

Modularity of Schur index

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Outline

- ◉ Quick review of SCFT/VOA
- ◉ Tool 1: new integration formula [Li, YP, WIP]
- ◉ Tool 2: flavored MLDE and quasi-modularity
- ◉ Example: $a = c$ theories [Li, YP, Yan, 25] [Guo, Satoshi, YP, Zhang, WIP] [YP, Yang, 25]

SCFT/VOA

[Beem, Lemos, Liendo, Peelaers, Rastelli, Rees]

- ◉ 4d $\mathcal{N} = 2$ SCFT $\mathcal{T}$
- ◉ local operators with special properties: Schur operators

$$E - 2R - j_1 - j_2 = 0, \quad r + j_1 - j_2 = 0$$

- ◉ restricted on a plane (+ twisted translation): associated chiral algebra/vertex operator algebra (VOA) $\mathbb{V}[\mathcal{T}]$

$$\text{SCFT/VOA } \mathbb{V} : \mathcal{T} \rightarrow \mathbb{V}[\mathcal{T}]$$

SCFT/VOA

4d side	VOA / 2d side
4d $\mathcal{N} = 2$ SCFT $\mathcal{T}$	associated VOA $\mathbb{V}[\mathcal{T}]$
Schur operator	VOA operator
$SU(2)_{\mathcal{R}}$ symmetry	Virasoro subalgebra $\text{Vir}_{c_{2d}}$
flavor symmetry $\mathfrak{f}$	affine Kac-Moody subalgebra $\hat{\mathfrak{f}}_{k_{2d}}$
$(-12c_{4d}, -\frac{1}{2}k_{4d}) > 0$	non-unitary $(c_{2d}, k_{2d}) < 0$
Schur index: $\text{str } q^{E-R}$	vacuum character: $\text{str } q^{L_0}$
Higgs branch	associated variety

SCFT/VOA

- ◉ $\mathbb{V}[\mathcal{T}]$: representation theory
- ◉ General idea: surface/line defects in $\mathcal{T}$ [Gaiotto, Rastelli, Razamat]
- ◉ Some precise relations [Cordova, Gaiotto, Shao, 16] [Nishinaka, Sasa, Zhu, 18]
[Guo, Li, YP, Wang, 23]

(A_{N-1}, A_{M-1}) vortex index = irr. chs of $\mathcal{W}_N/\mathcal{B}(k)_{A_{N-1}}$

$$\langle L \rangle_{(A_1, A_2)} = q^{-\frac{1}{2}} \chi_0(q) - q^{-\frac{1}{2}} \chi_1(q)$$

$$\langle W_{j \in \mathbb{Z}} \rangle_{SU(2) \mathcal{N}=4} = \text{ch}_0 - \frac{i}{2} \sum_{\substack{m=-j \\ m \neq 0}}^j \frac{b^m - b^{-m}}{q^{m/2} - q^{-m/2}} \text{ch}_{\text{Verma}}$$

Goal

- ◉ Understand representation theory of $\mathbb{V}[\mathcal{T}]$
 - Construct character space (from 4d)
 - Modular property
- ◉ connect to 4d mirror symmetry

Tool 1: Exact computation

Contour integral

- ◉ vacuum character of $\mathbb{V}[\mathcal{T}]:\mathcal{I}$
- ◉ non-vacuum character from defects: **not straightforward** except for vortex defect
- ◉ **closed-form**: highly convenient

Contour integral

- ◉ Lagrangian: Schur index as multivariate contour integral

$$\mathcal{I} = \oint_{|a|=1} \left[\frac{da}{2\pi i a} \right] \mathcal{Z}(\mathfrak{a}), \quad a_A = e^{2\pi i \mathfrak{a}_A}$$

- ◉ Superconformal invariance $\Rightarrow$ **ellipticity**

$$\mathcal{Z}(\mathfrak{a}_1, \dots, \mathfrak{a}_i + \tau, \dots, \mathfrak{a}_r) = \mathcal{Z}(\mathfrak{a}_1, \dots, \mathfrak{a}_i, \dots, \mathfrak{a}_r)$$

$$\mathcal{Z}(\mathfrak{a}_1, \dots, \mathfrak{a}_i + 1, \dots, \mathfrak{a}_r) = \mathcal{Z}(\mathfrak{a}_1, \dots, \mathfrak{a}_i, \dots, \mathfrak{a}_r)$$

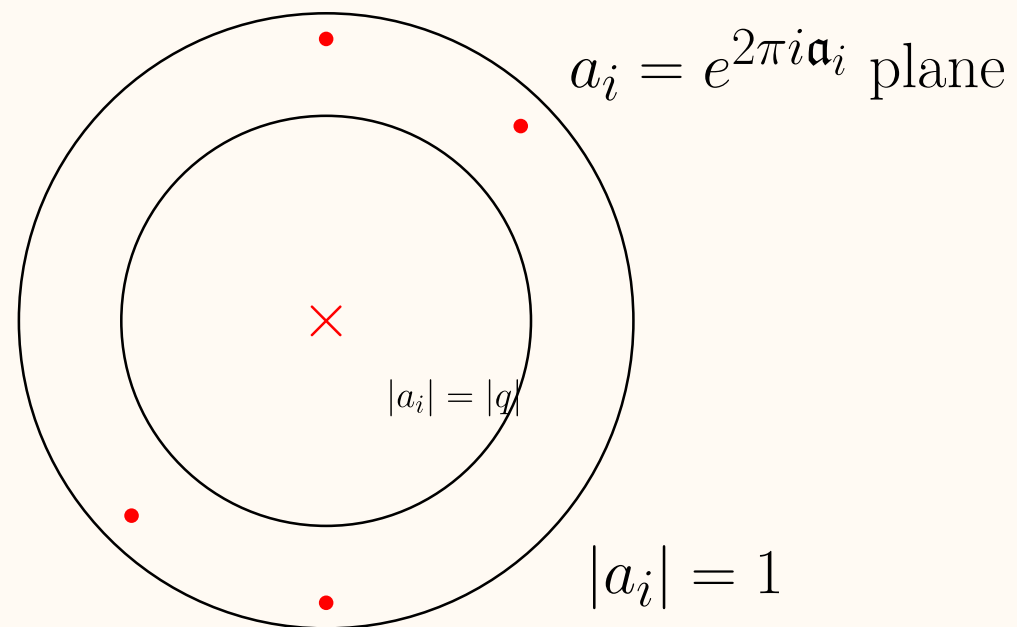
[Razamat, 12]

Contour integral

- ◉ Series of efforts in getting closed-form Schur index
- ◉ fermi-gas method: $\mathcal{N} = 4$ theories
[Bourdier, Drukker, Felix, 15] [Hatsuda, Okazaki, 22, 23] [Du, Huang, Wang, 23]
- ◉ (quasi-)elliptic integration formula: simple $\mathcal{N} = 2$ theories
[YP, Peelaers, 21] [Guo, Li, YP, Wang, 23] [YP, Yang, 24, 25]
- ◉ Residue theorem: $SU(2)$ theories, Macdonald index
[Leuven, Mathieson, Roy, 25]
- ◉ From modularity [Beem, Singh, Razamat, '21] [Huang, '22]
- ◉ Deformed index, symmetric polynomials [Hatsuda, 25] [Ren, Huang, 25, 26]

Contour integral

- Can we do better?
- One difficulty: **non-isolated** singularity ([Leuven, Mathieson, Roy, 25])



- avoid** the non-isolated singularity

Bethe ansatz formula

- ◉ Bethe ansatz formula for $\mathcal{N} = 1$ **superconformal index**:
quite general

[Benini, Milan, 18]

See also earlier work [Closset, Kim, Willet, 17]

- ◉ wide applicaiton to AdS_5 blackhole entropy from large N analysis

Bethe ansatz formula

- ◉ **Non-elliptic** integrand

$$\mathcal{Z} = \frac{\prod_{\rho, \omega} \Gamma((pq)^{\frac{r}{2}} z^{\rho} b^{\omega}; p, q)}{\prod_{\alpha} \Gamma(z^{\alpha}; p, q)}$$

$$\mathcal{Z}(\mathfrak{a}_i + \tau) = \mathcal{Z}(\mathfrak{a}_i) \mathcal{Q}_i(\mathfrak{a}), \quad \mathcal{Q}_i(\mathfrak{a}_j + \tau) = \mathcal{Q}_i(\mathfrak{a}_j)$$

Bethe ansatz formula

- Shift of contour

$$\oint \left[\frac{da}{2\pi i a} \right] \mathcal{Z}(a) = \oint \left[\frac{da}{2\pi i a} \right] \prod_i \frac{1 - \mathcal{Q}_i(a)}{1 - \mathcal{Q}_i(a)} \mathcal{Z}(a)$$

$$\text{expand numerator} = \sum_{\ell=0}^r \sum_{i_1 \neq \dots \neq i_\ell} \frac{(-1)^\ell}{\ell!} \oint_{\mathcal{C}_{i_1, \dots, i_\ell}} \left[\frac{da}{2\pi i a} \right] \frac{\mathcal{Z}(a)}{\prod_i (1 - \mathcal{Q}_i(a))} .$$

use ellipticity **for all variables, simultaneously**

- roughly: non-isolated singularity is "**walled off**" (outside the annulus region bounded by $\mathcal{C}_{i_1, \dots, i_\ell}$)

Bethe ansatz formula

- ◉ residue theorem (summing over Grothendieck residues)

$$= \sum_{a_* \in \text{BAE}} \text{Res}_{a=a_*} \left[\frac{1}{2\pi i a_1 \cdots a_r} \frac{\mathcal{Z}(a)}{\prod_i (1 - Q_i(a))} \right], \quad |q| \leq |a_*| \leq 1$$

- ◉ **Bethe ansatz solutions**

$$\text{BAE} = \{a \mid Q_i(a) = 1, \underbrace{|q| < |a_j| < 1}_{\text{annulus region}}, j = 1, \dots, r\}$$

- ◉ **Schur limit?**

Bethe ansatz formula

[Li, YP, WIP]

- ◉ 0th problem: **ellipticity** $\Rightarrow Q_i = 1$
- ◉ 1st problem: integration **breaks** periodicity behavior
- ◉ 2nd problem: higher dimensional residue theorem picks up poles **outside** the annulus region (**infinitely many** of them)

The result is **incomplete**, though **effective** for holography

- ◉ **Adapt the idea, identify the correction**

Bethe ansatz formula

- Schur index at **first integration**: $\mathcal{Z}$ is **elliptic**

$$\oint_{|a|=1} \frac{da}{2\pi i a} \mathcal{Z}(a) = \oint_{|a|=1} \frac{da}{2\pi i a} \frac{1 - Q(a)}{1 - Q(a)} \mathcal{Z}(a)$$

$Q(a)$ is **to be determined**

- Target: wall off non-isolated singularity**

$$\oint_{|a|=1} \frac{da}{2\pi i a} \frac{1 - Q(a)}{1 - Q(a)} \mathcal{Z}(a) \stackrel{\text{want}}{=} \left[\oint_{|a|=1} - \oint_{|a|=|q|} \right] \frac{da}{2\pi i a} \frac{\mathcal{Z}(a)}{1 - Q(a)}$$

$$\stackrel{\text{residue thm.}}{=} \sum_J \operatorname{Res}_{a=a_J} \frac{1}{a} \frac{\mathcal{Z}(a)}{1 - Q(a)}, \quad |q| \leq |a_J| < 1$$

Bethe ansatz formula

⦿ Difference equation

$$\frac{Q(a)Z(a)}{1-Q(a)} \stackrel{\text{want}}{=} \frac{Z(a+\tau)}{1-Q(aq)} = \frac{Z(a)}{1-Q(aq)} \Rightarrow \frac{Q(a)}{1-Q(a)} = \frac{1}{1-Q(aq)}$$

⦿ solution (not unique)

$$\frac{1}{1-Q(a)} = E_1 \left[\frac{-1}{\frac{a}{bq^{1/2}}} \right]$$

$$\Rightarrow \oint_{|a|=1} \frac{da}{2\pi i a} Z(a) = \sum_J \text{Res}_{a=a_J} \frac{1}{a} Z(a) E_1 \left[\frac{-1}{\frac{a}{bq^{1/2}}} \right]$$

reproduced [YP, Peelaers, 21]

ellipticity **braken** after one integration

Bethe ansatz formula

- Subsequent integration: elliptic $\times$ **polynomial** $P(a)$ **of Eisenstein**

$$P(a) = E_{k_1} \left[\begin{smallmatrix} \pm 1 \\ a^{K_1} b_1 \end{smallmatrix} \right] \cdots E_{k_n} \left[\begin{smallmatrix} \pm 1 \\ a^{K_n} b_n \end{smallmatrix} \right], \quad K_i \in \mathbb{N}$$

- difference equation (of $n(a)$)

$$\frac{Q(a)P(a)}{1 - Q(a)} = \frac{P(aq)}{1 - Q(aq)} \Rightarrow n(aq) - n(a) = P(a), \quad Q(a) = 1 + \frac{P(a)}{n(a)}$$

Bethe ansatz formula

- $n(a)$ can be solved easily

$$n(a) = \sum_{r=0}^{\infty} \frac{B_r}{r!} \partial^{r-1} P(a)$$

where ∂ is lowering operator of Eisenstein series (with linearity and Leibniz rule)

$$\partial E_k \left[\begin{smallmatrix} \pm 1 \\ a^K \end{smallmatrix} \right] = E_{k-1} \left[\begin{smallmatrix} \pm 1 \\ a^K \end{smallmatrix} \right]$$

if $P(a) = E_{k_1} \left[\begin{smallmatrix} \pm \\ a^{K_1} b_1 \end{smallmatrix} \right] E_{k_2} \left[\begin{smallmatrix} \pm \\ a^{K_2} b_2 \end{smallmatrix} \right]$

$$\Rightarrow n(a) = \sum_{r=0}^{k_1+k_2} \frac{B_r}{r!} \sum_{u=0}^r \sum_{s=0}^{k_2} (-1)^s \binom{r}{u} \frac{K_2^s}{K_1^{s+1}} K_1^u K_2^{r-u} E_{k_1+1+s-u} \left[\begin{smallmatrix} \pm \\ a^{K_1} b_1 \end{smallmatrix} \right] E_{k_2-s-r+u} \left[\begin{smallmatrix} \pm \\ a^{K_2} b_2 \end{smallmatrix} \right]$$

Bethe ansatz formula

- Integral

$$\begin{aligned} & - \oint \frac{da}{2\pi i a} \mathcal{Z}(a) P(a) \\ &= \sum_J \operatorname{Res}_{a=a_J} \frac{1}{a} \mathcal{Z}(a) n(a) \\ &= \sum \text{theta-ratio} \times \text{poly of Eisenstein} \end{aligned}$$

Bethe ansatz formula

- ◉ Elegant example

$$\mathcal{I} = \oint \frac{da_1}{2\pi i a_1} \frac{da_2}{2\pi i a_2} \frac{\vartheta_1(a_1 - a_2)^2 \vartheta_1(a_1 + a_2)^2}{\prod_{\alpha=\pm} \vartheta_4(\alpha(a_1 - a_2) + b_1) \vartheta_4(\alpha(a_1 + a_2) + b_2)} .$$

$$n(aq) - n(a) = E_1 \begin{bmatrix} 1 \\ ab \end{bmatrix} \Rightarrow n(a) = 1 - E_1 \begin{bmatrix} -1 \\ aq^{-1/2} \end{bmatrix} + E_2 \begin{bmatrix} -1 \\ abq^{1/2} \end{bmatrix} .$$

$$\Rightarrow \mathcal{I} = - \frac{4\vartheta_4(b_1)^2 \vartheta_4(b_2)^2}{\eta(\tau)^6 \vartheta_1(2b_1) \vartheta_1(2b_2)} E_1 \begin{bmatrix} -1 \\ b_1 \end{bmatrix} E_1 \begin{bmatrix} -1 \\ b_2 \end{bmatrix} .$$

Bethe ansatz formula

- ◉ Coefficient: Grothendieck residue of **some pole** in the annulus region

$$\frac{4\vartheta_4(b_1)^2\vartheta_4(b_2)^2}{\eta(\tau)^6\vartheta_1(2b_1)\vartheta_1(2b_2)}$$

- ◉ "Generalized residue"

$$\frac{4\vartheta_4(b_1)^2\vartheta_4(b_2)^2}{\eta(\tau)^6\vartheta_1(2b_1)\vartheta_1(2b_2)} E_1 \begin{bmatrix} -1 \\ b_1 \end{bmatrix} E_1 \begin{bmatrix} -1 \\ b_2 \end{bmatrix}$$

roughly: Eisenstein series keep tracks of the **homology class of the singular loci/deformation of contour**

Bethe ansatz formula

- ◉ Most(all?) of Lagrangian Schur index in closed form

$$\oint \left[\frac{da}{2\pi i a} \right] \mathcal{Z}(a) \\ = \sum \underbrace{\text{theta-ratio} \times \text{poly of Eisenstein}}_{\text{quasi-Jacobi}}$$

The only problem: computational complexity

Bethe ansatz formula

- Wilson line ('t Hooft line) Schur index

$$\oint \left[\frac{da}{2\pi i a} \right] \chi_{\mathcal{R}}(a) \mathcal{Z}(a) \\
 = \sum \text{rational fn} \times \underbrace{\text{theta-ratio} \times \text{poly of Eisenstein}}_{\text{quasi-Jacobi}}$$

- vortex surface defect Schur index

Bethe ansatz formula

- $USp(2r)$ with $2r + 2$ fundamental

$$\mathcal{I}_{USp(2r)} = \sum_{\substack{S \subset \{1, \dots, 2r+2\} \\ |S|=r}} \frac{i^{r^2} \eta(\tau)^{r(2r+3)} \prod_{i \in S} \vartheta_1(2\mathfrak{b}_i) E_1 \left[\frac{-1}{\mathfrak{b}_i} \right]}{\prod_{i \in S} \prod_{j \notin S} \vartheta_1(\mathfrak{b}_i - \mathfrak{b}_j) \vartheta_1(\mathfrak{b}_i + \mathfrak{b}_j)}$$

#

- associated VOA: conjectured to be $\widehat{SO}(4r + 4)_{-2r}$

[Chandra, Mukhi, Singh, 25] [YP, Yan, WIP]

- "generalized partition function" $\rightarrow D_2(SU(2N + 1)), (A_1, D_{2N+1}), (A_1, A_{2N})$

[Deb, Razamat, 25]

Bethe ansatz formula

- $SO(2r)$ with $2r - 2$ fundamentals

$$\mathcal{I}_{SO(2r), N_f=2r-2} = \sum_{\substack{S \subset F \\ |S|=r-1}} \mathcal{A}_S + \sum_{\substack{S \subset F \\ |S|=r}} \mathcal{B}_S,$$

where $F = \{1, \dots, 2r - 2\}$,

$$\mathcal{A}_S = \frac{i^{r-1} (-1)^{\frac{(r-1)(r-2)}{2}} \eta(\tau)^{2r^2-3r+3} \prod_{s \in S} \vartheta_4(\mathfrak{b}_s)^2 E_1[\frac{-1}{b_s}]}{\prod_{s \in S} \vartheta_1(2\mathfrak{b}_s) \prod_{t \in F \setminus S} \vartheta_4(\mathfrak{b}_t)^2 \prod_{\substack{s \in S \\ t \in F \setminus S}} \vartheta_1(\mathfrak{b}_s \pm \mathfrak{b}_t)},$$

$$\mathcal{B}_S = \frac{2i^r (-1)^{\frac{r(r-1)}{2}} \eta(\tau)^{r(2r-3)} \prod_{s \in S} E_1[\frac{-1}{b_s}]}{\prod_{s \in S} \vartheta_1(2\mathfrak{b}_s) \prod_{\substack{s \in S \\ t \in F \setminus S}} \vartheta_1(\mathfrak{b}_s \pm \mathfrak{b}_t)},$$

Bethe ansatz formula

- ◉ $\mathcal{N} = 4 SO(2N), SO(2N + 1)$ and $USp(2N)$
- ◉ $[SO(4)] - USp(2) - SO(4) - [USp(2)]$

$$\begin{aligned}
\mathcal{I} = & \frac{\eta(\tau)^6}{q^{5/4}} \left[- \frac{1}{\prod_{i=1}^2 \vartheta_4(-\mathfrak{b}_i + \mathfrak{c}) \vartheta_4(\mathfrak{b}_i + \mathfrak{c})} \left(\frac{1}{2} + E_2 \begin{bmatrix} 1 \\ c \end{bmatrix} \right) \right. \\
& + \sum_{i=1}^2 \left(\frac{(-1)^{i+1} \vartheta_1(\mathfrak{b}_i)^2}{\vartheta_1(\mathfrak{b}_1 \pm \mathfrak{b}_2) \vartheta_4(\mathfrak{c})^2 \vartheta_4(\pm \mathfrak{b}_i + \mathfrak{c})} \left(\frac{3}{8} + E_1 \begin{bmatrix} -1 \\ b_i \end{bmatrix} E_1 \begin{bmatrix} 1 \\ b_i \end{bmatrix} + E_2 \begin{bmatrix} -1 \\ b_i \end{bmatrix} \right) \right. \\
& \quad \left. + \frac{(-1)^{i+1} \vartheta_1(2\mathfrak{b}_i) \vartheta_4(\mathfrak{c})^2}{\vartheta_1(\mathfrak{b}_1 \pm \mathfrak{b}_2) \vartheta_1(\mathfrak{b}_i)^2 \vartheta_1(2\mathfrak{c}) \vartheta_4(\pm \mathfrak{b}_i + \mathfrak{c})} E_1 \begin{bmatrix} -1 \\ b_i \end{bmatrix} E_1 \begin{bmatrix} -1 \\ c \end{bmatrix} \right] \\
& - \frac{2i\eta(\tau)^3}{q^{5/4}} \sum_{i=1}^2 \frac{(-1)^{i+1}}{\vartheta_1(\mathfrak{b}_1 \pm \mathfrak{b}_2) \vartheta_1(2\mathfrak{c})} E_1 \begin{bmatrix} -1 \\ c \end{bmatrix} \left(E_1 \begin{bmatrix} -1 \\ b_i \end{bmatrix} E_1 \begin{bmatrix} 1 \\ b_i \end{bmatrix} + E_2 \begin{bmatrix} -1 \\ b_i \end{bmatrix} \right) .
\end{aligned}$$

Bethe ansatz formula

- ◉ $[1] - SU(2) \times SU(3) - [4]$ unflavored

$$\begin{aligned} \mathcal{I} = & \frac{2}{135\eta(\tau)^{15}\vartheta_4(0)^4} \left(-405E_2(\tau)E_4(\tau) + 1350E_2(\tau)^2E_4(\tau) - 20250E_4(\tau)^2 \right. \\ & - 4165E_6(\tau) - 1890E_2(\tau)E_6(\tau) - 4785E_4(\tau)E_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ & + 270E_2(\tau)^3E_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} - 8370E_2(\tau)E_4(\tau)E_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ & - 1080E_2(\tau)^2E_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}^2 + 13500E_4(\tau)E_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}^2 \\ & \left. + 1276E_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}^3 + 2592E_2(\tau)E_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}^3 - 2160E_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}^4 \right) \end{aligned}$$

Tool 2: Quasi-modularity

flavored MLDE and quasi-modularity

- ◉ $\mathbb{V}[\mathcal{T}]$: quasi-lisse (**non-rational**, non C_2 -cofinite/lisse) [Dongmin's talk] [Arakawa, Kawasetsu]
- ◉ ordinary/**non-ordinary** modules, logarithmic/**non-logarithmic** modules

flavored MLDE and quasi-modularity

- ◉ **all ordinary** unflavored character $\text{ch}(q)$: solve one **MLDE** [Arakawa, Kawasetsu]

$$\left[D_q^{(N)} + \sum_{i=0}^{N-1} \phi_r(\tau) D_q^{(N-r)} \right] \text{ch}(q) = 0$$

$\text{ch}(q)$ denotes any (unflavored) character, ϕ_r weight- $2r$ **modular forms**

$$D_q^{(n)} := \partial_{(2n-2)} \circ \cdots \circ \partial_{(2)} \circ \partial_{(0)}$$

$$\partial_{(k)} := q\partial_q + kE_2(\tau)$$

flavored MLDE and quasi-modularity

- ◉ MLDE has been extremely useful

- ◉ classifying RCFTs

[Mathur, Mukhi, Sen, 88] [Gaberdiel, Hampapura, Mukhi, 16] [Chandra, Mukhi, 18] [Bae, Duan, Lee, Lee, Sarkis, 20, 21] [Das, Gowdigere, Santara, 21] [Das, 23], many others

- ◉ classifying $\mathcal{N} = 2$ SCFT

[Beem, Rastelli] [Kaidi, Martone, Rastelli, Weaver, 22] [Deb, Meneghelli, Rastelli]

flavored MLDE and quasi-modularity

- For **non-ordinary** characters: flavor refinement is **necessary**
- analog of MLDE?**
- MLDEs (may) come from null states in VOA + Zhu's recursion

[Gaberdiel, Keller]

[Beem, Rastelli]

$$\begin{aligned}
 & \text{str}_M o(a_{[-h_a]}b) q^{L_0-c/24} \\
 &= \delta_{Q,0} \text{str}_M o(a)o(b) q^{L_0-c/24} \\
 & \sum_{n=1}^{+\infty} +E_n \begin{bmatrix} e^{2\pi i h_a} \\ 1 \end{bmatrix} \text{str}_M o(a_{[-h_a+n]}b) q^{L_0-c/24}.
 \end{aligned}$$

flavored MLDE and quasi-modularity

- ◉ null states + **flavored** Zhu's recursion

$$\begin{aligned} & \text{str}_M o(a_{[-h_a]} b) x^{J_0} q^{L_0 - c/24} \\ &= \delta_{Q,0} \text{str}_M o(a) o(b) x^{J_0} q^{L_0 - c/24} \\ &+ \sum_{n=1}^{+\infty} E_n \begin{bmatrix} e^{2\pi i h_a} \\ x^Q \end{bmatrix} \text{str}_M o(a_{[-h_a+n]} b) x^{J_0} q^{L_0 - c/24}. \end{aligned}$$

- ◉ **flavored MLDEs**

[Zhu, 96]

[Tuite, Mason, Zuevsky, 07]

[Peelaers]

[YP, Wang,]

[YP, Zeng]

[Arakawa, Ekeren, Li, 26]

$$o(\mathcal{N}) \Rightarrow \sum_{\substack{k, \ell_1, \dots, \ell_r \\ 2k + \ell_1 + \dots + \ell_r \leq 2n}} \phi_{k, \ell_1, \dots, \ell_r}(\tau, b_i) D_q^{(k)} D_{b_1}^{\ell_1} \dots D_{b_r}^{\ell_r} \text{ch}(q, b) = 0$$

flavored MLDE and quasi-modularity

- Use to check/constrain flavored characters
- ϕ are not modular forms, but **quasi-Jacobi forms**
- MLDE: **modular**, solution $\xrightarrow{\gamma}$ solution
- flavored MLDE: **quasi-modular**

$$S : \text{eq} \rightarrow \tau^k \text{eq} + \tau^{k-1} \mathfrak{b}_i \text{eq}_i + \tau^{k-2} \mathfrak{b}_i \mathfrak{b}_j \text{eq}_{ij} + \dots$$

[Zheng, YP, Wang]

flavored MLDE and quasi-modularity

- ◉ If space of solution is **modular inv**: all flavored characters solve all

$$eq = eq_i = eq_{ij} = \dots = 0$$

- ◉ **holomorphic quasi-modular bootstrap**: constrain parameters in OPE

[YP, Zeng]

flavored MLDE and quasi-modularity

- ◉ If all eq come from null states: **a modular action on the nulls**

$$\mathcal{N} \rightarrow \bigoplus_{n,\ell,i_1,\dots,i_\ell} \tau^{n+h[\mathcal{N}]-\ell} \mathfrak{b}_{i_1} \dots \mathfrak{b}_{i_\ell} \mathcal{N}_{i_1 \dots i_\ell}$$

looks similar to RCFT primaries: $\mathcal{O}_i \rightarrow S_{ij} \mathcal{O}_j$

flavored MLDE and quasi-modularity

- In many examples: **universal** modular structure

$$\mathcal{N} \xrightarrow{S,STS} \sum_{n,\ell,i_1,\dots,i_\ell} \binom{h[\mathcal{N}] - \ell}{n} (-1)^{h[\mathcal{N}] + n - \ell} \tau^{k-\ell} \mathfrak{b}_{i_1} \dots \mathfrak{b}_{i_\ell} h_1^{i_1} \dots h_1^{i_\ell} \mathcal{N}$$

[YP, Wang]

[YP, Zeng]

Examples

Examples

- ◉ A family of nice examples: **theories with $a = c$**

- ◉ $\mathcal{N} = 4$ SYM with gauge group G

- ◉ $\mathcal{N} = 3$ theories

[Aharony, Evtikhiev, 15]

[Garcia-Etxebarria, Regalado]

[Nishinaka, Tachikawa]

[Zafrir]

- ◉ $\widehat{\Gamma}(G)$ theory: $\mathcal{T}_{p,N}$

[Kang, Lawrie, Song]

[Jiang]

Examples

- ◉ Plan
- ◉ Discover the space of characters: from 4d, FMLDE
- ◉ Analyze (quasi-)modular property
- ◉ Compare with Zhu's algebra prediction/Coulomb branch geometry
- ◉ Find **relations** between representation theory of **different** $\mathbb{V}[\mathcal{T}]$

$\mathcal{N} = 4$ $SU(N)$ Wilson line index

- VOA: a $\mathcal{W}$ -algebra with known set of generators, $c_{2d} = -3(N^2 - 1)$

[Bonetti, Meneghelli, Rastelli]

[Bonetti, Meneghelli, 25]

[Gaberdiel, Li, 25]

$$J^a, G^\pm, \bar{G}^\pm, T, W^\cdots, G_{\ddot{W}}, \bar{G}_{\ddot{W}}, T_W$$

- representation theory?

$\mathcal{N} = 4$ $SU(N)$ Wilson line index

- ◉ (line defect) Schur index in closed form: **fermi-gas** or integration formula

[Bourdier, Drukker, Felix]

[Hatsuda, Okazaki]

[Du, Huang, Wang]

[Guo, Li,

YP, Wang]

- ◉ Representation-theoretic application: integration formula

$$\langle W_{\mathcal{R}} \rangle = \oint \left[\frac{da}{2\pi i a} \right] \chi_{\mathcal{R}}(a) \mathcal{Z}(a)$$

- ◉ A linear basis:

$$\mathcal{I}(\vec{n}) = (\dots) \oint \left[\frac{da}{2\pi i a} \right] \prod_{A=1}^N a_A^{n_A} \frac{\prod_{A \neq B} \vartheta_1(\mathfrak{a}_A - \mathfrak{a}_B)}{\prod_{A,B} \vartheta_4(\mathfrak{a}_A - \mathfrak{a}_B + \mathfrak{b})}, \quad \sum_{A=1}^N n_A = 0$$

$\mathcal{N} = 4$ $SU(N)$ Wilson line index

- Fact: The **ordering** in $\vec{n} = (n_1, \dots, n_N)$ is **unimportant**

symmetry in the integrand

$\mathcal{N} = 4$ $SU(N)$ Wilson line index

- Fact: the result is a linear combination of **quasi-Jacobi forms**

$$\mathcal{I}(\vec{n}) = \sum_{\alpha} R(\vec{n}, \alpha) J(\alpha)$$

α : some **labelling set**, to be introduced

$R(\vec{n}, \alpha)$: **rational** function in $b, q^{1/2}$, carries information of $\vec{n}$

$J(\alpha)$: **quasi-Jacobi** form in b, q

$\mathcal{N} = 4$ $SU(N)$ Wilson line index

$SU(3)$ example

$$\begin{aligned}
 & \mathcal{I}(2, -2, 0) \\
 = & - \frac{i(b^2 - b^{-2})[(b + b^{-1}) - (q^{1/2} + q^{-1/2})]}{12(q^{1/2} - q^{-1/2})^3} \frac{\eta(\tau)^3}{\vartheta_1(4\mathfrak{b})} \\
 & - \frac{i}{24} \frac{(b + b^{-1})(q^{1/2} + q^{-1/2}) - 2(b^2 + b^{-2})}{(q^{1/2} - q^{-1/2})^3} \frac{\eta(\tau)^3}{\vartheta_1(4\mathfrak{b})} \left(E_1 \begin{bmatrix} -1 \\ b \end{bmatrix} + E_1 \begin{bmatrix} -1 \\ b^3 \end{bmatrix} \right) \\
 & - \frac{i(b - b^{-1})}{q^{1/2} - q^{-1/2}} \frac{\eta(\tau)^3}{\vartheta_1(4\mathfrak{b})} \left(E_1 \begin{bmatrix} -1 \\ b \end{bmatrix}^2 + 2E_1 \begin{bmatrix} -1 \\ b \end{bmatrix} E_1 \begin{bmatrix} 1 \\ b^2 \end{bmatrix} - 2E_2 \begin{bmatrix} -1 \\ b^3 \end{bmatrix} - E_2 \begin{bmatrix} 1 \\ b^2 \end{bmatrix} \right)
 \end{aligned}$$

$\mathcal{N} = 4$ $SU(N)$ Wilson line index

- scan through all $\vec{n}$: a **complete basis** of quasi-Jacobi form

$$\mathcal{J}_{\ell}, \quad \ell = \{\ell_1, \dots, \ell_w\} \vdash N$$

- closed form

$$\mathcal{J}_{\ell} = \frac{\eta(\tau)^3}{\vartheta_{1 \text{ or } 4}(N\mathfrak{b})} \sum_{\sigma \in \mathcal{P}(\{1, \dots, w\} | w-1)} \prod_i \left(-(\mathbf{k}_i - 1)! E_{\mathbf{k}_i} \left[\begin{matrix} (-1)^{\sum_{p=1}^{k_i} \ell_{\sigma_{ip}}} \\ b^{\sum_{p=1}^{k_i} \ell_{\sigma_{ip}}} \end{matrix} \right] \right)$$

here $\mathcal{P}(\{1, \dots, w\} | w-1)$ is the set of **partitions** of a $(w-1)$ -element subset of $\{1, \dots, w\}$

- $\mathcal{J}_{\ell}$ has modular weight $w-1$

$\mathcal{N} = 4$ $SU(N)$ Wilson line index

partition

$$\begin{aligned} \{1, \dots, w\} &\rightarrow \{\sigma_1, \dots, \sigma_{w-1}\} \\ &\rightarrow \{\sigma_{11} \cdots \sigma_{1k_1}\} \{\sigma_{21} \cdots \sigma_{2k_2}\} \cdots \\ &\rightarrow k_1 + k_2 + \cdots = w - 1 \end{aligned}$$

any $(w - 1)$ -element
any partition

$$[\ell_1, \dots, \ell_w] \rightarrow \{\ell_{\sigma_{11}} \cdots \ell_{\sigma_{1k_1}}\} \{\ell_{\sigma_{21}} \cdots \ell_{\sigma_{2k_2}}\} \cdots$$

$\mathcal{N} = 4$ $SU(N)$ Wilson line index

- Fact: **flavored Schur index** closed-form, linear combination of $\mathcal{I}_\ell$

$$\mathcal{I}_{SU(N)}(b, q) = \sum_{\substack{\ell \\ \text{odd } \ell_i}} \left[\frac{1}{\prod_{A=1}^N m_A!} \prod_{j=1}^{w(\ell)} \left(\frac{(\ell_j - 2)!!}{(2i)^{(\ell_j - 1)/2}} \right) \frac{1}{\ell_j!} \right] \mathcal{I}_\ell$$

$\mathcal{N} = 4$ $SU(N)$ Wilson line index

- ◉ **Conjecture:** $\mathcal{J}_\ell$ are independent **module characters** of $\mathbb{V}[\mathcal{T}_{SU(N)}]$, in one to one correspondence with **integer partitions of N**
- ◉ Each $\mathcal{J}_\ell$ generates a $w(\ell)$ -**dimensional** space of quasi-Jacobi forms
- ◉ T^2 Jordan blocks

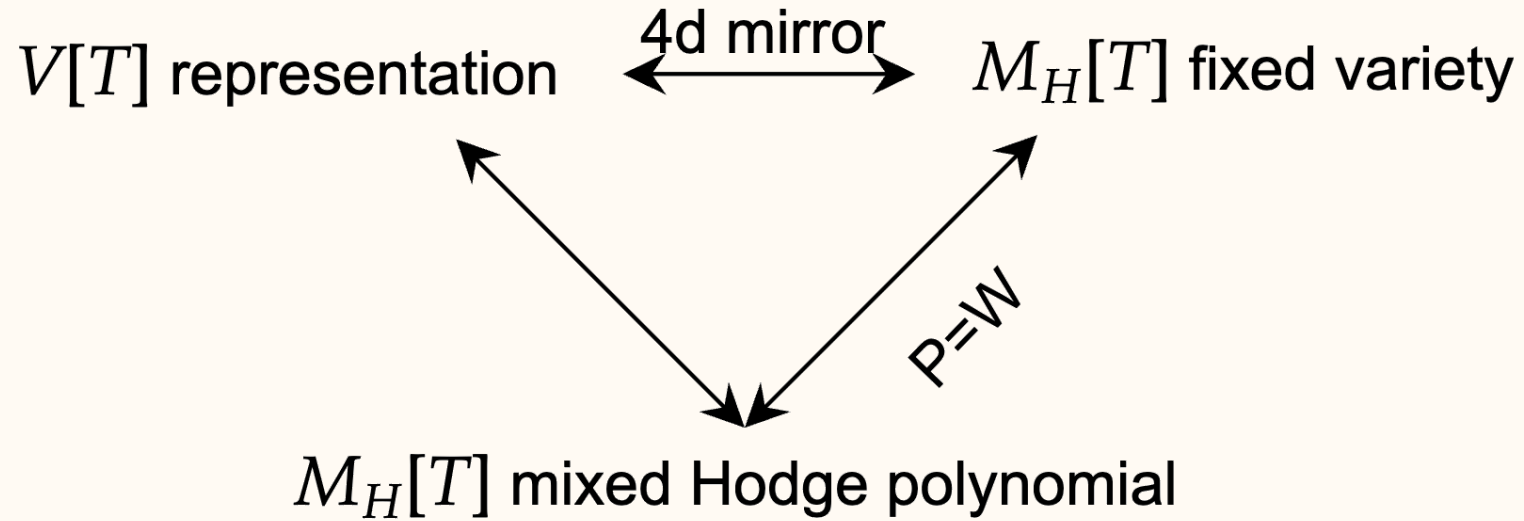
$$\oplus_{\ell} J_{w(\ell)}$$

- ◉ **Conjecture:** the modular orbit of $\{\mathcal{J}_\ell\}$ is full character space of $\mathbb{V}_{\mathcal{N}=4}$

$\mathcal{N} = 4$ $SU(N)$ Wilson line index

- ◉ Possible checks
- ◉ flavored MLDE constraints (to be done, computational heavy)
- ◉ 4d mirror symmetry
- ◉ relation to $\mathbb{V}[\mathcal{T}_{p,N}]$ modules

$\mathcal{N} = 4$ $SU(N)$ Wilson line index



[Gaiotto, Moore, Neitzke]

[Fredrickson, Neitzke]

[Fredrickson, Pei, Yan,

Ke, 17]

[Dedushenko, Gukov, Nakajima, Pei, Ye, 18]

[Shan, Xie, Yan, 23, 24]

[YP, Yan '24]

[Xie, '26]

$\mathcal{N} = 4$ $SU(N)$ Wilson line index

- coincides with **pure part of mixed Hoedge polynomial** of once-punctured $([N - 1, 1]$ type) Riemann surface [Li, YP, Yan, '25]

$$PH_c(\mathcal{M}_{B,N}; q) = \sum_{\ell \vdash N} q^{w(\ell)},$$

[Hausel, Rodriguez-Villegas]

[Hausel, Letellier, Rodriguez-Villegas]

- # similar phenomenon in **4d mirror symmetry conj.** of A_1 class- $\mathcal{S}$ theory [YP, Yan, '24]

$\mathcal{N} = 4$ $SU(N)$ Wilson line index

- ◉ $\mathcal{T}_{p,N}$ theory: non-Lagrangian theories with $a = c$

[Kang, Lawrie, Song] [Jiang]

- ◉ Known fact by [Kang, Lawrie, Song]

$$\text{ch}_0[\mathcal{T}_{p,N}](q) = \text{ch}_0[\mathcal{T}_{N=4}](b = q^{\frac{p}{2}-1}, q = q^p)$$

graded vector space isomorphism [from Wenbin's talk]

- ◉ **Generalization:** $\mathbb{V}[\mathcal{T}_{p,N}]$ [YP, Yang, '24] [Li, YP, Yan, 25]

$$\text{ch}[\mathcal{T}_{p,N}](q) = \mathcal{J}_\ell(b = q^{\frac{p}{2}-1}, q = q^p)$$

are characters

$\mathcal{N} = 4$ $SU(N)$ Wilson line index

- ◉ small summary

Wilson line index + closed form $\Rightarrow$ character space

$\mathcal{N} = 3$ theory

- 4d $\mathcal{N} = 4$ SCFT: 16 supercharges, rather **rigid**, Lagrangian

basically $\mathcal{N} = 4$ SYM with some gauge group G

- 4d $\mathcal{N} = 2$ SCFT: 8 supercharges, extremely **rich**

Lagrangian or non-Lagrangian

class- $\mathcal{S}$, Argyres-Douglas, $\mathcal{N} = 2$ quivers, etc.

- 4d $\mathcal{N} = 3$ SCFT: 12 supercharges, much less studied

no explicit construction until [García-Etxebarria, Regalado, '15] [Aharony, Tachikawa, '16]

$\mathcal{N} = 3$ theory

- ◉ Lagrangian $\mathcal{N} = 3$ SCFT = $\mathcal{N} = 4$ SYM
- ◉ pure $\mathcal{N} = 3 \Rightarrow$ non-Lagrangian
- ◉ Common property (rigidity): [Aharony, Evtikhiev, 15]
 - $c_{4d} = a_{4d}$
 - No $\mathcal{N} = 2$ exactly marginal deformations
 - **No** $\mathcal{N} = 3$ flavor symmetry
 - $U(1)_f$ flavor as $\mathcal{N} = 2$ theory (from $\mathcal{N} = 3$ $\mathcal{R}$ -symmetry)
 - **not accessible** through $\mathcal{N} = 2$ SUSY-preserving deformations
- ◉ Known construction: S-fold from F-theory, discrete gauging $\mathcal{N} = 4$
[García-Etxebarria, Regalado, 15], [Bourton, Pini, Pomoni, 18]

$\mathcal{N} = 3$ theory

- ◉ $\mathcal{N} = 2$ non-Lagrangian theories are **accessible** from $\mathcal{N} = 1$ theory through special RG flow

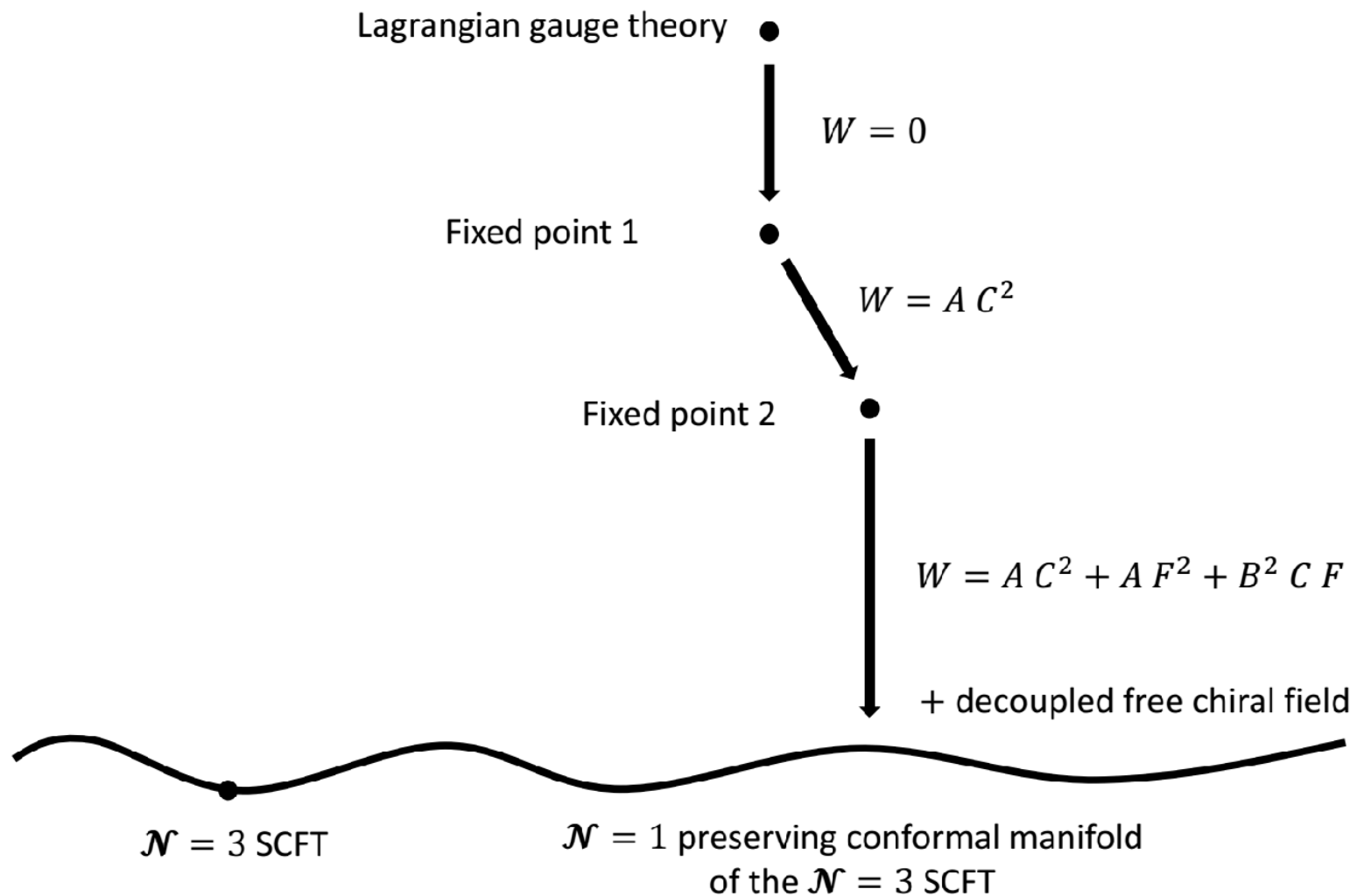
[Maruyoshi, Song, '16]

- ◉ Generalize to $\mathcal{N} = 3$

[Zafrir, '20]

- ◉ $\mathcal{N} = 1$ $SU(2) \times SU(2)$ quiver gauge theory

combination of RG flows and superpotential deformations

$\mathcal{N} = 3$ theory

$\mathcal{N} = 3$ theory

- RG-flow end point: $\mathcal{N} = 1$ **deformable** to rank-one $\mathcal{N} = 3$ theory with $c_{4d} = -\frac{5}{4}$

S-fold theory with moduli space $\mathbb{C}^3/\mathbb{Z}_3$, $\mathbb{Z}_3 = G(3, 1, 1)$

$$e^{\frac{2\pi i}{3}(r+s)} : (z_1, z_2, z_3) \rightarrow (e^{\frac{2\pi i}{3}} z_1, e^{\frac{2\pi i}{3}} z_2, e^{\frac{2\pi i}{3}} z_3) \in \mathbb{C}^3$$

index from counting states [Agarwal, Andriolo, Kántor, Papageorgakis, 21]

$\mathcal{N} = 3$ theory

[Guo, Satoshi, YP, Zhang, WIP]

- The **unflavored** Schur index from $\mathcal{N} = 1$ Lagrangian by [Zafrir, 20]

$$\mathcal{I} = \frac{\eta\left(\frac{\tau}{2}\right)^5}{4q\eta(\tau)} \oint \frac{dz}{2\pi iz} \frac{dy}{2\pi iy} \frac{\vartheta_1\left(2\eta\left|\frac{\tau}{2}\right.\right) \vartheta_1\left(2\eta|\tau\right) \vartheta_1\left(2\beta|\tau\right)^2 \vartheta_1\left(\beta \pm 2\eta|\tau\right)}{\vartheta_1\left(\beta + \frac{\tau}{2}|\tau\right)^2 \vartheta_1\left(\beta \pm 2\eta\left|\frac{\tau}{2}\right.\right) \vartheta_1\left(\beta \pm \eta - \frac{\tau}{4}\left|\frac{\tau}{2}\right.\right)}$$

#

$$\Gamma(qz) = \frac{-iq^{-\frac{1}{24}} \sqrt{z} \vartheta_1\left(\beta\left|\frac{\tau}{2}\right.\right)}{\eta\left(\frac{\tau}{2}\right)} \Gamma(z), \quad \Gamma(z)\Gamma(z^{-1}) = \frac{q^{-\frac{1}{4}} \eta(\tau) \eta\left(\frac{\tau}{2}\right)}{\vartheta_1\left(\beta\left|\frac{\tau}{2}\right.\right) \vartheta_1(\beta|\tau)}.$$

$\mathcal{N} = 3$ theory

- ◉ Integrand is **elliptic**: integration formula (+ a set of theta function identities)
- ◉ Compact closed form

$$\mathcal{I} = \frac{1}{3} \left(\frac{\eta(\tau)^4}{\eta(\frac{1}{2}\tau)^2} - \frac{\eta(\frac{\tau}{2})\eta(3\tau)^2}{\eta(\tau)^2\eta(\frac{3}{2}\tau)} \right)$$

$\mathcal{N} = 3$ theory

- ◉ "Wilson line index" from UV: insert $\chi_{j_1}(z)\chi_{j_2}(y)$ into integrand

$$\begin{aligned} \text{"}\langle W_{(j_1, j_2)} \rangle\text{"} &= \frac{\eta(\frac{\tau}{2})^5}{4q\eta(\tau)} \oint \frac{dz}{2\pi iz} \frac{dy}{2\pi iy} \chi_{j_1}(z) \chi_{j_2}(y) \\ &\quad \times \frac{\vartheta_1(2\eta|\frac{\tau}{2}) \vartheta_1(2\eta|\tau) \vartheta_1(2z|\tau)^2 \vartheta_1(z \pm 2\eta|\tau)}{\vartheta_1(z + \frac{\tau}{2}|\tau)^2 \vartheta_1(z \pm 2\eta|\frac{\tau}{2}) \vartheta_1(z \pm \eta - \frac{\tau}{4}|\frac{\tau}{2})} \end{aligned}$$

$$\text{"}\langle W_{(j_1 \in \mathbb{Z}, j_2 \in \mathbb{Z})} \rangle\text{"} = \mathcal{I} + \sum_{\substack{n=-j \\ n \neq 0}}^j \left(a_n \text{ch}_{\alpha=-\frac{1}{24}} + b_n \text{ch}_{\alpha=\frac{1}{8}} \right),$$

$$\text{"}\langle W_{\text{other}} \rangle\text{"} = 0, \quad \text{ch}_{\alpha=\frac{1}{8}} = \frac{\eta(q^{\frac{1}{2}})\eta(q^3)^2}{\eta(q)^2\eta(q^{\frac{3}{2}})}, \quad \text{ch}_{\alpha=-\frac{1}{24}} = \frac{\eta(q^{\frac{3}{2}})^2}{\eta(q)\eta(q^3)}$$

Examples

$\mathcal{N} = 3$ theory

- ◉ $\mathcal{I}, \text{ch}_{\alpha=\frac{1}{8}}, \text{ch}_{\alpha=-\frac{1}{24}}$ solve

$$\left(D_q^{(4)} + 4E_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} D_q^{(3)} - 10E_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}^2 D_q^{(2)} - 50E_4 D_q^{(2)} + 100E_4 E_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} D_q^{(1)} + 150E_4 E_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}^2 - 60E_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}^3 D_q^{(1)} - 375E_4^2 - 15E_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}^4 \right) \text{ch} = 0$$

- ◉ Logarithmic solution

$$\text{ch}_{\log} := STS\mathcal{I} \tag{1}$$

$$= -\tau \frac{2\eta(q)^4}{3\eta(\sqrt{q})^2} + \frac{\eta(q)^4}{3\eta(\sqrt{q})^2} - \frac{e^{\frac{i\pi}{3}} \eta(q^{\frac{1}{6}})^2}{9\eta(q^{\frac{1}{3}})\eta(q)} - \frac{\eta(q^{\frac{1}{3}})^2 \eta(\sqrt{q})}{9\eta(q^{\frac{1}{6}})\eta(q)^2} \tag{1}$$

$\mathcal{N} = 3$ theory◉ STS

$$STS = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 2 & \frac{1}{3}e^{-\frac{\pi i}{3}} & \frac{2}{3}e^{\frac{2\pi i}{3}} \\ 0 & 0 & \frac{1}{\sqrt{3}}e^{\frac{\pi i}{6}} & \frac{2}{\sqrt{3}}e^{-\frac{\pi i}{6}} \\ 0 & 0 & \frac{1}{\sqrt{3}}e^{-\frac{\pi i}{6}} & \frac{i}{\sqrt{3}} \end{pmatrix}.$$

◉ $\text{ch}_{\alpha=\frac{1}{8}}, \text{ch}_{\alpha=-\frac{1}{24}}$ are **unflavored characters**

$\mathcal{N} = 3$ theory

◉ Surprising

- associated VOA is $\mathcal{N} = 2$ structure
- The elliptic integral comes from $\mathcal{N} = 1$
- There is **no** Wilson line in $\mathcal{N} = 3$ theory

maybe UV Wilson line flows to IR line defect

$\mathcal{N} = 3$ theory

- ◉ To proceed, more detail of $\mathcal{W}_{\mathbb{Z}_3}$
- ◉ $\mathcal{W}_{\mathbb{Z}_3}$ generators: $T, J, G, \bar{G}, W, \bar{W}, G_W, \bar{G}_{\bar{W}}$
[Nishinaka, Tachikawa, 16] [Lemos, Liendo, Meneghelli, Mitev] [Bonetti, Meneghelli, Rastelli, 18]
- ◉ J : $U(1)$ flavor symmetry with no UV $\mathcal{N} = 1$ origin
- ◉ Free $bc\beta\gamma$ realization [Bonetti, Meneghelli, Rastelli, 18]

Examples

	h	m	r	R	$(-1)^F$
J	1	0	0	1	1
T	2	0	0	1	1
W	$\frac{3}{2}$	3	0	$\frac{3}{2}$	1
$\bar{W}$	$\frac{3}{2}$	-3	0	$\frac{3}{2}$	1
G	$\frac{3}{2}$	-1	$\frac{1}{2}$	1	-1
$\bar{G}$	$\frac{3}{2}$	1	$-\frac{1}{2}$	1	-1
G_W	2	2	$\frac{1}{2}$	$\frac{3}{2}$	-1
$\bar{G}_{\bar{W}}$	2	-2	$-\frac{1}{2}$	$\frac{3}{2}$	-1

$\mathcal{N} = 3$ theory

- ◉ singular vector at weight-3

$$\mathcal{N}_3 = (W\tilde{W}) - \frac{1}{27}(J(JJ)) + \frac{4}{9}(TJ) + \frac{1}{3}(J'J) - \frac{2}{9}T' - \frac{13}{27}J'' + \frac{2}{9}(G\tilde{G})$$

Higgs branch relation; generates the maximal null ideal

- ◉ null states $\rightarrow$ flavored MLDEs
 - 3 weight-three nulls $\Rightarrow$ 1 flavored MLDE
 - 10 weight-four nulls $\Rightarrow$ 4 flavored MLDEs
 - 14 weight-five nulls $\Rightarrow$ 9 flavored MLDEs
 - ...

$\mathcal{N} = 3$ theory

Example: weight-3 flavored MLDE

$$\begin{aligned}
0 = & D_b D_q^{(1)} \text{ch} - \frac{1}{12} D_b^2 D_b \text{ch} + \frac{9}{4} E_2(\tau) D_b \text{ch} \\
& + \frac{1}{2} E_1 \begin{bmatrix} -1 \\ b \end{bmatrix} D_q^{(1)} \text{ch} + \frac{3}{2} E_1 \begin{bmatrix} -1 \\ b^3 \end{bmatrix} D_q^{(1)} \text{ch} \\
& + \frac{1}{2} E_2 \begin{bmatrix} -1 \\ b \end{bmatrix} D_b \text{ch} + 3 E_2 \begin{bmatrix} -1 \\ b^3 \end{bmatrix} D_b \text{ch} \\
& - \frac{3}{4} E_1 \begin{bmatrix} -1 \\ b^3 \end{bmatrix} D_b^2 \text{ch} + \frac{15}{4} E_2(\tau) E_1 \begin{bmatrix} -1 \\ b^3 \end{bmatrix} \text{ch} \\
& - \frac{5}{2} E_3 \begin{bmatrix} -1 \\ b \end{bmatrix} \text{ch} - 5 E_3 \begin{bmatrix} -1 \\ b^3 \end{bmatrix} \text{ch}
\end{aligned}$$

$\mathcal{N} = 3$ theory

- four non-logarithmic **series** solutions: $\text{ch}_{\alpha=\frac{5}{8}}(q, b) = \mathcal{I}$, $\text{ch}_{\alpha=\frac{1}{8}}^{\pm}(q, b)$, $\text{ch}_{-\frac{1}{24}}(q, b)$
- closed-form (with some guess work)

$$\chi_{\alpha=-\frac{1}{24}}(b; q) = q^{-\frac{1}{3}} \frac{\eta(3\tau)^3}{\eta(\tau)^2} \frac{\vartheta_4(\mathfrak{b}|q)}{\vartheta_4(3\mathfrak{b} - \tau|3\tau) \vartheta_4(-3\mathfrak{b} - \tau|3\tau)},$$

$$\chi_{\alpha=\frac{1}{8}}^{+}(b; q) = q^{-\frac{1}{6}} b \frac{\eta(3\tau)^3}{\eta(\tau)^2} \frac{\vartheta_4(\mathfrak{b}|q)}{\vartheta_4(3\mathfrak{b} - \tau|3\tau) \vartheta_4(3\mathfrak{b}|3\tau)},$$

$$\chi_{\alpha=\frac{1}{8}}^{-}(b; q) = q^{-\frac{1}{6}} b^{-1} \frac{\eta(3\tau)^3}{\eta(\tau)^2} \frac{\vartheta_4(\mathfrak{b}|q)}{\vartheta_4(-3\mathfrak{b} - \tau|3\tau) \vartheta_4(-3\mathfrak{b}|3\tau)}.$$

checked against flavored MLDEs

$\mathcal{N} = 3$ theory

- ◉ What about the vacuum character $\mathcal{I}$?
- ◉ No refined contour integral from $\mathcal{N} = 1$ Lagrangian

all UV flavor symmetries break in RG flow

- ◉ $\mathcal{I}$ is **not** in the modular orbit of $\text{ch}_{\alpha = -\frac{1}{24}, [\frac{1}{8}]^{\pm}}$

$\mathcal{N} = 3$ theory

- Hint 1: if $\mathcal{I}$ comes from some mysterious elliptic contour integral

$$\mathcal{I} = \sum_{\text{non-vacuum } i} \text{ch}_i(q, b) P_i(q, b)$$

- Hint 2: from the unflavored modular orbit: only **one** logarithmic partner
- Hint 3: the Eisenstein series should captures the **charges** of the strong generators

$\mathcal{N} = 3$ theory

- Ansatz

$$\text{ch}_0(b, q) = \sum_{\alpha = -\frac{1}{24}, \frac{1}{8}, \frac{1}{8}} \chi_\alpha(b, q) P_\alpha(b, q) , \quad P_\alpha(b, q) = \sum_{m=1}^3 \lambda_{\alpha m} E_1 \left[\begin{matrix} \pm \\ b^m \end{matrix} \right] ,$$

- flavored MLDE fixes the coefficients: closed form

$$\begin{aligned} \mathcal{I} &= \frac{q^{-1/6} b \eta(3\tau)^3 \vartheta_4(\mathfrak{b})}{6\eta(\tau)^2 \vartheta_4(3\mathfrak{b}|3\tau) \vartheta_4(-\tau + 3\mathfrak{b}|3\tau)} \left(-1 + 2E_1 \left[\begin{matrix} -1 \\ b \end{matrix} \right] + 2E_1 \left[\begin{matrix} 1 \\ b^2 \end{matrix} \right] \right) \\ &\quad - \frac{q^{-1/6} b^{-1} \eta(3\tau)^3 \vartheta_4(\mathfrak{b})}{6\eta(\tau)^2 \vartheta_4(3\mathfrak{b}|3\tau) \vartheta_4(\tau + 3\mathfrak{b}|3\tau)} \left(1 + 2E_1 \left[\begin{matrix} -1 \\ b \end{matrix} \right] + 2E_1 \left[\begin{matrix} 1 \\ b^2 \end{matrix} \right] \right) \end{aligned}$$

$\mathcal{N} = 3$ theory

- Zhu's algebra: $h, j, g_w, \bar{g}_{\bar{w}}$, relations from null states

$$(g_w - j * g_w)v = (\bar{g}_{\bar{w}} + j * \bar{g}_{\bar{w}})v = (j * (12h - j^2 + 7))v = 0$$

$$(g_w * \bar{g}_{\bar{w}} - \frac{5}{9}j + \frac{11}{18}j * j + \frac{1}{18}j * j * j - \frac{1}{9}j * j * j * j - 2j * h + h * j + h * j * j)v = 0$$

$$(\bar{g}_{\bar{w}} * g_w - \frac{4}{3}j + \frac{1}{6}j * j * j + \frac{2}{9}j * h + \frac{4}{9}h - \frac{23}{9}h * j - \frac{1}{9}h * j * j + \frac{2}{3}h * h)v = 0$$

- solutions

(h, j)	$(0, 0)$	$(-\frac{2}{3}, 0)$	$(-\frac{1}{2}, 1)$	$(-\frac{1}{2}, -1)$
α	$\frac{5}{8}$	$-\frac{1}{24}$	$\frac{1}{8}$	$\frac{1}{8}$

$\mathcal{N} = 3$ theory

- free boson realization of non-vacuum modules

$$\beta = e^{\chi+\phi}, \quad \gamma = (\partial\chi)e^{-\chi-\phi}, \quad b = e^{-\sigma}, \quad c = e^{\sigma}$$

where $\chi\chi \sim \log(z-w)$, $\phi\phi \sim -\log(z-w)$, $\sigma\sigma \sim \log(z-w)$

- highest weight states

$$v_{-\frac{1}{24}} = e^{-\frac{1}{3}\chi + \frac{2}{3}\phi + \sigma}, \quad v_{\frac{1}{8}}^+ = e^{\phi + \sigma}, \quad v_{\frac{1}{8}}^- = e^{-\frac{1}{3}(\chi + \phi)}$$

$\mathcal{N} = 3$ theory

- conjectural irreducibility:

$$\text{ch}_{M_{\pm}} = \text{ch}_{\alpha=\frac{1}{8}}^{\pm} - \mathcal{I}$$

and $\text{ch}_{\alpha=\frac{1}{8}}^{\pm}$ are **Verma characters** with highest weight states $v_{\pm}$

$$q^{-\frac{1}{6}} b \underbrace{\frac{\eta(3\tau)^3}{\eta(\tau)^2}}_J \underbrace{\frac{\overbrace{\vartheta_4(\mathfrak{b}|q)}^{G, \bar{G}}}{\vartheta_4(3\mathfrak{b} - \tau|3\tau) \vartheta_4(3\mathfrak{b}|3\tau)}}_{W, \bar{W}},$$

$\mathcal{N} = 3$ theory

- Modular matrix: $(\text{ch}_0, \text{ch}_{\log}, \text{ch}_{\alpha=-\frac{1}{24}}, \text{ch}_{\alpha=\frac{1}{8}}^+, \text{ch}_{\alpha=\frac{1}{8}}^-)$

$$STS = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{3}e^{i\pi/6} & \frac{\sqrt{3}}{3}e^{-i\pi/6} & -\frac{\sqrt{3}}{3}e^{-i\pi/6} \\ 0 & 0 & \frac{\sqrt{3}}{3}e^{-i\pi/6} & \frac{\sqrt{3}}{3}e^{i\pi/6} & \frac{\sqrt{3}}{3}e^{-i\pi/6} \\ 0 & 0 & -\frac{\sqrt{3}}{3}e^{-i\pi/6} & \frac{\sqrt{3}}{3}e^{-i\pi/6} & \frac{\sqrt{3}}{3}e^{i\pi/6} \end{pmatrix}$$

- Jordan block sizes: $[1, 1, 1, 2]$

$\mathcal{N} = 3$ theory

- ◉ lesson: **Intrinsic relation?**

elliptic function integral (regardless of origin) $\leftrightarrow$ (partial) representation theory

also appear in "generalized partition function" [Deb, Razamat] : "Wilson line" $\Rightarrow$
 $(\widehat{E}_6)_{-3}$ character

$$\text{vacuum character} = \sum \text{non-vacuum character} \times \text{Eisenstein}$$

Open questions

- ◉ free field realization of $\mathcal{N} = 4$ $SU(N)$ VOA

[Bonetti, Meneghelli, Rastelli, 18]

$\Rightarrow$ null states $\Rightarrow$ flavored MLDEs $\Rightarrow$ verify $\mathcal{J}_\ell$ are independent characters

- ◉ same for $\mathcal{N} = 3$ VOAs
- ◉ Understand the underlying relation between **elliptic function** integral and **representation theory**
- ◉ Understand the homological nature of the "generalized residue"
- ◉ Apply to other $\mathcal{N} = 3$ SCFT from discrete gauging $\mathcal{N} = 4$ SYM
- ◉ Graded vector space isomorphism between $\mathcal{N} = 4$ and $\mathcal{T}_{p,N}$

Thank you

$\mathcal{T}_{p,N}$ theory

- Verify a map from

$\mathcal{N} = 4$ chiral algebra "characters" (?) $\rightarrow \mathcal{T}_{p,N}$ chiral algebra characters

- First task: understand $\mathbb{V}[\mathcal{T}_{p,N}]$ characters

$\mathcal{T}_{p,N}$ theory

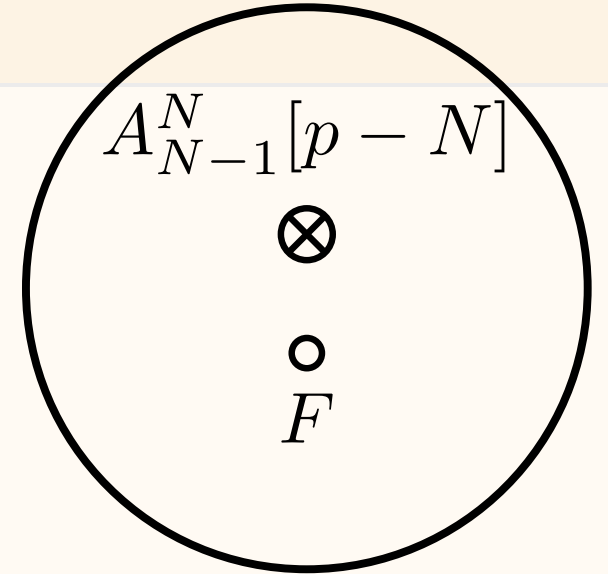
- AD theory: p, N **coprime**

$$D_p(SU(N)) = (A_{N-1}^N[p - N], F)$$

[Cecotti, Del Zotto]

[Cecotti, Del Zotto, Giacomelli]

- One irregular puncture $J^b[\kappa] = A_{N-1}^N[p - N]$
- One full regular puncture F : $SU(N)$ **flavor symmetry**
- p, N **coprime** $\Rightarrow$ irregular puncture carries **no** flavor symmetry



$\mathcal{T}_{p,N}$ theory

- The Schur index of $D_p(SU(N))$

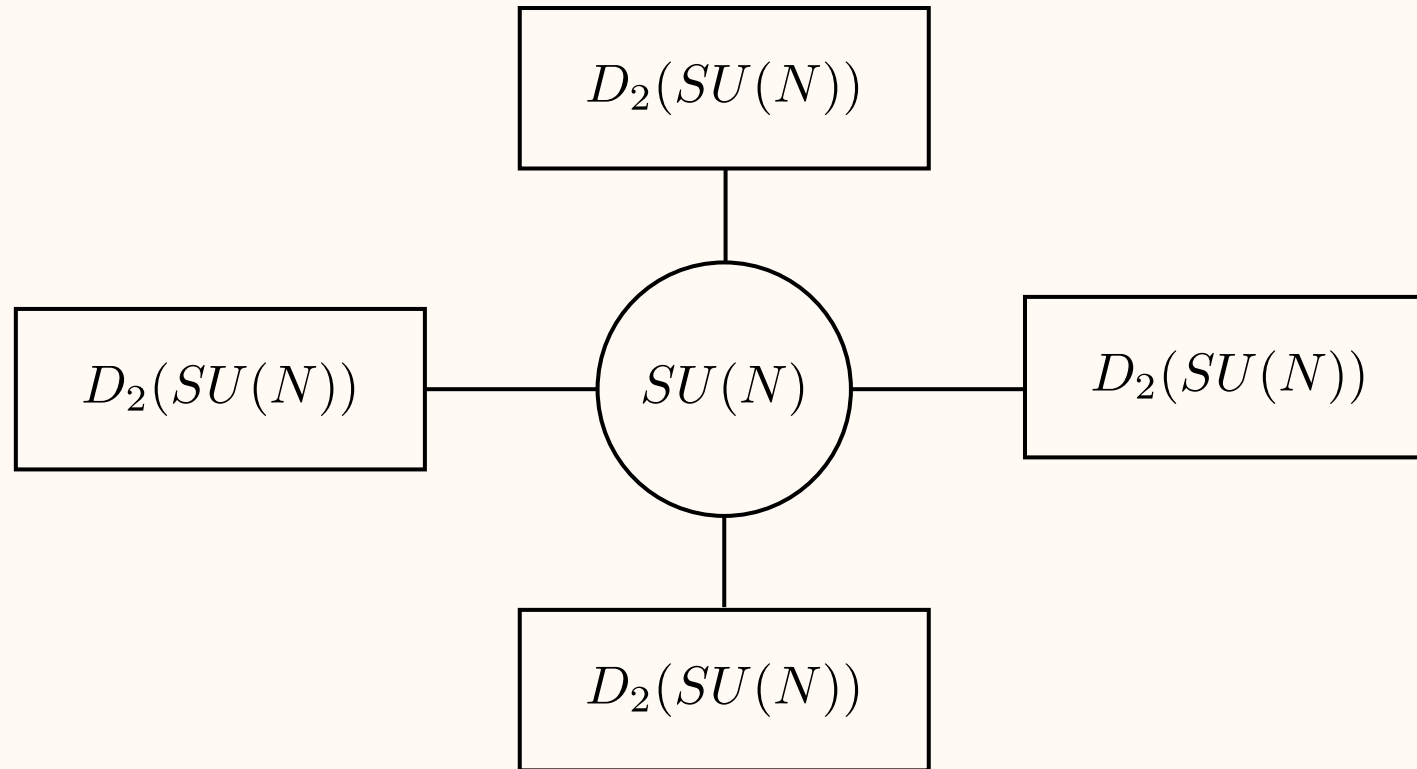
$$\begin{aligned} \mathcal{I}_{D_p(SU(N))}(q, b) &= \text{PE} \left[\frac{q - q^p}{(1 - q)(1 - q^p)} \chi_{\text{adj}}^{SU(N)}(b) \right] \\ &= \left[\frac{\eta(p\tau)}{\eta(\tau)} \right]^{\frac{1}{2}(3r - \dim \mathfrak{g})} \prod_{\substack{A, B=1 \\ A \neq B}}^N \frac{\vartheta_1(\mathfrak{b}_A - \mathfrak{b}_B | p\tau)}{\vartheta_1(\mathfrak{b}_A - \mathfrak{b}_B | \tau)} \end{aligned}$$

$\mathcal{T}_{p,N}$ theory

- New theories from **gauging** $D_p(SU(N))$
- Conformal gauging several $D_{p_i}(SU(N_i))$: **4 special possibilities** (among many)

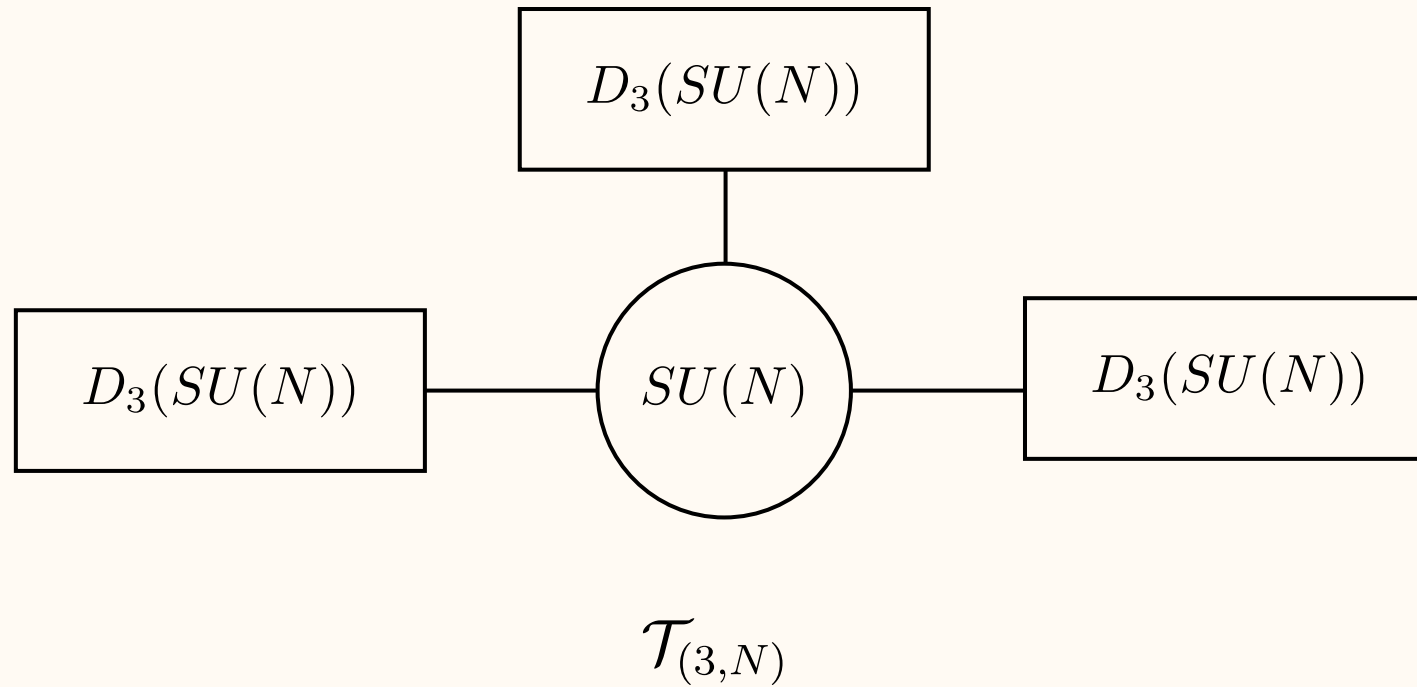
[Buican, Nishinaka] [Del Zotto, Vafa, Xie] [Kang, Lawrie, Song] [Closset,
Giacomelli, Schafer-Nameki, Wang] [Jiang] ...

$$\mathcal{T}_{2,N}, \quad \mathcal{T}_{3,N}, \quad \mathcal{T}_{4,N}, \quad \mathcal{T}_{6,N} .$$

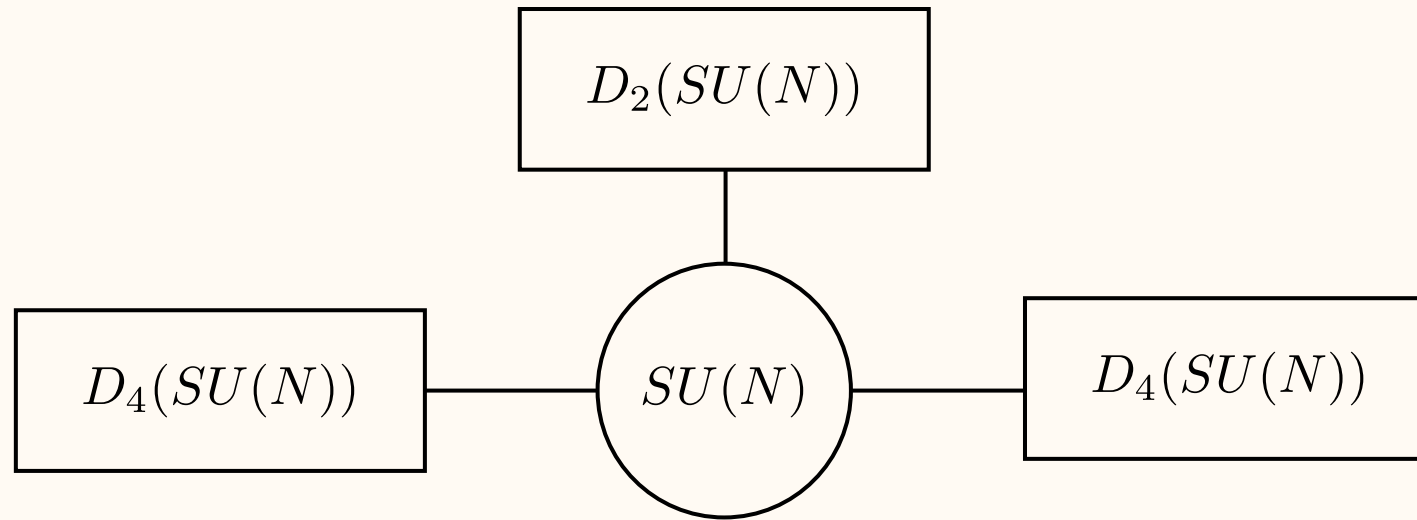
$\mathcal{T}_{p,N}$ theory

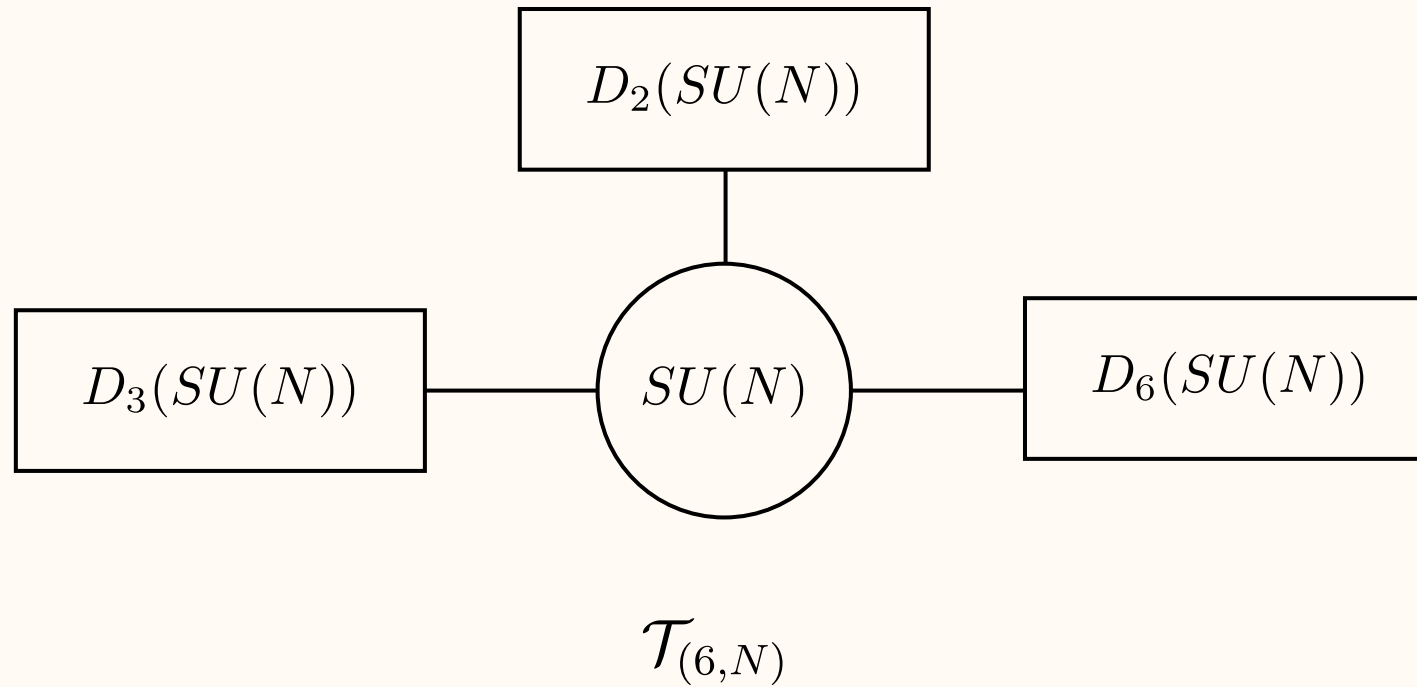
$$\mathcal{T}_{(2,N)}$$

$$N = \text{odd}$$

$\mathcal{T}_{p,N}$ theory

$$N = 1, 2 \pmod{3}$$

$\mathcal{T}_{p,N}$ theory $\mathcal{T}_{(4,N)}$ $N = \text{odd}$

$\mathcal{T}_{p,N}$ theory

$$N = 1, 5 \pmod{6}$$

$\mathcal{T}_{p,N}$ theory

- Known relation between **Schur index**

$$\mathcal{I}_{\mathcal{T}_{p,N}}(q) = \underbrace{q^{-\frac{c_{2d}[\mathcal{T}_{p,N}] - pc_{2d}[\mathcal{T}_{SU(N)}]}{24}}}_{\text{relative coefficient}} \mathcal{I}_{SU(N)}(b \rightarrow q^{\frac{p}{2}-1}, q \rightarrow q^p) .$$

[Kang, Lawrie, Song, 21]

- Vector space isomorphism** between $\mathbb{V}[\mathcal{T}_{SU(2)}]$ and $\mathbb{V}[\mathcal{T}_{3,2}]$

[Buican, Nishinaka]

vacuum module of $\mathbb{V}[\mathcal{T}_{SU(2)}] \rightarrow$ **vacuum** module of $\mathbb{V}[\mathcal{T}_{3,2}]$

$\mathcal{T}_{p,N}$ theory

- ◉ **First task:** $\mathbb{V}[\mathcal{T}_{p,N}]$ **spectrum** and modularity
- ◉ MLDE/Modularity study of $\mathcal{T}_{p,N}$ initiated in [Jiang, 24]
- ◉ To go further: **3 steps**
 - **closed-form index:** use $\mathcal{N} = 4$ $SU(N)$ Schur index closed-form
 - **MLDE:** find the **minimal** order (in differentiation) MLDE
 - **Modularity:** construct modular orbit

$\mathcal{T}_{p,N}$ theory

- Closed form
- Example: just polynomials of Eisenstein series

$$\mathcal{I}_{\mathcal{T}_{2,5}} = \frac{1}{640} \left[3 + 20 \left(E_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + E_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) + 20 \left(E_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + E_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^2 - 80 \left(E_4 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + E_4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \right].$$

$$\mathcal{I}_{\mathcal{T}_{3,4}} = -\frac{1}{72} - \frac{1}{6} E_2(3\tau) + \frac{1}{24} E_1 \begin{bmatrix} -1 \\ \sqrt{q} \end{bmatrix} (3\tau)$$

$\mathcal{T}_{p,N}$ theory

Theory	$\dim \mathcal{V}$	T^2 Jordan block number	Jordan block size
$\mathcal{T}_{2,2\ell+1}$	$1 + 3\ell(\ell + 1)$	$3\ell + 1$	$[2\ell + 1, 2\ell + 1, \dots, 5, 5, 5, 3, 3, 3, 1]$
$\mathcal{T}_{3,2}$	5	3	$[2, 2, 1]$
$\mathcal{T}_{3,4}$	33	11	$[2, 1, 3, 3, 4, 4, 3, 4, 2, 3, 4]$
$\mathcal{T}_{4,3}$	13	6	$[3, 3, 2, 2, 2, 1]$

$\mathcal{T}_{p,N}$ theory

- ◉ **Second task:** compare with $\mathbb{V}[\mathcal{T}_{SU(N)}]$ modules
- ◉ **Rephrase:**

$$\mathcal{I}_{\mathcal{T}_{p,N}}(q) = q^{-\frac{c_{2d}[\mathcal{T}_{p,N}] - pc_{2d}[\mathcal{T}_{SU(N)}]}{24}} \mathcal{I}_{SU(N)}(q^{\frac{p}{2}-1}, q^p)$$

vacuum character of $\mathbb{V}[\mathcal{T}_{SU(N)}] \rightarrow$ vacuum character of $\mathbb{V}[\mathcal{T}_{p,N}]$

- ◉ **Other "characters" of $\mathbb{V}[\mathcal{T}_{SU(N)}]$?**

$\mathcal{T}_{p,N}$ theory

- ◉ Candidates: $\mathcal{I}_\ell$
- ◉ $\mathcal{I}_\ell$ map to characters of $\mathbb{V}[\mathcal{T}_{p,N}]$?
- ◉ Answer seems to be "yes"

$\mathbb{V}[\mathcal{T}_{SU(2)}]$ and $\mathbb{V}[\mathcal{T}_{3,2}]$ modules

- Simplest $SU(2)$
- Wilson line index

$$\langle W_j \rangle = \delta_{j \in \mathbb{Z}} \frac{i\vartheta_4(\mathfrak{b})}{\vartheta_1(2\mathfrak{b})} E_1 \begin{bmatrix} -1 \\ b \end{bmatrix} - \frac{1}{2} \frac{i\vartheta_4(\mathfrak{b})}{\vartheta_1(2\mathfrak{b})} \sum_{\substack{m=-j \\ m \neq 0}}^j \frac{b^m - b^{-m}}{q^{m/2} - q^{-m/2}}.$$

- Basis for $\mathbb{V}[\mathcal{T}_{SU(N)}]$

$$\mathcal{J}_{[2]} = \frac{i\vartheta_4(\mathfrak{b})}{\vartheta_1(2\mathfrak{b})}, \quad \mathcal{J}_{[1,1]} = \frac{i\vartheta_4(\mathfrak{b})}{\vartheta_1(2\mathfrak{b})} E_1 \begin{bmatrix} -1 \\ b \end{bmatrix} = \mathcal{I}_{SU(2)}(q, b)$$

$\mathbb{V}[\mathcal{T}_{SU(2)}]$ and $\mathbb{V}[\mathcal{T}_{3,2}]$ modules

- correspond to two irreducible modules + one "logarithmic" module

[Adamovich]

- Trivial limit at $p = 3$: $b \rightarrow q^{\frac{1}{2}}, q \rightarrow q^3$

$$q^{-\frac{c_{2d}[\mathcal{T}_{3,2}] - 3c_{2d}[\mathcal{T}_{SU(2)}]}{24}} \left[\mathcal{J}_{[2]} \right]_{b \rightarrow q^{\frac{1}{2}}, q \rightarrow q^3} = 1$$

A solution to the 5-th order MLDE; also true for $\text{ch}_{\log}$

$\mathbb{V}[\mathcal{T}_{SU(3)}]$ and $\mathbb{V}[\mathcal{T}_{4,3}]$ modules

- $\mathcal{T}_{SU(3)} \rightarrow \mathcal{T}_{4,3}$
- Wilson line basis vector

$$\mathcal{J}_{[2,1]} = \frac{\vartheta_4(\mathfrak{b})}{\vartheta_4(3\mathfrak{b})} \left(E_1 \begin{bmatrix} -1 \\ b \end{bmatrix} + E_1 \begin{bmatrix} 1 \\ b^2 \end{bmatrix} \right)$$

- Simple verification for $p = 4, N = 3$:

$$q^{-\frac{c_{2d}[\mathcal{T}_{3,4}] - pc_{2d}[\mathcal{T}_{SU(N)}]}{24}} \left[\mathcal{J}_1([2, 1]) \right]_{b \rightarrow q^{\frac{p}{2}-1}, q \rightarrow q^p}$$

solves the 13-order MLDE

equivalently: linear combination of $\text{ch}_{0,1,\dots,12}$

$\mathbb{V}[\mathcal{T}_{SU(5)}]$ and $\mathbb{V}[\mathcal{T}_{2,5}]$ modules

- 4d $\mathcal{N} = 4$ $SU(5) \rightarrow \mathcal{T}_{p=2, N=5}$
- Wilson line basis vector

$$\mathcal{J}_{[4,1]} = \frac{\vartheta_4(\mathfrak{b})}{\vartheta_4(5\mathfrak{b})} \left(E_1 \begin{bmatrix} -1 \\ b \end{bmatrix} + E_1 \begin{bmatrix} 1 \\ b^4 \end{bmatrix} \right)$$

- Example: all solve both the 37-order and 25-th order MLDE

$$(\mathcal{J}_{[4,1]})_{b \rightarrow q^{\frac{p}{2}-1}, q \rightarrow q^p} = \mathcal{J}_{[3,2]}|_{b \rightarrow q^{\frac{p}{2}-1}, q \rightarrow q^p} = 0$$

$$(\mathcal{J}_{[3,1,1]})_{b \rightarrow q^{\frac{p}{2}-1}, q \rightarrow q^p} = -3E_2(\tau)$$

$$(\mathcal{J}_{[1,1,1,1,1]})_{b \rightarrow q^{\frac{p}{2}-1}, q \rightarrow q^p} = 15E_2(\tau)^2 - 30E_4(\tau)$$

$\mathbb{V}[\mathcal{T}_{SU(7)}]$ and $\mathbb{V}[\mathcal{T}_{2,7}]$ modules

- $4d \mathcal{N} = 4 SU(7) \rightarrow \mathcal{T}_{p=2, N=7}$
- Spectral flow (subset of $\{\mathcal{J}_w\}$)

$$\sigma(\text{ch}_0(q, b)) := \frac{\text{ch}_0(bq, q)}{b^{48}q^{24}} - \text{ch}_0(b, q)$$

- Example:

$$\begin{aligned} \sigma^4(\text{ch}_0(q, b))|_{b \rightarrow 1} &\rightarrow -\frac{4067294 \text{ch}_{36}}{3} + \frac{1925 \text{ch}_{41}}{74} - \frac{9675 \text{ch}_{42}}{74} + \frac{75186889 \text{ch}_{45}}{888} \\ \sigma^5(\text{ch}_0(b, q))|_{b \rightarrow 1} &\rightarrow -\frac{4705960}{3} \text{ch}_{36} + \frac{588245}{6} \text{ch}_5, \quad \dots \end{aligned}$$

$\mathcal{T}_{p,N}$ theory

- Summary: a simple correspondence between $\mathbb{V}[\mathcal{T}_{SU(N)}]$ and $\mathcal{T}_{p,N}$ modules

$$\mathcal{J}_\ell \text{ of } \mathbb{V}[\mathcal{T}_{SU(N)}] \rightarrow \text{characters of } \mathbb{V}[\mathcal{T}_{p,N}]$$

4d Mirror symmetry

- ◉ "4d mirror symmetry": relation between HB and CB of the **same** 4d theory $\mathcal{T}$
- ◉ $\mathcal{T} = \text{AD} (J^b[k], Y)$, associated VOA is **boundary admissible** $\mathcal{W}(\mathfrak{g}, f)$

irreducible/simple module of $\mathbb{V}[\mathcal{T}] \leftrightarrow$ Hitchin moduli space fixed **point**

[Gaiotto, Moore, Neitzke]

[Fredrickson, Neitzke]

[Fredrickson, Pei, Yan,

Ke, 17]

[Dedushenko, Gukov, Nakajima, Pei, Ye, 18]

[Shan, Xie, Yan, 23, 24]

4d mirror symmetry

- Generalization to non-admissible affine/ $\mathcal{W}$ -algebra

[Shan, Xie, Yan, 23] [Xie]

- Generalization to A_1 class- $\mathcal{S}$ theories

[YP, Yan] ; fixed points data from [Boden, Yokogawa]

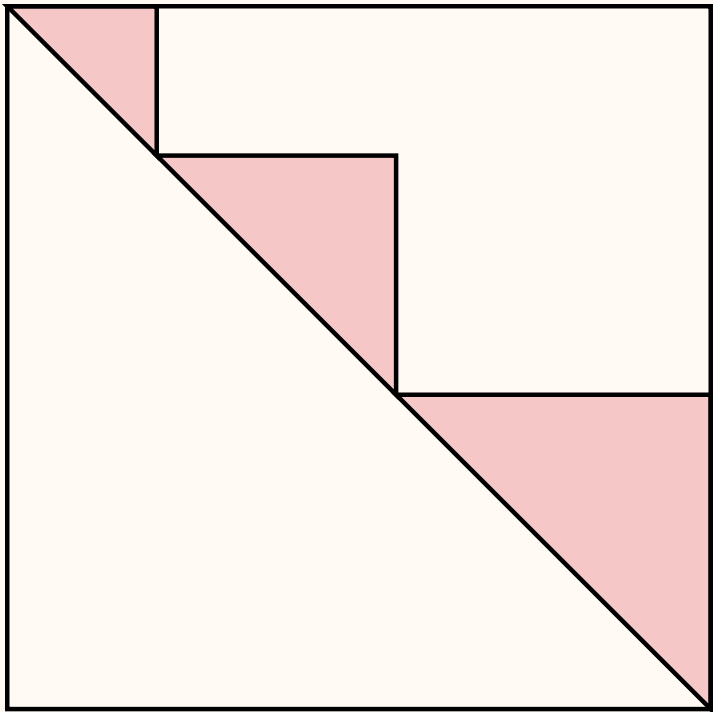
- logarithmic** characters: T^2 is **non-diagonalizable**

Jordan blocks of $T^2 \leftrightarrow$ simple modules $\leftrightarrow$ fixed varieties

Jordan block **sizes** $\leftrightarrow$ **dimension/cohomology** of fixed varieties

Jordan block info $\leftrightarrow$ pure part of mixed Hodge polynomial

4d Mirror symmetry



simple₁

log₁₁

simple₂

log₂₁

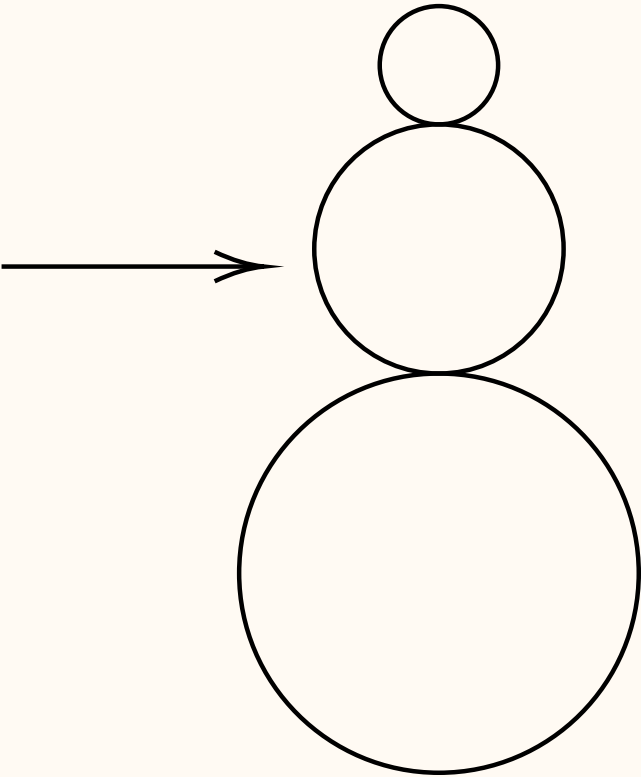
log₂₁

simple₃

log₃₁

log₃₂

log₃₃



4d Mirror symmetry

- For some of the $\mathcal{T}_{p,N}$: are $(J^b[k], \mathfrak{p})$ type theories
- number** of fixed points can be computed relatively easily

[Kang, Laurie, Lee, Sacchi, Song] *[Jiang]*

$$\mathcal{T}_{3,2} = (D_4^6[3], \mathfrak{p}), \quad \mathcal{T}_{4,3} = (E_6^{12}[4], \mathfrak{p}), \quad \mathcal{T}_{6,5} = (E_8^{30}[6], \mathfrak{p})$$

$\mathcal{T}_{p,N}$ theory

Theory	$\dim \mathcal{V}$	T^2 Jordan block number	Jordan block size
$\mathcal{T}_{3,2}$	5	3	$[1, 2, 2]$
$\mathcal{T}_{4,3}$	13	6	$[1, 2, 2, 2, 3, 3]$

- affine Springer-fibre technique $(J^b[k], f)$: counting special elements in the Weyl group
- $\mathcal{T}_{3,2}$: 3 fixed varieties; dimension $[0, 1, 1]$
- $\mathcal{T}_{4,3}$: 6 fixed varieties; dimension $[0, 1, 1, 1, 2, 2]$

[YP, Yang, 24]

[Xie, 26]

Open questions

- ◉ check/clarify 4d mirror symmetry dictionary for non-admissible $\mathcal{W}$ -algebras, 4d $\mathcal{N} = 4$ VOA
- ◉

4d Mirror symmetry

- ◉ **Conjecture:** CB fixed point data from mixed Hodge polynomial

CB fixed points $\leftrightarrow$ integer partitions of N

CB fixed points dimensions $\leftrightarrow$ width of integer partitions of N

Dimensions

$N = 2: \{2, 1\}$

$N = 3: \{3, 2, 1\}$

$N = 4: \{4, 3, 2, 2, 1\}$

$N = 5: \{5, 4, 3, 3, 2, 2, 2, 1\}$

$N = 6: \{6, 5, 4, 4, 3, 3, 3, 2, 2, 2, 1\}$

4d Mirror symmetry

[WIP with Yutong Li, Wenbin Yan]

- Wilson line index basis $\mathcal{J}_\ell$: determined precisely by integer partitions ℓ of N

$$\mathcal{J}_\ell \leftrightarrow \text{simple modules}$$

- The **dimension** of modular orbit of $\dim \mathcal{O}_{\mathcal{J}_\ell} = w(\ell)$
- The Jordan block sizes of $T^2 = w(\ell)$

$$\text{dim fixed var.} \quad \leftrightarrow \quad \text{The set of widths of integer partitions of } N$$

$$\updownarrow$$

$$\dim \mathcal{O}_{\mathcal{J}_\ell}$$

Modularity

- ◉ **Summary** of results
- ◉ The $SL(2, \mathbb{Z})$ **orbit** of the vacuum character $\Rightarrow$ complete space $\mathcal{V}_{p,N}$ of characters
- ◉ There is an **order-dim** $\mathcal{V}_{p,N}$ unflavored MLDE $\mathcal{E}_{\min}$ that annihilates all characters
- ◉ $\mathcal{E}_{\min}$ is **non-monic** in general
- ◉ There is a **monic** but **higher order** unflavored MLDE, with $\dim \mathcal{V}_{p,N}$ **rational** indicial roots
- ◉ Modular matrices T, S and Jordan decompositions

Modularity

- ◉ The $\mathcal{T}_{2,2\ell+1}$ theory: $p = 2, N = 2\ell + 1$
- ◉ The flavor fugacity $b \rightarrow q^{\frac{p}{2}-1} = 1$: **just the unflavoring limit**
- ◉ The Schur index examples

$$\mathcal{I}_{\mathcal{T}_{2,3}}(q) = \frac{1}{24} + \frac{1}{4} \left(E_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + E_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) .$$

$$\begin{aligned} \mathcal{I}_{\mathcal{T}_{2,5}} = & \frac{1}{640} \left[3 + 20 \left(E_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + E_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \right. \\ & \left. + 20 \left(E_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + E_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^2 - 80 \left(E_4 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + E_4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \right] . \end{aligned}$$

Modularity

- $\mathcal{I}_{\mathcal{T}_{2,3}}$ under $SL(2, \mathbb{Z})$

$$1 \xrightarrow{SL(2, \mathbb{Z})} 1$$

$$\begin{aligned} \left(E_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + E_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) &\xrightarrow{SL(2, \mathbb{Z})} \tau^{\ell=0,1,2} \left(E_2 \begin{bmatrix} +1 \\ +1 \end{bmatrix} + E_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right) , \\ &\tau^{\ell=0,1,2} \left(E_2 \begin{bmatrix} +1 \\ +1 \end{bmatrix} + E_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) , \\ &\tau^{\ell=0,1,2} \left(E_2 \begin{bmatrix} +1 \\ +1 \end{bmatrix} + E_2 \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right) . \end{aligned}$$

- In total **10** linear independent objects: $\dim \mathcal{V}_{\mathcal{T}_{2,3}} = 10$

Modularity

- ◉ **12-th** order MLDE in: **10** rational and **2** irrational indicial roots

[Jiang]

$$\begin{aligned}
 0 = & \left[D_q^{(12)} - 1510E_4D_q^{(10)} - 55440E_6D_q^{(9)} - 233400E_4^2D_q^{(8)} + 2364600E_4E_6D_q^{(7)} \right. \\
 & + 2000(31228E_4^3 - 41013E_6^2)D_q^{(6)} + 1422624000E_4^2E_6D_q^{(5)} \\
 & + (3925360000E_4^4 + 40438916000E_4E_6^2)D_q^{(4)} + \dots \\
 & \left. + (23905224000000E_4^4E_6 + 31682361200000E_4E_6^3)D_q^{(1)} \right] \text{ch}_0(q)
 \end{aligned}$$

- ◉ **10 < 12? Did we miss two characters?**

Modularity

- ◉ **We didn't**
- ◉ A 10-th order equation, but at **modular weight 36**

$$\left[\left(\frac{55}{196} E_4^4 + E_4 E_6^2 \right) D_q^{(10)} - \frac{405}{11} E_4^3 E_6 D_q^{(9)} + \dots \right] \text{ch}_0(q) = 0$$

non-monic: polynomial coefficient $\frac{55}{196} E_4^4 + E_4 E_6^2$

- ◉ span $\mathcal{V}$ of the modular orbit gives the full spectrum: 10 rational indicial roots

Modularity

◉ S, T matrices

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} \\ -8 & 0 & 8 & 0 & -5 & 0 & 4 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -8 & 0 & 8 & 0 & -4 & 0 & 3 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -8 & 8 & 0 & -4 & 0 & 2 & 0 & 0 & 0 & \frac{1}{2} \\ -16 & 16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Modularity

- ◉ T : 4 Jordan blocks
- ◉ Jordan block size

$[3, 3, 3, 1]$

Modularity

- $\mathcal{T}_{2,5}$ theory

$$\mathcal{I}_{\mathcal{T}_{2,5}} = \frac{1}{640} \left[3 + 20 \left(E_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + E_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) + 20 \left(E_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + E_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^2 - 80 \left(E_4 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + E_4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \right] .$$

Modularity

- Modular transformation: linear independent objects

1

$$\tau^{\ell=0,1,2} \left(E_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + E_2 \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right) \Big|_{\substack{(\alpha,\beta)=(1,-1), \\ (-1,1),(-1,-1)}}$$

$$\tau^{\ell=0,1,2,3,4} \left[\left(E_2 \begin{bmatrix} 1 \\ +1 \end{bmatrix} + E_2 \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right)^2 - 4 \left(E_4 \begin{bmatrix} 1 \\ +1 \end{bmatrix} + E_4 \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right) \right] \Big|_{\substack{(\alpha,\beta)=(1,-1), \\ (-1,1),(-1,-1)}}$$

- Dimension of $SL(2, \mathbb{Z})$

$$\dim \mathcal{V}_{\mathcal{T}_{2,5}} = 1 + 3 \times 3 + 3 \times 5 = 25$$

Modularity

- ◉ MLDE?
- ◉ first MLDE: order 37: **25** rational and **12** irrational indicial roots
- ◉ $37(> 25)$; **miss?**
- ◉ **No we didn't**: at **modular weight 254**, found a 25-th order **non-monic** MLDE
- ◉ $\mathcal{V}$ is the full spectrum: 25 **rational** indicial roots
- ◉ S, T matrix
- ◉ T -matrix: 7 Jordan blocks, block sizes $[1, 3, 3, 3, 5, 5, 5]$

Modularity

- For all $\mathcal{T}_{2,2\ell+1}$
- Dimension of $SL(2, \mathbb{Z})$ -orbit

$$1 + 3 \times 3 + 3 \times 5 + 3 \times 7 + \cdots + 3 \times (2\ell + 1) = 1 + 3\lfloor \ell \rfloor (2 + \lfloor \ell \rfloor)$$

- Conjecture:** minimal order of MLDE: $1 + 3\lfloor \ell \rfloor (2 + \lfloor \ell \rfloor)$
- T -matrix Jordan block numbers: $1 + 3\ell$, block sizes

$$[1, 3, 3, 3, 5, 5, 5, \cdots, 2\ell + 1, 2\ell + 1, 2\ell + 1]$$

Modularity

- $\mathcal{T}_{3,2}$ theory

$$\text{ch}_0 = q^{-1/3} \frac{i\vartheta_4\left(\frac{\tau}{2} \middle| 3\tau\right)}{\vartheta_1(\tau \middle| 3\tau)} E_1 \begin{bmatrix} -1 \\ \sqrt{q} \end{bmatrix} (3\tau) = \frac{1}{3} \sum_{\ell=0}^2 E_1 \begin{bmatrix} -1 \\ e^{\frac{2\pi i \ell}{3}} \sqrt{q} \end{bmatrix} (\tau)$$

- Modular orbit: complete basis of $\mathcal{V}_{\mathcal{T}_{3,2}}$

$$\text{ch}_0, \quad \text{ch}_1 = S \text{ch}_0$$

$$\text{ch}_2 = T \text{ch}_1, \quad \text{ch}_3 = T \text{ch}_2, \quad \text{ch}_4 = T \text{ch}_3 - \text{ch}_1$$

Different basis in [Jiang]

Modularity

- 5-th order unflavored MLDE

[Jiang]

$$[D_q^{(5)} - 140E_4D_q^{(3)} - 700E_6D_q^{(2)} - 2000E_4^2D_q^{(1)}] \text{ch}_0 = 0$$

- $\mathcal{V}_{\mathcal{T}_{3,2}}$ is the full spectrum
- Constant** $\text{ch} = 1$ is a solution

Modularity

- ◉ S, T matrices

see also *[Jiang]*

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 3\sqrt[3]{-1} & -3 & 3 - 3\sqrt[3]{-1} & 2\sqrt[3]{-1} \end{pmatrix}$$

- ◉ 3 Jordan blocks, sizes $[2, 2, 1]$

Modularity

- ◉ $\mathcal{T}_{4,3}$
- ◉ basis of length 13

ch_0

$$\text{ch}_1 = S \text{ch}_0, \quad \text{ch}_2 = T \text{ch}_1, \quad \text{ch}_3 = T \text{ch}_2 - \text{ch}_1, \quad \text{ch}_4 = T \text{ch}_3, \quad \text{ch}_5 = T \text{ch}_4,$$

$$\text{ch}_6 = i \text{ch}_3 - \text{ch}_4 - i \text{ch}_5 + T \text{ch}_5, \quad \text{ch}_7 = T \text{ch}_6, \quad \text{ch}_8 = T \text{ch}_7,$$

$$\text{ch}_9 = S \text{ch}_8, \quad \text{ch}_{10} = T \text{ch}_9,$$

$$\text{ch}_{11} = T \text{ch}_{10}, \quad \text{ch}_{12} = T \text{ch}_{11},$$

Modularity

- ◉ 17-th order unflavored MLDE in [Jiang] : 13 rational indicial roots and 4 irrational roots
- ◉ $17 > 13$
- ◉ **Guess: 13-th order unflavored MLDE?**
- ◉ **Yes:** 13-th order unflavored MLDE at modular weight 72
- ◉ $\mathcal{V}$ is the full spectrum: 13 rational indicial roots
- ◉ Compute S, T matrix
- ◉ T has 6 Jordan blocks, of sizes

3, 3, 2, 2, 2, 1

Modularity

- ◉ $\mathbb{V}$ and ordinary modules

$$w_{\min}, n_{\min}, \dim \mathcal{V}_{\text{ord}}, \dim \mathcal{O}_{\text{ch}_0}, h(\mathcal{N}_T)$$

- ◉ **What are their relations?**

Summary

- ◉ Wilson line index $\rightarrow \mathbb{V}[\mathcal{T}_{SU(N)}]$ characters
- ◉ $\mathcal{T}_{p,N}$ modules and modularity
- ◉ Establishes map from $\mathbb{V}[\mathcal{T}_{SU(N)}]$ characters to $\mathbb{V}[\mathcal{T}_{p,N}]$ characters
- ◉ more evidence from 4d mirror symmetry

Outlook

- ◉ More direct proof of $\mathcal{I}_\ell$ being characters
- ◉ 't Hooft line
- ◉ More examples of 4d mirror symmetry, relation to 3d
- ◉ Gukov-Witten defects index in 4d $\mathcal{N} = 4$
- ◉ Relation between the $w_{\min}, \dots$

Thank you!