



Quantum Entanglement of Bethe States

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For a complete list of references, please see the preprint

Solving the spectrum is not enough

$$H|\psi_n\rangle = E_n|\psi_n\rangle$$

Quantum mechanics gives two kinds of data.

Eigenvalue

$$E_n$$

Level statistics

Wavefunction

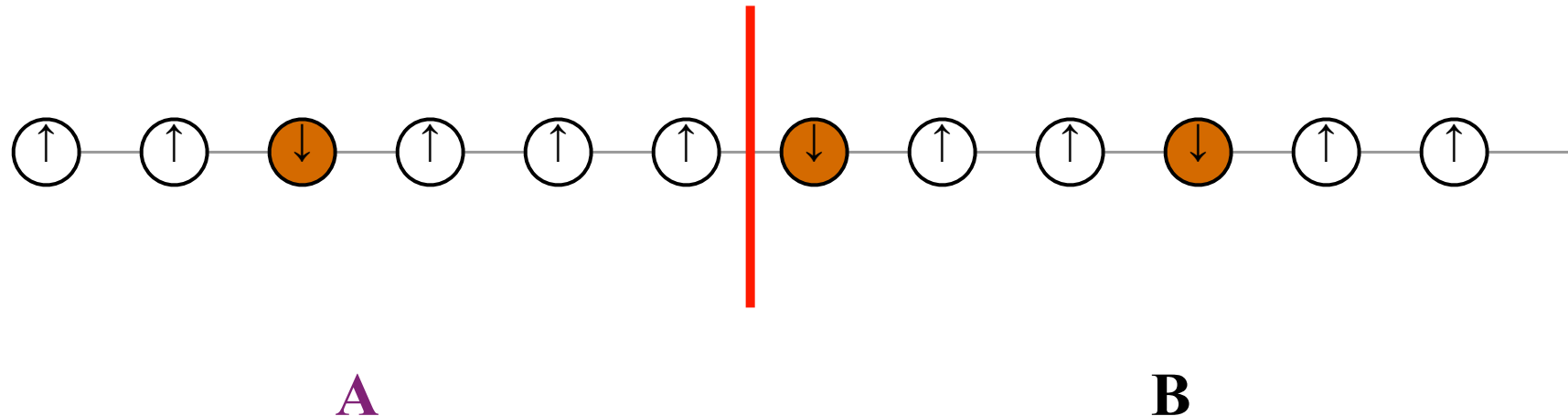
$$|\psi_n\rangle$$

contains correlations,
structure, and entanglement.

- Eigenvalues: numbers, we study by statistics. Poisson vs Wigner-Dyson.
- Wavefunction: harder, because it is not a number
- Idea: cut the system and measure entanglement entropy $S(\ell)$

Entanglement data from a cut

- ❖ Choose a spatial cut A|B and trace out B



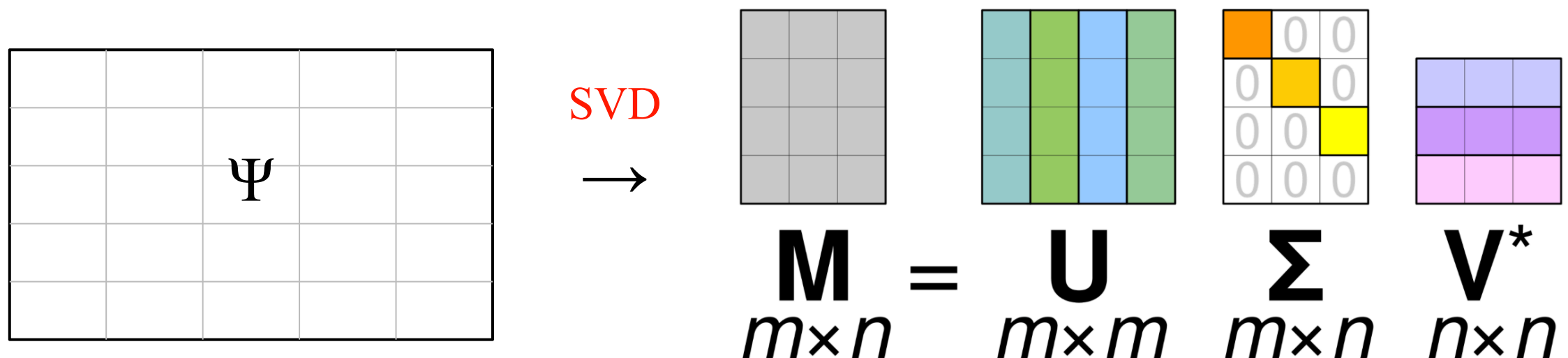
$$\rho_A = \text{Tr}_B |\psi\rangle\langle\psi|, \quad S_A = -\text{Tr} \rho_A \log \rho_A$$

- ❖ If ρ_A is mixed, A is entangled with B. Its eigenvalues tell how much information is lost across the cut.

Computation method: Schmidt decomposition

❖ Reshape the wavefunction coefficients into a matrix $\Psi_{i,j}$

$$|\Psi\rangle = \sum_{i \in A, j \in B} \Psi_{i,j} |i\rangle_a \otimes |j\rangle_b$$



$$\Psi_{ij} = U \text{diag}(s_a) V^\dagger, \quad s_a \geq 0$$

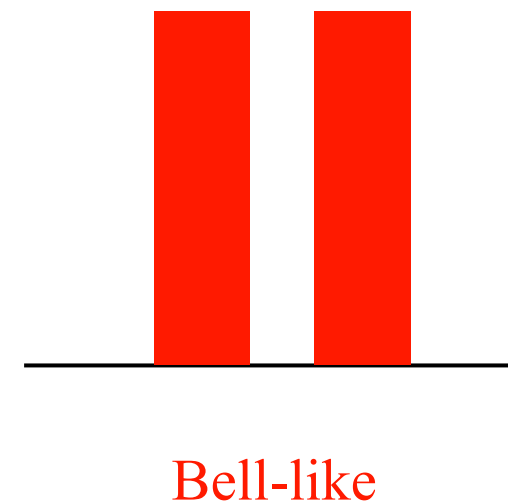
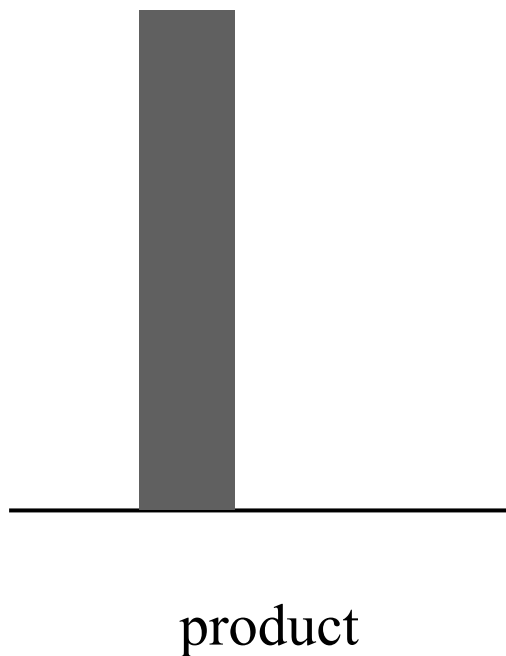
The strictly positive s_a are called the Schmidt coefficients.

Schmidt coefficients by example

❖ Product state: one Schmidt coefficient

❖ Bell-like state: two equal Schmidt coefficients

$$|00\rangle : s = (1, 0, \dots), \quad \frac{|00\rangle + |11\rangle}{\sqrt{2}} : s = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

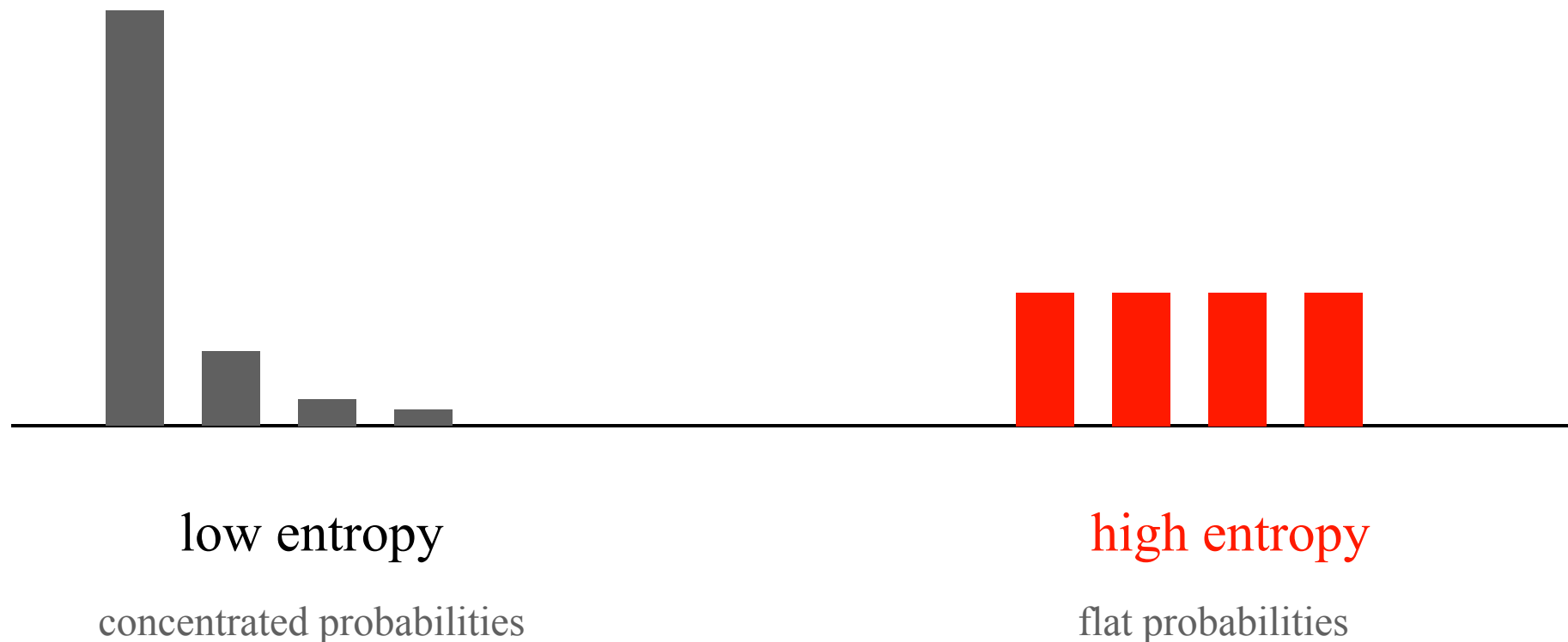


Entropy from Schmidt coefficients

❖ Square the Schmidt coefficients to get probabilities

$$\lambda_a = s_a^2, \quad \sum_a \lambda_a = 1, \quad S_A = - \sum_a \lambda_a \log_2 \lambda_a$$

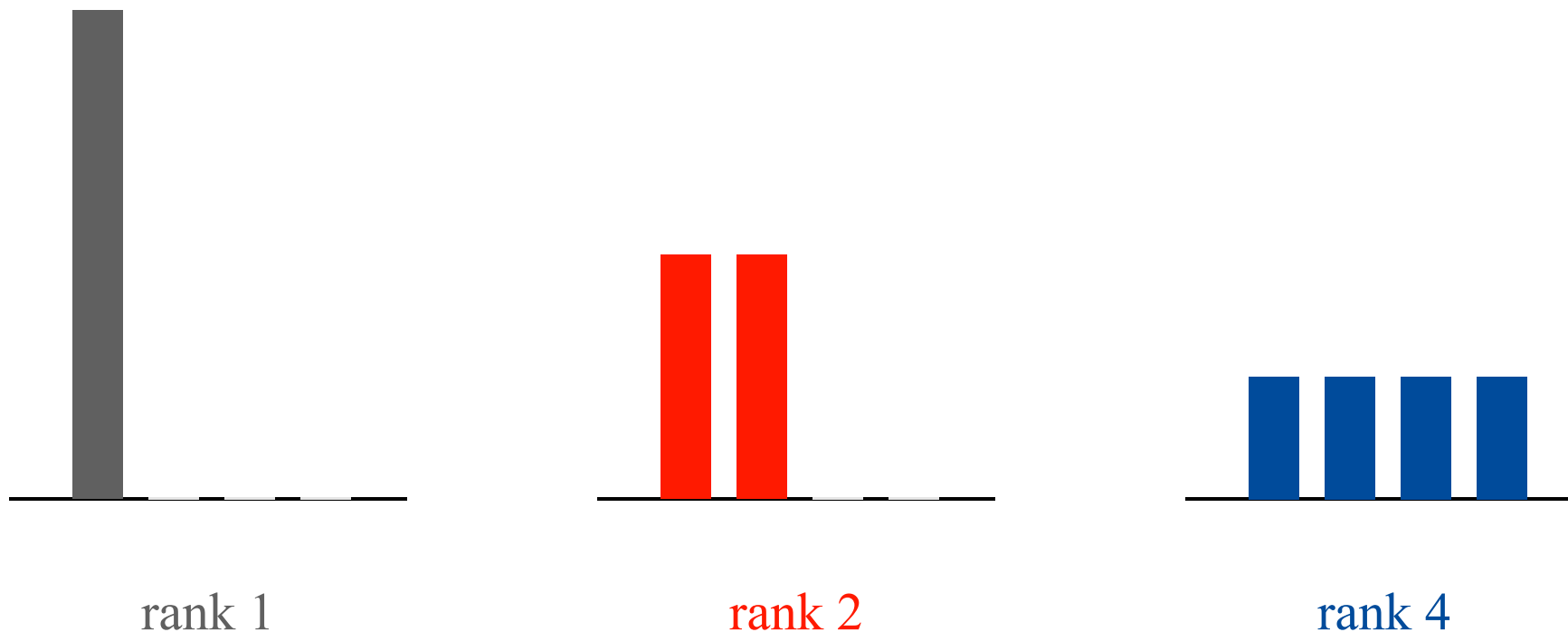
The list $\{\lambda_a\}$ is called the Schmidt spectrum.



Schmidt rank

❖ Schmidt rank counts how many Schmidt probabilities are nonzero

$$r = \#\{a : \lambda_a > 0\}, \quad S_A \leq \log_2 r$$



This rank will later give an upper bound on entanglement.

Spatial scaling distinguishes many-body states

- ❖ The entropy depends on subsystem size:
- ❖ Area law: $S(\ell) = O(1)$ for gapped ground states
- ❖ Log law: $S(\ell) \propto \log \ell$ at one-dimensional critical points
- ❖ Volume law: $S(\ell) \propto \ell$ for typical or highly excited pure states
- ❖ Later: vary L at a middle cut, or vary charge M at fixed L

Integrable spin chain

- ❖ For generic quantum systems we are limited to ground state
- ❖ Integrable systems: we know the spectrum and the wave function!
- ❖ Integrable theory: BAE $\Rightarrow$ algebraic equations that determine the spectrum

E.g. 1D Heisenberg XXXs chain: rapidities u_j satisfies

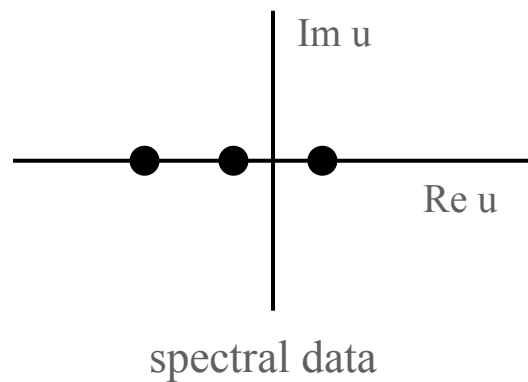
- rational constraints
$$\left(\frac{u + is}{u - is}\right)^L = \prod_{\substack{k=1 \\ k \neq j}}^M \frac{u_j - u_k + i}{u_j - u_k - i}, \quad j = 1, \dots, M.$$
- energy is a rational function of them
$$E = - \sum_{j=1}^M \frac{2s}{u_j^2 + s^2}$$
- wave function known explicitly

- ❖ We can ask questions on the full set of states!

Can Bethe roots predict entanglement?

$$\{u_j\} \text{ or } \{I_j\} \longrightarrow |\psi_{\text{Bethe}}\rangle \longrightarrow \{\lambda_a\}, S_\ell$$

- ❖ Which root patterns minimize the entropy?
- ❖ Which root patterns maximize it?
- ❖ What changes if rapidities move off shell?



Three plus one take-home messages

Minima are simple

Compact or string-organized root patterns give low entanglement.

Maxima are richer

Root spreading helps, but scattering phases and representation matter.

Off shell reveals bounds

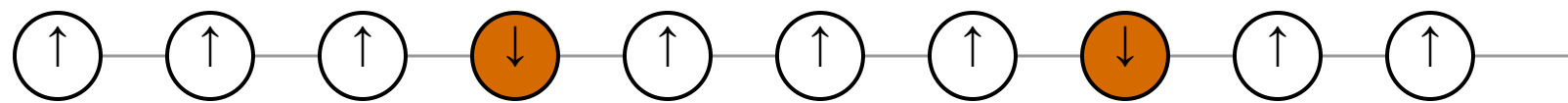
The Bethe-vector manifold exposes rank bounds above and product loci below.

+ New exact method

No need to construct the full wavefunction.

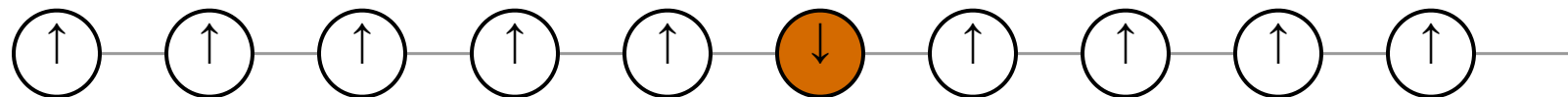
Integrability primer: $XX_{1/2}$ spin chain

❖ A spin chain has one Hilbert space at each site



local Hilbert space: $|\uparrow\rangle, |\downarrow\rangle$

❖ Start from all spins up; a flipped spin is a magnon

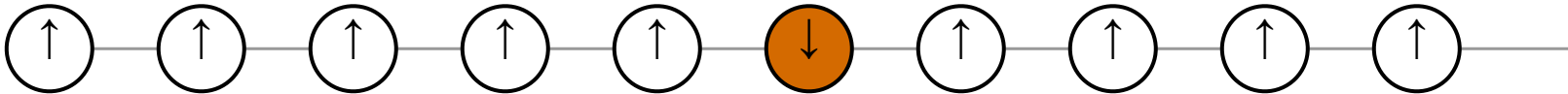


magnon

$$M \text{ magnons} = M \text{ spin flips}$$

❖ Translational invariance: must be a momentum eigenstate!

Bethe ansatz in one picture

- ❖ One magnon is a wave 
magnon

$$\psi(x) \sim e^{ipx} \quad e^{ip} = \frac{u + i/2}{u - i/2}$$

- ❖ Two magnon wave function: superposition of plane waves

$$\Psi(x_1, x_2) = e^{ip_1 x_1 + ip_2 x_2} + S(p_1, p_2) e^{ip_2 x_1 + ip_1 x_2}, \quad x_1 < x_2$$

- ❖ Many magnons are waves plus two-body scattering phases

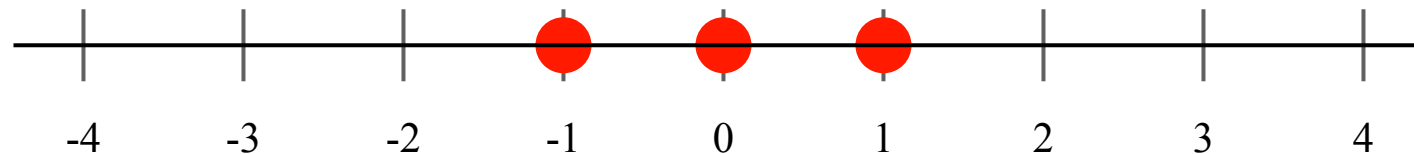
$$\Psi(x_1, \dots, x_M) = \sum_{\sigma \in S_M} A_\sigma \exp \left(i \sum_j p_{\sigma_j} x_j \right)$$

Bethe equations label eigenstates

❖ Rapidities u_j are momentum-like coordinates for the magnons

A Bethe root is a rapidity that solves the periodicity condition.

$$e^{ip_j L} \prod_{k \neq j} S(u_j, u_k) = 1$$



For real roots, Bethe quantum numbers I_j look like occupied momentum slots.

On shell versus off shell

on-shell

Bethe equations satisfied

$\Rightarrow$ **eigenstate**

physical energy eigenstate

off-shell

rapidities free

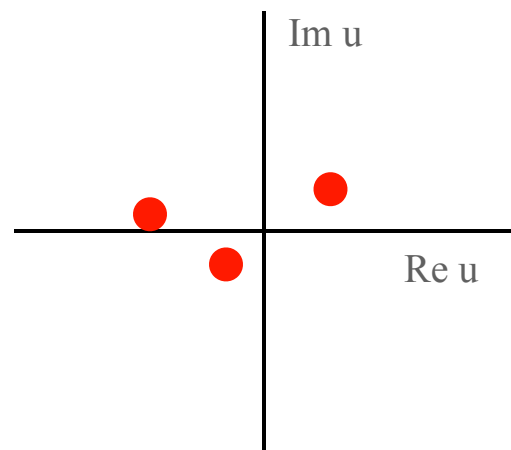
$\Rightarrow$ **Bethe vector**

structured state, not necessarily an
eigenstate

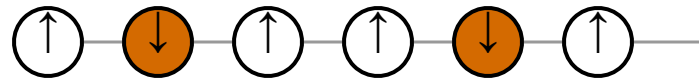
We compare physical eigenstates with the larger Bethe-vector manifold.

Can roots organize entanglement?

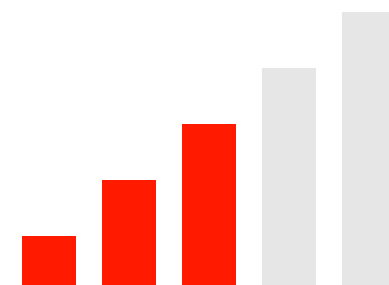
$$\{u_j\} \longrightarrow |\{u_j\}\rangle \xrightarrow{\text{SVD}} \{\lambda_a\} \longrightarrow S_\ell$$



roots



Bethe state



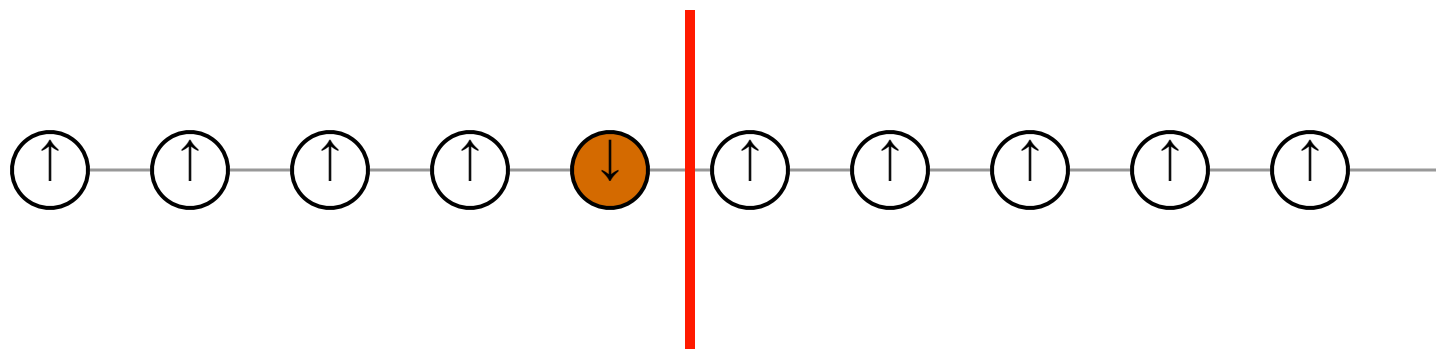
Schmidt spectrum

❖ Which root patterns organize minima, maxima, and scaling?

One magnon across the cut

❖ The magnon is either on the left or on the right

$$\begin{aligned}
 |p\rangle &= \frac{1}{\sqrt{L}} \sum_{x=1}^L e^{ipx} |x\rangle \\
 &= \sqrt{\frac{\ell}{L}} |p\rangle_A |0\rangle_B + e^{ip\ell} \sqrt{\frac{L-\ell}{L}} |0\rangle_A |p\rangle_B
 \end{aligned}$$



left probability

right probability

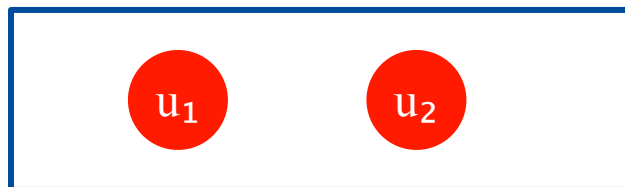
$$S(L, 1; \ell) = -x \log_2 x - (1 - x) \log_2(1 - x), \quad x = \ell/L$$

Cutting an M-magnon state

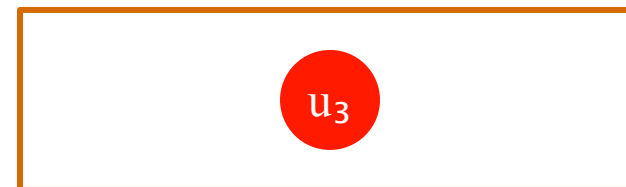
$$|\{u_j\}\rangle = \sum_{\alpha \cup \bar{\alpha} = \{u_j\}} H(\alpha, \bar{\alpha}) |\alpha\rangle_A \otimes |\bar{\alpha}\rangle_B$$

$$|p\rangle = \frac{1}{\sqrt{L}} \sum_{x=1}^L e^{ipx} |x\rangle = \sqrt{\frac{\ell}{L}} |p\rangle_A |0\rangle_B + e^{ip\ell} \sqrt{\frac{L-\ell}{L}} |0\rangle_A |p\rangle_B$$

❖ Each rapidity is assigned to the left or right subchain, e.g.,



subchain A



subchain B

This is why the Schmidt rank is controlled by partitions of the roots.

The method: overlaps to Schmidt data

- ❖ The cut formula resembles Schmidt form

$$|\{u_j\}\rangle = \sum_{\alpha \cup \bar{\alpha} = \{u_j\}} H(\alpha, \bar{\alpha}) |\alpha\rangle_A \otimes |\bar{\alpha}\rangle_B$$

- ❖ But subchain Bethe states are NOT orthonormal $|\Psi\rangle = \sum_{i \in A, j \in B} \Psi_{i,j} |i\rangle_a \otimes |j\rangle_b$
- ❖ Do orthogonalization of the off-shell Bethe state: using off-shell overlap formula

Bethe overlaps $\rightarrow$ orthonormal bases $\rightarrow$ block SVD $\rightarrow$ entropy

This avoids brute-force construction of the full wavefunction and works across the rank-one chains studied here.

Computation strategy

- ❖ On shell: solve Bethe equations and scan physical eigenstates
- ❖ Off shell: optimize the cut entropy directly over complex rapidities

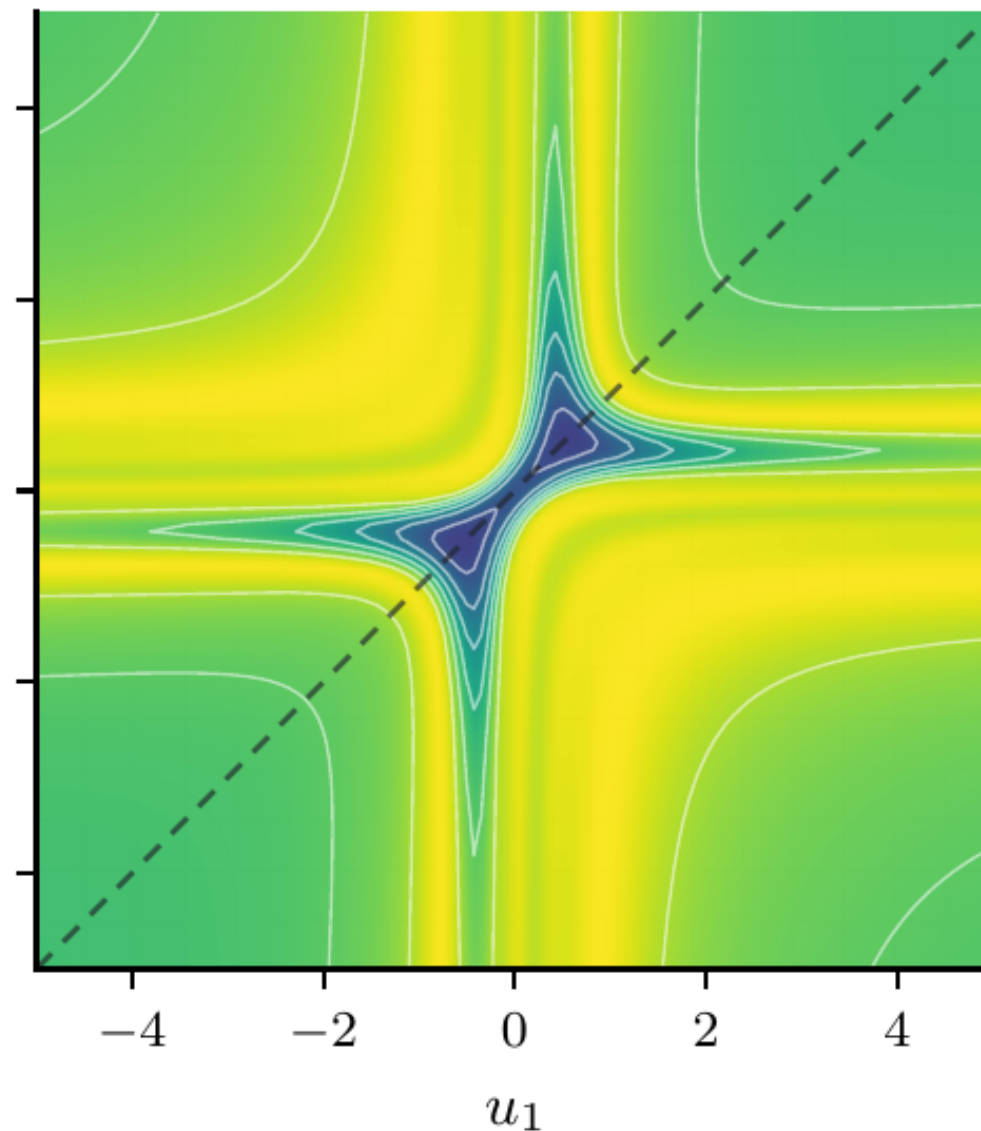
$$u_j = x_j + i y_j, \quad q = (x_1, \dots, x_M, y_1, \dots, y_M)$$

gradient search in these $2M$ real variables

The optimization is over the Bethe-vector manifold, not over arbitrary Hilbert-space vectors.

Offshell: entropy is not a coordinate on root space

- ❖ Parity-related roots have identical Schmidt spectra
- ❖ Off shell, equal-S sets become continuous curves or surfaces

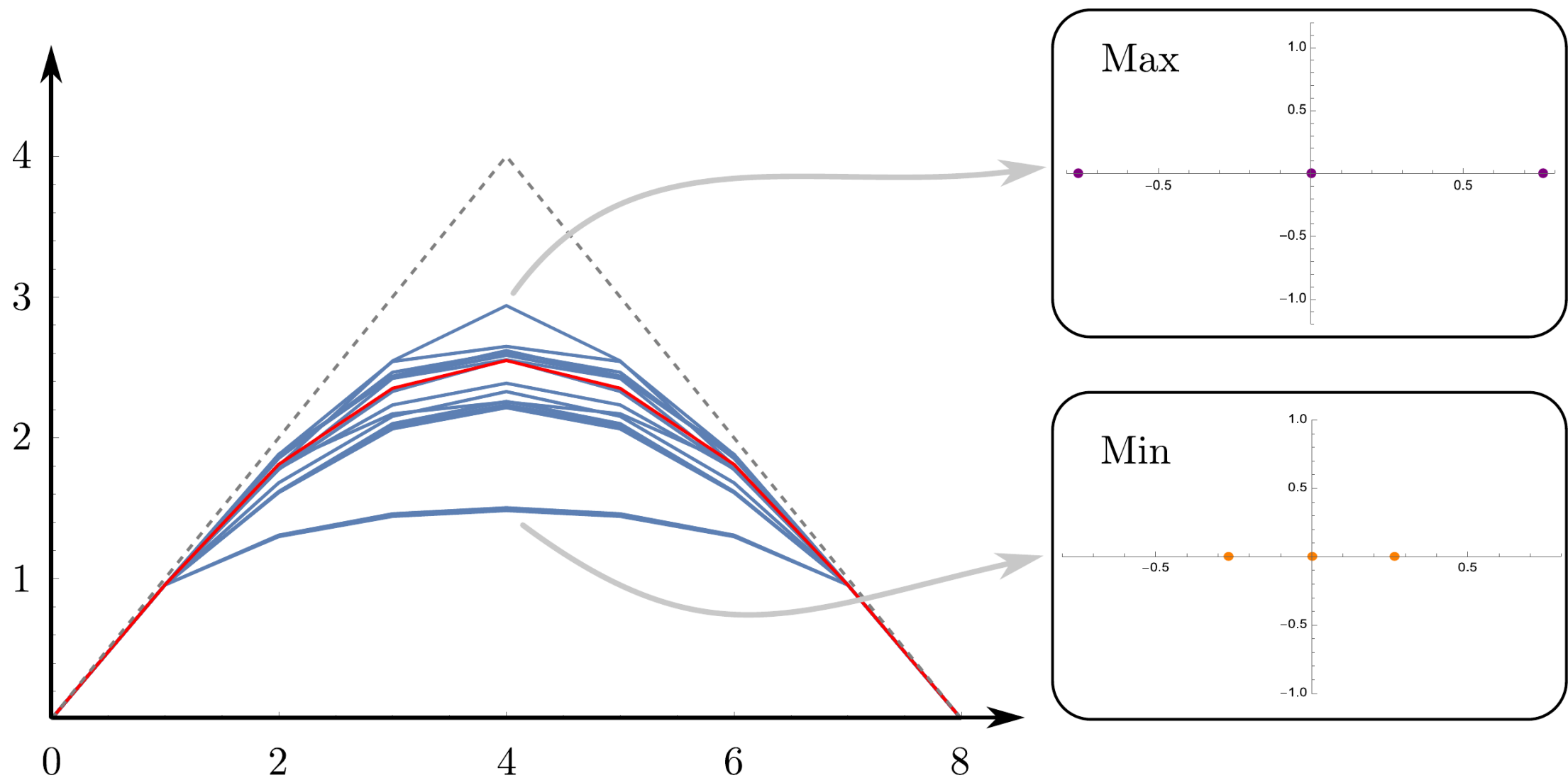


Example real slice: $L=6$, $M=2$. White curves are equal-entropy contours.

Onshell: A broad entropy range at fixed L, M

❖ Example: representative Bethe states at $L=8$, $M=3$

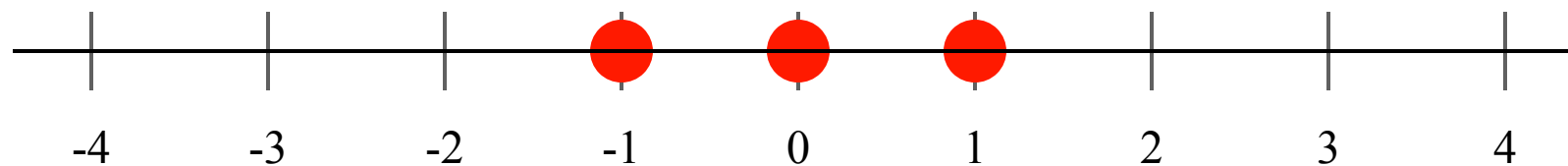
Even at fixed size and magnon number, eigenstates span a broad entanglement range.



Low entropy = compact coherent modes

❖ Minimum entropy is tied to compact, centered Bethe quantum numbers

$$I_j = -\frac{M-1}{2}, \dots, \frac{M-1}{2}$$

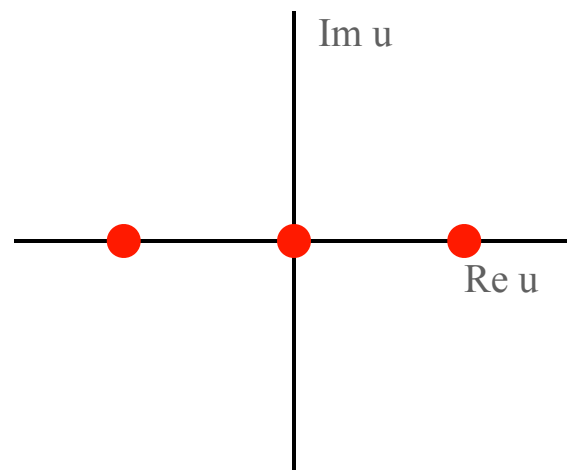


❖ Nearby quantum numbers mean nearby momenta and smoother restricted wavefunctions

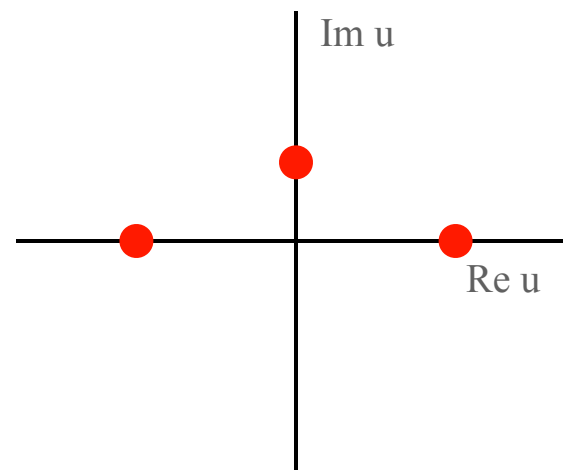
For $XXX_{1/2}$, these are also the lowest-energy states in each fixed-M sector.

High entropy: diagnostic, not rule

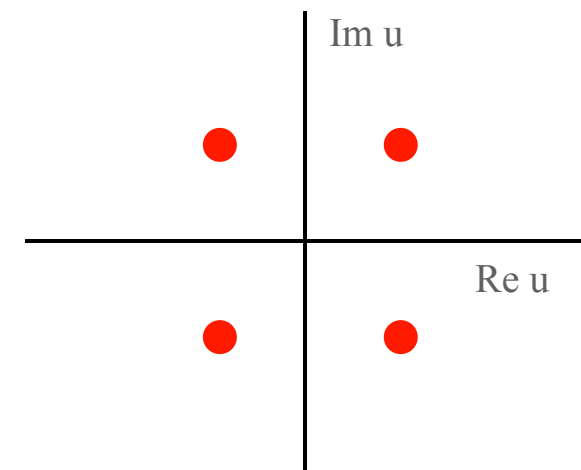
❖ Mode-number spread is useful, but not a variational principle



spread modes
 $\{-2, 0, 2\}$



edge or hole
patterns



complex or
string-like roots

High entanglement depends on mode spread, scattering phases, and the cut.

Off-shell again: an exact bound from cutting

❖ Cutting an M-magnon Bethe state gives two choices for each rapidity

$$\text{rank}_{\text{Schmidt}} \leq 2^M, \quad S_\ell \leq \log_2(2^M) = M$$



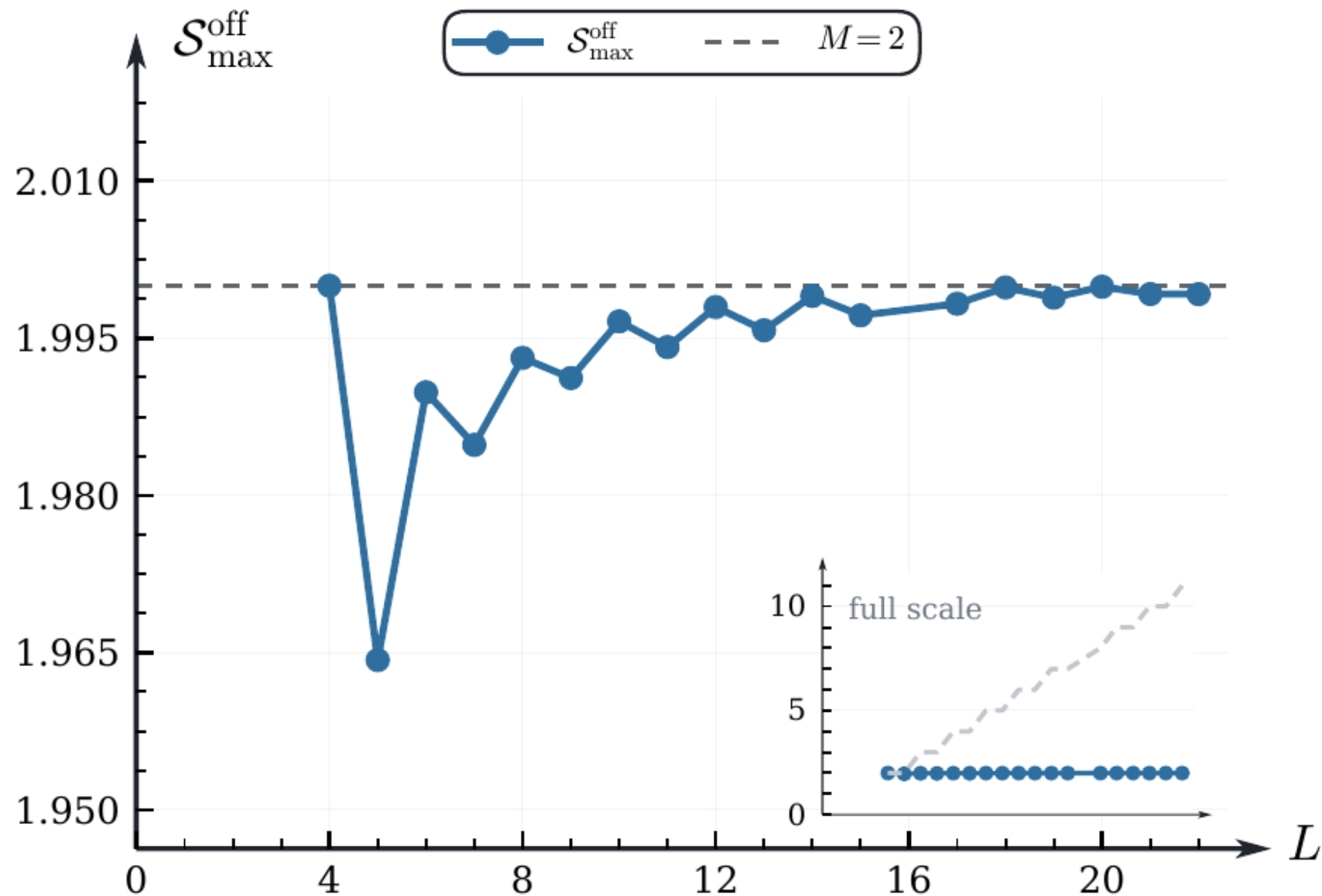
each root chooses A or B

This partition bound is much smaller than the generic Hilbert-space dimension.

Fixed M: saturation

- ❖ At fixed M, the off-shell maximum approaches the partition bound

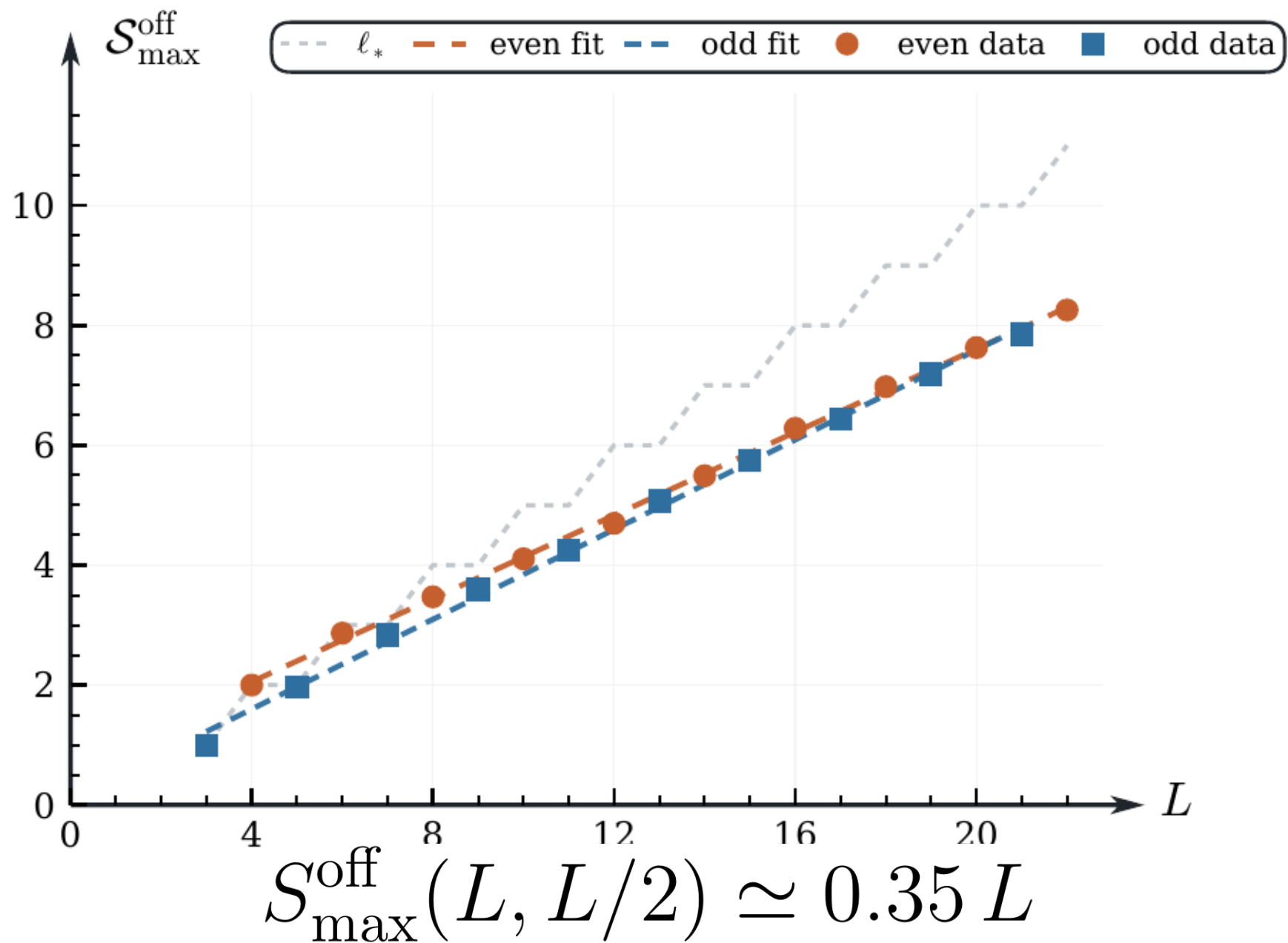
partition bound = the limit from all possible left/right assignments of roots.



Half filling: volume law

- ❖ When M grows with L , the maximum grows linearly

Finite-range fit through $L \leq 22$; extensive but below the local-dimension bound.

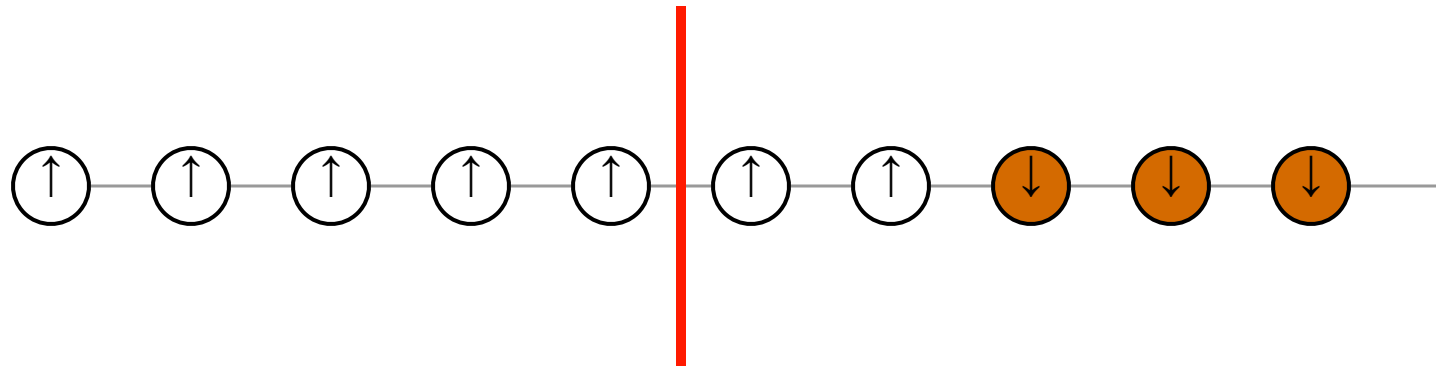


Off-shell minima are product states

❖ The minimum can reach zero entropy after normalization

$$u_j \longrightarrow +\frac{i}{2} \quad \text{or} \quad -\frac{i}{2}$$

$$B(i/2)^M |0\rangle \propto S_{L-M+1}^- \cdots S_L^- |0\rangle$$



$$|\psi\rangle \longrightarrow |\psi_A\rangle \otimes |\psi_B\rangle, \quad S_\ell \longrightarrow 0$$

The Bethe equations exclude much of this low-entanglement boundary.

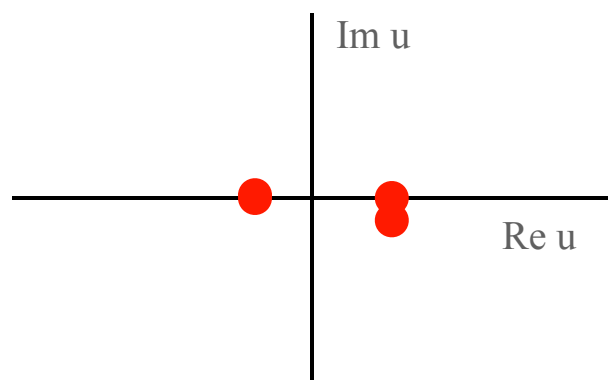
Higher spin changes the allowed roots

❖ Each site has $d=2s+1$ local states

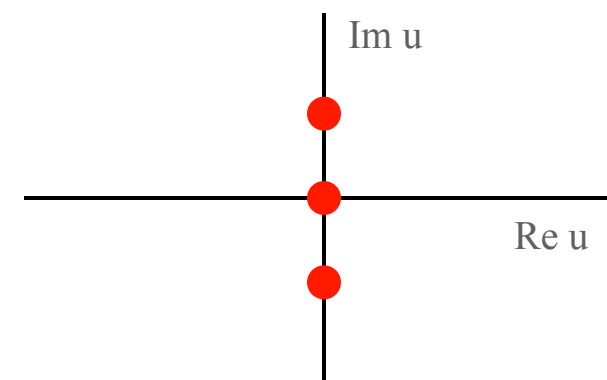
spin 1 : $|0\rangle, |1\rangle, |2\rangle$ spin 3/2 : $|0\rangle, |1\rangle, |2\rangle, |3\rangle$

❖ A site can carry more lowering excitations

❖ Repeated roots and singular strings become physical features



repeated roots

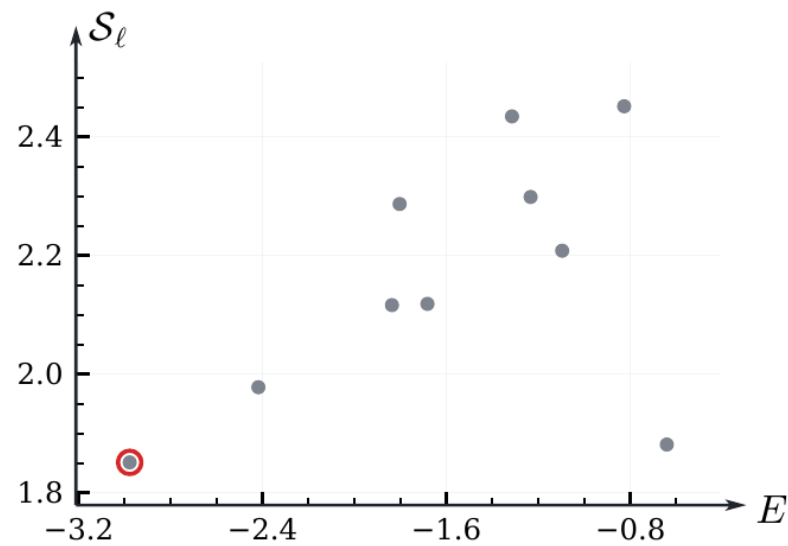


singular string

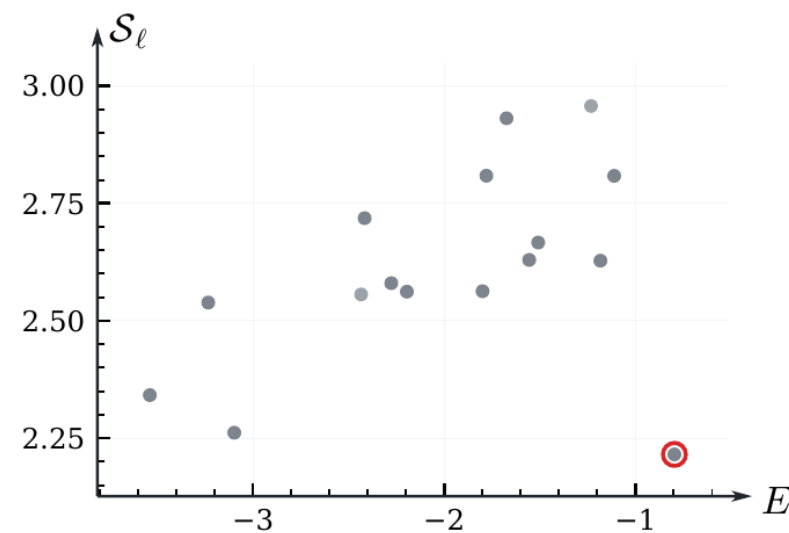
Higher spin separates energy and EE

❖ The lowest-entanglement state is not always the energy minimum

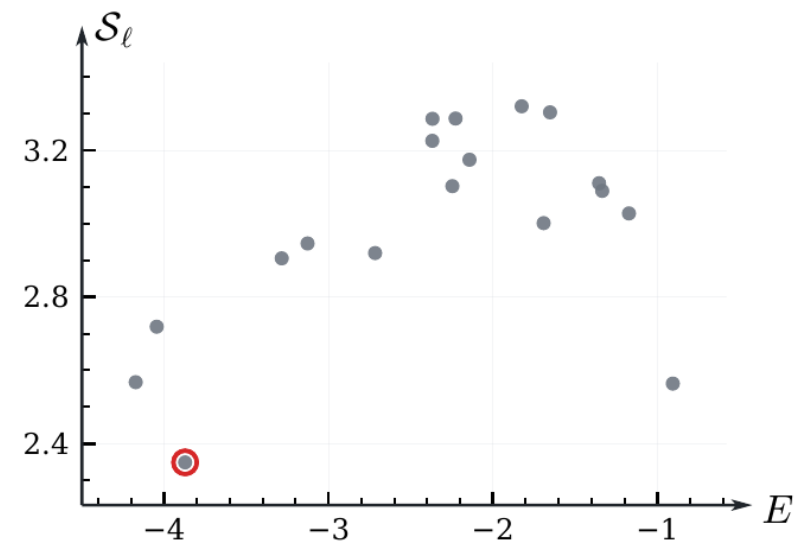
Spin 3/2, L=5, l=2



min EE = min E



min EE = max E



min EE = neither

For higher spin, energy and entanglement orderings separate.

Why non-compact is different

- ❖ Each site has infinitely many local states

non-compact here means the one-site Hilbert space is infinite dimensional.

$$|0\rangle, |1\rangle, |2\rangle, \dots$$

- ❖ But if total magnon number M is fixed, the Hilbert space is finite

fixed charge = keep the total number of excitations M fixed.

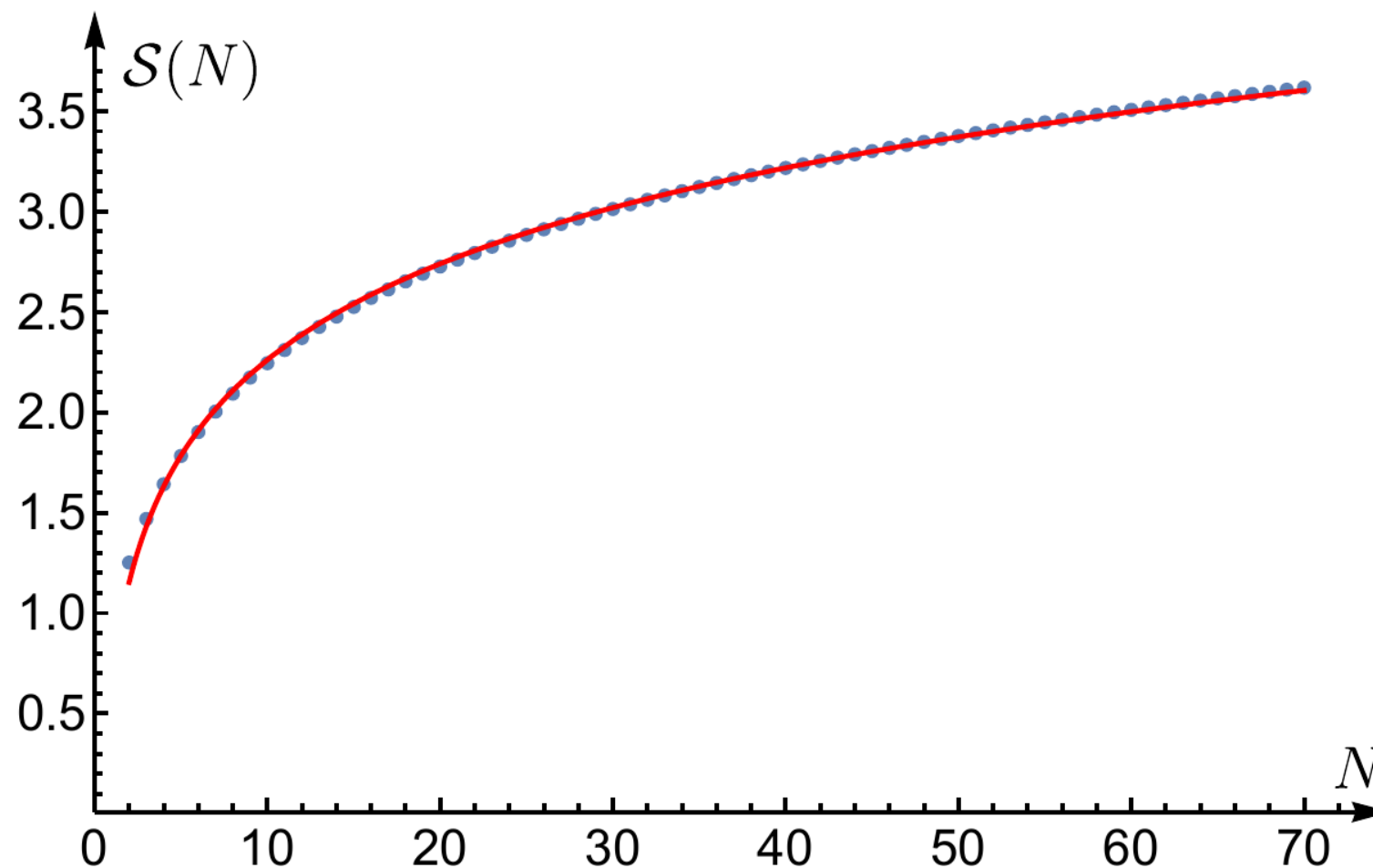
$$R(L, \ell, M) = \sum_{m=0}^M \min \left[\binom{m + \ell - 1}{\ell - 1}, \binom{M - m + L - \ell - 1}{L - \ell - 1} \right]$$

The bound now depends on how a fixed charge can be divided across the cut.

On-shell logarithmic growth

❖ For $L=2$ there is one on-shell state for each M

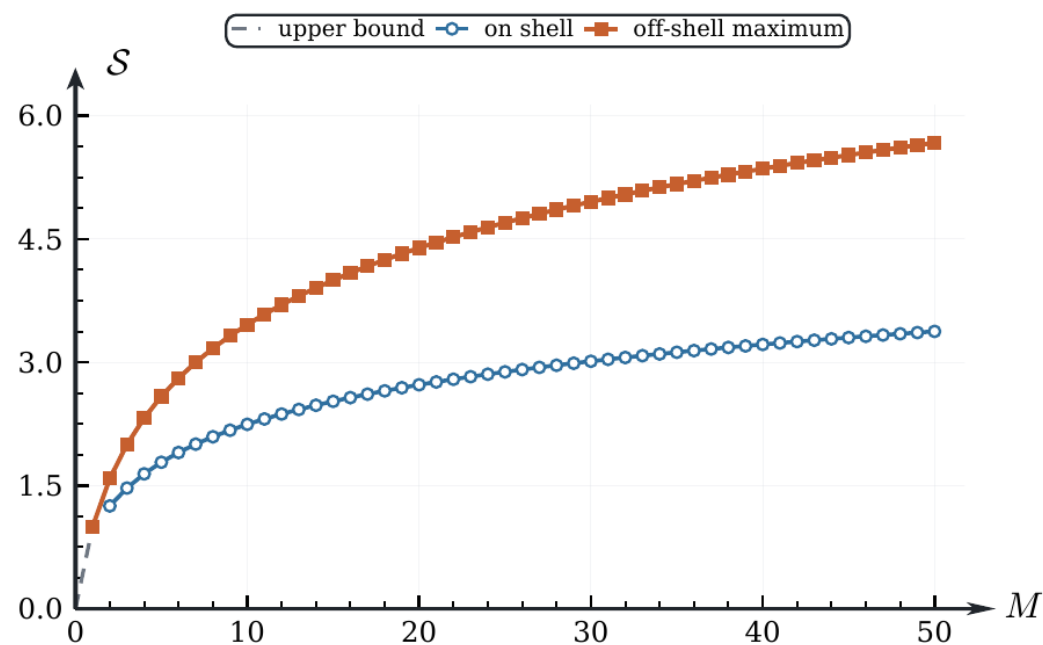
Evidence: entropy grows logarithmically in M over the sampled range.



$$S^{\text{on}}(2, M; 1) \simeq 0.48 \log_2 M + 0.68$$

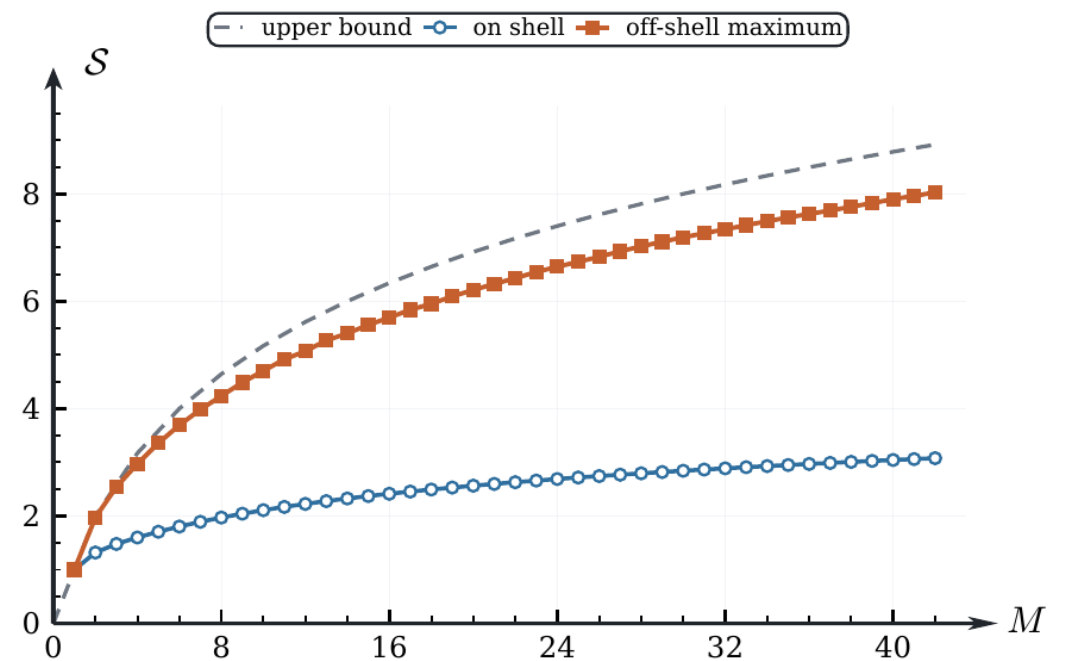
Non-compact off-shell maxima

❖ Off-shell maxima sit much higher than the on-shell values



Two sites, one-site cut: nearly saturates the bound

$$S_\ell \leq \log_2 R(L, \ell, M),$$



Four sites, two-site cut: remains below the bound

$$R(2, 1, M) = M + 1$$

Non-compact off-shell minima

❖ Zero-entropy boundaries also appear, but through a different rapidity mechanism

$$u_j \longrightarrow \frac{i}{2}(2j - 1) \quad \text{or} \quad -\frac{i}{2}(2j - 1)$$

❖ Half-integer imaginary towers localize the fixed charge on one side of the cut

$$|\psi\rangle \longrightarrow |\psi_A\rangle \otimes |\psi_B\rangle, \quad S_\ell \longrightarrow 0$$

Same information-theory endpoint as compact chains; different root-space path.

What survives across the three models

The method is exact

Cuts and Bethe overlaps yield Schmidt data without constructing the full wavefunction.

Minima are structured

Simple on-shell root patterns and exact off-shell product limits organize low entanglement.

Maxima obey bounds

Root patterns are model-dependent; optimized values approach representation-dependent limits.

Outlooks

- ❖ Prove or classify maximal-entanglement root patterns
- ❖ Understand the geometry of equal-entropy root sets
- ❖ Extend to XXZ and higher-rank
- ❖ Beyond bi-partite EE: mutual information etc

Which entanglement features can be inferred directly from root patterns?

谢谢！

Backup

Why off shell matters

- ❖ The same Bethe-state formula exists before imposing Bethe equations
- ❖ Off-shell vectors appear naturally in overlaps, form factors, and correlation functions
- ❖ They define a constrained variational manifold inside the full Hilbert space
- ❖ We can ask how much entanglement the Bethe form can express before eigenstate conditions select states

On shell: spectrum of the Hamiltonian. Off shell: geometry of the Bethe ansatz itself.

How we build the state

- ❖ Coordinate Bethe ansatz gives the state but it is tedious to write it down

$$\Psi(x_1, \dots, x_M) = \sum_{\sigma \in S_M} A_\sigma \exp\left(i \sum_j p_{\sigma_j} x_j\right)$$

- ❖ Algebraic Bethe ansatz gives the state without listing all amplitudes
- ❖ The B-operator is an algebraic magnon creation operator

$$|\{u_j\}\rangle = B(u_1)B(u_2) \cdots B(u_M)|0\rangle$$

- ❖ Simpler to do the cutting!