

What Are Bosonic String Effective Amplitudes? Uniqueness and Analytic Structures

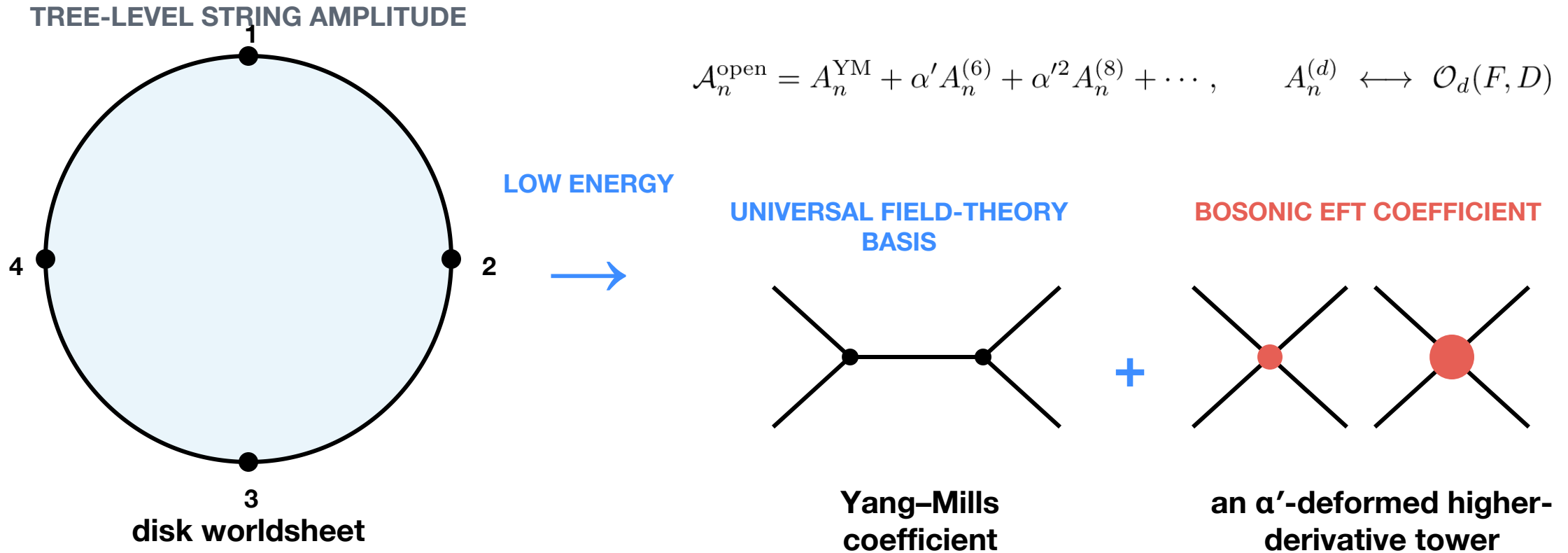
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Joint work with Fan Zhu (朱凡) arXiv:2607.01790

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Bosonic strings retain a nontrivial EFT sector

Refs: Huang, Schlotterer & Wen, JHEP 2016; Azevedo et al., JHEP 2018



Zero transcendentality leaves Yang-Mills for superstrings, but a nontrivial bosonic EFT.

One universal basis for super- and bosonic strings

Refs: Huang, Schlotterer & Wen, JHEP 2016.

$$\mathcal{A}_n^{\text{super}}(1, \rho, n-1, n; \alpha') = \sum_{\sigma \in S_{n-3}} F_\rho^\sigma(\alpha') A_{\text{YM}}(1, \sigma, n-1, n),$$

$$\mathcal{A}_n^{\text{bos}}(1, \rho, n-1, n; \alpha') = \sum_{\sigma \in S_{n-3}} F_\rho^\sigma(\alpha') B(1, \sigma, n-1, n; \alpha').$$

01 COMMON BASIS

same F basis

02 SUPERSTRING

coefficient A_{YM}

03 BOSONIC STRING

$B(\alpha')$ adds rational terms

At each order in α' , the transcendental term comes entirely from F.

At four points, the bosonic information sits in B_4

$$\mathcal{A}_4^{\text{bos}}(1, 2, 3, 4; \alpha') = F_4(\alpha') B(1, 2, 3, 4; \alpha'),$$

$$B(1, 2, 3, 4; \alpha') = A_{\text{YM}}(1, 2, 3, 4) + (2\alpha')^2 s_{13} \left[\left(\frac{f_{12} f_{34}}{s_{12}^2 (1 - s_{12})} + \text{cyc}(2, 3, 4) \right) - \frac{g_1 g_2 g_3 g_4}{s_{12}^2 s_{13}^2 s_{14}^2} \right].$$

$$F_4 = s_{12} B(s_{12}, 1 + s_{23}) \\ = \frac{\Gamma(1 + s_{12}) \Gamma(1 + s_{23})}{\Gamma(1 + s_{12} + s_{23})} = 1 + \mathcal{O}(\alpha'^2).$$

universal Euler–Beta factor

&

$$f_{ij} = (e_i \cdot e_j)(k_i \cdot k_j) - (k_i \cdot e_j)(k_j \cdot e_i), \\ g_i = (k_{i-1} \cdot e_i) s_{i, i+1} - (k_{i+1} \cdot e_i) s_{i-1, i}.$$

gauge-invariant kinematic building blocks

Setting ζ -data aside isolates the EFT coefficient

Refs: Huang, Schlotterer & Wen, JHEP 2016; Azevedo et al., JHEP 2018

$$\mathcal{A}_n = \sum_{\rho} F_{\rho}^{\sigma}(\alpha') A_n(1, \rho, n-1, n)$$

WORLD-SHEET FACTOR

$\zeta_2, \zeta_3, \dots$

The transcendental tower is already well organized.

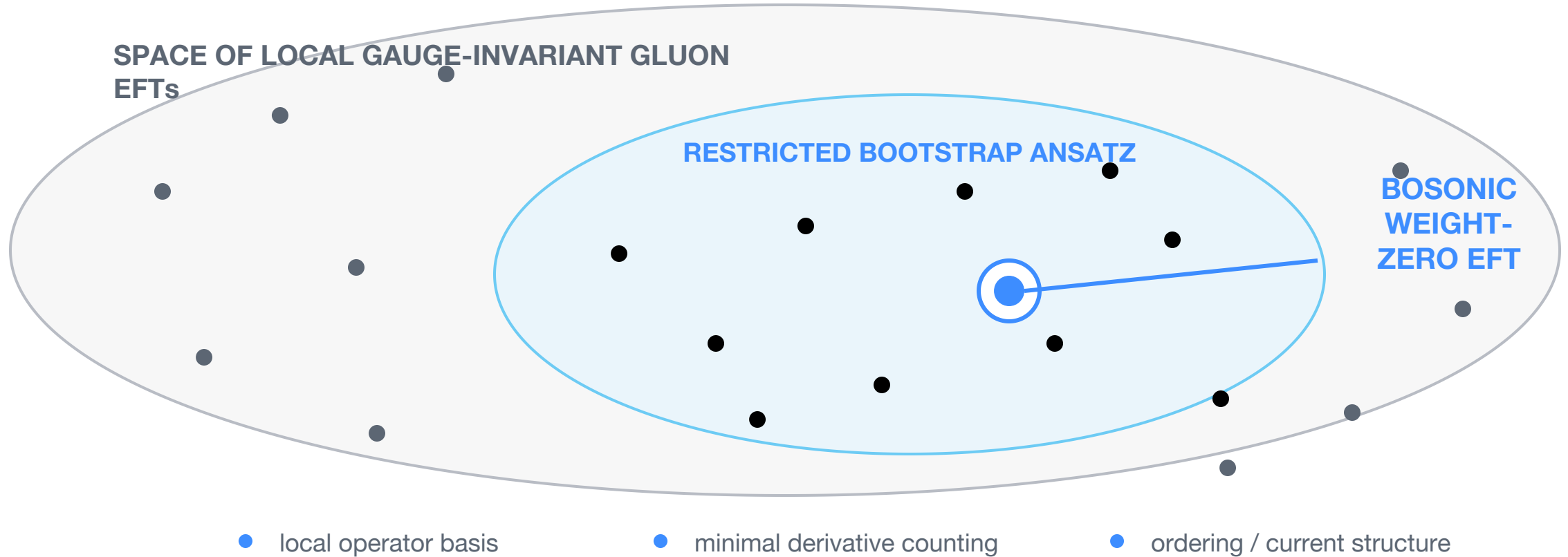
COEFFICIENT THEORY

$\zeta \rightarrow 0$

Rational α' corrections remain nontrivial.

Target: the bosonic weight-zero effective gauge theory

Gauge invariance leaves a large EFT landscape



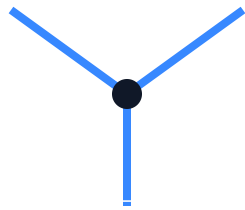
The bootstrap asks whether this point is isolated inside the restricted theory space.

A mass deformation controls the EFT coefficient

Refs: Johansson & Nohle, 2017; Azevedo et al., JHEP 2018

$$\mathcal{L}_{(DF)^2+YM} = \mathcal{L}_{(DF)^2} - \frac{1}{2}m^2\varphi^2 - \frac{1}{4}m^2F^2, \quad m^2 = -\frac{1}{\alpha'}$$

cubic gauge



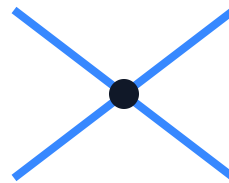
D-F and F^3 vertices

LARGE $|m|$

Higher-derivative modes decouple

Yang-Mills

quartic gauge



contact interactions

FINITE m

Two- and four-derivative modes coexist

Bosonic coefficient

scalar-gauge



ϕ F^2 coupling

ZERO MASS

Mass deformation is removed

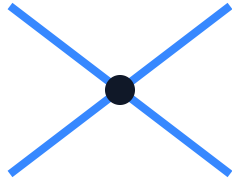
Pure $(DF)^2$ theory

One scale links the Lagrangian deformation to the tachyon pole.

Tachyon poles organize the deformation

$$t_\rho = \frac{-2\alpha'}{1 - \alpha' s_\rho}$$

local contact



A^4 · polynomial in momenta

massless exchange



gauge propagator · massless pole

tachyon exchange



ϕ propagator · tachyon pole

blue: gauge line orange dashed: massive scalar line

The pole occurs at $s = 1/\alpha' = -m^2$: the mass deformation fixes its location.

Two structural questions remain hidden

AMPLITUDES CAN BE COMPUTED

Feynman rules work at low multiplicity

World-sheet integrals work after IBP

Both approaches become complicated quickly

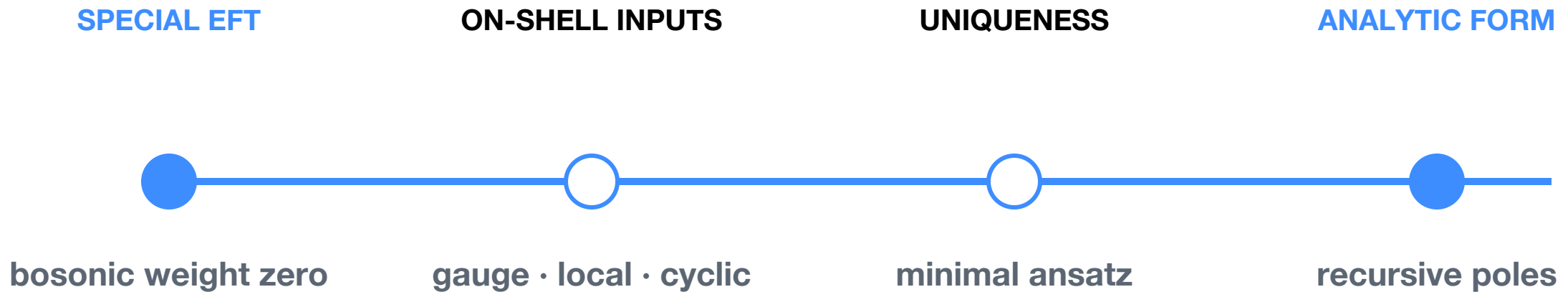
THE STRUCTURE IS STILL HIDDEN

Are the EFT amplitudes uniquely fixed?

Can they be constructed systematically?

Is there a simple all-multiplicity form?

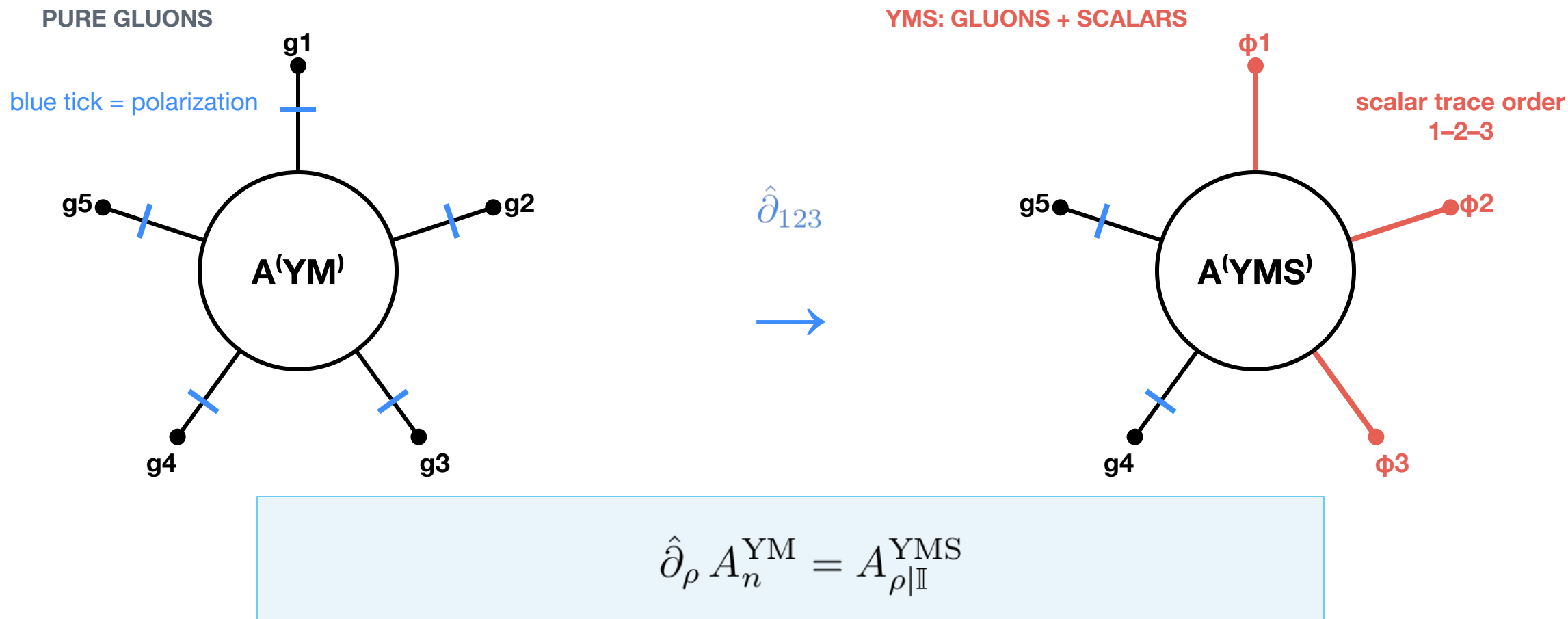
Start from amplitudes, not the world-sheet



Can on-shell consistency recover the string-selected theory?

Transmutation trades spin data for flavor data

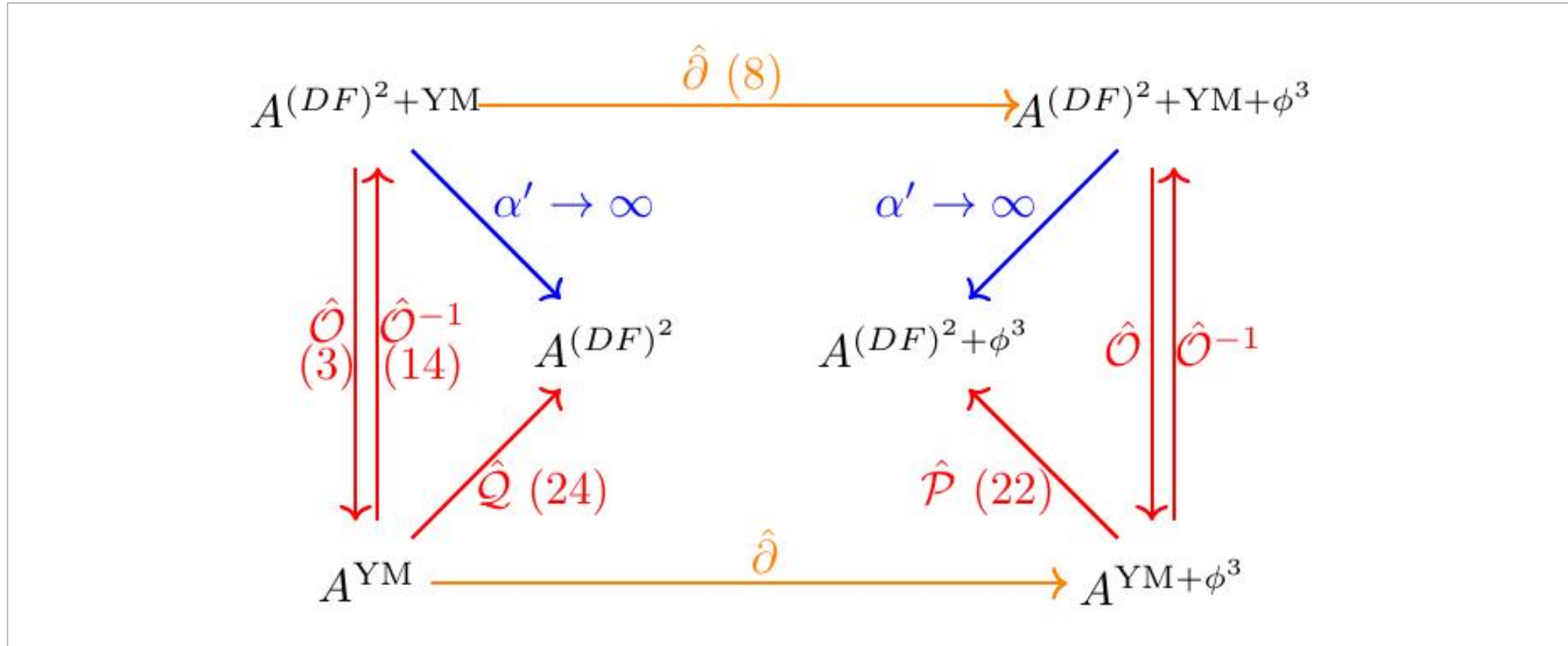
Refs: Cheung, Shen & Wen, JHEP 2018; Cachazo, He & Yuan, JHEP 2014



All momentum-carrying external legs remain; selected polarization data are removed.

One operator web connects four gauge theories

Refs: Johansson & Nohle, 2017; Cao et al., arXiv:2504.21676



Red arrows denote transmutation and inverse maps; blue arrows take the undeformed higher-derivative limit.

Four points contain the whole strategy

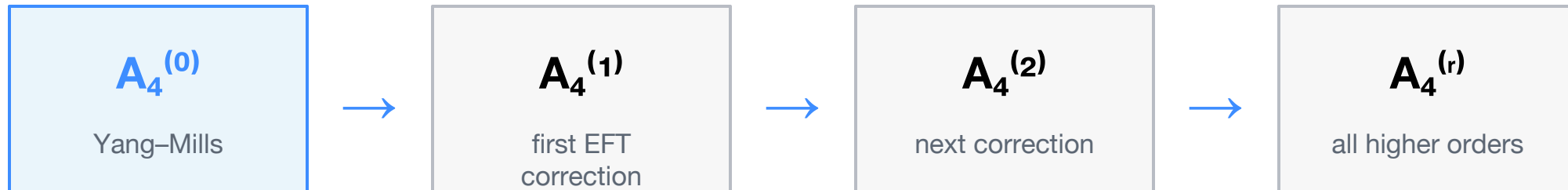
Start from the four-point low-energy expansion

$$A_4^{(DF)^2+YM}(\alpha') = A_4^{(0)} + \alpha' A_4^{(1)} + \alpha'^2 A_4^{(2)} + \cdots, \quad A_4^{(0)} = A_4^{YM}$$

KINEMATIC WEIGHT

$$w(\varepsilon \cdot \varepsilon) = w(\varepsilon \cdot p) = w(p \cdot p) = 1$$

ε and p carry equal mass dimension; $s_j = 2p_i \cdot p_j$ sets the unit.



Question: are these genuinely new functions at every order?

Two local derivatives build transmutation

Ref: Cheung, Shen & Wen, JHEP 2018

$$\mathcal{T}_{ij} = \partial_{\epsilon_i \cdot \epsilon_j}, \quad \mathcal{T}_{lmn} = \partial_{p_n \cdot \epsilon_m} - \partial_{p_l \cdot \epsilon_m}$$

TRACE OPERATOR

Select two gluon legs



Create a scalar trace (i, j)

INSERTION OPERATOR

Select one more gluon leg m



Insert it into the trace: (l, m, n)

ρ = scalar flavor order | l = original gauge-color order

Why is the four-point operator basis so small?

Ref: Cheung et al., JHEP 2017

1 BOTTOM-UP + DIMENSION

Use polarization derivatives that genuinely raise the EFT order.

2 GAUGE + LOCAL

Dress each derivative with a local gauge-invariant trace of field strengths.

3 CURRENT + KK BASIS

Fix leg 4 and reduce to four independent ordered channels.

THE MOST GENERAL MINIMAL ANSATZ

$$D_4 = a_{12} \text{tr}_{12} \hat{\partial}_{12} + a_{13} \text{tr}_{13} \hat{\partial}_{13} + a_{23} \text{tr}_{23} \hat{\partial}_{23} + a_{123} \text{tr}_{123} \hat{\partial}_{123},$$
$$f_i^{\mu\nu} := p_i^\mu \epsilon_i^\nu - \epsilon_i^\mu p_i^\nu, \quad \text{tr}_{12} := \frac{1}{2} \text{Tr}(f_1 f_2), \quad \text{tr}_{123} := \text{Tr}(f_1 f_2 f_3).$$

12

13

23

123

Three pair channels and one ordered trace leave four coefficients to determine.

The assumptions reduce the problem to finite-dimensional linear algebra.

Cyclicity fixes the unique relative combination

RELATIVE COEFFICIENTS

$$a_{12} = a_{13} = a_{23} = a_{123} \equiv a, \quad a \xrightarrow{\text{normalization}} 2\alpha'$$

UNIQUE OPERATOR

$$D_4 = \text{tr}_{12} \hat{\partial}_{12} + \text{tr}_{13} \hat{\partial}_{13} + \text{tr}_{23} \hat{\partial}_{23} + \text{tr}_{123} \hat{\partial}_{123}$$

THE FIRST EFT CORRECTION

$$A_4^{(1)} = -2 \left(\text{tr}_{12} A_{12|1234}^{\text{YMS}} + \text{tr}_{13} A_{13|1234}^{\text{YMS}} + \text{tr}_{23} A_{23|1234}^{\text{YMS}} + \text{tr}_{123} A_{123|1234}^{\text{YMS}} \right)$$

Uniqueness means one relative combination inside the minimal local ansatz.

Five points begin a richer operator hierarchy

FOUR POINTS

$$A_4^{(r+1)} = -2D_4 A_4^{(r)}, \quad A_4^{(r)} = (-2D_4)^r A_4^{\text{YM}}$$

one generator

Repeated action builds every higher EFT order.

FIVE POINTS

ONE-STEP BASIS

$$\mathcal{D}_5^{(1)} = \sum_{\tau \in [4]} 2\alpha' \tilde{\text{tr}}_\tau \hat{\partial}_\tau$$

A NEW TWO-STEP PRODUCT

$$\mathcal{D}_5^{(2)} \supset \text{tr}_{12} \text{tr}_{34} \hat{\partial}_{12} \hat{\partial}_{34}$$

Two disjoint actions raise mass dimension by two steps.

More points add operator products, not arbitrary freedom: minimality and cyclicity fix each layer.

Resumming the tower gives a forward operator

Ref: Cao, Dong, He & Zhu, arXiv:2504.21676

$$\begin{aligned} A_4^{(DF)^2+YM} &= \sum_{r \geq 0} (\alpha')^r A_4^{(r)} = \sum_{r \geq 0} (-2\alpha' D_4)^r A_4^{YM} \\ &= (1 + 2\alpha' D_4)^{-1} A_4^{YM} \\ A_4^{YM} &= \hat{\mathcal{O}}_{[3]} A_4^{(DF)^2+YM}, \quad \hat{\mathcal{O}}_{[3]} = 1 + 2\alpha' D_4 \end{aligned}$$

forward operator

BOSONIC EFT

all rational α' corrections



removes the tower

YANG-MILLS

the familiar amplitude

This relation is entirely field-theoretic: it acts only on polarization data.

The constructive question is to reverse the arrow

BOSONIC EFT

unknown target

forward operator $\rightarrow$

$\leftarrow$ inverse operator ?

YANG-MILLS

known input

- 1 Choose the same local transmutation basis.
- 2 Leave its four coefficients undetermined.
- 3 Impose inverse $\times$ forward = identity.

Write the most general four-point inverse ansatz

$$\hat{\mathcal{O}}_{\{1,2,3\}}^{-1} = 1 + \tilde{c}_{12}\hat{\partial}_{12} + \tilde{c}_{13}\hat{\partial}_{13} + \tilde{c}_{23}\hat{\partial}_{23} + \tilde{c}_{123}\hat{\partial}_{123}$$

12

pair coefficient

13

pair coefficient

23

pair coefficient

123

ordered-trace coefficient

No string integral is used here: only the local operator basis and four unknown functions.

The inverse identity becomes four equations

IMPOSE

INVERSE $\times$ FORWARD = IDENTITY

$$(1 - \alpha' s_{ij}) \tilde{c}_{ij} + 2\alpha' \text{tr}_{ij} = 0,$$
$$(1 - \alpha' s_{123}) \tilde{c}_{123} + 2\alpha' (\text{tr}_{123} + \tilde{c}_{12} V_{231} + \tilde{c}_{23} V_{312} + \tilde{c}_{31} V_{123}) = 0$$

PAIR CHANNELS

Each pair coefficient closes on itself.

LONG TRACE

The 123 coefficient mixes with three shorter channels.

WHY LINEAR?

Independent transmutation operators can be matched separately.

Solving the identity exposes the tachyon poles

$$\tilde{c}_{12} = t_{12} \text{tr}_{12}, \quad \tilde{c}_{123} = t_{123} (\text{tr}_{123} + \tilde{c}_{12} V_{231} + \tilde{c}_{23} V_{312} + \tilde{c}_{31} V_{123})$$

$$t_\rho = \frac{-2\alpha'}{1 - \alpha' s_\rho}$$

universal channel pole

×

$$V_{ijk} := -p_i \cdot f_j \cdot p_k, \quad f_j^{\mu\nu} := p_j^\mu \epsilon_j^\nu - \epsilon_j^\mu p_j^\nu$$

gauge-invariant momentum–field-strength chain

The string-scale pole is generated by inversion; it was not inserted by hand.

Four YMS amplitudes reconstruct the answer

$$A_{1234}^{(DF)^2+YM} = A_{1234}^{YM} + \tilde{c}_{12} A_{12|1234}^{YMS} + \tilde{c}_{13} A_{13|1234}^{YMS} \\ + \tilde{c}_{23} A_{23|1234}^{YMS} + \tilde{c}_{123} A_{123|1234}^{YMS}$$

KNOWN BASE

Yang–Mills amplitude

THREE PAIRS

YMS amplitudes weighted by
the three pair coefficients

LONG TRACE

YMS amplitude weighted by
the recursive 123 coefficient

One familiar amplitude and four transmuted amplitudes reconstruct the full α' dependence.

The four-point pattern extends to all multiplicities

$$A_n^{(DF)^2 + \text{YM}} = \sum_{\tau \in [n-1]} \prod_{\sigma \in \tau} \tilde{c}_\sigma A_{\sigma|\mathbb{I}}^{\text{YMS}}$$

01 Minimal basis

trace-dressed transmutations

02 Cyclicity

fixes relative coefficients

03 Recursion

builds every operator layer

From five points onward new products appear, yet the same three constraints close the recursion.

Three physical tests constrain the answer

$$\alpha' \rightarrow 0 : (DF)^2 + \text{YM} \rightarrow \text{YM}, \quad \alpha' \rightarrow \infty : (DF)^2 + \text{YM} \rightarrow (DF)^2$$

LOW ENERGY

Return to Yang–Mills

The mass deformation decouples smoothly.

HIGHER DERIVATIVE

Reach the pure theory

The same operator web survives the undeformed limit.

FACTORIZATION

Split on every pole

Residues reproduce lower-point amplitudes and channel data.

Gauge invariance and cyclicity are built in; these limits test the analytic structure.

The construction extends beyond one theory

GAUGE-THEORY FAMILY

$(DF)^2$ gauge theory

$(DF)^2 + \phi^3$

$(DF)^2 + \text{YM} + \phi^3$

The inverse-operator logic extends across the family.

DOUBLE COPY & STRINGS

Color–kinematics-compatible gauge factors

Conformal-gravity and heterotic-string double copies

A route toward closed bosonic-string structures

Three ideas to take away

1 The bosonic weight-zero sector is a meaningful EFT.

2 Minimality and cyclicity fix an operator hierarchy.

3 **The inverse operator reconstructs the exact EFT amplitude and exposes universal tachyon poles.**

Three directions beyond the tree-level result

$$\alpha' \rightarrow 0 : (DF)^2 + \text{YM} \rightarrow \text{YM}, \quad \alpha' \rightarrow \infty : (DF)^2 + \text{YM} \rightarrow (DF)^2$$

Higher-level external states

Extend to higher-spin massive theories

What are the corresponding EFT for these string amplitudes?

MIXED DISK SECTORS

Connect to mixed disk string amplitudes

Can the similar basis organize open-closed interactions?

LOOP LEVEL

Search for an operator analogue

Do the coefficients organize the integrand?

The question is how far tree-level uniqueness and factorization persist.

Technical details and examples

What enters the operator ansatz

DERIVATIVE OPERATORS

Polarization derivatives

Trace/transmutation operators

Dimension-raising combinations

Cyclic sums over color orderings

CONSTRAINTS ON COEFFICIENTS

Gauge-invariant prefactors

Allowed propagator poles

Correct mass dimension

Compatibility with lower multiplicity

Coefficients recurse over ordered subsets

$$c_{12\dots m} = t_{12\dots m} \times$$

The diagram illustrates the recursion of coefficients over ordered subsets. It shows two circular diagrams representing channel structures. The first diagram is a circle with points labeled 1, 2, 3, 4, 5, 6, 7, 8, ..., m, m-1, 1. The label $\text{tr}_{12\dots m}$ is inside the circle. The second diagram is a circle with points labeled 1, 2, 3, 4, 5, 6, 7, 8, ..., m, m-1, 1. The label V_{123} is inside the circle. The label V_{678} is next to point 7. The label $\tilde{c}_{3456}$ is above the arc between points 3 and 4. The label $\tilde{c}_{(m-1)m1}$ is below the arc between points m and m-1. The entire expression is enclosed in large parentheses, with an ellipsis between the two diagrams.

The recursion peels off ordered subsets while preserving the channel structure.

Equivalent four-point decompositions

$$\left[\tilde{c}_{1234} + 1 \cdot \text{tr}_{1234} + 3 + \left(1 \cdot \tilde{c}_{12} + V_{2341} + 3 + 1 \cdot \tilde{c}_{123} + V_{341} + 3 + 1 \cdot \tilde{c}_{4123} + V_{34} + 3 \text{ cyclic.} \right) + \left(1 \cdot \tilde{c}_{12} + V_{23} + V_{41} + \tilde{c}_{34} + 1 \text{ cyclic.} \right) \right] - (3 \leftrightarrow 4)$$

$$\tilde{c}_{1234} = c_{12[34]} = t_{1234} \times \left[1 \cdot \text{tr}_{1234} + 3 + \left(1 \cdot \tilde{c}_{12} + V_{2341} + 3 + 1 \cdot \tilde{c}_{123} + V_{341} + 3 + 1 \cdot \tilde{c}_{4123} + V_{34} + 3 \text{ cyclic.} \right) + \left(1 \cdot \tilde{c}_{12} + V_{23} + V_{41} + \tilde{c}_{34} + 1 \text{ cyclic.} \right) \right] - (3 \leftrightarrow 4)$$

Different cyclic presentations agree after the coefficient identities are imposed.

KK relations keep the transmutation basis finite

Refs: Cheung, Shen & Wen, JHEP 2018; Kleiss & Kuijf, NPB 1989

FOR ONE m -LEG SCALAR TRACE



FIX THE ENDS

Hold the first and last labels fixed.

PERMUTE THE INTERIOR

Only the $m-2$ interior labels remain.

BOOTSTRAP CONSEQUENCE

The local ansatz is finite before cyclicity fixes its coefficients.

This basis reduction is the counting input behind the uniqueness argument.

Independent checks support the coefficients

Refs: Chen et al., 2024; He et al., JHEP 2019

BCJ / HOPF ALGEBRA

The recursive coefficients match $W'(\rho)$.

The inverse operator gives an amplitude-level origin for the same coefficient data.

STRING-INTEGRAND IBP

The coefficients match T_p after normalization.

The tachyon-pole factors agree with the bosonic-string integration-by-parts basis.

EXPLICIT LOW-POINT TESTS

Equivalent amplitudes agree through seven points.

Different bases and cyclic presentations give the same physical result.

The operator construction reorganizes known results in a gauge-invariant basis.

Let's continue the conversation in Hangzhou

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Questions, ideas, and collaborations are welcome — come visit us at Westlake University.