

格点杨-米尔斯场的动力学和自举

Gang Yang

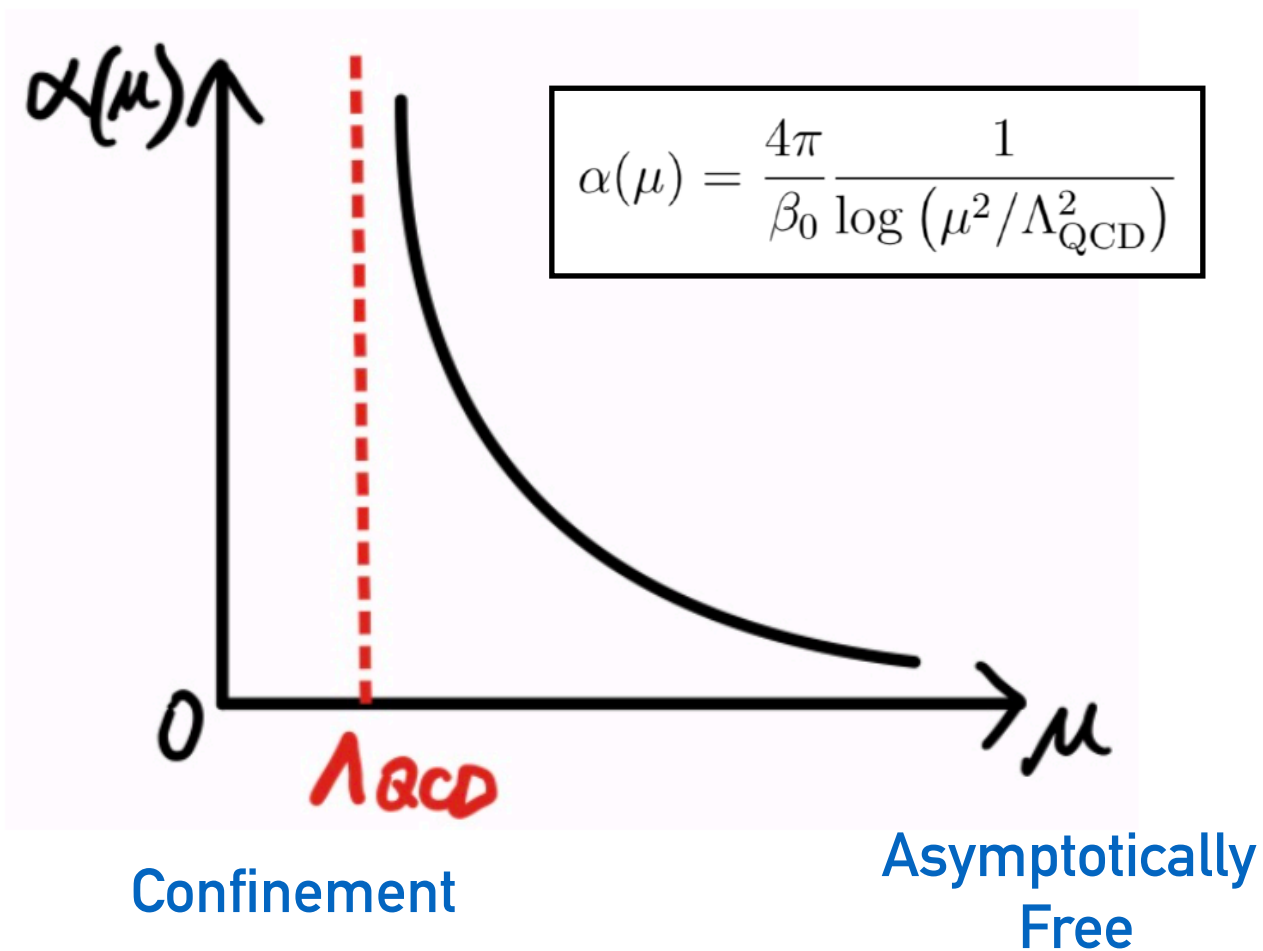
ITP, CAS

*Based on 2502.14421 with Yuanhong Guo, Zeyu Li, Guorui Zhu,
and 2601.04316 with Xizhe Liu*

第七届全国“场论与弦论”学术研讨会
USTC, Aug 03-07, Hefei

Non-perturbative QCD

$$\mathcal{L} = -\frac{1}{4} (F_{\mu\nu}^a)^2 + \bar{q}_L^f i \not{D} q_L^f + \bar{q}_R^f i \not{D} q_R^f$$



The key point:

$$\beta_0 > 0$$

$$\beta_0 = -\frac{4}{3}T_F n_q + \frac{11}{3}C_A$$

Millennium Prize Problem

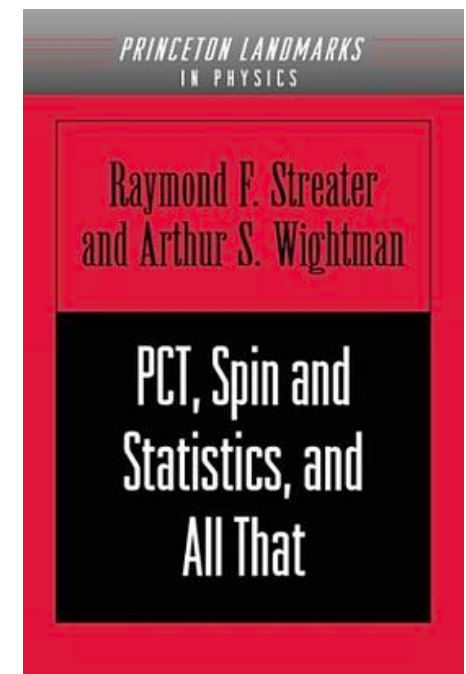
Yang–Mills Existence and Mass Gap. by Jaffe and Witten

Prove that for any compact simple gauge group G , a non-trivial quantum Yang–Mills theory exists on $\mathbb{R}^4$ and has a mass gap $\Delta > 0$.

Existence includes establishing axiomatic properties at least as strong as those cited in [45, 35].

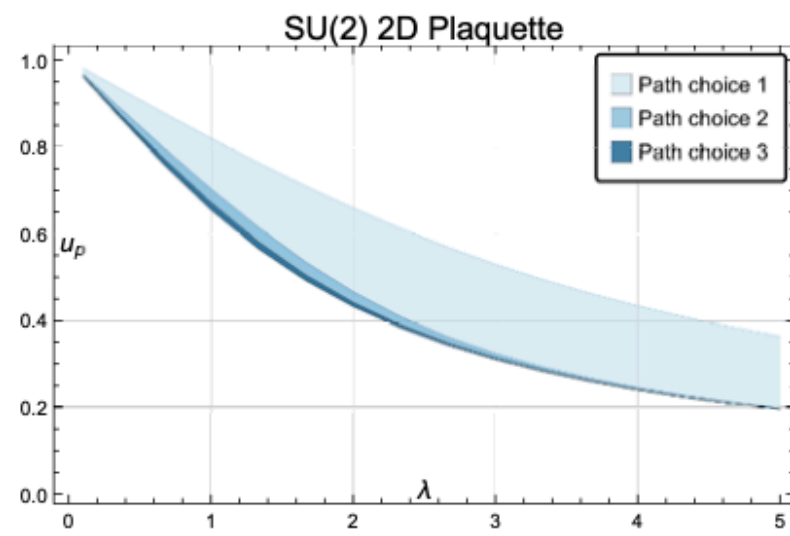
[45] R. Streater and A. Wightman, “PCT, Spin and Statistics and all That”, W. A. Benjamin, New York, 1964.

[35] **K. Osterwalder and R. Schrader**, “Axioms for Euclidean Green’s functions”, *Comm. Math. Phys.* 31 (1973), 83–112, and *Comm. Math. Phys.* 42 (1975), 281–305.

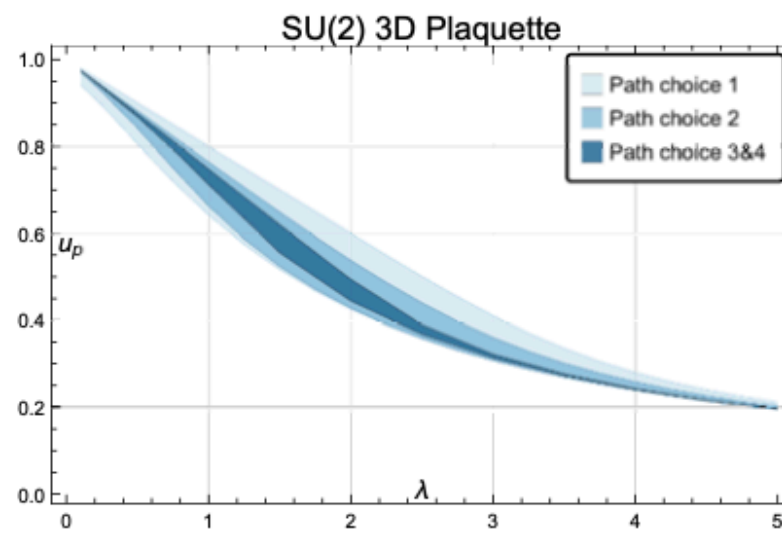


Results

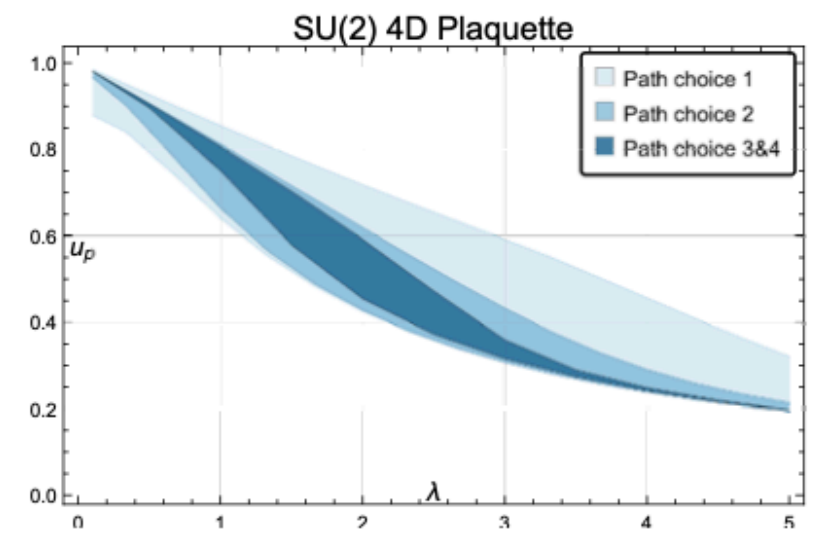
Result: Plaquette in pure YM



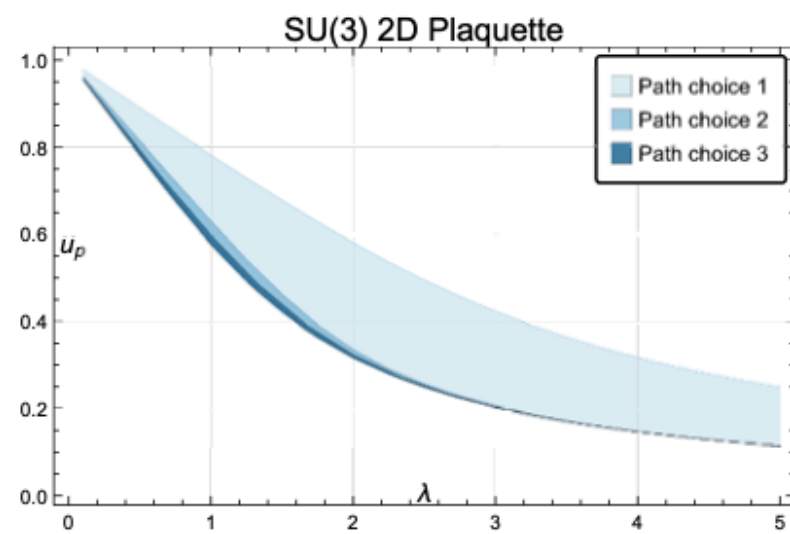
(a)



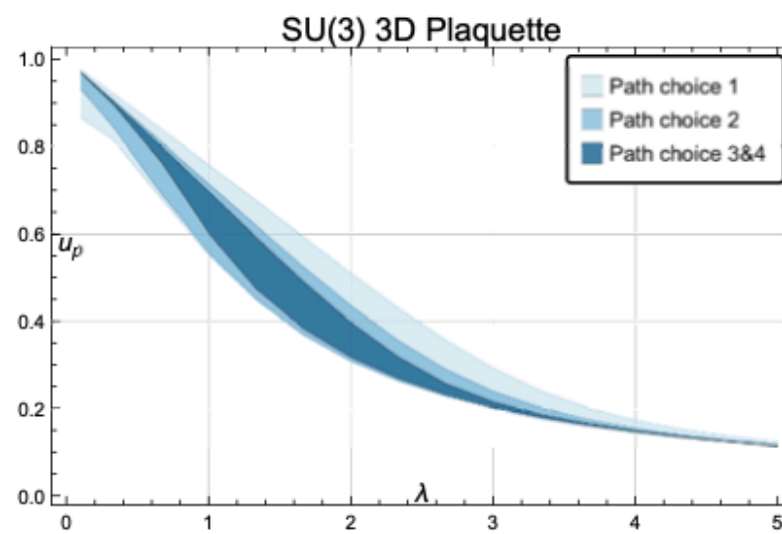
(b)



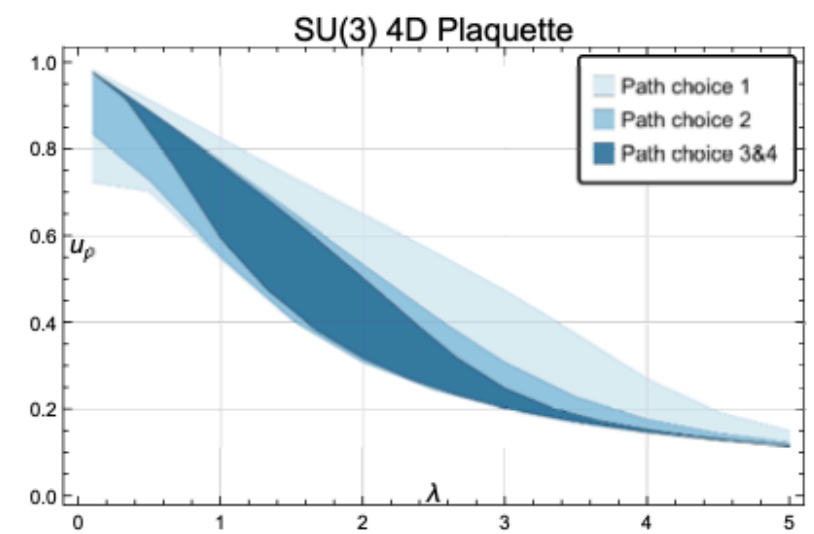
(c)



(d)



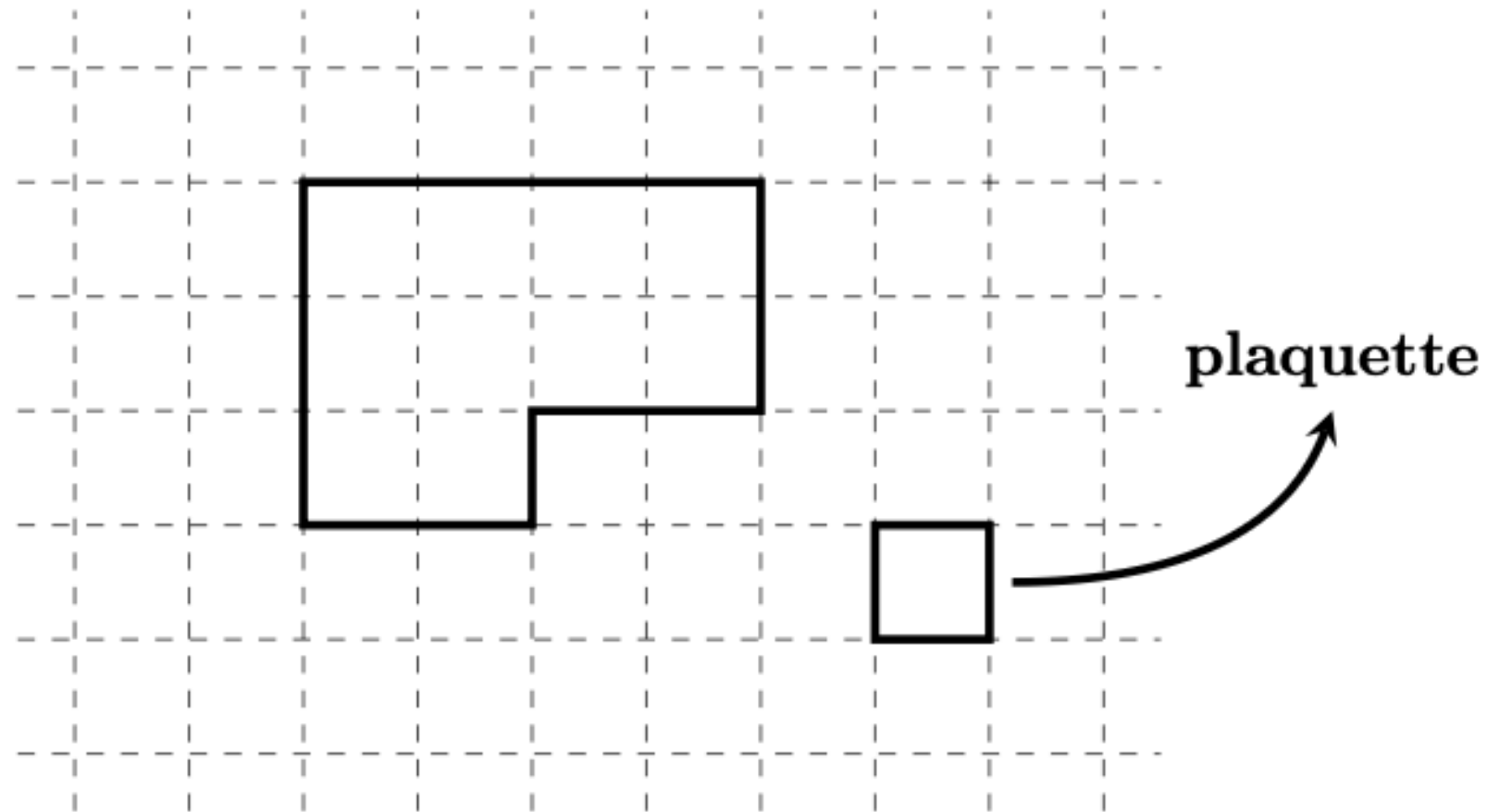
(e)



(f)

Plaquette: fundamental Wilson loop in lattice YM

Expectation value of plaquette: $u_P = \langle \square \rangle = \int \mathcal{D}U e^{-S_{\text{YM}}} \square$



$$S_{\text{YM}} = -\frac{N}{2\lambda} \sum_P \text{tr}(U_P + U_P^\dagger)$$

$$U_P = U_1 U_2 U_3 U_4 =$$

A diagram of a square plaquette with four directed edges. The bottom edge is labeled u_1 with an arrow pointing right. The right edge is labeled u_2 with an arrow pointing up. The top edge is labeled u_3 with an arrow pointing left. The left edge is labeled u_4 with an arrow pointing down.

Challenges

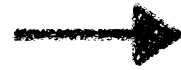
- No SUSY
- No conformal
- Finite N , e.g., $SU(3)$
- General dim, e.g., 4D
- Non-perturbative quantitative results

Lattice MC simulation: most successful quantitative method

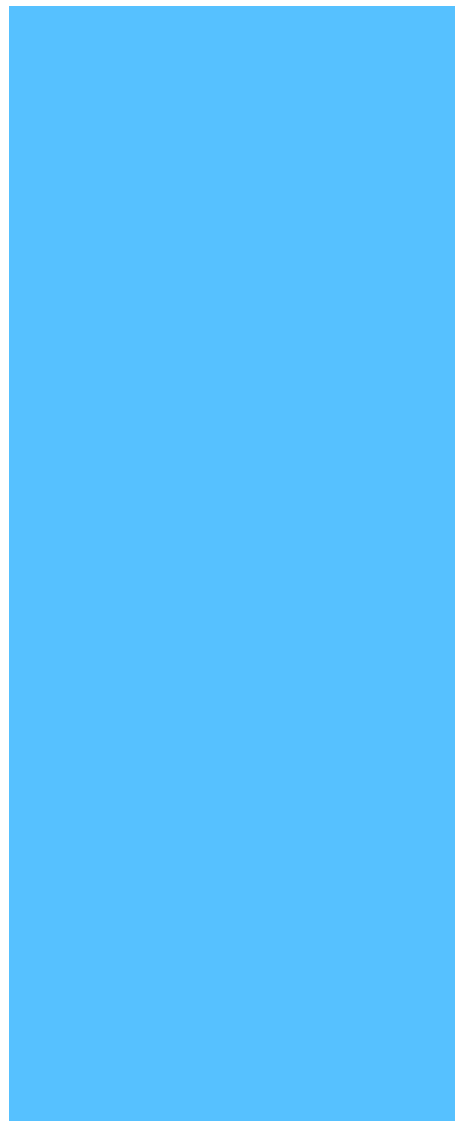
New “positivity matrix bootstrap” strategy

SDP (Semi-definite programming)

Inequality constraints



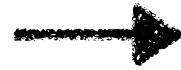
Allowed space



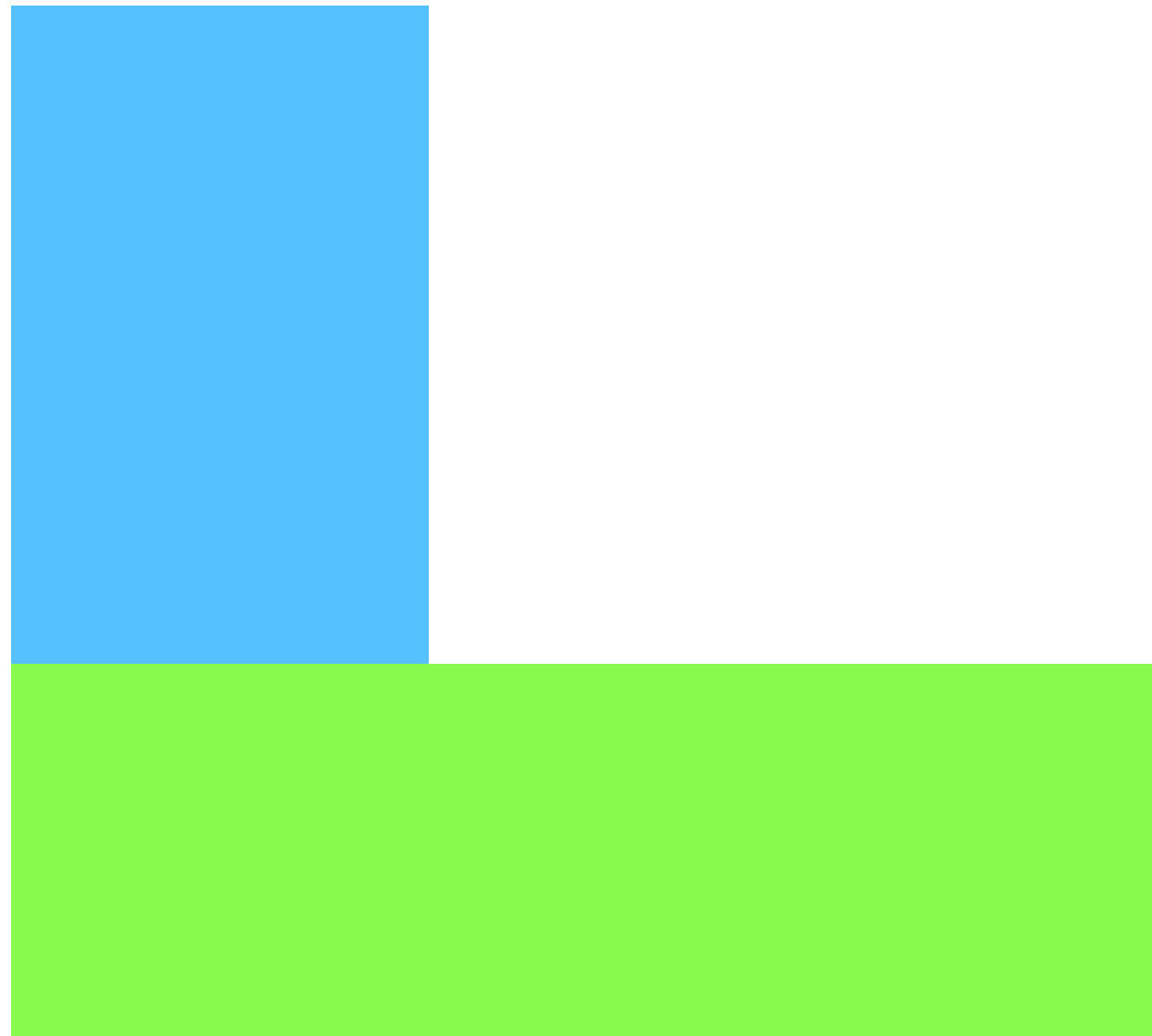
$$x \geq 0$$

SDP (Semi-definite programming)

Inequality constraints



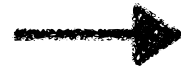
Allowed space



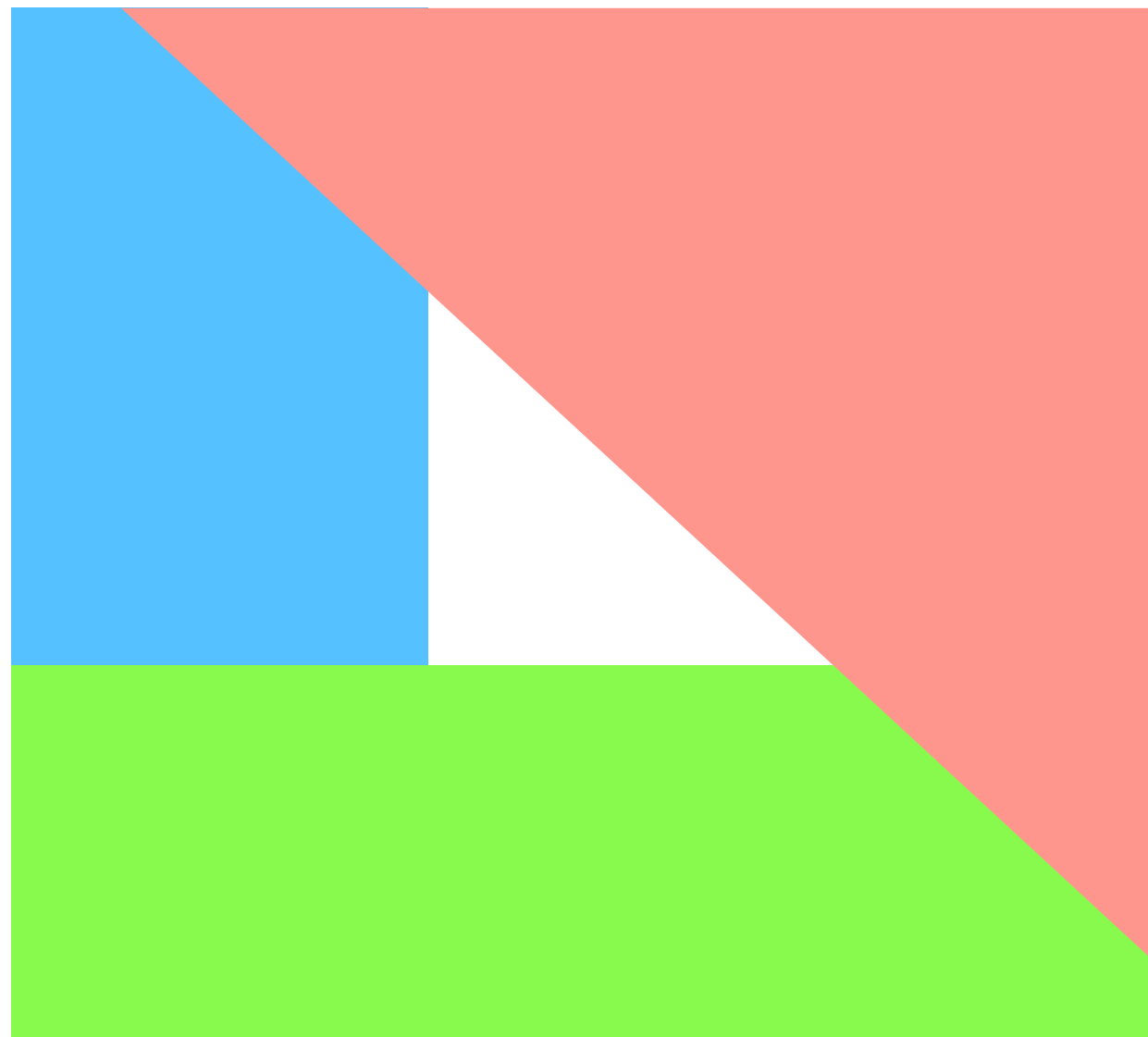
$$x \geq 0 \text{ \& } y \geq 0$$

SDP (Semi-definite programming)

Inequality constraints



Allowed space



$$x \geq 0 \ \& \ y \geq 0 \\ \& \ x + y \leq 1$$

Positivity bootstrap

First proposed by Anderson and Kruczenski in 2016, and later improved by Kazakov and Zheng.

[Anderson, Kruczenski, arXiv:1612.08140](#)

[Kazakov, Zheng, arXiv:2203.11360, 2404.16925](#)

[U\(1\) case also studied by Li, Zhou, arXiv:2024.17071](#)

[See also: H. W. Lin 2020 \(matrix model\), Han et.al, 2020 \(matrix QM\), Cho et.al, 2022, \(Ising Lattice\), and other applications](#)

Previous studies focus on large N or $SU(2)$ case, where **only single-trace** Wilson loops operators are necessary.

Positivity bootstrap

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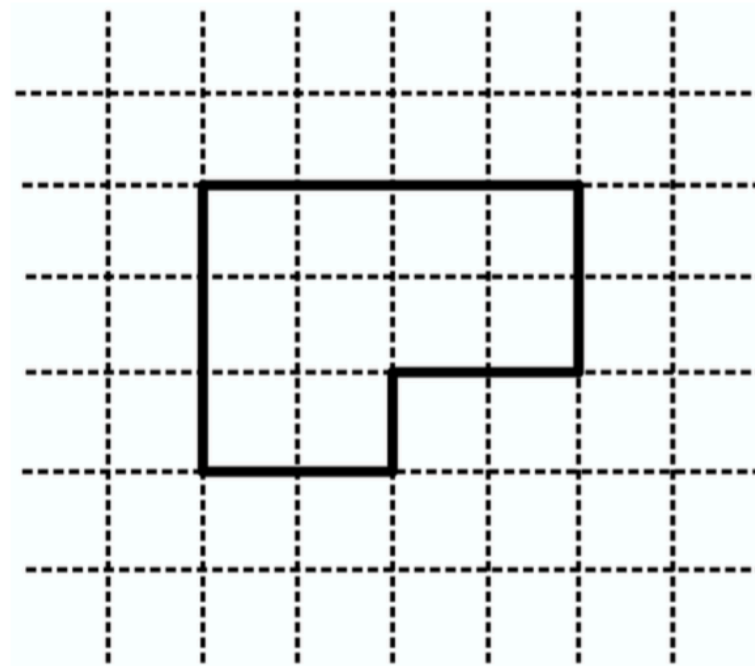
We consider for the first time the **$SU(3)$ case** which involves single- and **double-trace** operators.

[Guo, Li, GY, Zhu, arXiv:2502.14421](#)

These studies show that the new method should work by principle to general pure YM theories in general dimensions.

Setup

Setup of the theory



Action:

$$S = -\frac{N}{2\lambda} \sum_P \text{tr}(U_P + U_P^\dagger) \quad U_P = U_1 U_2 U_3 U_4 = \begin{array}{c} \xleftarrow{u_3} \\ \downarrow u_4 \quad \uparrow u_2 \\ \xrightarrow{u_1} \end{array}$$

Partition function:

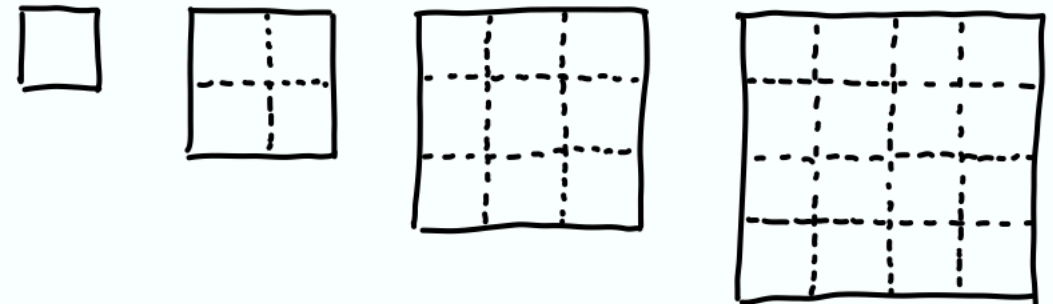
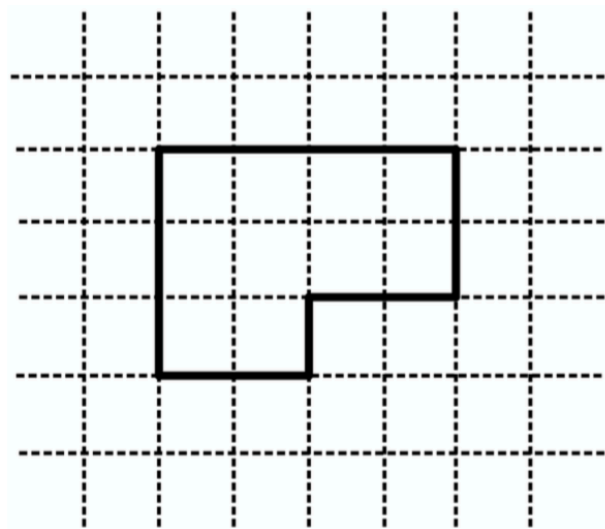
$$Z = \int \mathcal{D}U e^{-S} = \int \prod_{x, \hat{\mu}} dU_{\hat{\mu}}(x) e^{+\frac{N}{2\lambda} \sum_P \text{tr}(U_P + U_P^\dagger)}$$

Physical quantity

Expectation value:

$$\langle \mathcal{O} \rangle = Z^{-1} \int \mathcal{D}U e^{-S} \mathcal{O} = Z^{-1} \int \prod_{x, \hat{\mu}} dU_{\hat{\mu}}(x) e^{+\frac{N}{2\lambda} \sum_P \text{tr}(U_P + U_P^\dagger)} \mathcal{O}$$

The fundamental quantities are the Wilson loops.



- Area law for confinement
- Spectrum from loop correlators

Bootstrap strategy

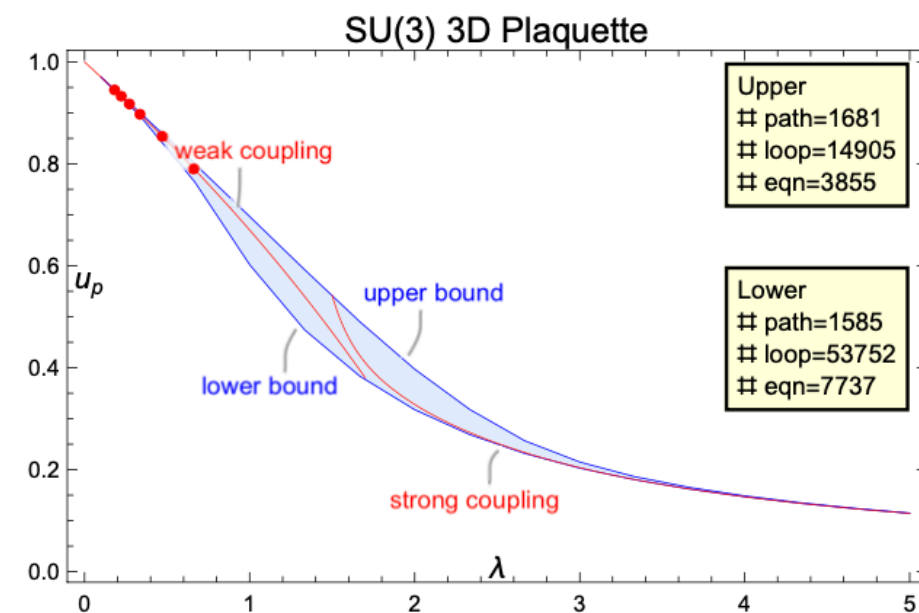
Positivity property

+

Schwinger-Dyson
equations

SDP
→

Constrain the value of
Wilson loop operators



Main strategy

Positivity property

+

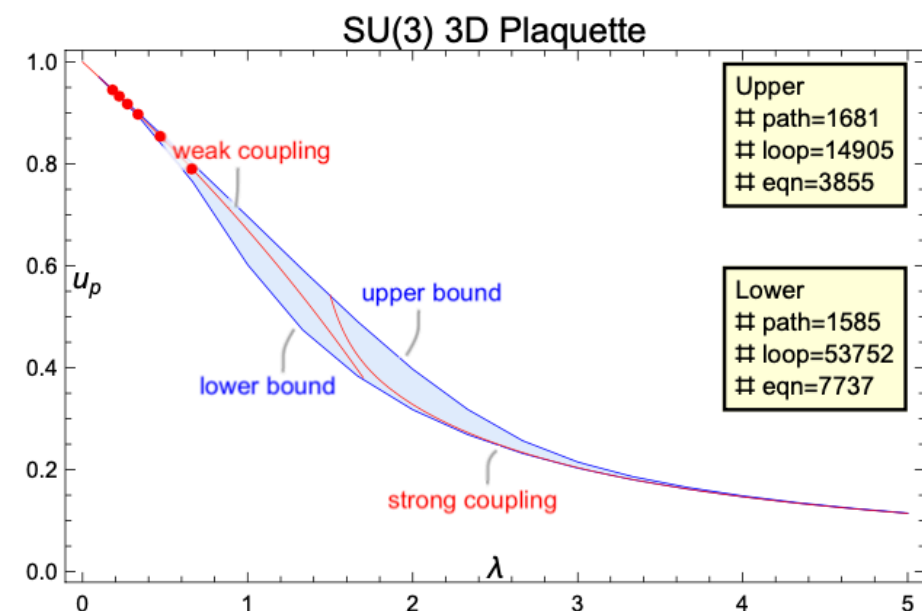
Schwinger-Dyson equations

SDP
→

Hermitian positivity

$$\langle \mathcal{O}^\dagger \mathcal{O} \rangle \geq 0$$

Constrain the value of
Wilson loop operators



Reflection positivity

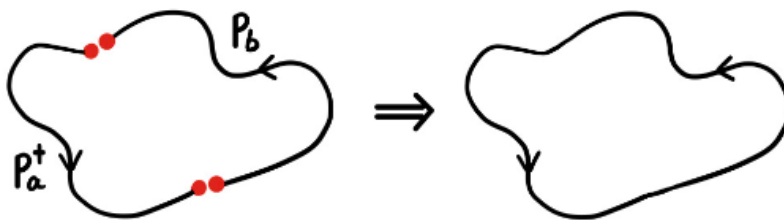
$$\langle \mathcal{O}_+^R \mathcal{O}_+ \rangle \geq 0$$

Hermitian positivity

$$\langle \mathcal{O}^\dagger \mathcal{O} \rangle \geq 0$$

$$\int \mathcal{D}U e^{-S} |f(U)|^2 \geq 0$$

Transform this to a matrix language for Wilson loops:

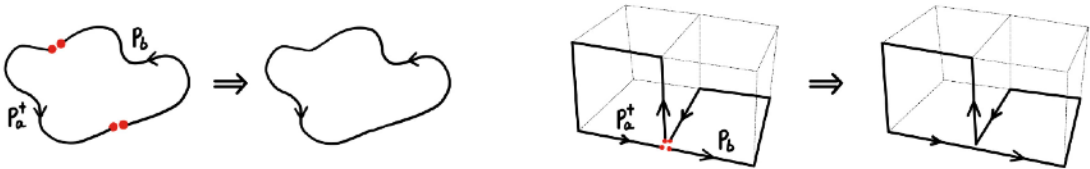


$$U = \sum_{a=1}^M c_a P_a$$

$$\langle \text{tr}(U^\dagger U) \rangle = \sum_{a,b} c_a c_b \langle \text{tr}(P_a^\dagger P_b) \rangle \geq 0$$

$\longrightarrow$
 $A \geq 0, \quad A_{ab} := \langle \text{tr}(P_a^\dagger P_b) \rangle$

Example



$$A \geq 0, \quad A_{ab} := \langle \text{tr}(P_a^\dagger P_b) \rangle$$

$$U = \sum_{a=1}^M c_a P_a$$

$$\mathcal{O} = \{ :, \square_{\rightarrow}, \square_{\leftarrow}, \square_{\leftarrow}, \square_{\rightarrow} \}$$

$$\mathcal{O}^\dagger = \{ :, \square_{\leftarrow}, \square_{\rightarrow}, \square_{\rightarrow}, \square_{\leftarrow} \}$$

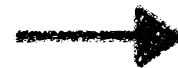
Sewing a set of paths gives the positivity matrix:

$$\begin{pmatrix} : & \square_{\rightarrow} & \square_{\leftarrow} & \square_{\leftarrow} & \square_{\rightarrow} \\ : & 1 & \square_{\rightarrow} & \square_{\leftarrow} & \square_{\rightarrow} \\ \square_{\leftarrow} & \square_{\leftarrow} & 1 & \square_{\leftarrow} & \square_{\rightarrow} \\ \square_{\rightarrow} & \square_{\rightarrow} & \square_{\leftarrow} & 1 & \square_{\rightarrow} \\ \square_{\leftarrow} & \square_{\leftarrow} & \square_{\rightarrow} & \square_{\rightarrow} & 1 \end{pmatrix} \succeq 0 \quad \xrightarrow{\left\{ \begin{matrix} \square_{\rightarrow} & \square_{\leftarrow} & \square_{\rightarrow} & \square_{\leftarrow} \\ w_1 & w_2 & w_3 & w_4 \end{matrix} \right\}} \begin{pmatrix} 1 & w_1 & w_1 & w_1 & w_1 \\ w_1 & 1 & w_4 & w_2 & w_3 \\ w_1 & w_4 & 1 & w_3 & w_2 \\ w_1 & w_2 & w_3 & 1 & w_4 \\ w_1 & w_3 & w_2 & w_4 & 1 \end{pmatrix} \succeq 0$$

Reflection positivity (RP)

K. Osterwalder and R. Schrader,
“Axioms for Euclidean Green’s functions”, 1973

Euclidean theory
reflection positivity

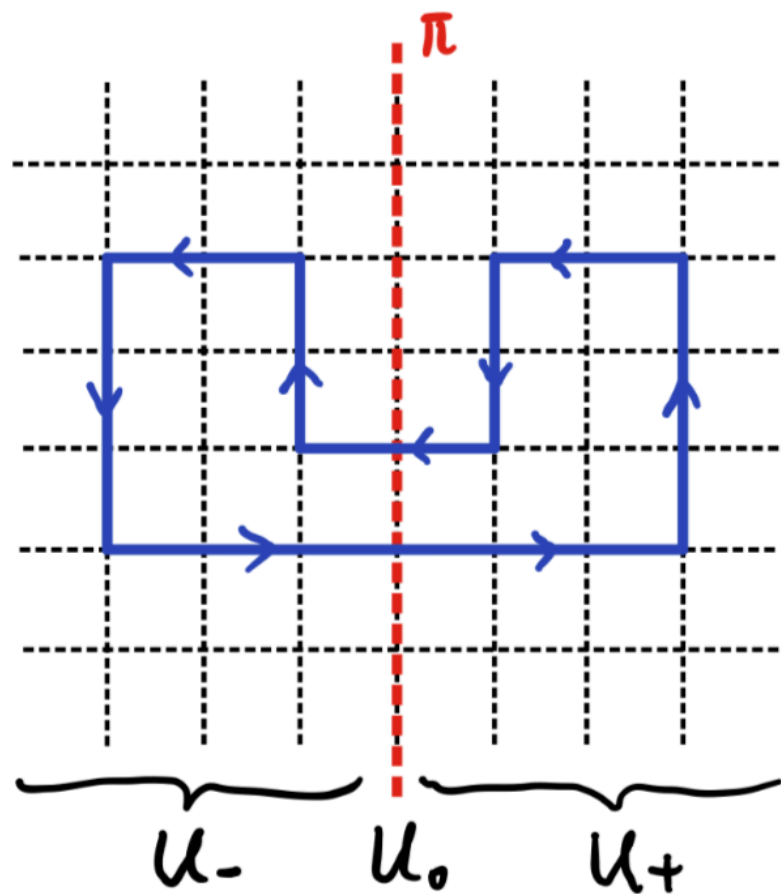


Minkowski theory
unitarity

Reflection positivity (RP)

K. Osterwalder and R. Schrader,

“Axioms for Euclidean Green’s functions”, 1973



$$\Theta_t[U(\{x, y, z\})] = U^\dagger(\{-x, y, z\})$$

$$\int \mathcal{D}U e^{-S} P(U_+, U_0) \Theta[P(U_+, U_0)]$$

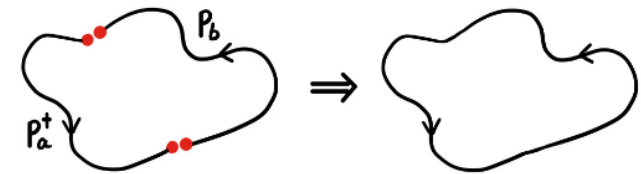
$$= \int \mathcal{D}U_0 e^{-S_{U_0}} \left| \int \mathcal{D}U_+ e^{-S_{U_+} - S_{U_+ \cup U_0}} P(U_+, U_0) \right|^2 \geq 0$$

$$\langle \mathcal{O}_+^R \mathcal{O}_+ \rangle \geq 0$$

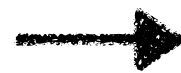
Reflection positivity

$$\langle \mathcal{O}_+^R \mathcal{O}_+ \rangle \geq 0$$

Transform to matrix in a similar way:



$$U = \sum_{a=1}^M c_a P_a \quad \langle \text{tr}(\Theta(U)U) \rangle = \sum_{a,b} c_a c_b \langle \text{tr}(\Theta(P_a)P_b) \rangle \geq 0$$

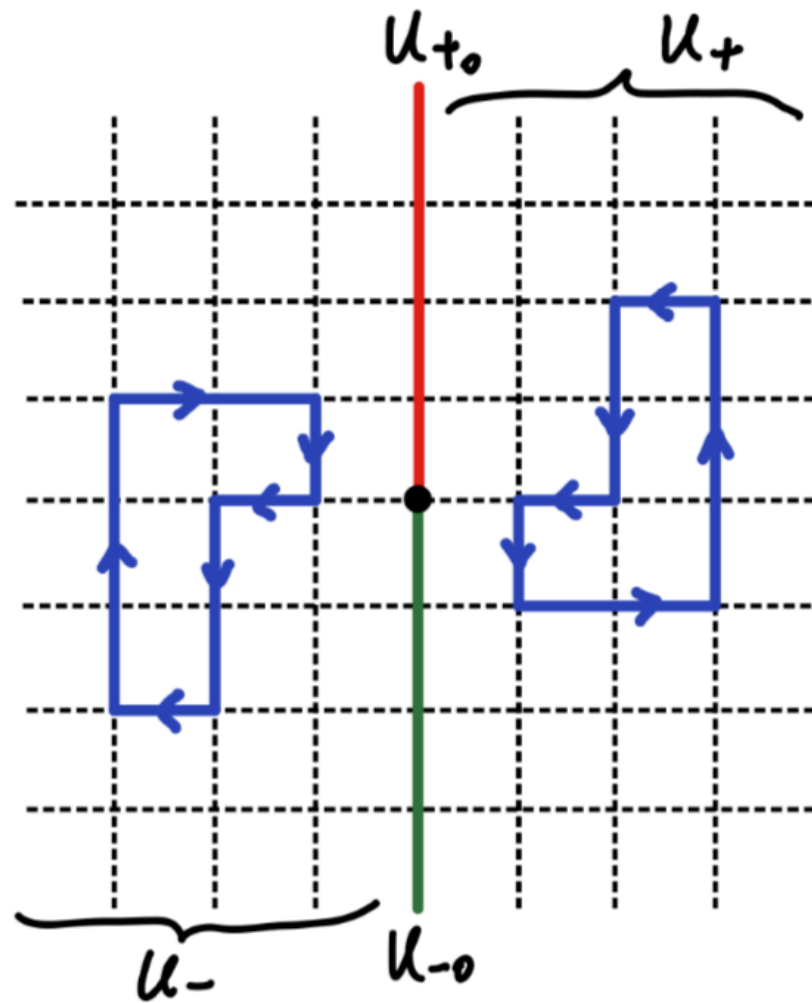


$$A \succcurlyeq 0, \quad A_{ab} := \langle \text{tr}(\Theta(P_a)P_b) \rangle$$

The diagram shows a matrix A represented by a large pair of parentheses containing several loop diagrams. To the left of the matrix is a vertical ellipsis $:$, and to the right is a vertical ellipsis $:$. The matrix elements are represented by various loop configurations: a single box with a dot, a box with a horizontal line, a box with a vertical line, a box with a diagonal line, and a box with a cross. The matrix is followed by a symbol $\succcurlyeq 0$, indicating that the matrix is positive semi-definite.

Twist-reflection

We consider a twist version of Reflection positivity (**twist-RP**):



$$\Theta_t[U(\{x, y, z\})] = U^\dagger(\{-x, -y, z\})$$

$$\langle \mathcal{O}_+^R \mathcal{O}_+ \rangle \geq 0$$

True in 2D, but
not in higher D

$$\begin{array}{c} : \\ : \\ \begin{array}{c} \rightarrow \cdot \\ \leftarrow \cdot \end{array} \end{array} \left(\begin{array}{ccc} 1 & \begin{array}{c} \square \rightarrow \\ \square \leftarrow \end{array} & \begin{array}{c} \square \leftarrow \\ \square \rightarrow \end{array} \\ \begin{array}{c} \square \rightarrow \\ \square \leftarrow \end{array} & \begin{array}{c} \square \rightarrow \\ \square \leftarrow \end{array} & \begin{array}{c} \square \leftarrow \\ \square \rightarrow \end{array} \end{array} \right) \approx 0$$

Positivities matrices

$$\left(\begin{array}{c} \text{matrices involving} \\ \text{a large number of} \\ \text{Wilson loops} \end{array} \right) \succcurlyeq 0$$



Reduce to a set of **independent** basis



Makeenko-Migdal loop equations

Loop equations

Schwinger-Dyson equation for Wilson loops:

Makeenko and Migdal, 1980

(Quantum EOM for Wilson loops)

$$\int \mathcal{D}U \, \delta_{\epsilon(\mu)} \left[e^{-S} W_{\vec{x}}^{ab}(\mu, C_\mu) \right] = 0 \quad \longrightarrow \quad - \langle W_{\vec{x}}^{ab} \delta_{\epsilon} S \rangle + \langle \delta_{\epsilon} W_{\vec{x}}^{ab} \rangle = 0$$

$$\delta_{\epsilon} S : \quad \text{---} \cdot \text{---} \Rightarrow \quad \frac{N}{2\lambda} \left(\begin{array}{c} \text{---} \cdot \text{---} \\ \text{---} \cdot \text{---} \end{array} - \begin{array}{c} \text{---} \cdot \text{---} \\ \text{---} \cdot \text{---} \end{array} \right) \\ \frac{1}{2\lambda} \left(\begin{array}{c} \text{---} \cdot \text{---} \\ \text{---} \cdot \text{---} \end{array} - \begin{array}{c} \text{---} \cdot \text{---} \\ \text{---} \cdot \text{---} \end{array} \right)$$

$$\delta_{\epsilon} W : \quad \begin{array}{l} n_+ \\ n_- \end{array} \quad \begin{array}{c} \text{---} \cdot \text{---} \\ \text{---} \cdot \text{---} \end{array} \Rightarrow \quad \begin{array}{c} \text{---} \cdot \text{---} \\ \text{---} \cdot \text{---} \end{array}$$

(overlapping edges)

Loop equations

Schwinger-Dyson equation for Wilson loops:

Makeenko and Migdal, 1980

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$$\int \mathcal{D}U \delta_{\epsilon(\mu)} \left[e^{-S} W_{\vec{x}}^{ab}(\mu, C_\mu) \right] = 0 \quad \longrightarrow \quad - \langle W_{\vec{x}}^{ab} \delta_\epsilon S \rangle + \langle \delta_\epsilon W_{\vec{x}}^{ab} \rangle = 0$$

Examples:

$$\begin{aligned} \square \uparrow : & \quad \left\{ (\square \square - \square \rightarrow) + (\square \uparrow - 1) \right\} \\ & - \left\{ (\square \uparrow \square - \square \uparrow \square) + (\square \uparrow - \square \uparrow) \right\} \\ & + 2\lambda \left(1 - \frac{1}{N^2} \right) \square \uparrow = 0 \end{aligned}$$

$$\begin{aligned} \square \uparrow \uparrow (n_+ = 1) : & \quad \left\{ (\square \square \square - \square \uparrow) + (\square \uparrow \square - \square \uparrow) \right\} \\ & - \left\{ (\square \uparrow \square - \square \uparrow \square) + (\square \uparrow \square - \square \uparrow \square) \right\} \\ & + 2\lambda \left(1 - \frac{2}{N^2} \right) \square \uparrow \square + 2\lambda \square \uparrow \uparrow = 0 \end{aligned}$$

$$\square \uparrow \epsilon \xrightarrow[\text{SU}(2)]{2\text{D}} \frac{3}{2}\lambda \square + \square \square - \square \rightarrow + \square \uparrow - 1 = 0$$

Example

An explicit example

$$\begin{pmatrix} 1 & w_1 & w_1 & w_1 & w_1 \\ w_1 & 1 & w_4 & w_2 & w_3 \\ w_1 & w_4 & 1 & w_3 & w_2 \\ w_1 & w_2 & w_3 & 1 & w_4 \\ w_1 & w_3 & w_2 & w_4 & 1 \end{pmatrix} \succeq 0 \quad \begin{pmatrix} 1 & w_1 & w_1 \\ w_1 & w_2 & w_3 \\ w_1 & w_3 & w_2 \end{pmatrix} \succeq 0$$

$$\square \uparrow : \frac{3}{2}\lambda \square \uparrow + \square \downarrow \uparrow - \square \uparrow \downarrow + \square \uparrow \downarrow - 1 \stackrel{su(z)}{=} 0$$

$$\frac{3}{2}\lambda w_1 + w_3 - w_2 + w_4 - 1 = 0$$

Here only 1 loop equation is related.

Constrain the value
of Wilson loops

$$\left\{ \square, \square\square, \square\square, \square\square \right\}$$

$w_1 \quad w_2 \quad w_3 \quad w_4$

“Semi-definite Programming”

An explicit example

$$\begin{pmatrix} 1 & w_1 & w_1 & w_1 & w_1 \\ w_1 & 1 & w_4 & w_2 & w_3 \\ w_1 & w_4 & 1 & w_3 & w_2 \\ w_1 & w_2 & w_3 & 1 & w_4 \\ w_1 & w_3 & w_2 & w_4 & 1 \end{pmatrix} \succeq 0 \quad \begin{pmatrix} 1 & w_1 & w_1 \\ w_1 & w_2 & w_3 \\ w_1 & w_3 & w_2 \end{pmatrix} \succeq 0$$

$$\square \uparrow : \frac{3}{2}\lambda \square \uparrow + \square \square - \square \square + \square - 1 \stackrel{su(2)}{=} 0$$

$$\frac{3}{2}\lambda w_1 + w_3 - w_2 + w_4 - 1 = 0$$

Constrain the value of Wilson loops

$$\{ \square, \square, \square, \square \}$$

$$w_1 \quad w_2 \quad w_3 \quad w_4$$

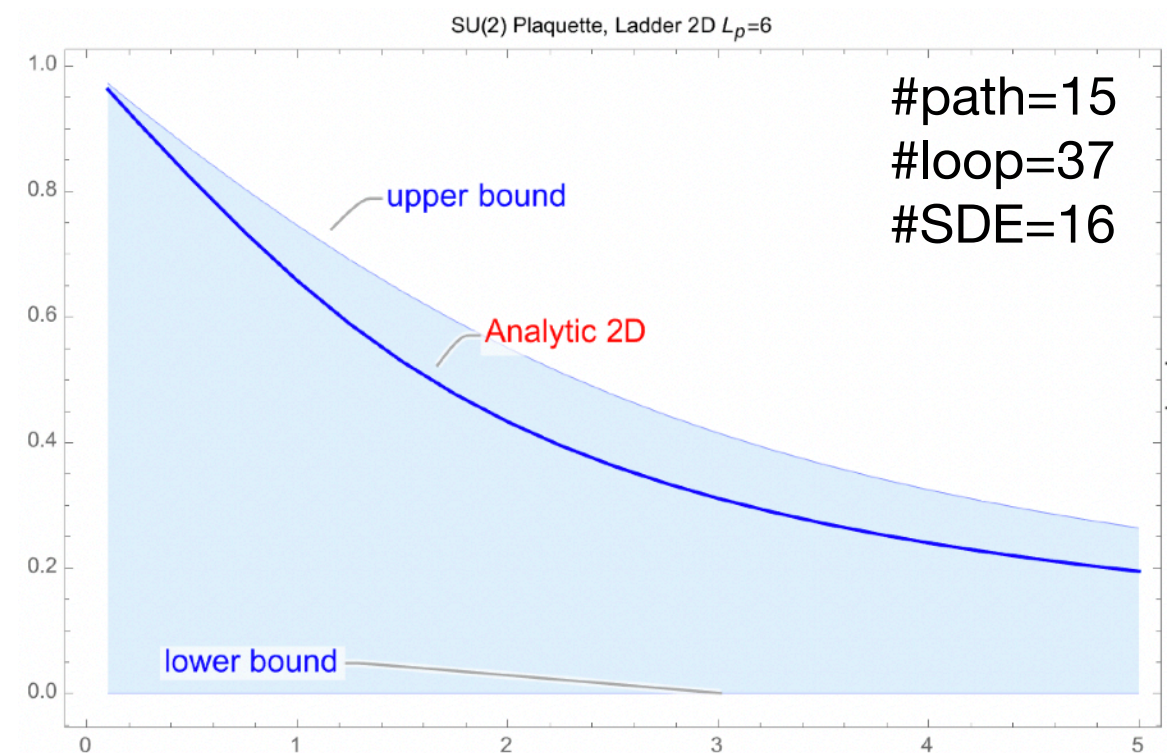
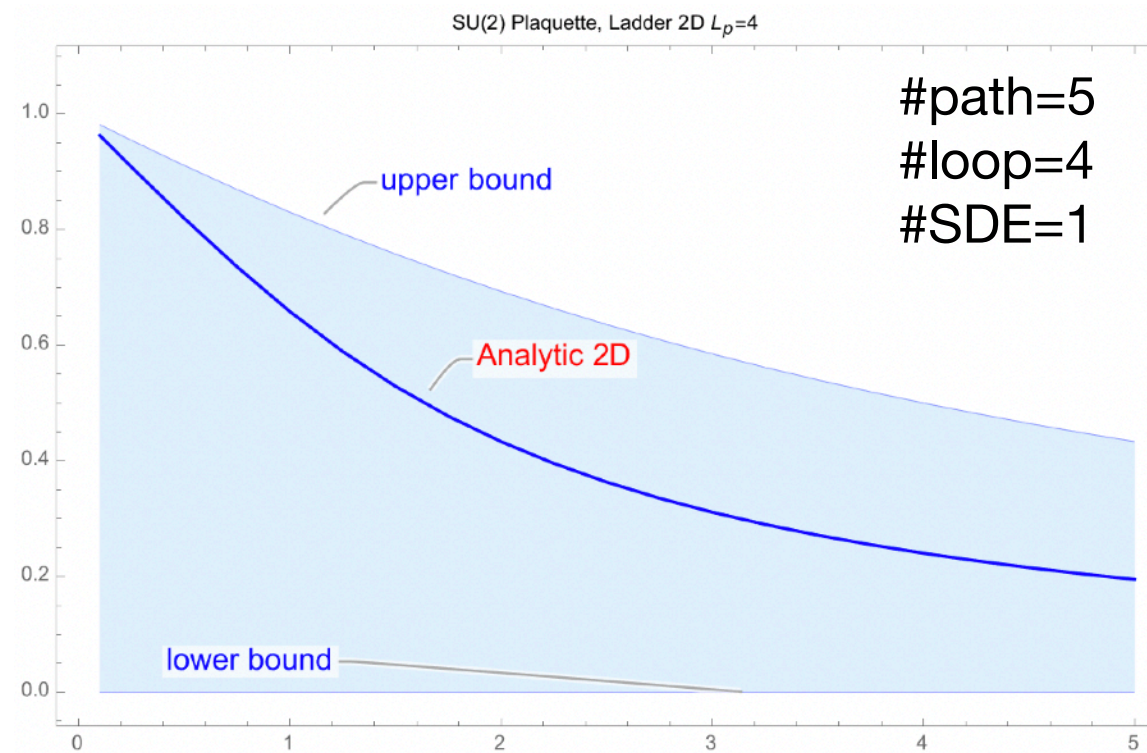
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“Semi-definite Programming”

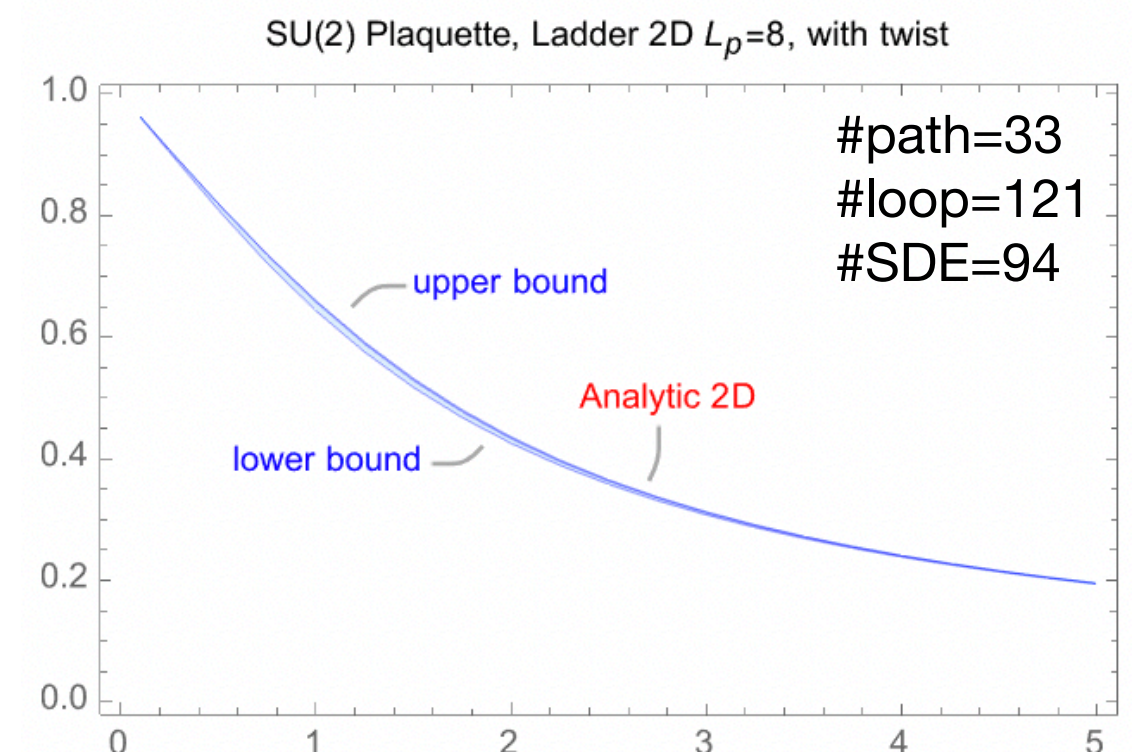
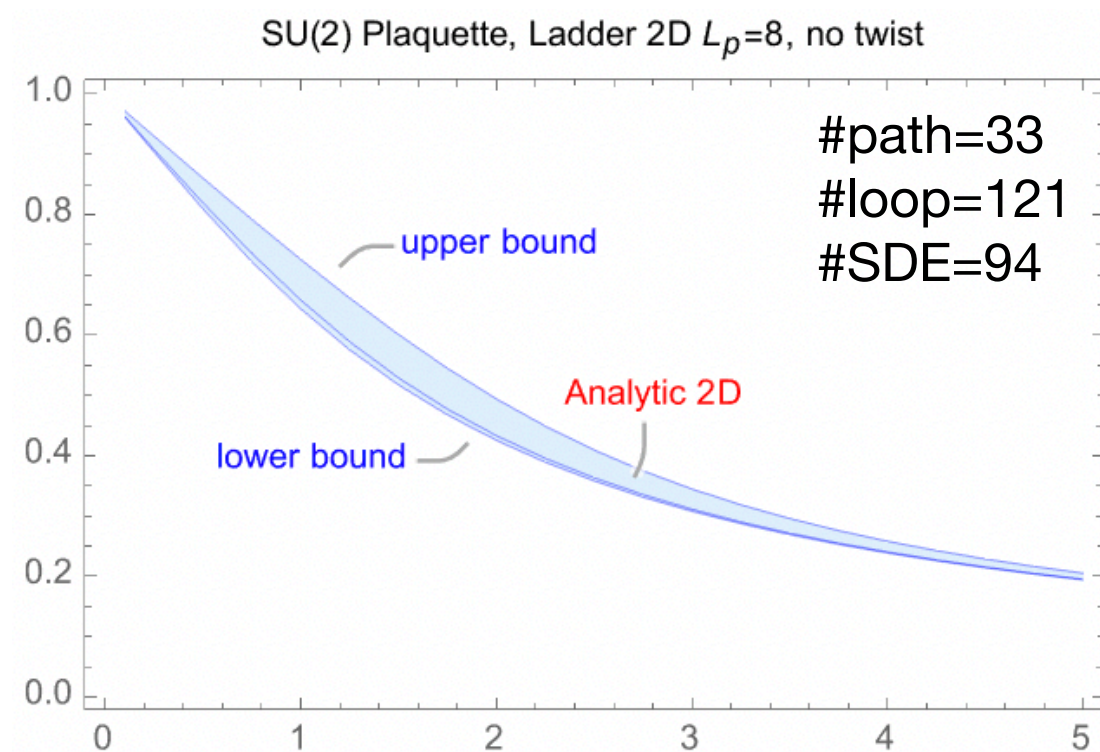
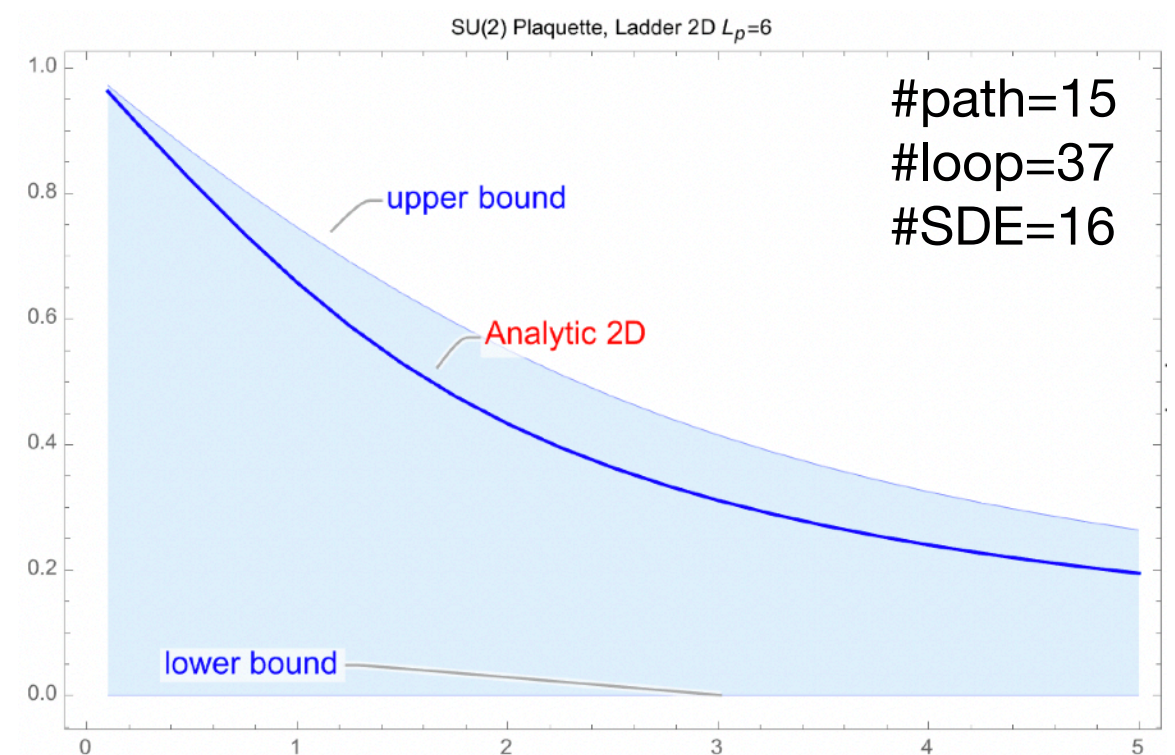
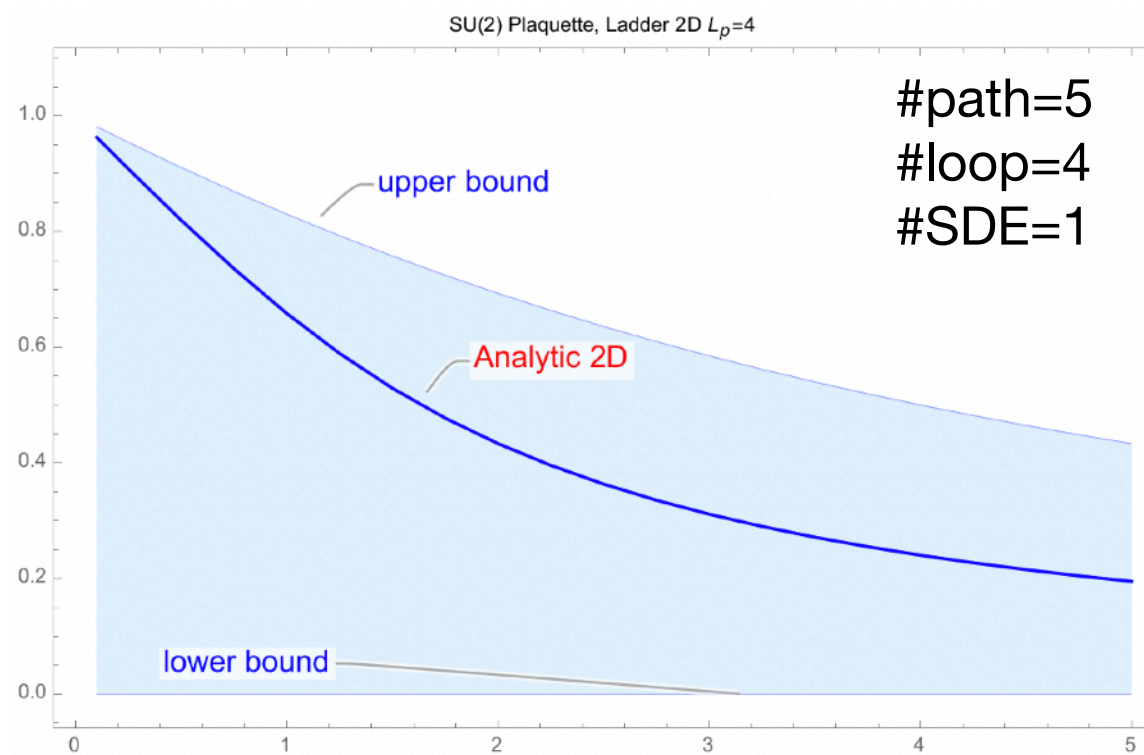
$$\text{Semidefinite Optimization} \left[\underbrace{\pm w_1}_{\text{object function}}, \underbrace{\{M \succeq_{S_+^{\text{Im}} 0}\}, \{\text{Eqns}\}}_{\text{constraints}}, \underbrace{\text{vars}}_{\text{all appeared variables}} \right]$$

More advanced tools: MOSEK

An explicit example



An explicit example



Challenges in SU(3)

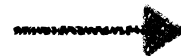
Large N:

$$\langle w(C_{xx'})w(C_{x'x}) \rangle = \langle w(C_{xx'}) \rangle \langle w(C_{x'x}) \rangle + \mathcal{O}(1/N^2)$$

SU(2):

$$w(C_1)w(C_2) = \frac{1}{2}w(C_1 C_2) + \frac{1}{2}w(C_1 \bar{C}_2)$$

Single trace



Double trace

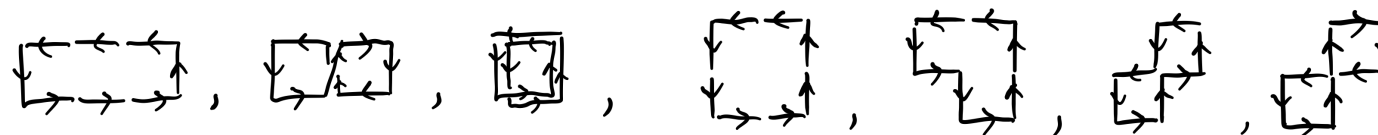
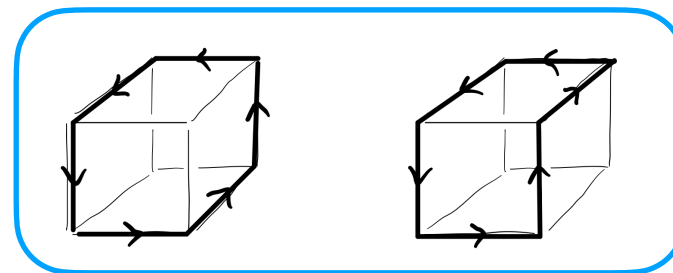
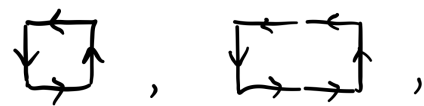
SU(3):

$$\begin{aligned} w(C_1)w(C_2)w(C_3) = & -\frac{1}{9} [w(C_1 C_2 C_3) + w(C_3 C_2 C_1)] \\ & + \frac{1}{3} [w(C_1)w(C_2 C_3) + w(C_2)w(C_3 C_1) + w(C_3)w(C_1 C_2)] \\ & + \frac{1}{3} w(\bar{C}_1 C_2)w(\bar{C}_1 C_3) - \frac{1}{9} w(\bar{C}_1 C_2 \bar{C}_1 C_3), \end{aligned}$$

Challenges in SU(3)

Independent (no SDE) single-trace loops:

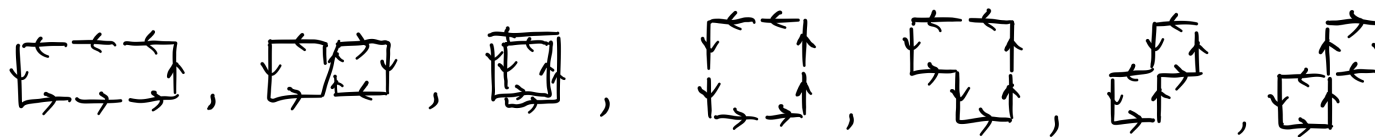
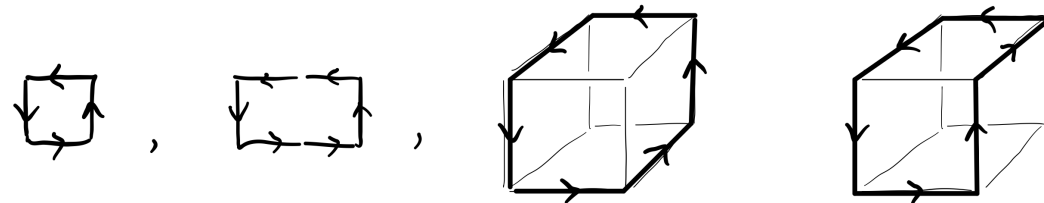
Length of loops	4	6	8	10	12	14	16
# of 2D loops	1	1	7	15	95	465	3,217
# of 3D loops	0	2	11	117	1,657	27,012	488,300
# of 4D loops	0	0	7	106	3,304	109,304	3,849,514



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The number of loop operators increases significantly from the single-trace case to the double-trace case.

Does SU(3) has good convergence as in SU(2)?

SDE is more complicated in SU(3)

General SDE for double-trace operators:

$$\begin{aligned}
 & \sum_{\pm\nu\neq\mu} \left[\langle w_x(\mu\nu\bar{\mu}\bar{\nu}C_1)w_y(C_2) \rangle - \langle w_x(\nu\mu\bar{\nu}\bar{\mu}C_1)w_y(C_2) \rangle \right] \\
 & - \frac{2\lambda}{9} (\check{n}_{1,+} - n_{1,-} + n_{2,+} - n_{2,-}) \langle w_x(C_1)w_y(C_2) \rangle \\
 & + \frac{2\lambda}{9} \left[\sum_{j_+=1}^{n_{2,+}} \langle w(C_{1,x}C_{2,x_+}) \rangle - \sum_{j_-=1}^{n_{2,-}} \langle w(C_{1,x}C_{2,x_-}) \rangle \right] \\
 & + 2\lambda \left[\sum_{j_+=1}^{\check{n}_{1,+}} \langle w_x(C_{1,x,x_+})w_x(C_{1,x_+,x})w_y(C_2) \rangle - \sum_{j_-=1}^{n_{1,-}} \langle w_x(C_{1,x,x_-})w_x(C_{1,x_-,x})w_y(C_2) \rangle \right] \\
 & - \sum_{\pm\nu\neq\mu} \left[\langle w_x(\mu\nu\bar{\mu}\bar{\nu})w_x(C_1)w_y(C_2) \rangle - \langle w_x(\nu\mu\bar{\nu}\bar{\mu})w_x(C_1)w_y(C_2) \rangle \right] = 0.
 \end{aligned}$$

Examples:

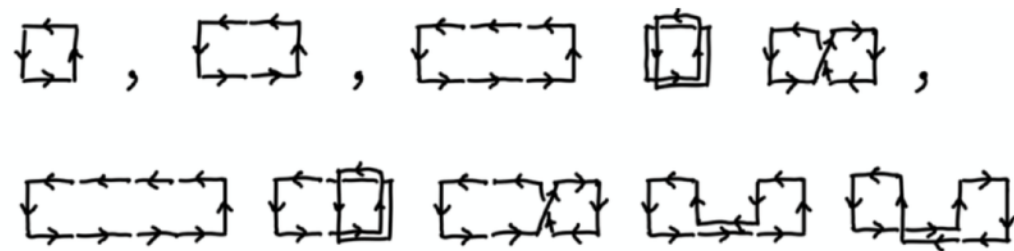
$$\begin{aligned}
 \begin{array}{c} \square \square \\ \rightarrow \end{array} \epsilon \xrightarrow[\text{SU}(3)]{2D} & \left(\begin{array}{c} \square \square \square \\ \rightarrow \end{array} - \begin{array}{c} \square \square \square \\ \rightarrow \end{array} + \begin{array}{c} \square \square \square \\ \rightarrow \end{array} - \begin{array}{c} \square \square \square \\ \rightarrow \end{array} \right) + 2\lambda \left(1 - \frac{1}{9} \right) \left(\begin{array}{c} \square \square \\ \rightarrow \end{array} \right) \\
 & - \left(\begin{array}{c} \square \square \square \square \\ \rightarrow \end{array} - \begin{array}{c} \square \square \square \square \\ \rightarrow \end{array} + \begin{array}{c} \square \square \square \square \\ \rightarrow \end{array} - \begin{array}{c} \square \square \square \square \\ \rightarrow \end{array} \right) = 0.
 \end{aligned}$$

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 \begin{array}{c} \square \square \\ \rightarrow \end{array} \epsilon \xrightarrow[\text{SU}(3)]{2D} & \left(\begin{array}{c} \square \square \square \\ \rightarrow \end{array} - \begin{array}{c} \square \square \square \\ \rightarrow \end{array} + \begin{array}{c} \square \square \square \\ \rightarrow \end{array} - \begin{array}{c} \square \square \square \\ \rightarrow \end{array} \right) + 2\lambda \left(1 - \frac{2}{9} \right) \begin{array}{c} \square \square \\ \rightarrow \end{array} \\
 & - \left(\begin{array}{c} \square \square \square \square \\ \rightarrow \end{array} - \begin{array}{c} \square \square \square \square \\ \rightarrow \end{array} + \begin{array}{c} \square \square \square \square \\ \rightarrow \end{array} - \begin{array}{c} \square \square \square \square \\ \rightarrow \end{array} \right) + \frac{2\lambda}{9} \begin{array}{c} \square \square \\ \rightarrow \end{array} = 0.
 \end{aligned}$$

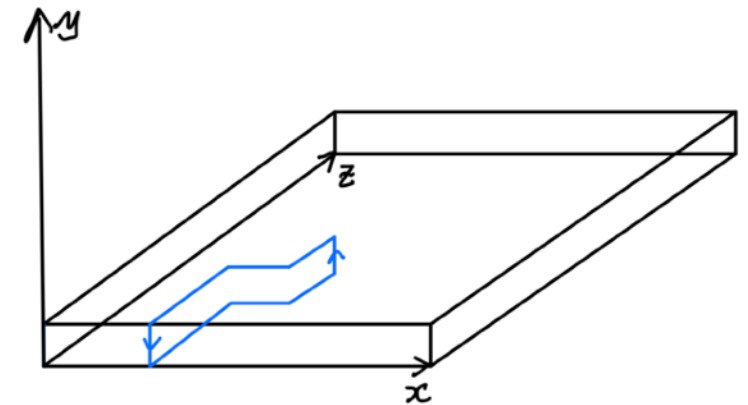
Choice of paths and loops

A simplification: consider operators that are in **(d-1) sub-plane**.

2d loops, extending in only one direction:



3d loops in a 2d sub-plane

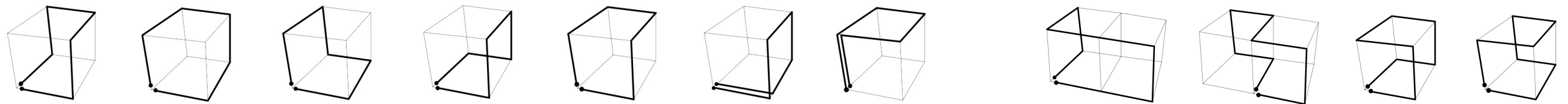
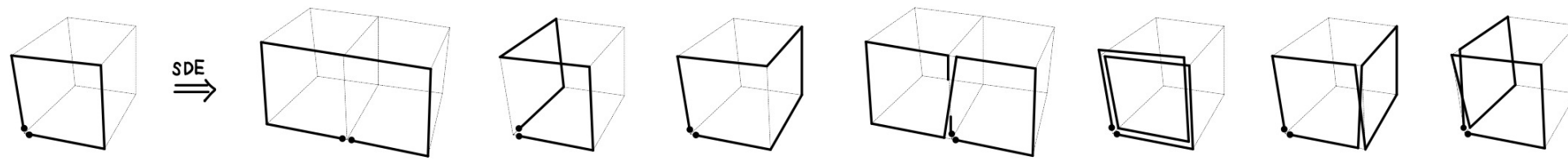


Compare number of loops:

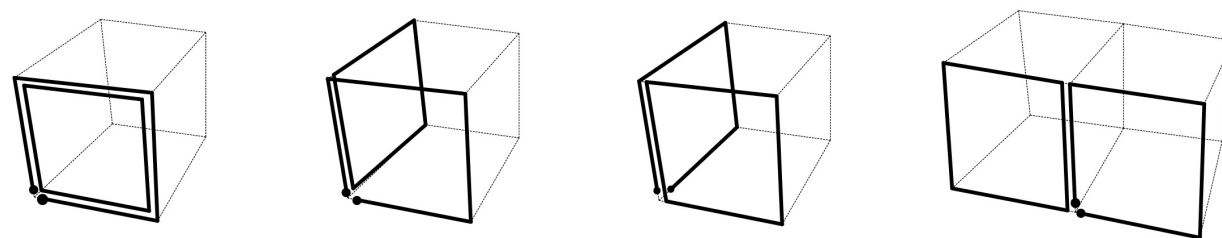
Length of loops	4	6	8	10	12	14	16	18	20
# of general loops	1	1	7	15	95	465	3,217	21,762	159,974
# of ladder-type loops	1	1	3	5	13	32	90	268	867

Choice of paths and loops

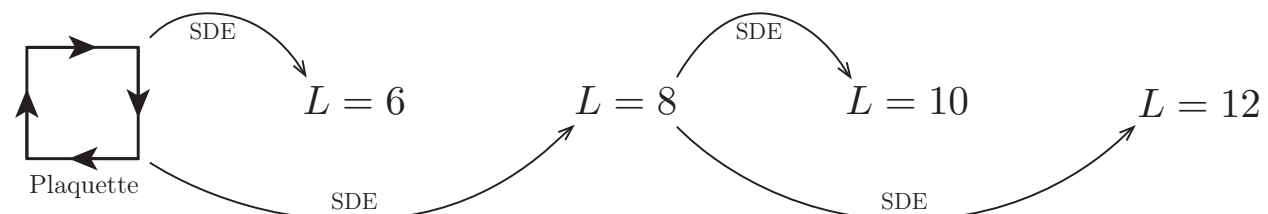
Example of 3D paths:



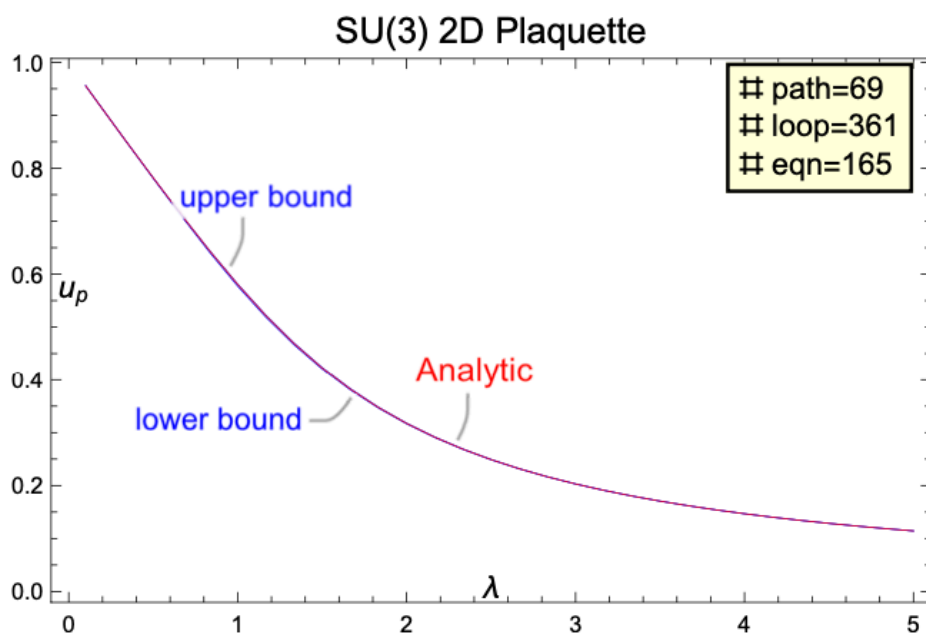
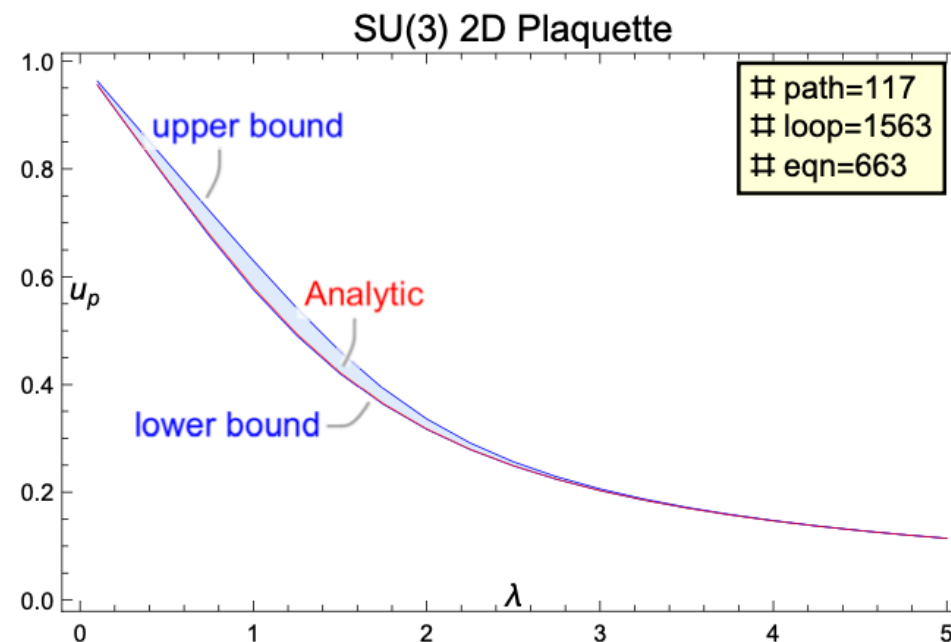
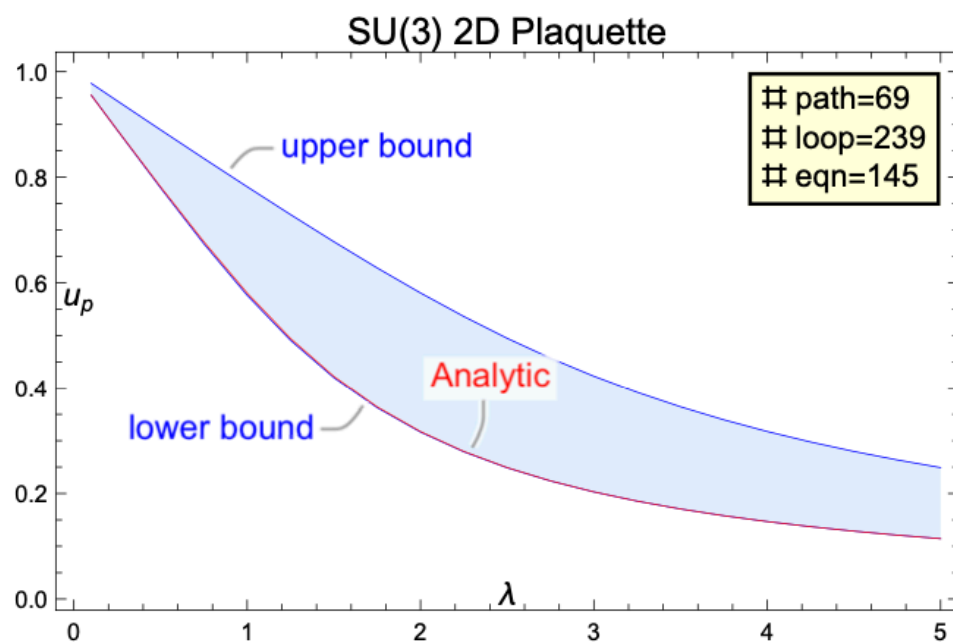
Double-trace ones:



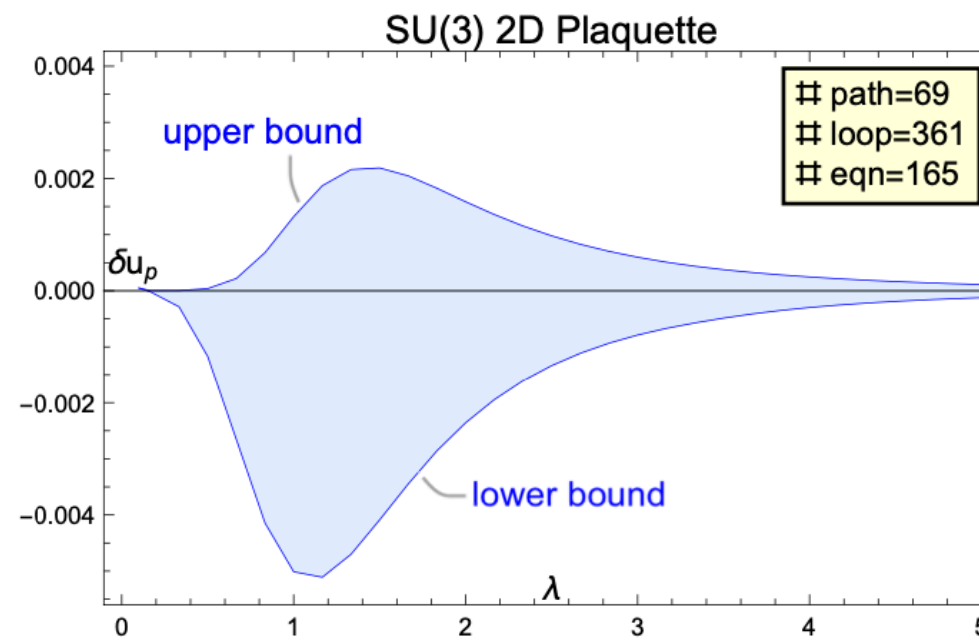
4D path strategy:



Results: 2d SU(3)



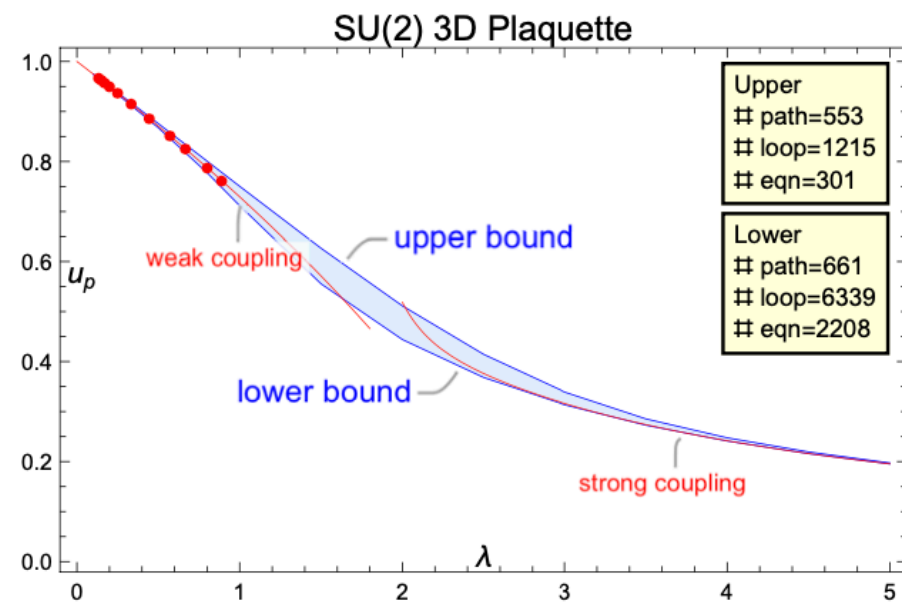
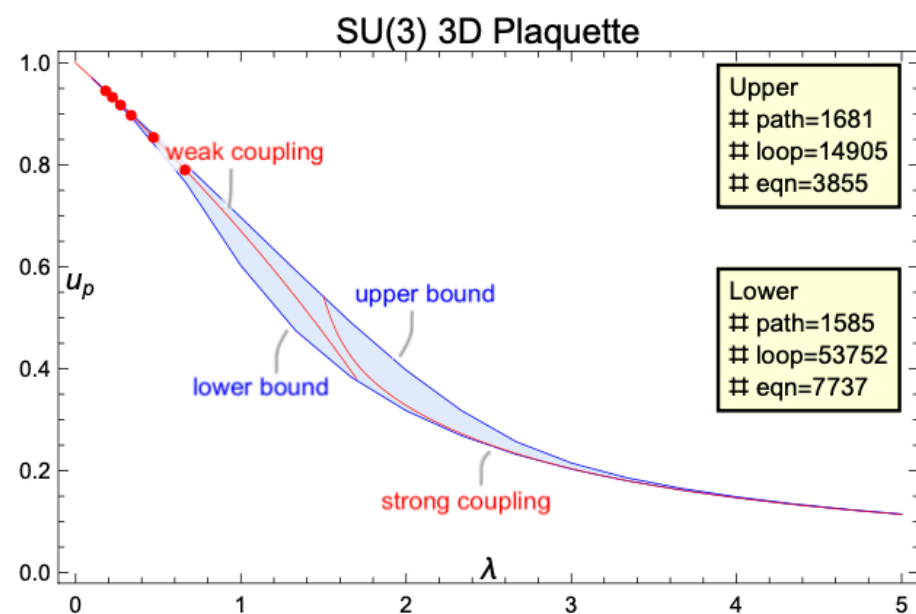
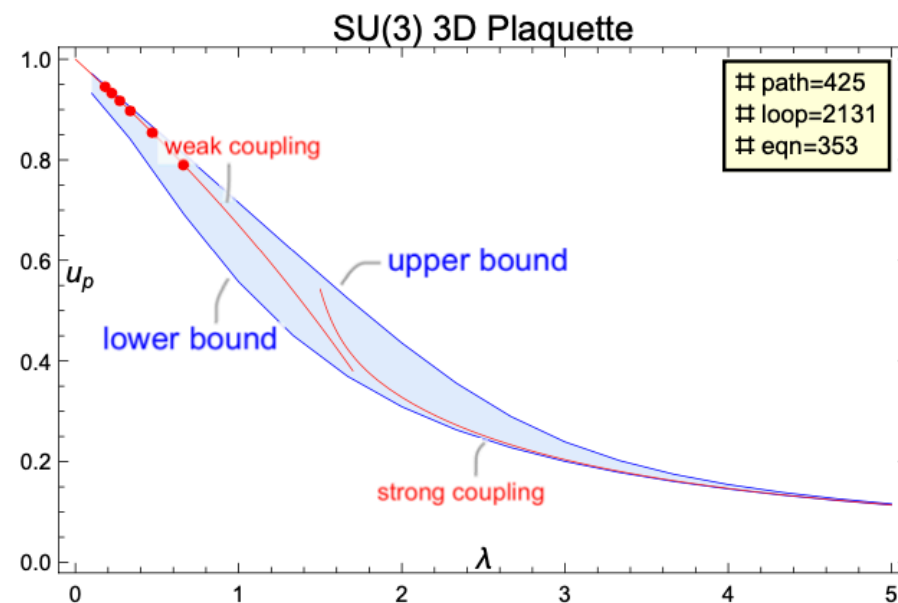
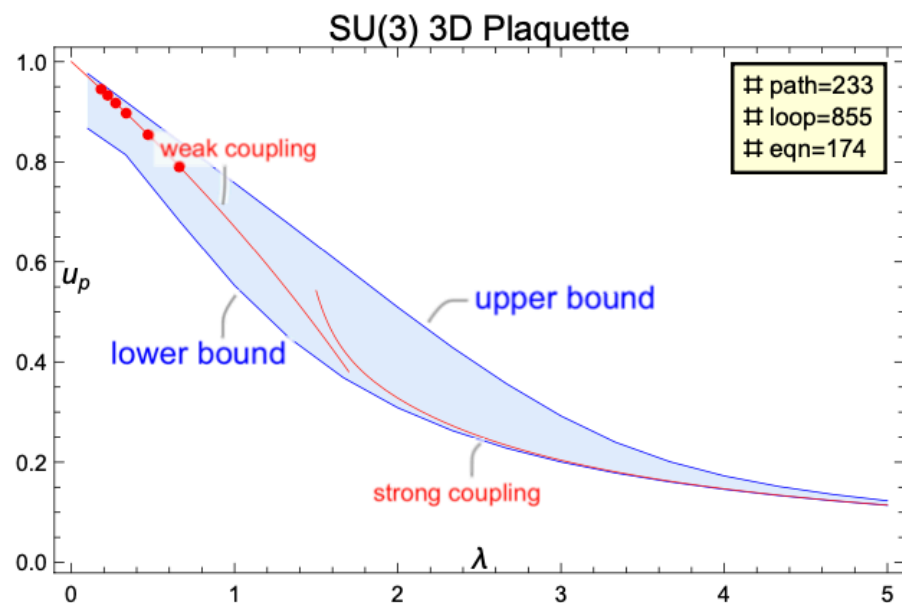
(a) Bounds.



(b) Error of bounds: $\frac{\text{bound} - \text{analytic}}{\text{analytic}}$.

With twist-reflection, 3 digits precision

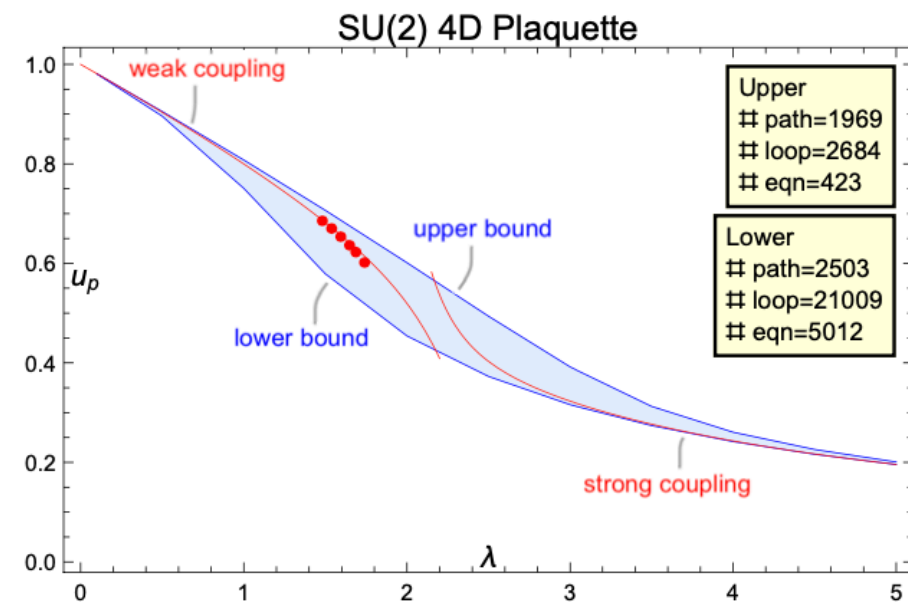
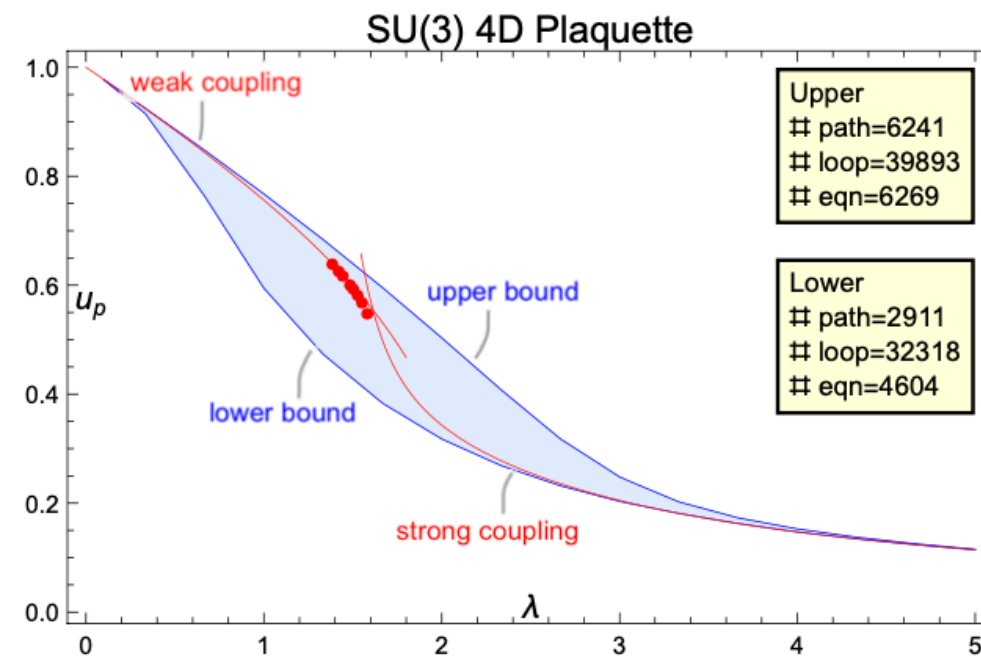
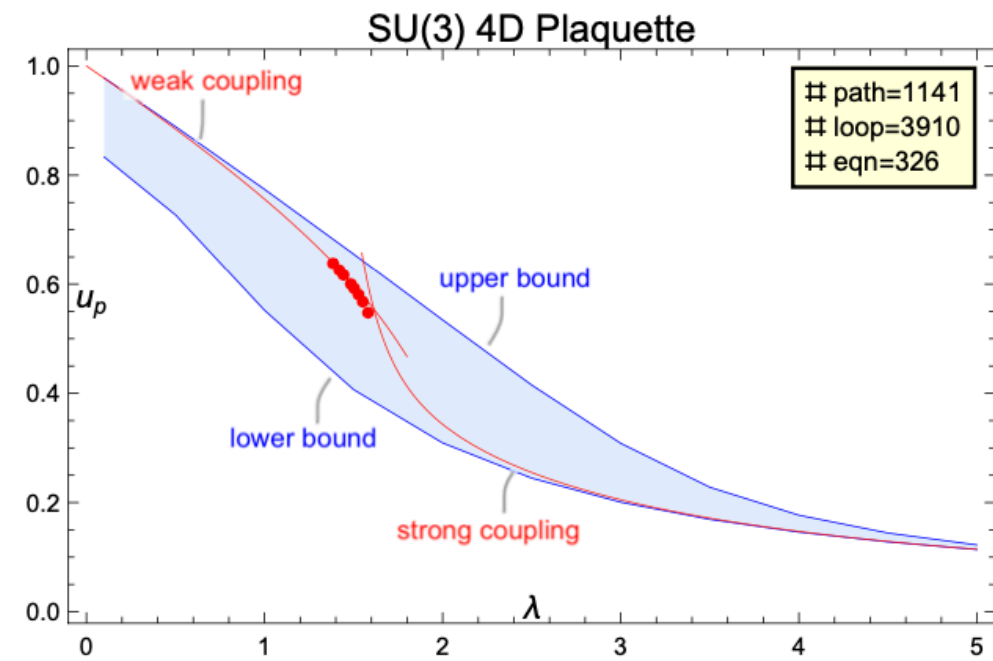
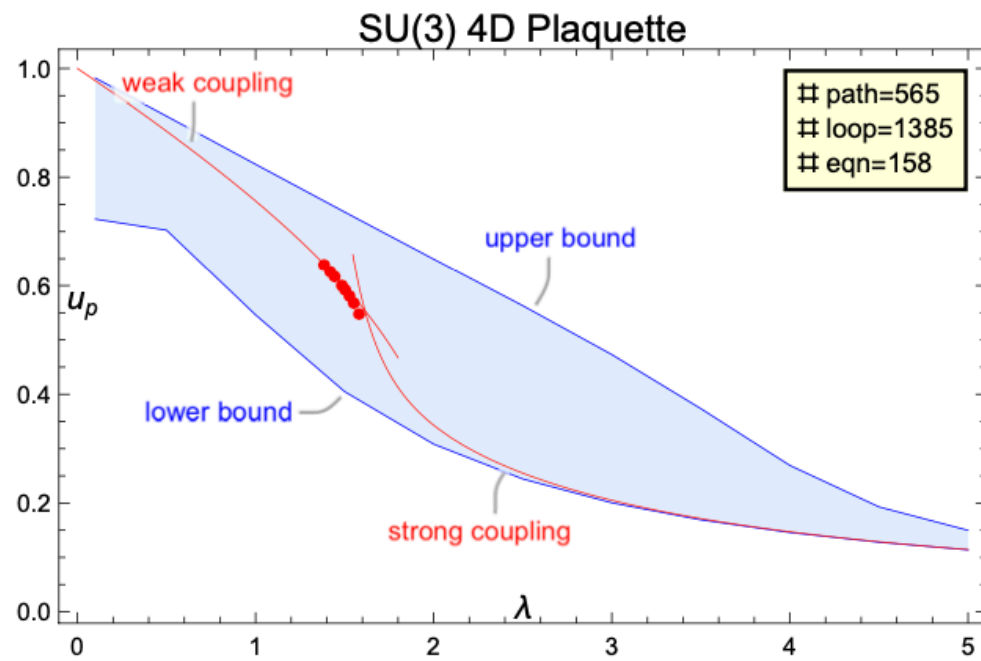
Results: 3d SU(3)



Compare with SU(2)

Red dots from MC data: [Athenodorou, Teper, arXiv:1609.03873 \(3d\), 2106.00364 \(4d\)](#)

Results: 4d SU(3)

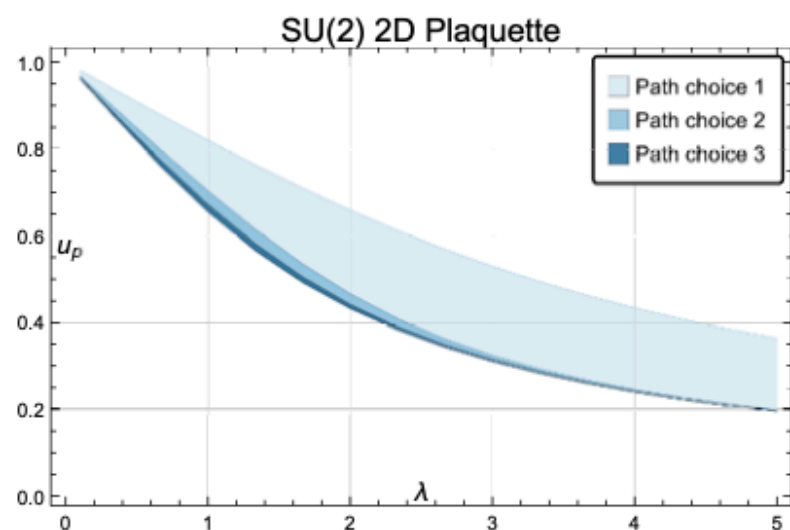


Compare with SU(2)

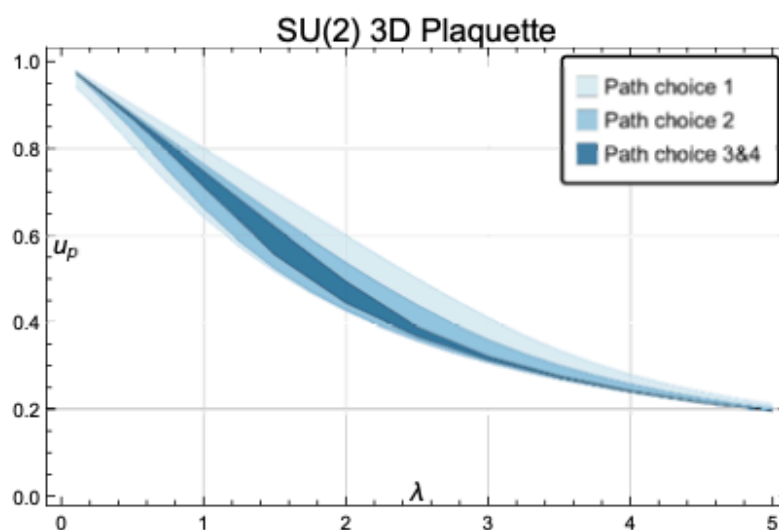
Red dots from MC data: [Athenodorou, Teper, arXiv:1609.03873 \(3d\), 2106.00364 \(4d\)](#)

Bootstrapping SU(3)

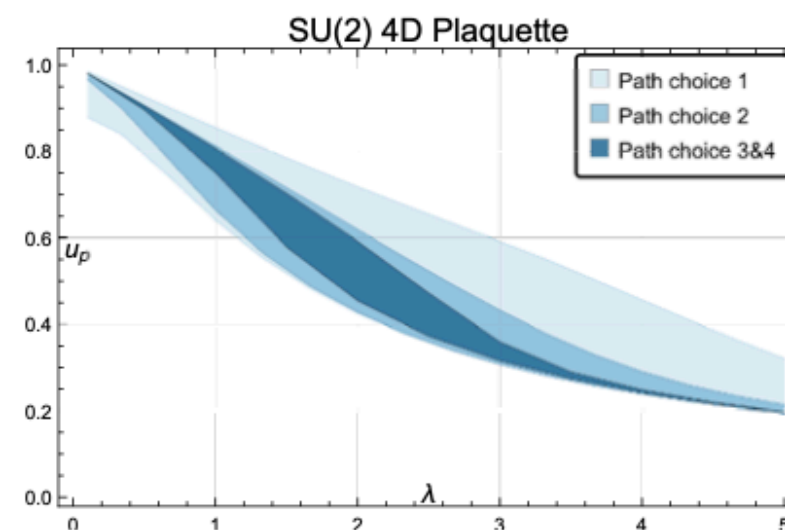
Guo, Li, GY, Zhu, 2025



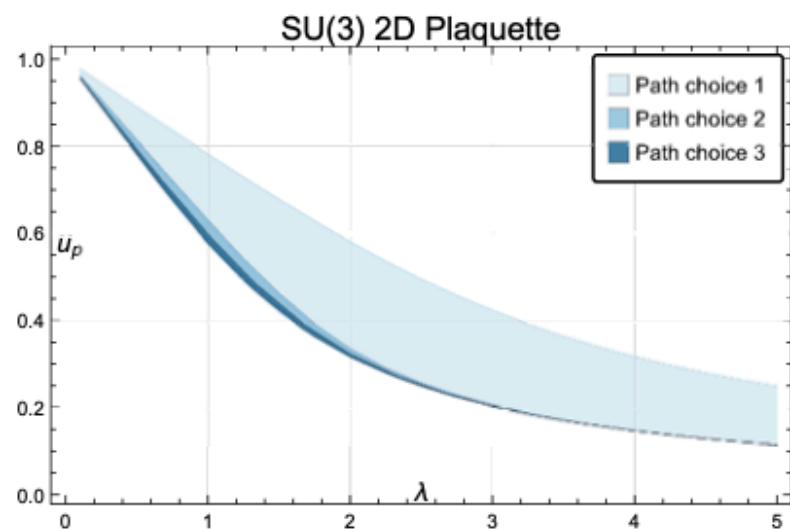
(a)



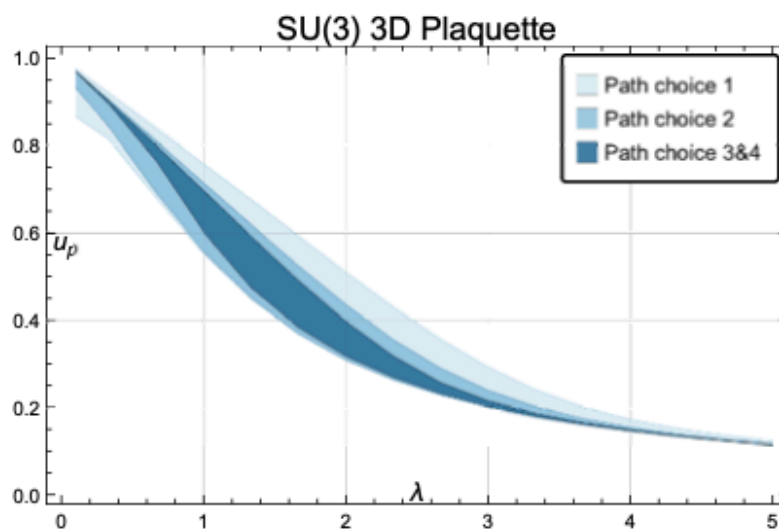
(b)



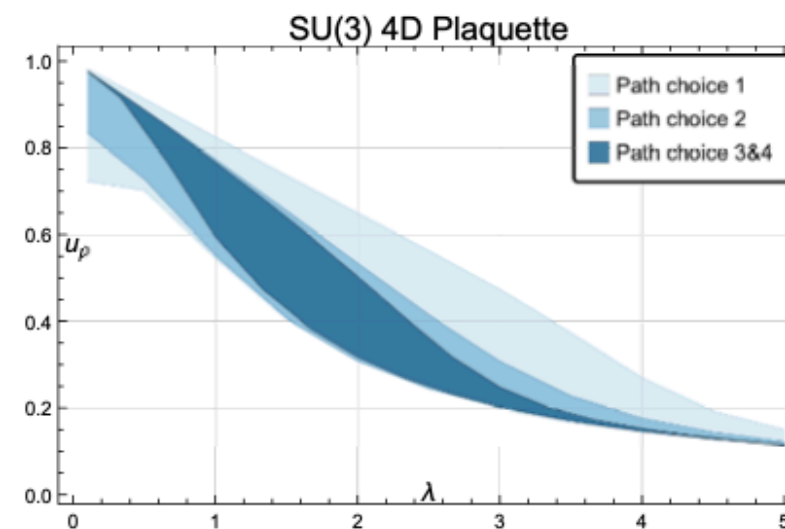
(c)



(d)



(e)



(f)

Loop equations

Analogy with integration-by-parts (IBP) reduction in amplitudes:

$$\int \mathcal{D}U \delta[e^{-S} W(c)] \xrightarrow{\text{SDE}} \sum \text{Wilson loops} = 0$$

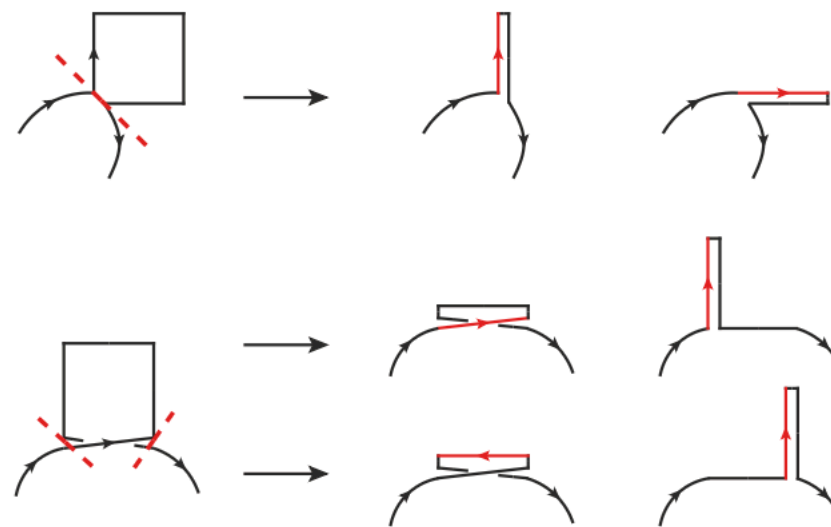
$$\int d^D \ell \, p^\mu \frac{\partial I}{\partial \ell^\mu} = 0 \xrightarrow{\text{IBP}} \sum \text{integrals} = 0$$

Goal: reduce a large number of blocks (loops or integrals) to a basis

Constructing loop equations efficiently

Xizhe Liu, GY 2026

We introduce **plaquette-cut and subloop-cut strategy**, based on detecting sub-loop structures. This is very efficient to obtain all SDEs (including backtrack types) as well as SU(2) trace relations.



We also discussed the hidden ‘**indirect**’ loop equations, due to the finite truncation of higher-length loops.

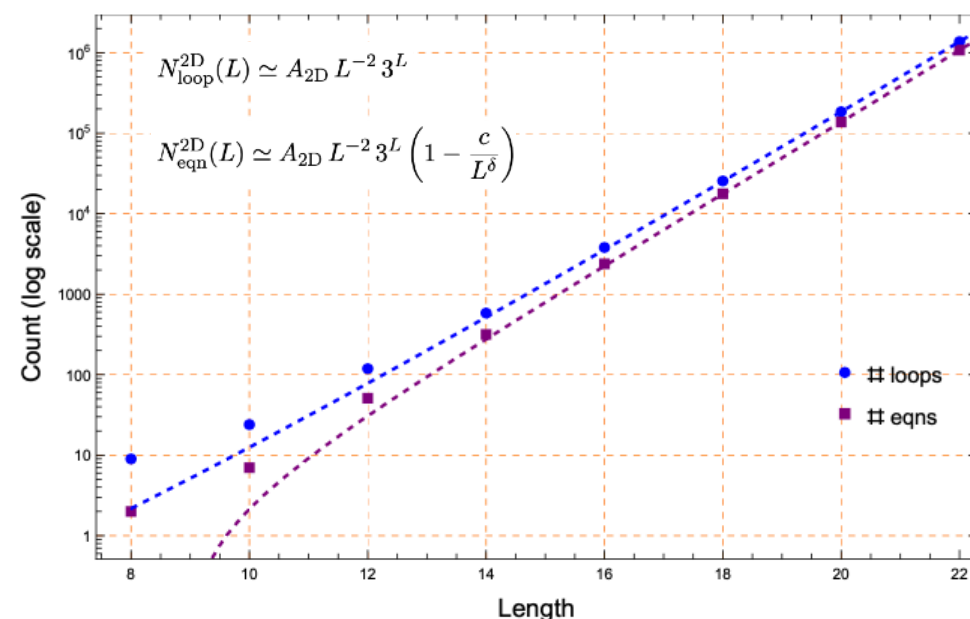
$$\text{Diagram 1} - \text{Diagram 2} + \text{Diagram 3} - \text{Diagram 4} = 0$$

Counting of loops and equations

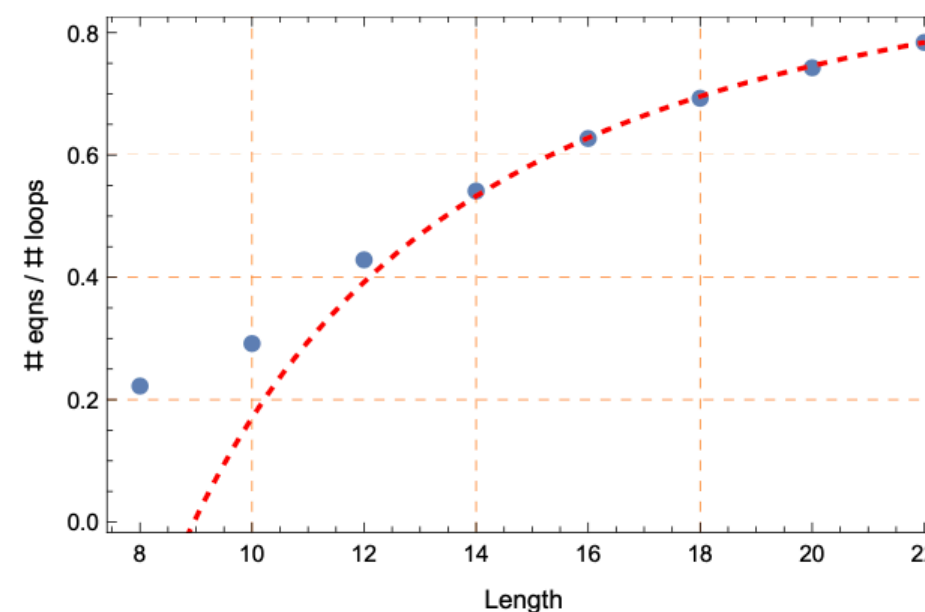
Xizhe Liu, GY 2026

2D loops:

Length(2D)	8	10	12	14	16	18	20	22
# loops	9	24	119	584	3,801	25,563	185,537	1,374,414
# AllE	2	7	51	316	2,383	17,709	137,775	1,077,512
# AllE/# loops	0.222	0.292	0.429	0.541	0.627	0.693	0.743	0.784



(a) Loops and equations



(b) Ratios

* in self-avoid-walking, $N_{\text{loop}}(L) \simeq A L^{-\alpha} \mu^L$

$\mu \simeq 2.638$ for square lattice

$\mu = \sqrt{2 + \sqrt{2}}$ for honeycomb lattice

Constructing loop equations efficiently

Xizhe Liu, GY 2026

3D loops:

Length(3D)	8	10	12	14	16
# Loops	22	154	1,906	29,383	520,900
#AllE	2	19	344	6,785	141,881
# AllE/# loops	0.091	0.123	0.180	0.231	0.272

4D loops:

Length(4D)	8	10	12	14	16
# Loops	29	267	5,323	142,104	4,483,135
# AllE	2	20	542	17,253	612,103
# AllE/# loops	0.069	0.075	0.102	0.121	0.137

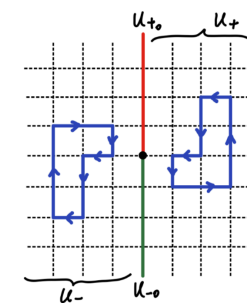
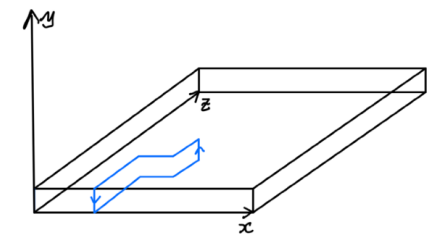
Ratio grows as length increases, but less constraining than 2D.

Summary

Recent studies of lattice YM bootstrap provide a new framework to probe non-perturbative realistic gauge theories.

New progress for bootstrapping lattice gauge theory:

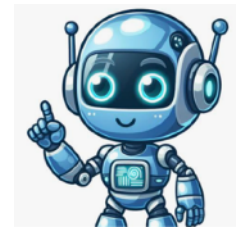
- Consider $SU(3)$ that involves multi-trace operators for the first time. Show that the method works well.
- “Dimension reduction” by consider $(d-1)$ sub-plane operators.
- Introduce new **twist-reflection** positivity, which is exact in 2d lattice theory.
- Efficient strategy to find SDEs.



Outlook

A new non-perturbative quantitative method work by principle.

- Compute more general loops.
- Make “smart” truncations?
- Main bottleneck: the SDP computation (AI?)



Goal: understanding mass gap and confinement in (lattice) YM.

Thank you for your attention!

Backup slides

Reduction of high-trace operators

SU(2): only single-trace operators

$$\text{tr}U\text{tr}V = \text{tr}UV + \text{tr}UV^\dagger, \quad \forall U, V \in SU(2)$$

SU(3): need single- and double-trace operators

$$\begin{aligned} \text{tr}U\text{tr}V\text{tr}W &= -\text{tr}WVU - \text{tr}UVW + \text{tr}U\text{tr}VW + \text{tr}UV\text{tr}W + \text{tr}UW\text{tr}V \\ &+ \text{tr}U^\dagger V\text{tr}U^\dagger W - \text{tr}U^\dagger VU^\dagger W, \quad \forall U, V, W \in SU(3) \end{aligned}$$

The SU(3) matrix satisfies

$$1 = \det(X) = \frac{1}{3!} \epsilon^{a_1 a_2 a_3} \epsilon_{b_1 b_2 b_3} X_{a_1}^{b_1} X_{a_2}^{b_2} X_{a_3}^{b_3}, \quad \forall X \in SU(3). \quad (\text{A.1})$$

Expanding the RHS we have

$$\text{tr}(X)\text{tr}(X)\text{tr}(X) - 3\text{tr}(X)\text{tr}(XX) + 2\text{tr}(XXX) = 6, \quad \forall X \in SU(3). \quad (\text{A.2})$$

For general triple-trace operator, using

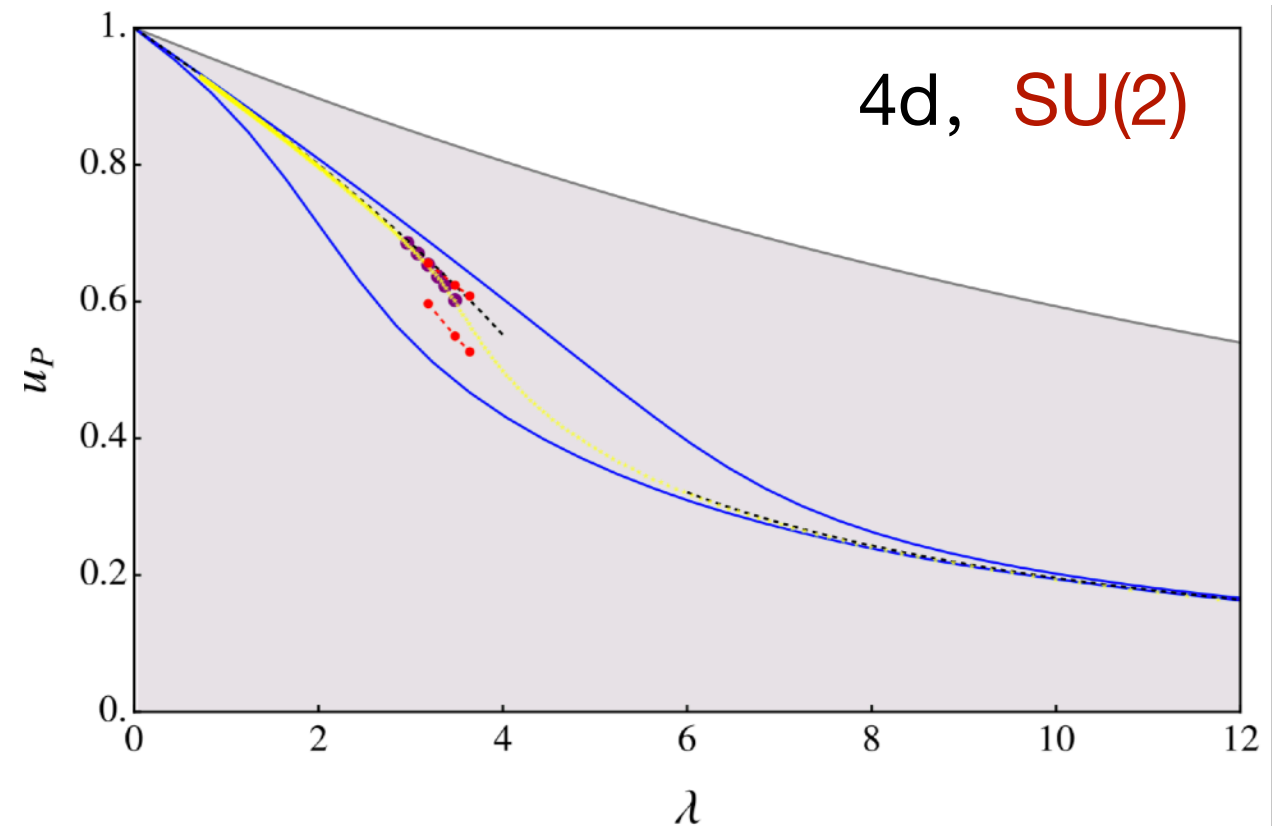
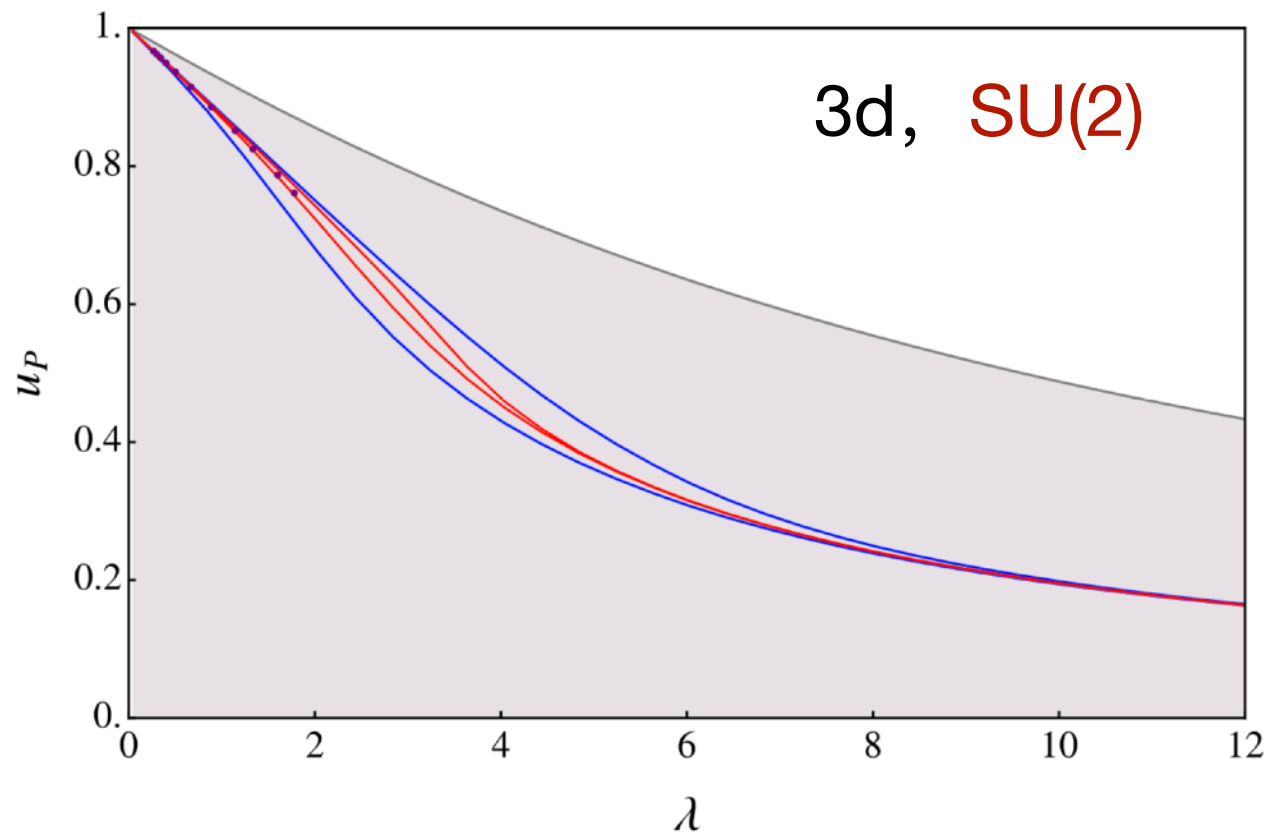
$$\delta_{b_1 b_2 b_3}^{a_1 a_2 a_3} X_{a_1}^{b_1} Y_{a_2}^{b_2} Z_{a_3}^{b_3} = \delta_{b_1 b_2 b_3}^{a_1 a_2 a_3} X_{a_1}^{b_1} Y_{a_2}^{b_2} Z_{a_3}^{b_3} \det(X^\dagger), \quad \forall X, Y, Z \in SU(2), \quad (\text{A.3})$$

we can prove the following identity

$$\begin{aligned} \text{tr}(X)\text{tr}(Y)\text{tr}(Z) &= -\text{tr}(XYZ) - \text{tr}(ZYX) + \text{tr}(X)\text{tr}(YZ) + \text{tr}(Y)\text{tr}(ZX) + \text{tr}(Z)\text{tr}(XY) \\ &+ \text{tr}(X^\dagger Y)\text{tr}(X^\dagger Z) - \text{tr}(X^\dagger YX^\dagger Z), \quad \forall X, Y, Z \in SU(3), \quad (\text{A.4}) \end{aligned}$$

SU(2) results

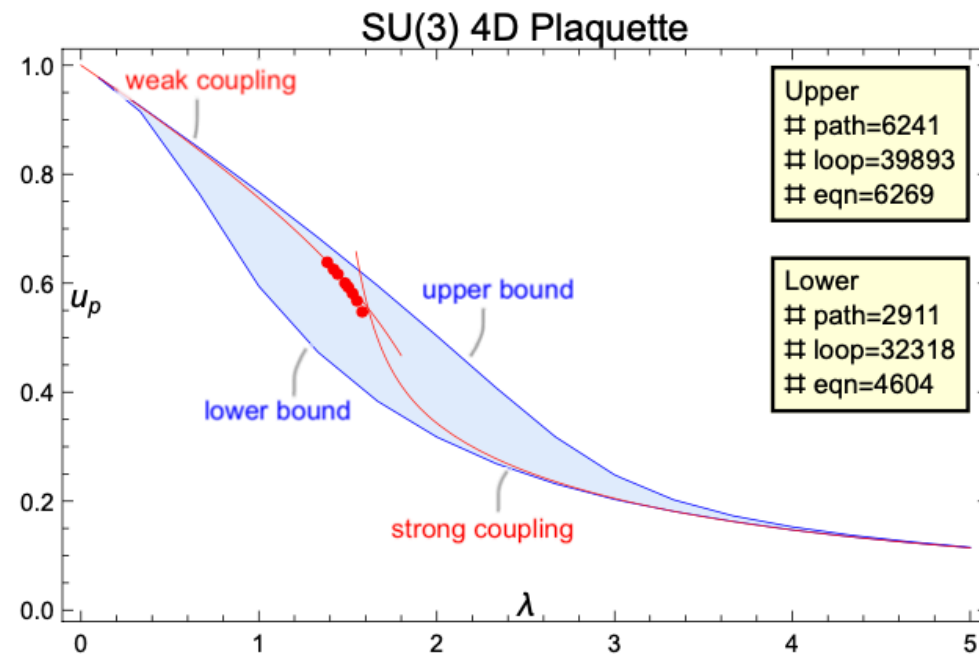
Kazakov, Zheng, arXiv:2404.16925



Dim	#var	#LE	#SDPvar	#Block	Blocksize	Memory	Time
2	8335	14591	1044	18	~ 50	~ 100 MB	~ 1 s
3	174387	98032	93561	38	~ 200	220 GB	~ 1 days
4	343851	152149	211912	40	~ 300	1 TB	~ 20 days

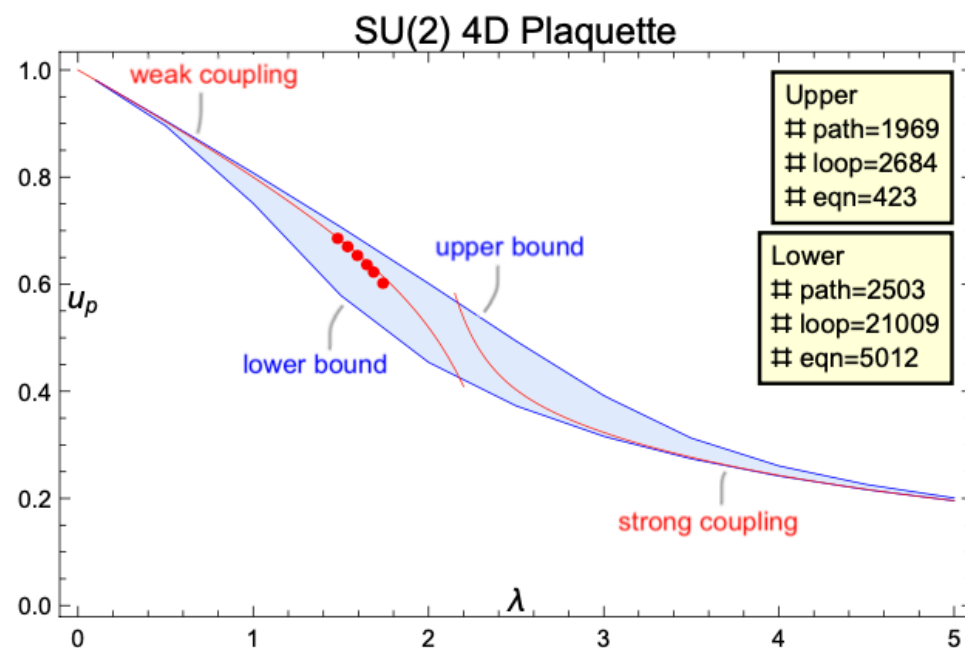
SDP computation

MacBook Pro with Apple M3 Max (using single thread)



2 hours, 33GB

40 min, 18GB



30 seconds

30 min, 8.5GB

Loop equations

Schwinger-Dyson equation for Wilson loops:

Makeenko and Migdal, 1980

$$\int \mathcal{D}U \, \delta_{\epsilon(\mu)} \left[e^{-S} W_{\vec{x}}^{ab}(\mu, C_\mu) \right] = 0 \quad \longrightarrow \quad - \langle W_{\vec{x}}^{ab} \delta_{\epsilon} S \rangle + \langle \delta_{\epsilon} W_{\vec{x}}^{ab} \rangle = 0$$

$$\delta_{\epsilon} S : \quad \text{---} \bullet \text{---} \bullet \text{---} \text{---} \Rightarrow \quad \frac{N}{2\lambda} \left(\begin{array}{c} \text{---} \bullet \text{---} \bullet \text{---} \text{---} \\ \text{---} \bullet \text{---} \bullet \text{---} \text{---} \end{array} - \begin{array}{c} \text{---} \bullet \text{---} \bullet \text{---} \text{---} \\ \text{---} \bullet \text{---} \bullet \text{---} \text{---} \end{array} \right)$$

$$\frac{1}{2\lambda} \left(\begin{array}{c} \text{---} \bullet \text{---} \bullet \text{---} \text{---} \\ \text{---} \bullet \text{---} \bullet \text{---} \text{---} \end{array} - \begin{array}{c} \text{---} \bullet \text{---} \bullet \text{---} \text{---} \\ \text{---} \bullet \text{---} \bullet \text{---} \text{---} \end{array} \right)$$

$$\delta_{\epsilon} W : \quad \begin{array}{l} n_+ \quad \begin{array}{c} \text{---} \bullet \text{---} \bullet \text{---} \text{---} \\ \text{---} \bullet \text{---} \bullet \text{---} \text{---} \end{array} \Rightarrow \begin{array}{c} \text{---} \bullet \text{---} \bullet \text{---} \text{---} \\ \text{---} \bullet \text{---} \bullet \text{---} \text{---} \end{array} \\ n_- \quad \begin{array}{c} \text{---} \bullet \text{---} \bullet \text{---} \text{---} \\ \text{---} \bullet \text{---} \bullet \text{---} \text{---} \end{array} \Rightarrow \begin{array}{c} \text{---} \bullet \text{---} \bullet \text{---} \text{---} \\ \text{---} \bullet \text{---} \bullet \text{---} \text{---} \end{array} \end{array}$$

(overlapping edges)

Loop equations

Backtrack SDE $\begin{array}{c} \square \\ \text{red arrow } \epsilon \end{array} \xrightarrow[\text{SU}(2)]{2D} \left(\square - \square \times \square \right) + \left(\begin{array}{c} \square \\ \square \end{array} - \begin{array}{c} \square \\ \square \end{array} \right) = 0$

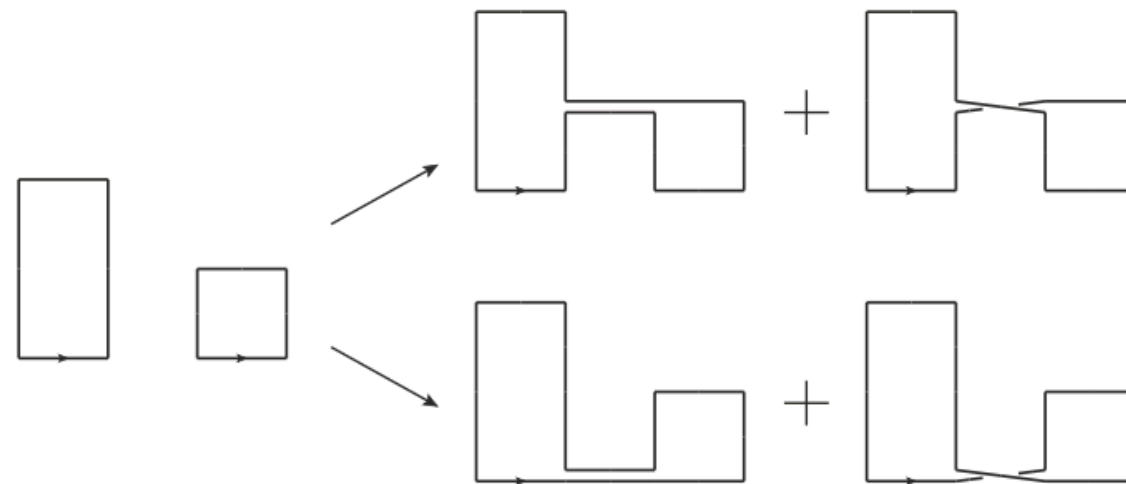


Figure 2. SU(2) trace relations.

Indirect equations:

$$\begin{array}{c} \square \\ \square \end{array} - \begin{array}{c} \square \\ \square \end{array} + \begin{array}{c} \square \\ \square \end{array} - \begin{array}{c} \square \\ \square \end{array} = 0,$$

$$1 + 2\lambda \left(\begin{array}{c} \square \\ \square \end{array} - \square \right) + \lambda \left(\begin{array}{c} \square \\ \square \end{array} - \begin{array}{c} \square \\ \square \end{array} \right) - \square - 2 \begin{array}{c} \square \\ \square \end{array} + 2 \begin{array}{c} \square \\ \square \end{array} + \begin{array}{c} \square \\ \square \end{array} - \begin{array}{c} \square \\ \square \end{array} = 0.$$