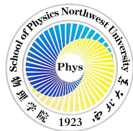


第七届
全国场论与弦论学术研讨会



Complexification of Black Hole Thermodynamic Phase Transition

许震明

西北大学

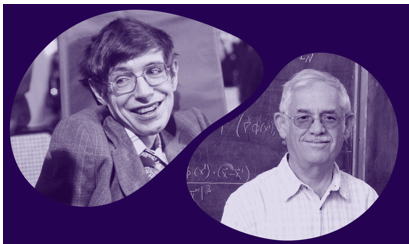
2026-08-07 @ 合肥

Contents

- **Introduction: Black hole thermodynamics**
- **Complex counterpart of phase transition**
- **Emergence: Supercriticality**
- **Summary**

Introduction

Black Hole Thermodynamics



◆ **1971/1972** **Hawking's area theorem** PRL(1971) 、 CMP(1972)

◆ **1972** **Bekenstein: Black hole has entropy?**

1972 Les Houches Summer School on Black Holes

◆ **1973** **Hawking: Black hole mechanics** CMP(1973)

Bekenstein: Black holes and entropy PRD(1973)

The Four Laws of Black Hole Mechanics

J. M. Bardeen*

Department of Physics, Yale University, New Haven, Connecticut, USA

B. Carter and S. W. Hawking

Institute of Astronomy, University of Cambridge, England

This work was carried out while the authors were attending the 1972 Les Houches Summer School on Black Holes. The authors would like to thank Larry Smarr, Bryce de Witt and other participants of the school for valuable discussions.

Bardeen, Carter & Hawking (1973)

"It can be seen that $\kappa/8\pi$ is analogous to temperature in the same way that A is analogous to entropy. It should however be emphasized that $\kappa/8\pi$ and A are distinct from the temperature and entropy of the black hole."

PHYSICAL REVIEW D

VOLUME 7, NUMBER 8

15 APRIL 1973

Black Holes and Entropy*

Jacob D. Bekenstein†

Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540
and Center for Relativity Theory, The University of Texas at Austin, Austin, Texas 78712†
(Received 2 November 1972)

There are a number of similarities between black-hole physics and thermodynamics. Most striking is the similarity in the behaviors of black-hole area and of entropy: Both quantities tend to increase irreversibly. In this paper we make this similarity the basis of

In those days in 1973 when I was often told that I was headed the wrong way, I drew some comfort from Wheeler's opinion that "black-hole thermodynamics is crazy, perhaps crazy enough to work."

Wheeler's suggestion led² to a formula for black hole entropy independent of the properties of matter

$$S_{\text{bh}} = \eta k A / L_{\text{PW}}^2$$

Here η is a pure number, presumably of order unity, to be determined separately.

$$L_{\text{PW}} = (\hbar G / c^3)^{1/2} = 1.61 \times 10^{-33} \text{ cm}$$



John Archibald Wheeler

Introduction

Black Hole Thermodynamics

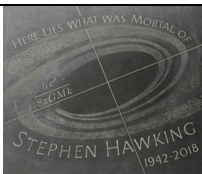
Letter | Published: 01 March 1974

Black hole explosions?

[S. W. HAWKING](#)

[Nature](#) 248, 30–31 (1974) | [Cite this article](#)

97k Accesses | 4966 Citations | 993 Altmetric | [Metrics](#)



Beckenstein* suggested on thermodynamic grounds that some multiple of κ should be regarded as the temperature of a black hole. He did not, however, suggest that a black hole could emit particles as well as absorb them. For this reason Bardeen, Carter and I considered that the thermodynamical similarity between κ and temperature was only an analogy. The present result seems to indicate, however, that there may be more to it than this. Of course this calculation ignores the back reaction of the particles on the metric, and quantum fluctuations on the metric. These might alter the picture.

Hawking also gave an argument showing that charge and rotation do not alter the thermal nature of the emission.⁵ They only affect the form of the characteristic temperature T . In every case it is given by the same formula as T_{bh} with the choice $\eta = 1/4$.



◆ 1974 Black Holes Ain't So Black

Nature(1974)

◆ 1975 Particle creation by black holes

CMP(1975)

In the classical theory black holes can only absorb and not emit particles. However it is shown that quantum mechanical effects cause black holes to create and emit particles as if

PHYSICAL REVIEW D

VOLUME 13, NUMBER 2

15 JANUARY 1976

Black holes and thermodynamics*

S. W. Hawking[†]

California Institute of Technology, Pasadena, California 91125

and Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge, England

(Received 30 June 1975)

Hawking Temperature

$$T = \frac{\hbar \kappa}{2\pi k c}$$

Bekenstein-Hawking Entropy

$$S_{bh} = \frac{k c^3 A}{4 G \hbar}$$

Black Hole

Introduction

Black Hole Thermodynamics

The 4 Laws of Black Hole (thermo-)Mechanics

◆ **0th Law** $\kappa = \text{constant}$

-- surface Gravity is constant over the event horizon

◆ **1st Law** $dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ + \dots$

-- differences in mass between nearby solutions are equal to differences in area times the surface gravity plus additional work terms

◆ **2nd Law** $dA \geq 0$

-- area of the event horizon never decreases in any physical process

◆ **3rd Law** $\kappa_n > \kappa_{n+1} > 0, n < \infty$

-- No procedure can reduce the surface gravity to 0 in a finite number of steps

定律	热力学
0 th	在热平衡状态下, 系统各处都具有相同的温度。
1 st	$dE = TdS + \sum_i \mu_i dN_i$
2 nd	$dS \geq 0$
3 rd	不可能通过任何物理过程达到绝对零度。

Testing

Introduction

Black Hole Thermodynamics

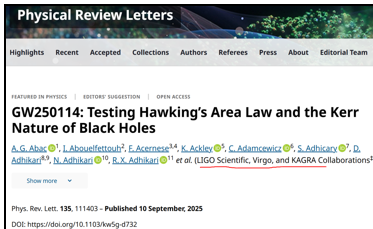
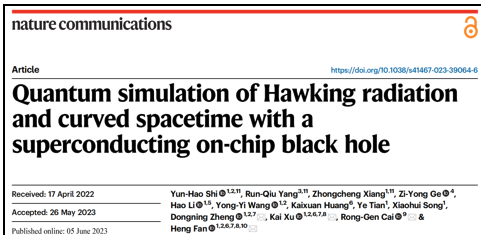
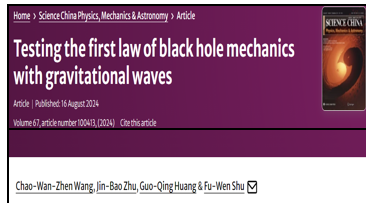
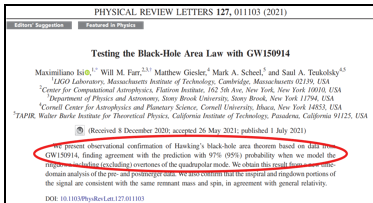
Superconducting
quantum simulation

Hawking radiation and
dynamics the curved
spacetime

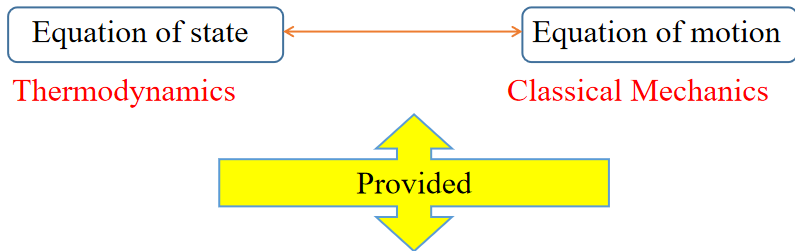
Gravitational wave test



Laws of Black Hole
Mechanics



Generalized Free energy Landau potential



The process of a system from an unknown state to an equilibrium state:

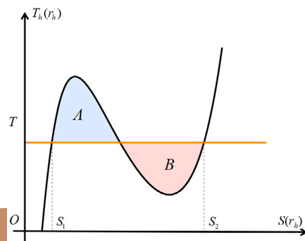
selecting a relation (equation of state in equilibrium) from all possible relations

$$L = \int F(X, T, P) dX$$

$$F(X, T, P) \equiv P - f(X, T)$$

Generalized Free energy

Thermal potential

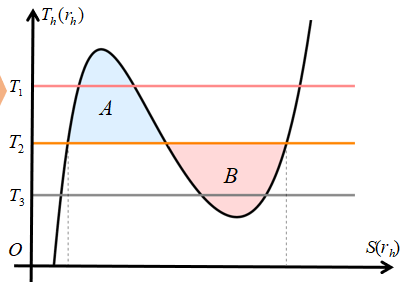


$$\int_{S_1}^{S_2} T_h dS = T \cdot (S_2 - S_1)$$

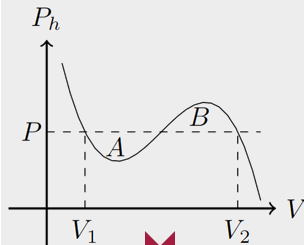
$$\Rightarrow \int_{S_1}^{S_2} (T - T_h) dS = 0.$$



$$U^{(T)} = \int (T_h - T) dS,$$



Z.-M. Xu, PRD (2021)

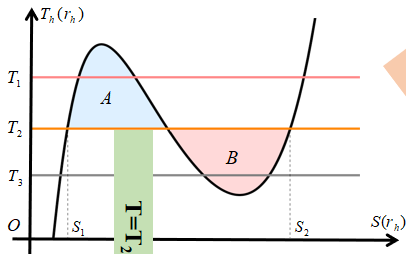


$$U^{(P)} = \int (P_h - P) dV,$$

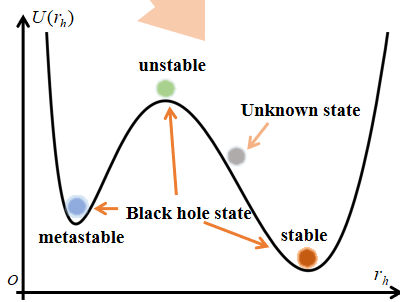
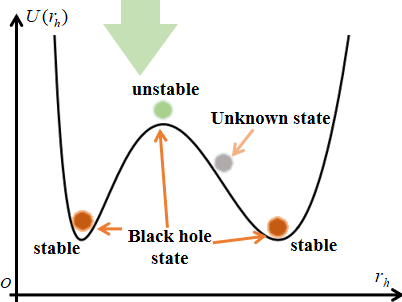
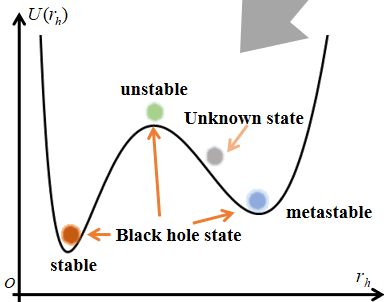
Generalized Free energy

$$\frac{dU(r_h)}{dS(r_h)} = 0$$

$$T = T_h(r_h)$$

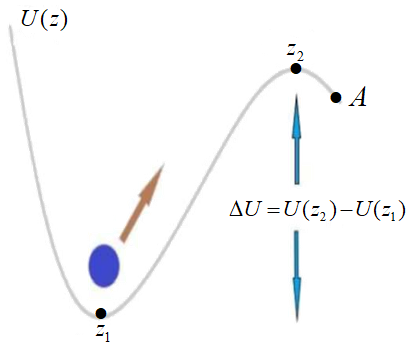


Different states in the system at the extremum of the potential function



Small/Large phase transition picture

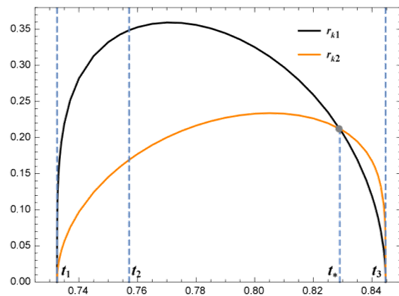
Kramer escape rate



$$v = \frac{1}{2\pi} \sqrt{|U''(z_1)U''(z_2)|} e^{-\Delta U/D}, \quad \Delta U = U(z_2) - U(z_1)$$

RN-AdS BH

The overall process is dominated by the transition from a **small** black hole to a **large** black hole.

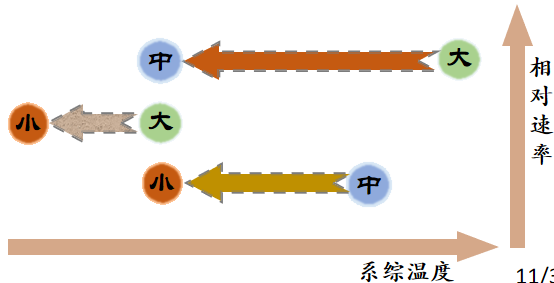
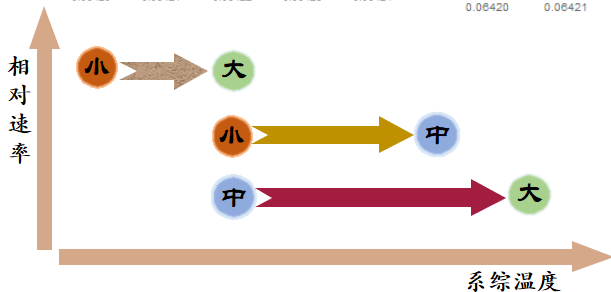
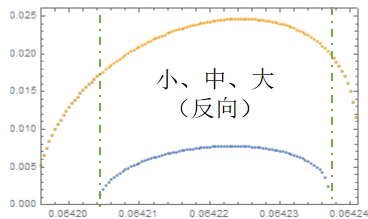
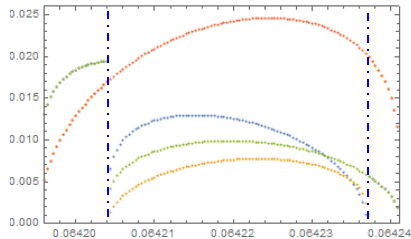
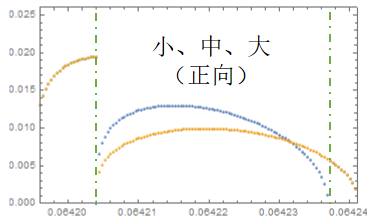


Triple phases transition picture

Small/Middle/Large phases

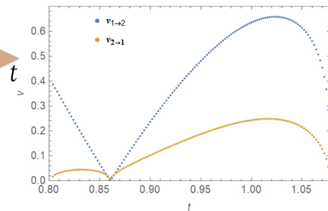
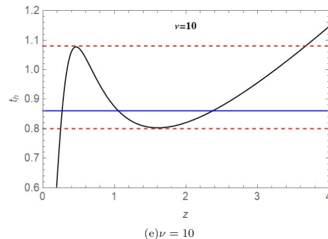
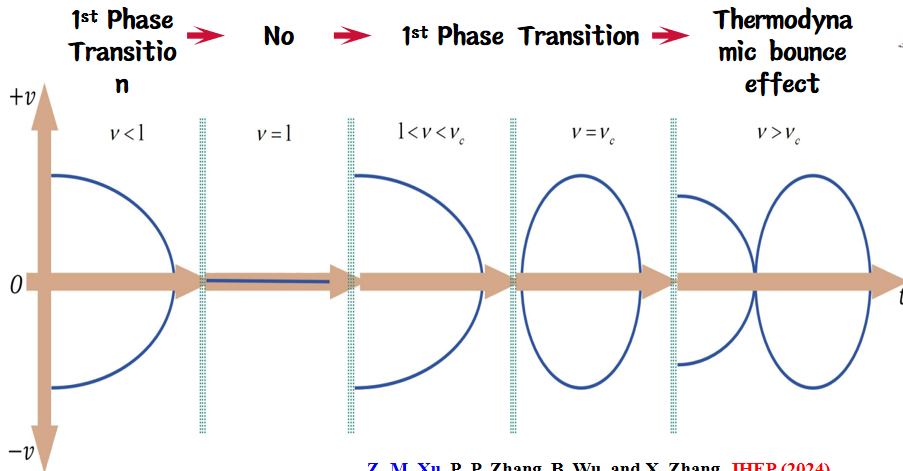
6D GB-AdS BH

C. Ma, B. Wu, and Z.-M. Xu, *PLB* (2025)



Thermodynamic bounce effect

qBTZ BH



Z.-M. Xu, P.-P. Zhang, B. Wu, and X. Zhang, *JHEP* (2024)

Complexification

- Mapping the different states in the system into the extremum of the potential function

zeros of the derivative of potential function

- Analytical properties of thermodynamic phase transitions

the Lee-Yang Formalism

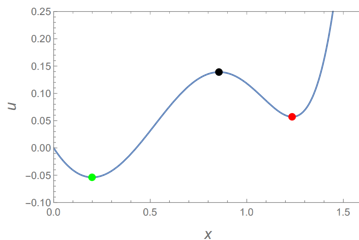


zeros of the partition function

Complex Domain

Complexification

$$U(r_h) = \int (T_h(r_h) - T) dS(r_h) \quad \Longrightarrow \quad U(z) = u(x, y) + iv(x, y)$$



Black hole state at $f(z) = \frac{dU}{dz}$



**Zeroes of
Analytic Function**

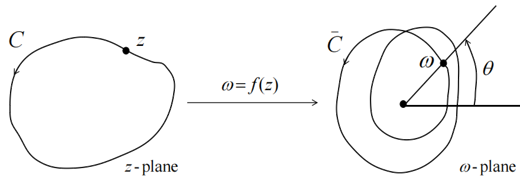
Complexification

Argument Principle

Let C be a simple closed contour lying entirely within a domain D . Suppose f is analytic in D except at a finite number of poles inside C , and that $f(z) \neq 0$ on C . Then

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz = N_0 - N_p,$$

where N_0 is the total number of zeros of f inside C and N_p is the total number of poles of f inside C . In determining N_0 and N_p , zeros and poles are counted according to their order or multiplicities.



Winding number

$$W := \frac{1}{2\pi i} \oint_{\bar{C}} \frac{d\omega}{\omega} = \frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz$$

$$N_0 - N_p = \frac{1}{2\pi} [\text{change in } \arg(f(z)) \text{ as } z \text{ traverses } C \text{ once in the positive direction}].$$

Complexification

4D Kerr-AdS BH

complex r_h to z



$$U(r_h) = \int (T_h(r_h) - T) dS(r_h).$$

$$f(z) = \frac{dU}{dz}$$

Hawking Temperature:

$$T_h = \frac{1}{2\pi} \left[r \left(\frac{r^2}{l^2} + 1 \right) \left(\frac{1}{a^2 + r^2} + \frac{1}{2r^2} \right) - \frac{1}{r} \right]$$

Bekenstein Entropy:

$$S = \frac{\pi(a^2 + r^2)}{(1 - \frac{a^2}{l^2})}$$

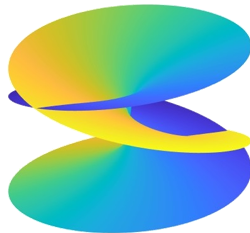
Generalized free energy:

$$U = \int (T_h - T) dS = \int \frac{1}{2\pi} \left[r \left(\frac{r^2}{l^2} + 1 \right) \left(\frac{1}{a^2 + r^2} + \frac{1}{2r^2} \right) - \frac{1}{r} \right] d \frac{\pi(a^2 + r^2)}{(1 - \frac{a^2}{l^2})}$$

$$= - \frac{r(-2a^2 + l^2 + r^2) + 2(a^3 - al^2) \text{Aractan}[\frac{r}{a}]}{2(a^2 - l^2)}$$

Analytical function:

$$f(z) = \frac{a^2(-l^2 + z^2) + z^2(l^2 + 3z^2)}{4l^2\pi z(a^2 + z^2)}$$

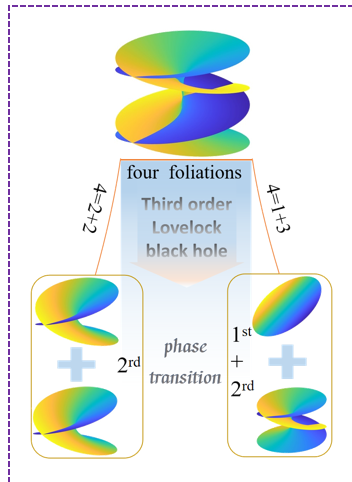


$$W_{\text{local-max}} = 3$$

Complexification

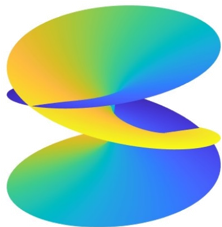
Complex correspondence

Black holes	Phase transition	Winding number		Complex counterpart (Riemann surface)
		local-max	global	
Schwarzschild	No	1	0	one foliation 
Reissner-Nordström	2 nd	2	0	two foliations 
Schwarzschild-AdS	2 nd	2	1	two foliations 
Charged AdS	1 st and 2 nd	3	1	three foliations 
6D charged Gauss-Bonnet	1 st and 2 nd	5	1	five foliations 

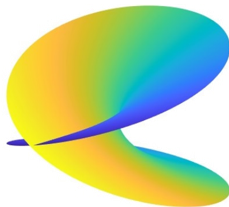


Complexification

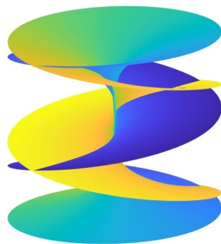
Riemann surface = Phase transition?



一阶相变
三叶黎曼面结构



二阶相变
二叶黎曼面结构



三相态
五叶黎曼面结构

Complexification

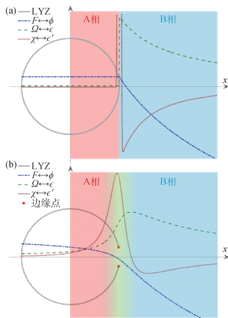
● Analytical properties of thermodynamic phase transitions

● The Lee-Yang Formalism

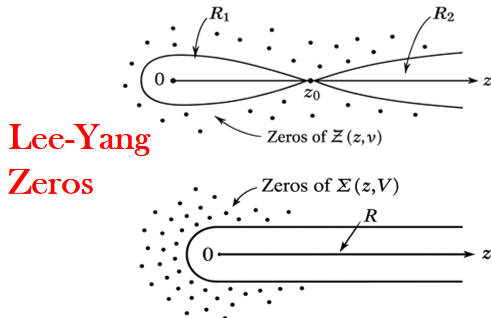
C.-N. Yang and T.-D. Lee, **Phys. Rev. (1952)**

T. -D. Lee and C. -N. Yang, **Phys. Rev. (1952)**

从统计物理中最基本的配分函数出发，将系统的强度量视为复数，指出配分函数在复平面上的零点分布决定了系统的全部相变行为。



静电学类比
配分函数的零点分布对自由能的影响，相当于电荷分布对实轴上的静电场的影响



Complexification

Experimental Observation of Lee-Yang Zeros

用亚磷酸三甲酯分子实现该系统：充当探针的是磷原子的核自旋、充当铁磁伊辛模型的是周围九个氢原子的核自旋。

实验上首次测量出单个李-杨零点的位置。

PRL 109, 185701 (2012) PHYSICAL REVIEW LETTERS week ending 2 NOVEMBER 2012

Lee-Yang Zeros and Critical Times in Decoherence of a Probe Spin Coupled to a Bath

Bo-Bo Wei^{1,2} and Ren-Bao Liu^{1,2,3,*}

¹Department of Physics, The Chinese University of Hong Kong, Hong Kong, China
²Center for Quantum Coherence, The Chinese University of Hong Kong, Hong Kong, China
³Institute of Theoretical Physics, The Chinese University of Hong Kong, Hong Kong, China
 (Received 11 July 2012; revised manuscript received 22 August 2012; published 31 October 2012)

Phase transitions are not usually seen in the time domain. Here, we report on the finding of critical times at which a physical observable, in the thermodynamic limit, becomes nonanalytic as a function of time. We find that the coherence of a probe spin coupled to a many-body system vanishes at times in one-to-one correspondence to the Lee-Yang zeros of the partition function of the many-body system. In the thermodynamic limit, the Lee-Yang zeros form a continuum cut in the complex plane of fugacity and the probe spin coherence presents sudden death and birth at the critical times corresponding to the Yang-Lee singularities. These results provide new experimental possibilities in many-body physics.

DOI: 10.1103/PhysRevLett.109.185701 PACS numbers: 64.60.Bd, 03.65.Yz, 05.50.+g, 64.60.De

PRL 114, 010601 (2015) PHYSICAL REVIEW LETTERS week ending 9 JANUARY 2015

Experimental Observation of Lee-Yang Zeros

Xinhua Peng,^{1,2} Hui Zhou,¹ Bo-Bo Wei,² Jiangyu Cui,¹ Jiangfeng Du,^{1,2} and Ren-Bao Liu^{2,3}

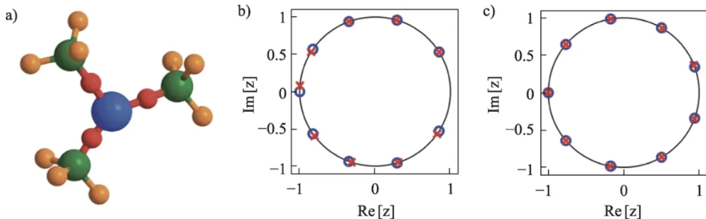
¹Hefei National Laboratory for Physical Sciences at Microscale, Department of Modern Physics, and Synergetic Innovation Center of Quantum Information & Quantum Physics, University of Science and Technology of China, Hefei 230026, China
²Department of Physics, Centre for Quantum Coherence, and Institute of Theoretical Physics, The Chinese University of Hong Kong, Shatin, New Territories, Hong Kong, China
 (Received 11 September 2014; published 5 January 2015)

Lee-Yang zeros are points on the complex plane of physical parameters where the partition function of a system vanishes and hence the free energy diverges. Lee-Yang zeros are ubiquitous in many-body systems and fully characterize their thermodynamics. Notwithstanding their fundamental importance, Lee-Yang zeros have never been observed in experiments, due to the intrinsic difficulty that they would occur only at complex values of physical parameters, which are generally regarded as unphysical. Here we report the first observation of Lee-Yang zeros, by measuring quantum coherence of a probe spin coupled to an Ising-type spin bath. The quantum evolution of the probe spin introduces a complex phase factor and therefore effectively realizes an imaginary magnetic field. From the measured Lee-Yang zeros, we reconstructed the free energy of the spin bath and determined its phase transition temperature. This experiment opens up new opportunities of studying thermodynamics in the complex plane.

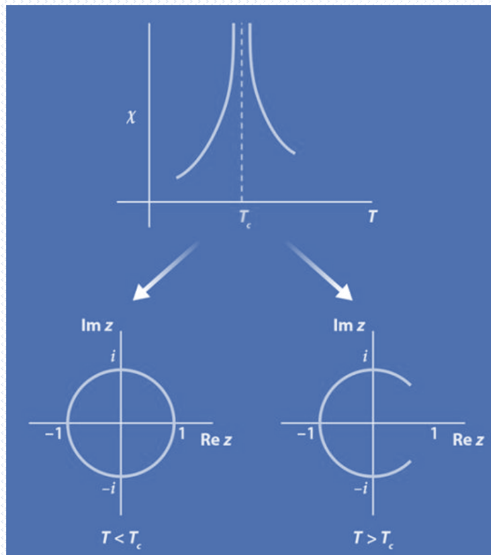
DOI: 10.1103/PhysRevLett.114.010601 PACS numbers: 05.70.-a, 03.65.Yz, 33.25.+k, 64.60.De

设计了一个探针-热库复合系统：探针是一个双态系统，热库是一个铁磁伊辛模型。

系统中，时间轴相当于虚轴，位于复空间的李-杨零点信息，被转化为时间演化信息这一可观测量。



Complexification



配分函数的零点分布

欧式作用量 I_E

黑洞系统配分函数 Z

黑洞系统自由能 F

$$F = -T \ln Z = T I_E$$

黑洞自由能的奇点分布

Complexification

4D Charged AdS black hole

$$t = \frac{3p}{8}z + \frac{3}{4z} - \frac{1}{8z^3},$$

$$g = \frac{3}{4}\left(z - \frac{p}{6}z^3 + \frac{1}{2z}\right),$$

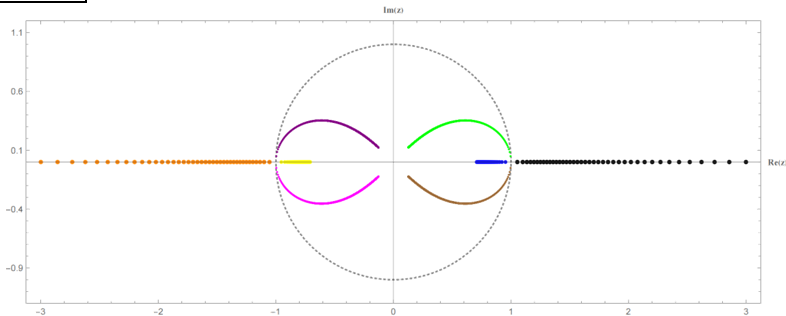
$$c_p = \frac{z^2(3pz^4 + 6z^2 - 1)}{24(pz^4 - 2z^2 + 1)},$$

$$z = \frac{r_h}{r_c}$$

$$r_h = \text{Re}(r_h) + i \text{Im}(r_h)$$

Lee-Yang zeros

$$pz^4 - 2z^2 + 1 = 0$$



Complexification

4D Charged AdS black hole

$$t = \frac{3p}{8}z + \frac{3}{4z} - \frac{1}{8z^3},$$

$$g = \frac{3}{4}\left(z - \frac{p}{6}z^3 + \frac{1}{2z}\right),$$

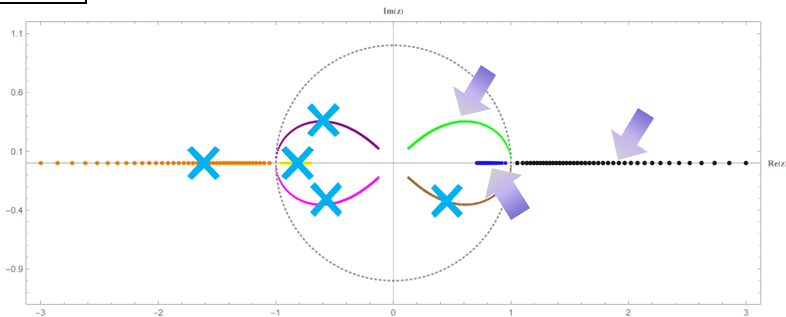
$$c_p = \frac{z^2(3pz^4 + 6z^2 - 1)}{24(pz^4 - 2z^2 + 1)},$$

$$z = \frac{r_h}{r_c}$$

$$r_h = \text{Re}(r_h) + i \text{Im}(r_h)$$

Lee-Yang zeros

$$pz^4 - 2z^2 + 1 = 0$$



Complexification

Charged AdS black hole: Complex Plane Phase Diagram

.....

Spinodal line

.....

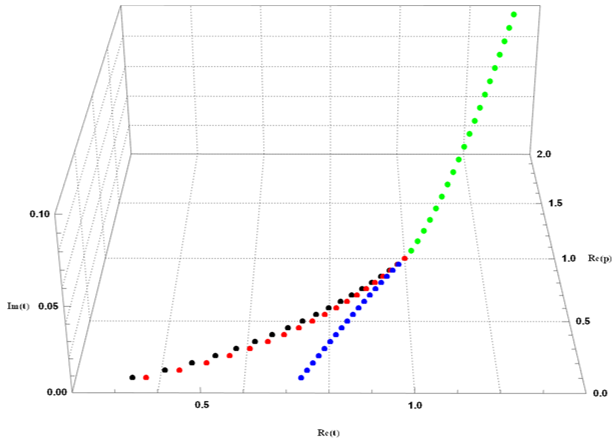
.....

Complex phase line

.....

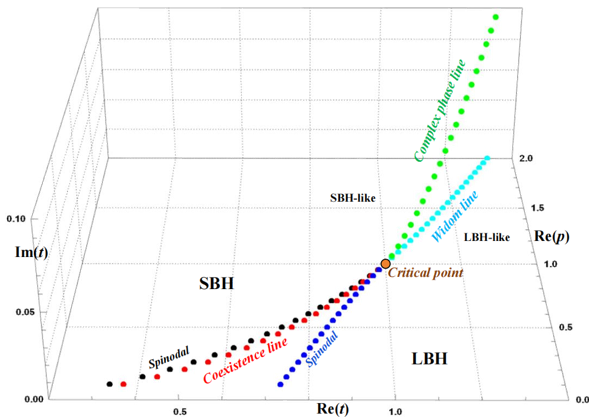
Coexistence line

$$t = \sqrt{\frac{p(3 - \sqrt{p})}{2}}$$



Complexification

Charged AdS black hole: Complex Plane Phase Diagram

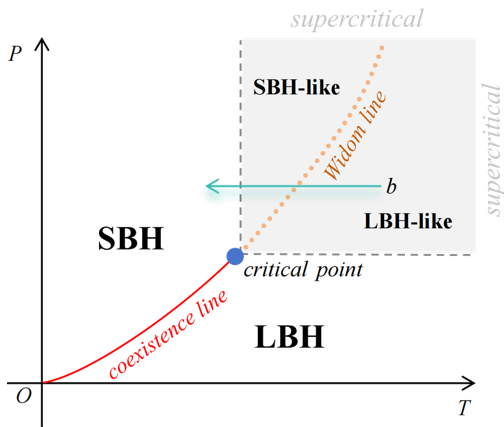


Widom line



Projection of complex Lee-Yang zeros on the real phase plane

Complexification



Widom line extends the concept of phase separation into the supercritical regime, beyond the critical point where no true phase transition occurs. It marks a transition between qualitatively different liquid-like and gas-like regimes.

Widom line is commonly defined as the **locus of maxima in thermodynamic response functions**, such as heat capacity, compressibility, or correlation length, but its precise and **universal definition remains an open question.**

Supercriticality—Widom line

Relation between the Widom line and the dynamic crossover in systems with a liquid–liquid phase transition

Limei Xu^{*,†}, Pradeep Kumar^{*}, S. V. Buldyrev^{*,‡}, S.-H. Chen[§], P. H. Poole[¶], F. Sciortino^{||}, and H. E. Stanley^{*,†}

^{*}Center for Polymer Studies and Department of Physics, Boston University, Boston, MA 02215; [†]Department of Physics, Yeshiva University, 500 West 185th Street, New York, NY 10033; [‡]Department of Nuclear Science and Engineering, Massachusetts Institute of Technology, Cambridge, MA 02139; [§]Department of Physics, St. Francis Xavier University, Antigonish, NS, Canada B2G 2W5; [¶]Department of Physics and Istituto Nazionale Fisica della Materia Unità di Ricerca and Soft-Complex Dynamics in Structured Systems, Università di Roma “La Sapienza,” Piazzale Aldo Moro 2, I-00185 Rome, Italy

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Behavior of the Widom Line in Critical Phenomena

Jiayuan Luo,¹ Limei Xu,^{2,3,*} Erik Lascaris,¹ H. Eugene Stanley,¹ and Sergey V. Buldyrev^{4,1}

¹Center for Polymer Studies and Department of Physics, Boston University, Boston, Massachusetts 02215, USA

²International Center for Quantum Materials, Peking University, Beijing 100871, China

³Collaborative Innovation Center of Quantum Matter, Beijing 100871, China

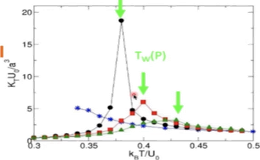
⁴Department of Physics, Yeshiva University, 500 West 185th Street, New York, New York 10033, USA

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Phase diagram (traditional view)

Two phase coexistence region
Critical point



Phase diagram (New view)

Two phase coexistence region
Critical point
The Widom line

L. Xu et. al, PNAS (2005); Nature Physics (2009)



ARTICLE

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Widom line and dynamical crossovers as routes to understand supercritical water

P. Gallo¹, D. Corradini^{2,†} & M. Rovere¹

PHYSICAL REVIEW E 109, 024118 (2024)

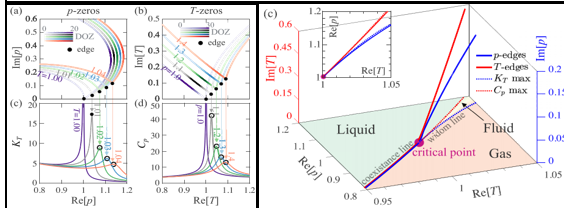
Complex phase diagram and supercritical matter

Xiao-Yu Ouyang^{1,†}, Qi-Jun Ye^{1,2,*} and Xin-Zheng Li^{1,2,3,†}

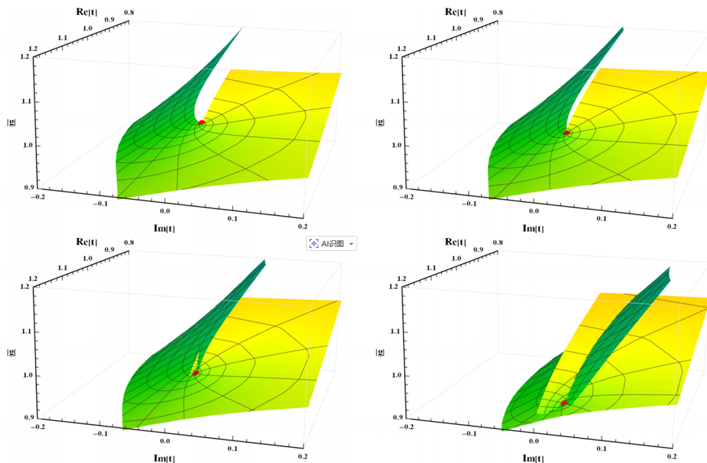
¹State Key Laboratory for Artificial Microstructure and Mesoscopic Physics, Frontier Science Center for Nano-optoelectronics and School of Physics, Peking University, Beijing 100871, People's Republic of China

²Interdisciplinary Institute of Light-Element Quantum Materials, Research Center for Light-Element Advanced Materials, and Collaborative Innovation Center of Quantum Matter, Peking University, Beijing 100871, People's Republic of China

³Peking University Yangtze Delta Institute of Optoelectronics, Nantong, Jiangsu 226010, People's Republic of China



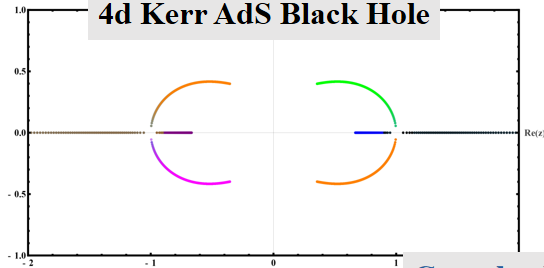
Complexification



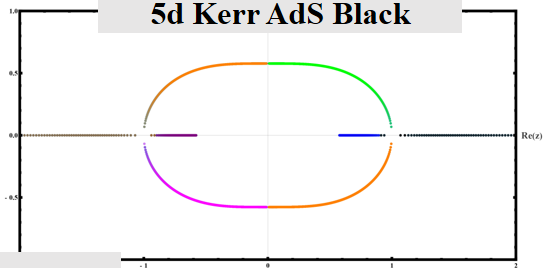
The moduli of the complex Gibbs free energy g with respect to complex temperature t

Complexification

4d Kerr AdS Black Hole



5d Kerr AdS Black



Complex Phase Diagram

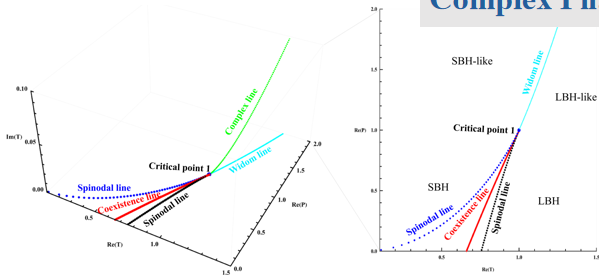


FIG. 4: Complex phase diagram of Kerr-AdS black hole at $d = 4$.

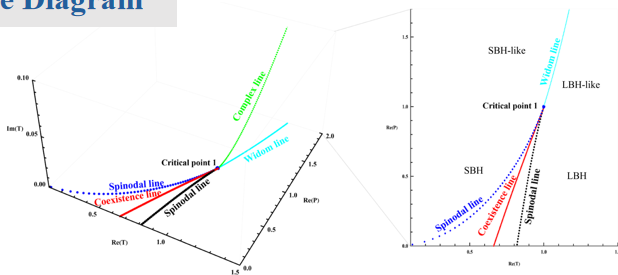
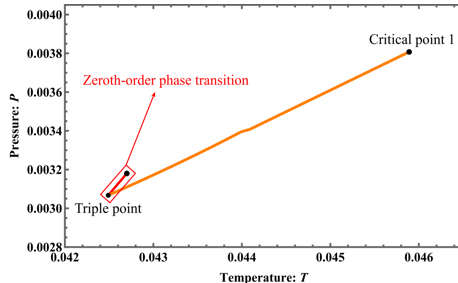
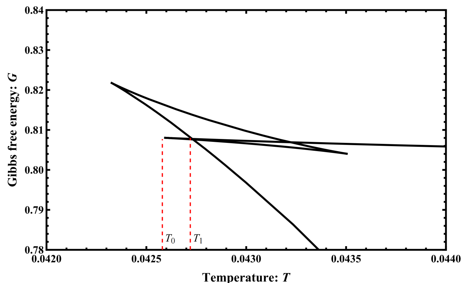


FIG. 6: Complex phase diagram of Kerr-AdS black hole at $d = 5$.

Complexification

Reentrant phase transition



Kerr AdS black hole ($D \geq 6$)

Natacha Altamirano et al, PRD(2013)

6D Gauss-Bonnet black hole (Modify gravity)

Lovelock black hole thermodynamics Frassino et al, JHEP(2014)

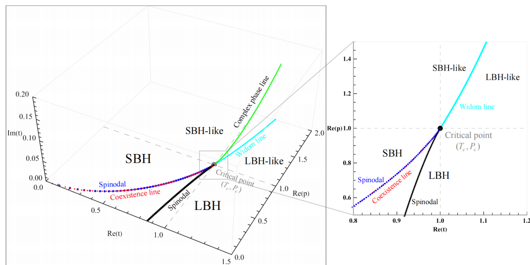
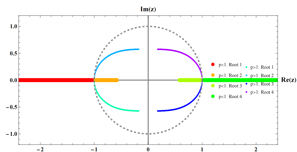
4D Born-Infeld AdS black hole/ $D \geq 6$ dRGT gravity (Modify the matter field)

S. Gunasekaran et al, JHEP(2012)/ D.-C. Zou et al, EPJC(2017)

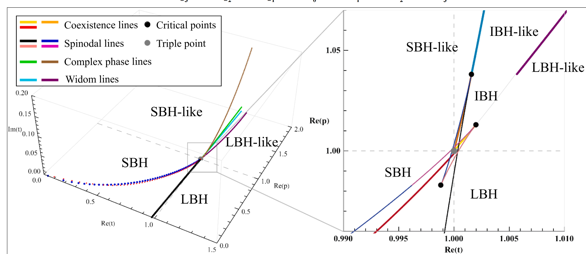
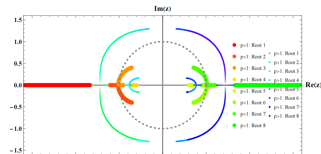
Complexification

Charged Gauss-Bonnet AdS black hole

5D



6D



Complexification

Born-Infeld AdS black hole

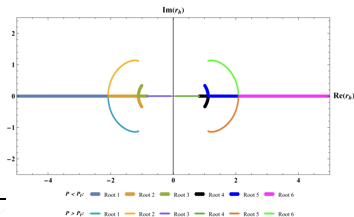


FIG. 4: Complex phase diagram of BI-AdS black hole at $b = 0.4$.

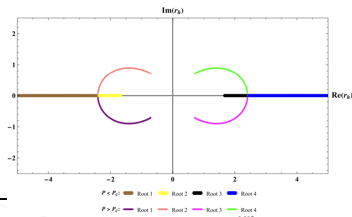


FIG. 8: Complex phase diagram of BI-AdS black hole at $b = 1$.

Summary

- Complex structure of the 1st 、 2nd phase transitions for black holes

Phase transition



Riemann surface

- A supercritical description of thermodynamic phase transitions in black holes was established using the Lee-Yang phase transition theorem

Widom line $\mapsto$ Projection of complex Lee-Yang zeros on the real phase plane.

- In the thermodynamic limit

$$\text{Black Hole Thermodynamics} = \lim_{\substack{V \rightarrow \infty \\ N \rightarrow \infty}} \text{Thermodynamics}$$