{2} & & & u + \frac{i(1+\gamma)}{2} \end{pmatrix}$$

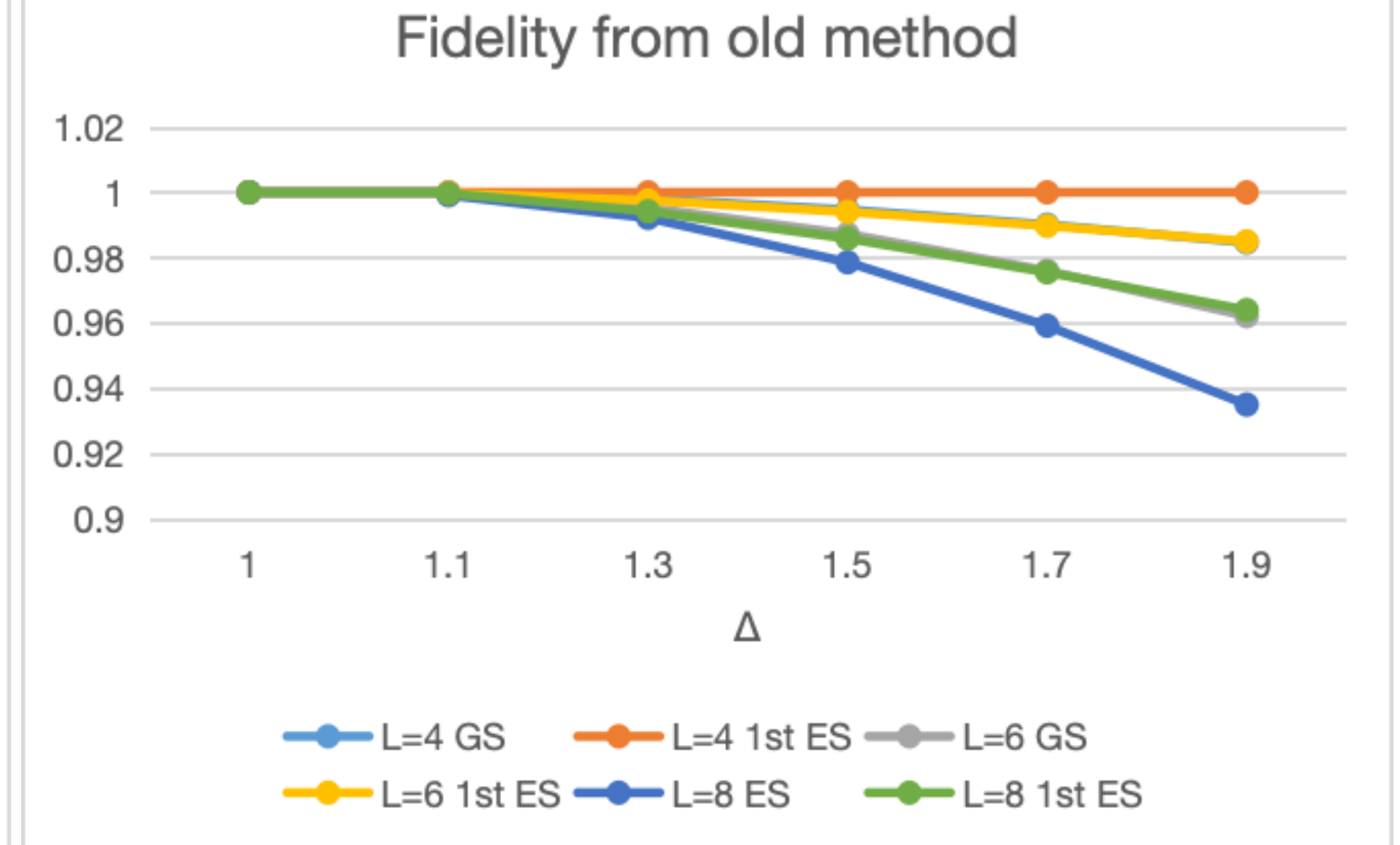
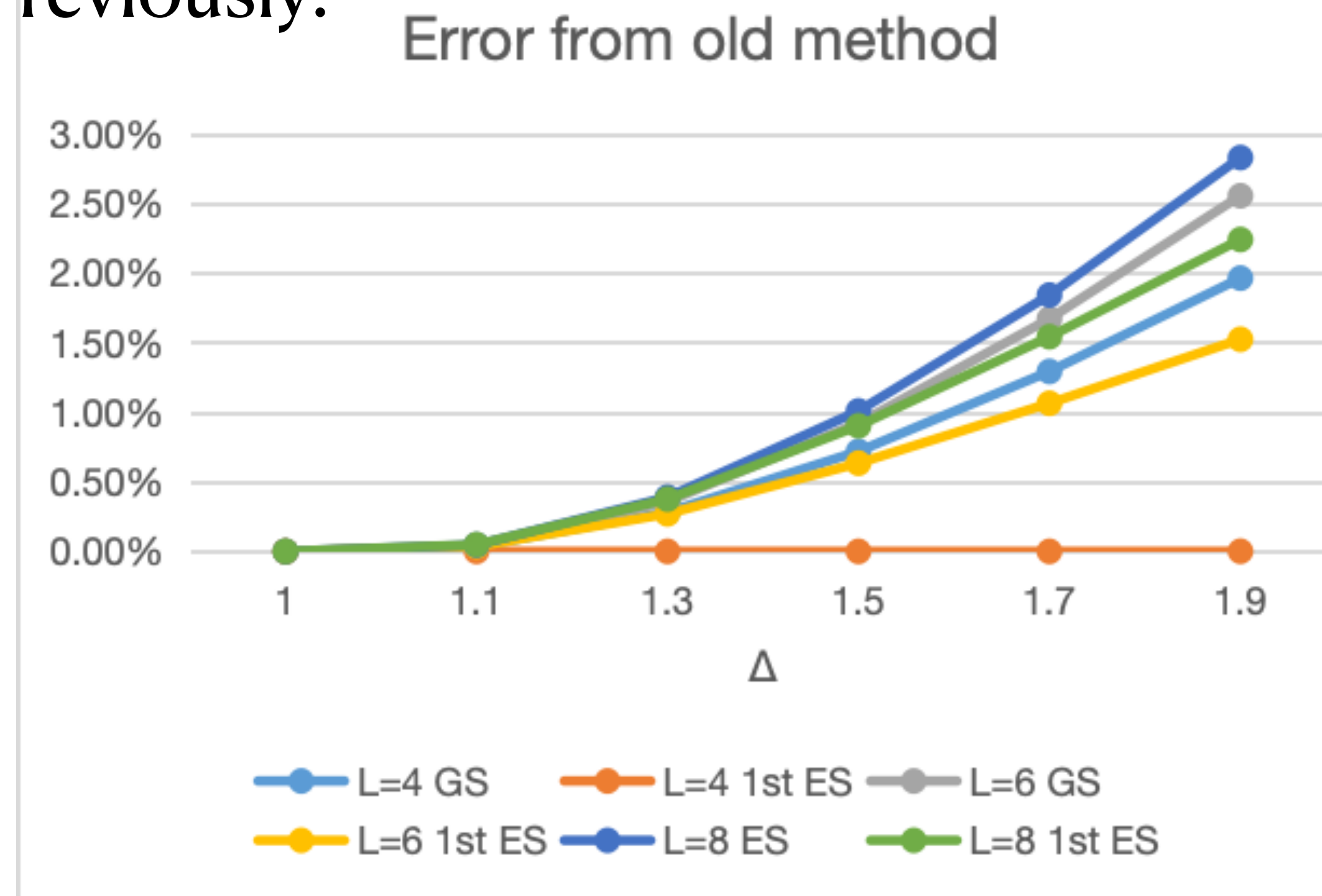
*In our approach, the above “R-matrix” does not necessarily satisfy the Yang-Baxter equation.

We let β, γ to be free parameters in our ansatz and optimize them.

Example: XXZ spin chain

$$H_{XXZ} = - \sum_{i=1}^L (X_i X_{i+1} + Y_i Y_{i+1} + \Delta Z_i Z_{i+1}) = -H_{XXX} - \delta\Delta \sum_i Z_i Z_{i+1} \quad (\Delta = 1 + \delta\Delta)$$

Previously:



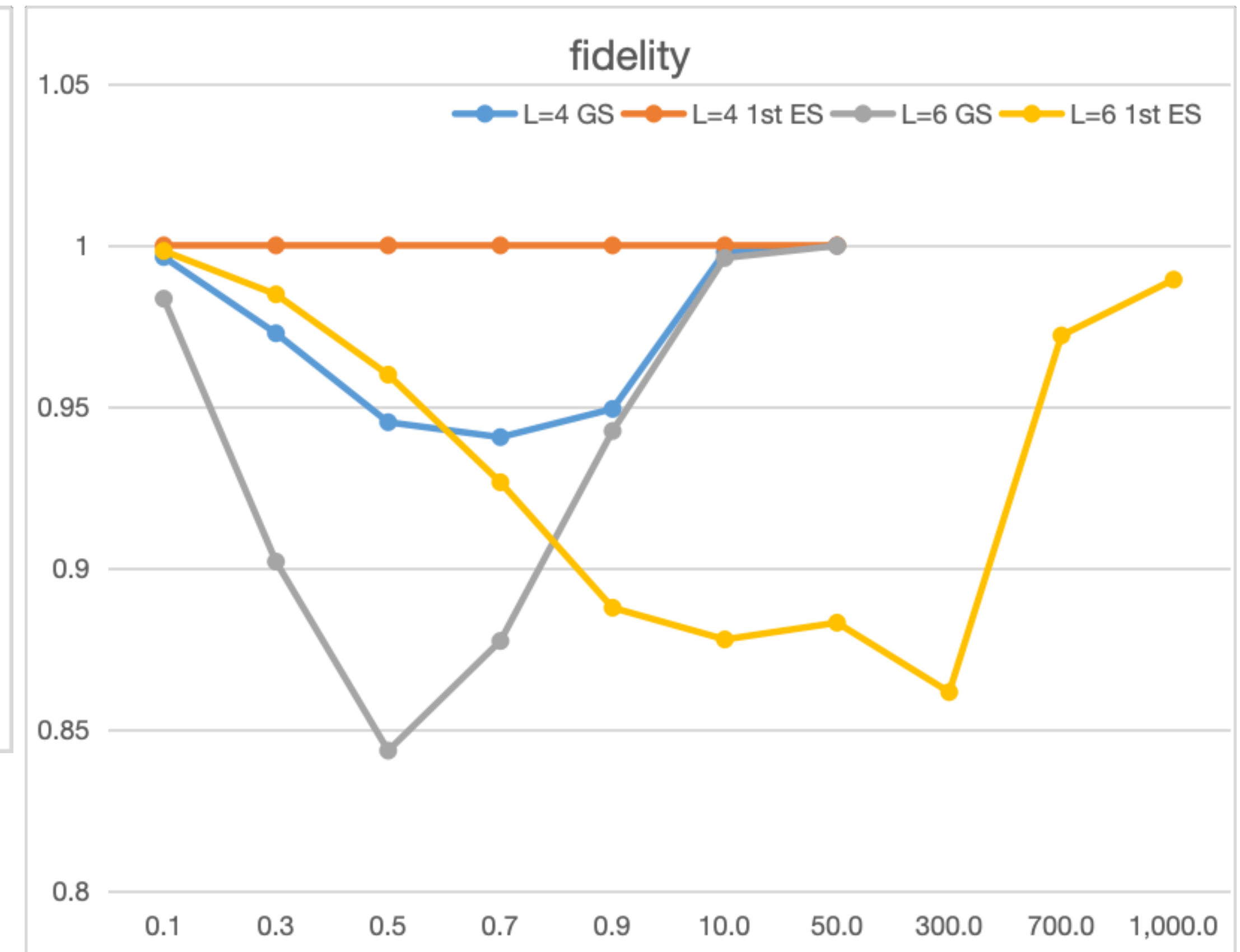
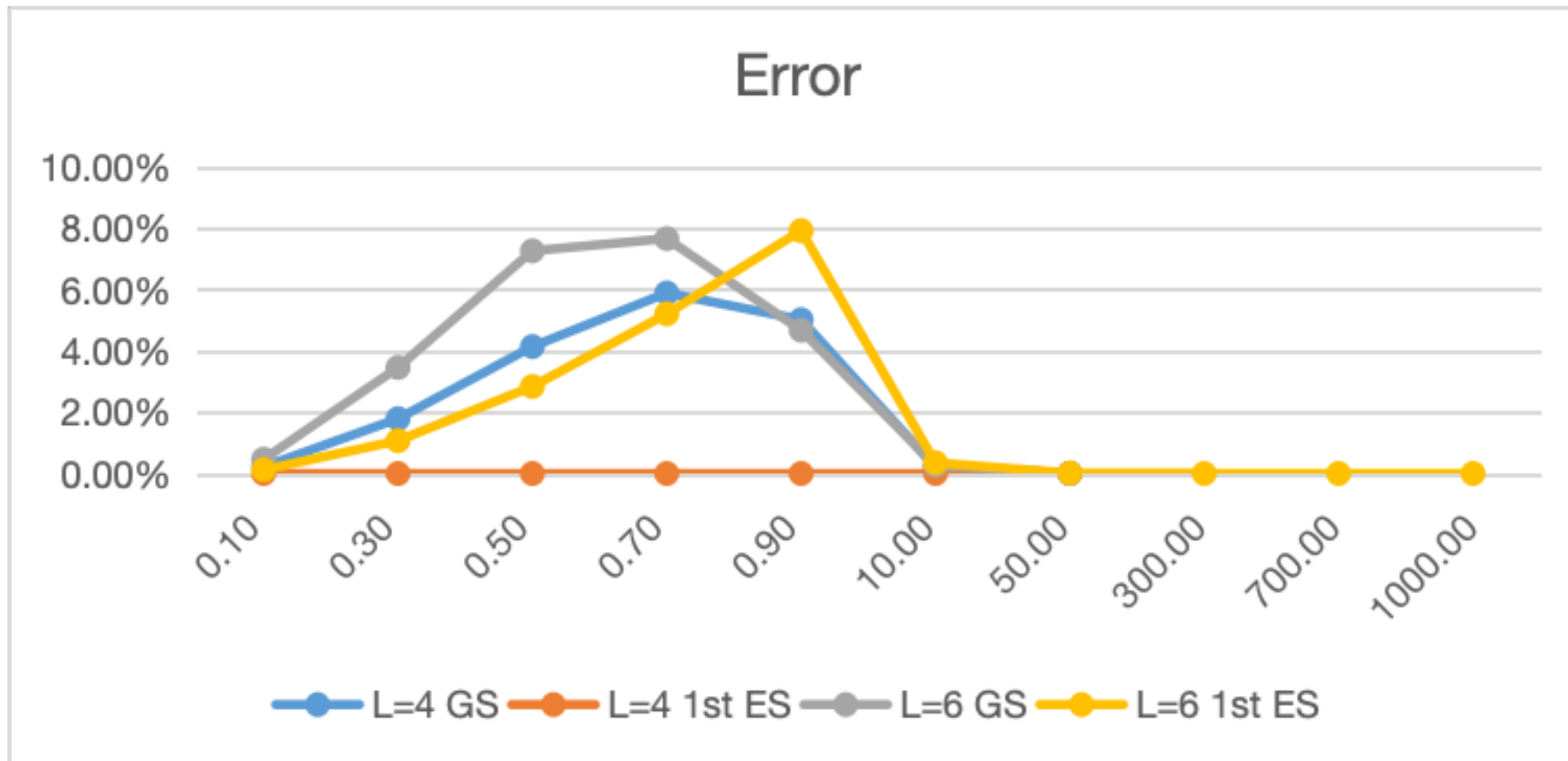
Improved method:

Error: $\sim 10^{-13}$, fidelity: $\sim 1 - \mathcal{O}(10^{-16})$, $\beta \sim 1$

Example: XXX spin chain with staggered magnetic field (completely breaks the integrability)

$$H = H_0 + hH_z$$

$$H_z = \sum_{n=1}^L (-1)^n S_n^z$$



*The results improve a lot.

*The 8-vertex type integrability revives in the large magnetic field limit.

Example: XXX spin chain with weak breaking of the integrability)

$$H = H_{\text{XXX}} + \lambda H^{(1)} \qquad H^{(1)} = \sum_{n=1}^L \vec{S}_i \cdot \vec{S}_{i+2}$$

Fidelity

λ	0.1	0.9	50	130
L=4 GS	1	0.985822	0.992457	0.990436
L=4 1st ES	1	1	1	1

λ	0.1	0.5
L=6 GS	0.961541	0.971426
L=6 1st ES	0.999406	0.964236

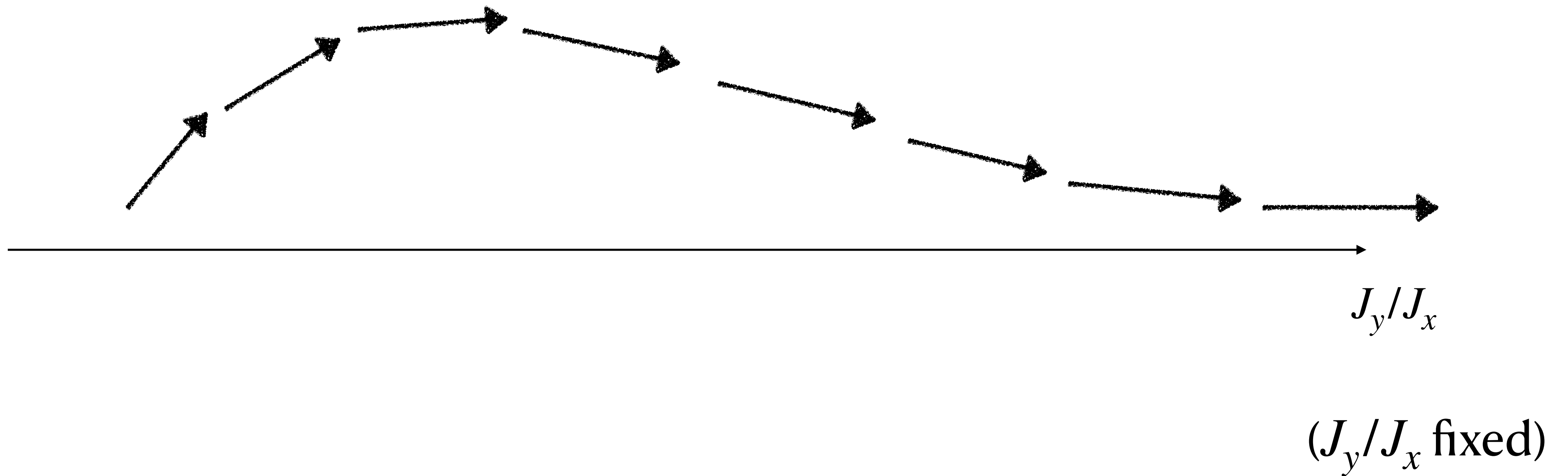
Error $\lesssim 1\%$

The results stay at high fidelity and accuracy.

Example: XYZ model

$$H = J_x \sum_i X_i X_{i+1} + J_y \sum_i Y_i Y_{i+1} + J_z \sum_i Z_i Z_{i+1}$$

Error



A plane-wave ansatz

$$|\psi_m(\vec{k})\rangle = \sum_{I \subset I_L} \sum_{\sigma \in S(M)} A_{\sigma(I)} e^{\sum_{j=1}^M i k_{\sigma(j)} j} |I\rangle$$

I is a set of numbers denoting the sites with spin down.

Translating to the Coordinate Bethe Ansatz will restrict $A_{\sigma(I)}$.

By setting all $A_{\sigma(I)}$'s to be free parameters to optimize,

- Fidelity extremely closed to 1
- The number of parameters to optimize is exponentially growing.

Improvement II

Including inhomogeneous parameters

$$t(u) := \text{tr}_0 \left(\mathbf{R}_{0L}(u - \theta_L) \mathbf{R}_{0(L-1)}(u - \theta_{L-1}) \dots \mathbf{R}_{01}(u - \theta_1) \right)$$

$$|\psi_{BA}(u_i)\rangle = \prod_{i=1}^M B(u_i) |\Omega\rangle$$

gives fidelity ~ 0.99 in all models considered!

Even when the perturbation is very large!

At least some sectors of non-integrable models can be described very well in the integrable language.

The cost we have to pay as a numerical approach:

L+M parameters to optimize

As a classical algorithm

A brute-force/faithful implement:

memory-saving: only needs $\sim 1/8$ memory compared to ED with sparse matrix

Managed to run $L=26$ (CPU only) on 16GB workstation while ED failed (limitation ~ 28)

Matrix-product state (MPS) form:

$$\psi(s_1, s_2, \dots, s_L) = \text{tr} (A^{s_1} A^{s_2} \dots A^{s_L})$$

Each component of wavefunction

Optimize the wavefunction to find A

→ DMRG

Complexity reduced to a polynomial-like level! Time complexity $\sim O(L^{2.5} \chi^3 \log L)$

The Bethe ansatz can be rewritten in an MPS manner

Theoretical time complexity

$$\sim O(ML\chi^3)$$

χ : bond dimension

But χ also grows rapidly... The algorithm numerically works much better for small M .

Potential advantage in quantum computer

A systematic way to prepare a Bethe state has been proposed.

*coordinate Bethe Ansatz [Van Dyke, Barron, Mayhall, Barnes, Economou (2021)]

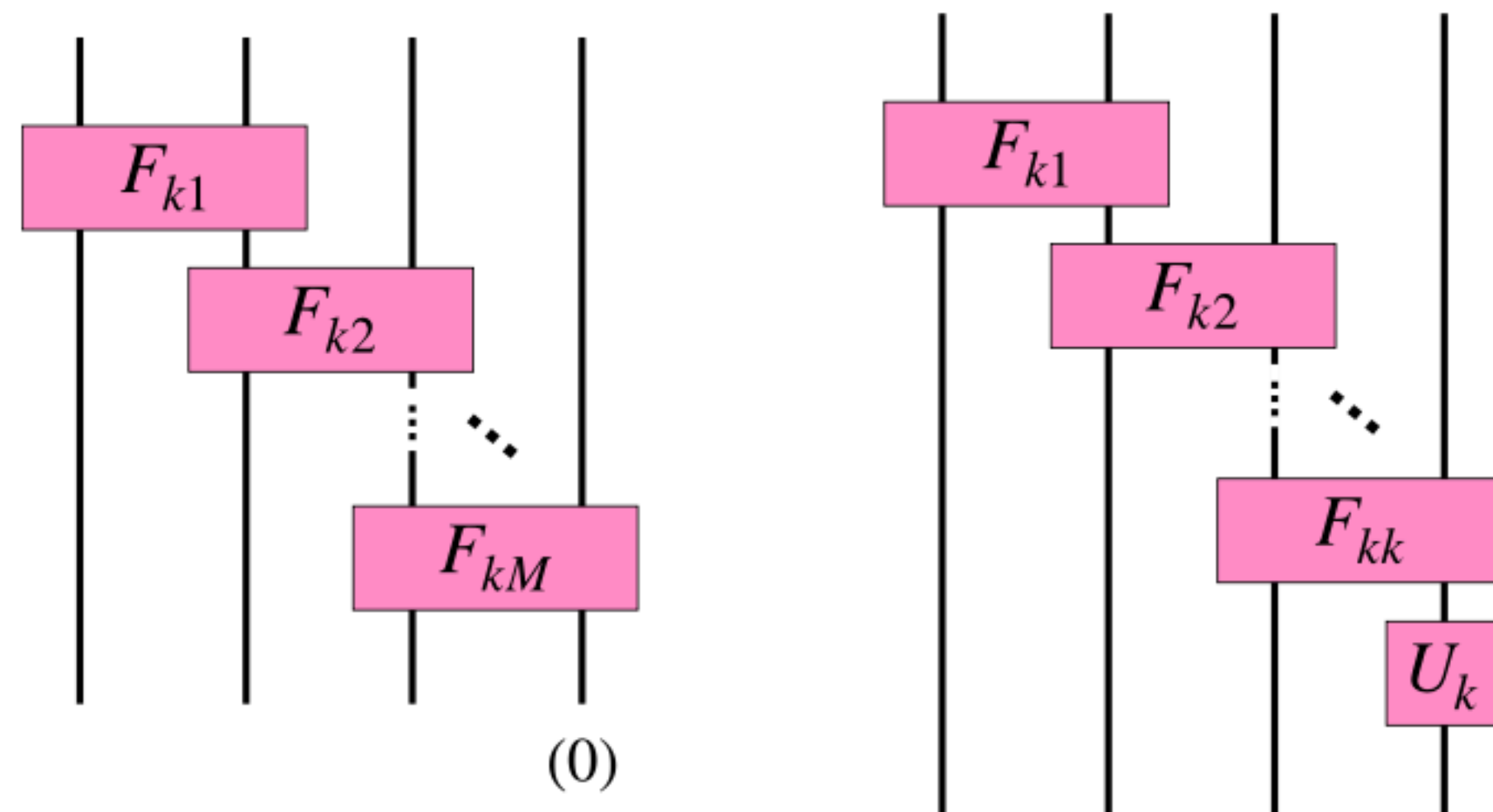
*algebraic Bethe Ansatz

Advantage: much fewer parameters, thus very fast!

Reduce optimization complexity from $O(KL 2^L)$ (classical) to $O(KM L)$ measurements

K: iteration steps

Preparation of Bethe state



At least for XX model, $O(ML)$ gates are needed.

[Ruiz, Sopena, Gordon, Sierra, Lopez (2024)]

For XXZ, $O(ML)$ gates can give fidelity ~ 0.99

Truly go beyond classical algorithms
represented by DMRG in the near future!

c.f. discussions in [Wang, RZ (2026)]

No barren plateau issue

Summary and Outlook

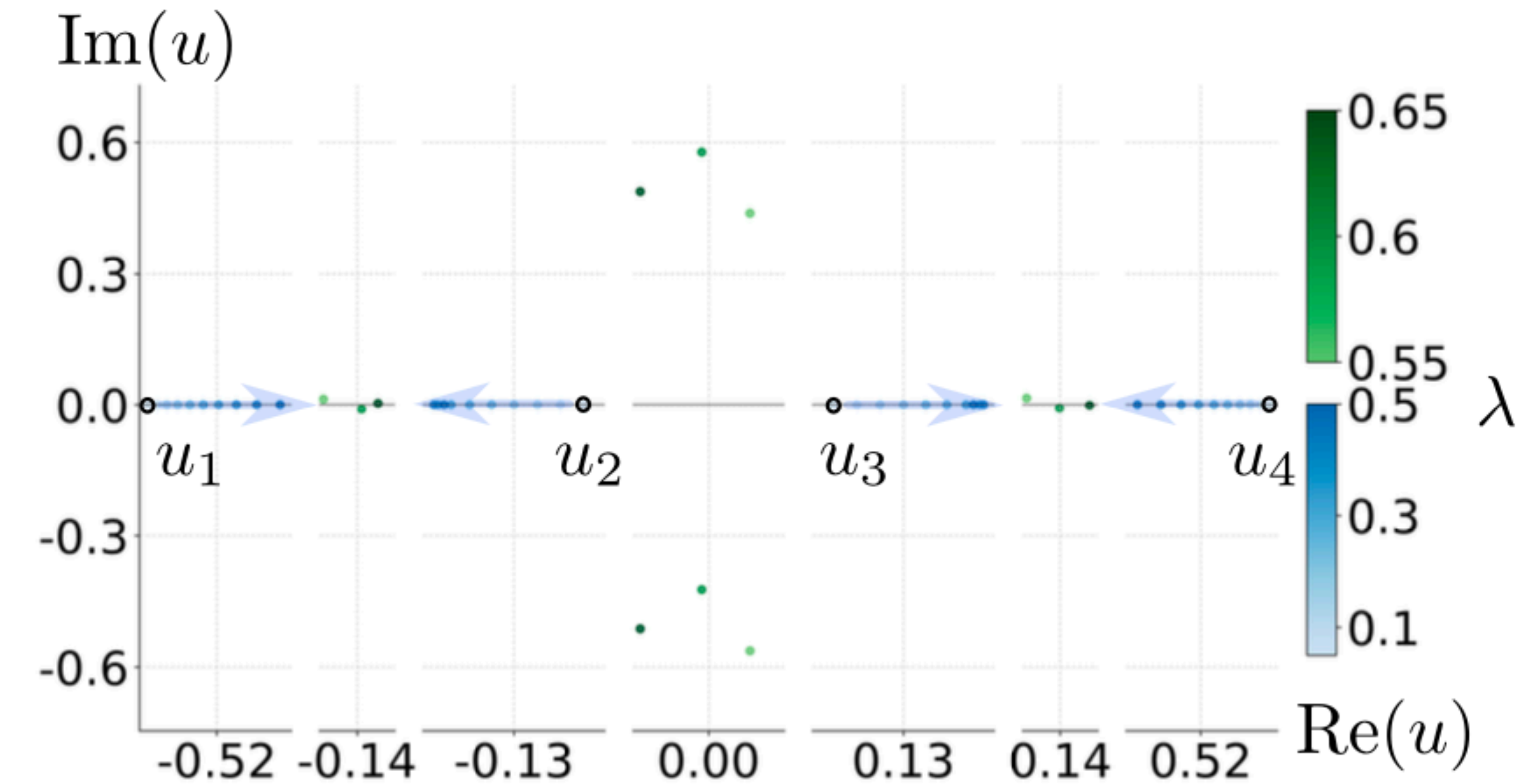
Variational Quantum Algorithms + Bethe ansatz

- Approximation method but still works well for integrability broken cases
- Efficient, relatively accurate. Some of the excited states are easily accessible.
- Magnon number symmetry ($U(1)$) is assumed.

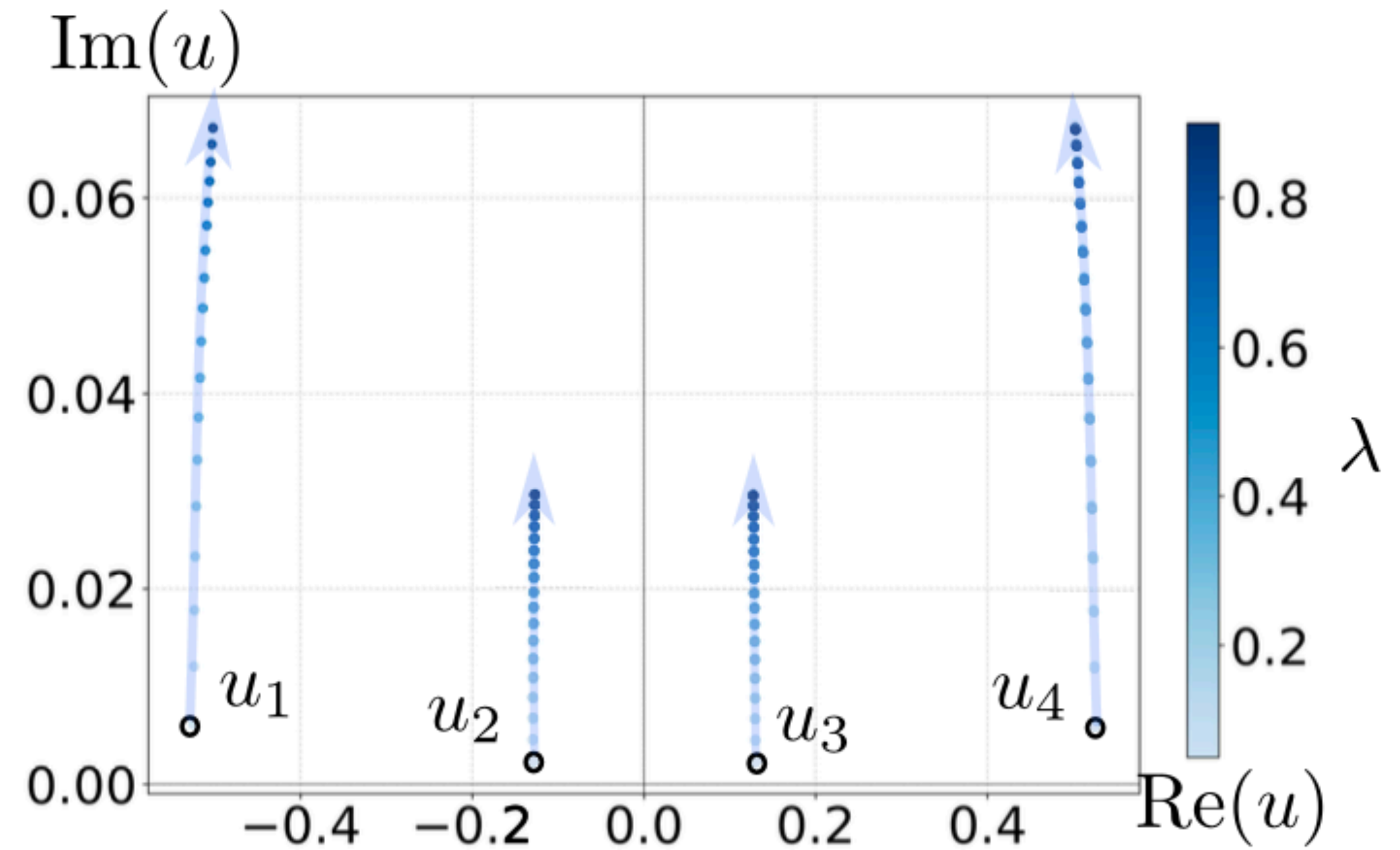
Open problems:

- Off-shell/modified Bethe equation? For $M=1$, yes
- With open boundaries?
- $U(1)$ symmetry broken cases?
- Combining with Bethe/Gauge correspondence?

The roaming of Bethe roots 流浪的贝特根



(a) In weak integrability-breaking model



(b) In strong integrability-breaking model

A clear pattern, but how to use it?