



中国科学技术大学

University of Science and Technology of China

Recent Progress in S-matrix Bootstrap

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BOOTSTRAP: MANY PLACES, MANY MEANINGS

At its core: Using a small set of inputs or principles to generate, determine, or improve the rest.



COMPUTER SCIENCE

Bootsstrapping (Computer)

Starting a system using a small initial program to load the rest (e.g., OS bootstrapping)



STATISTICS

Bootstrap (Resampling)

Resampling with replacement to estimate errors, confidence intervals, and test statistics



FINANCE

Bootstrap (Finance)

Building a business or portfolio with minimal capital; also used in risk/return resampling



ENTREPRENEURSHIP

Bootstrap (Business)

Start and grow a company with limited resources; self-funding and frugality



BOOTSTRAP (LITERAL)

The strap on a boot used to pull it on

"pull yourself up by your bootstraps" (idiom)



ENGINEERING

Bootstrap Design

Design systems that are self-starting or require minimal external support



ECOLOGY

Bootstrap Population

A new population started by a small number of founders



SPORTS

Bootstrap Training

Build strength or skill progressively using minimal equipment



PHILOSOPHY / METHODOLOGY

Bootstrap Method

Justify or explain something using its own internal structure; self-consistency



GENERAL MEANING

Bootstrap (General)

Any self-help or self-reinforcing process: "pull yourself up by your bootstraps" (idiom)

BOOTSTRAP IN PHYSICS

S-matrix Bootstrap

Constrain scattering amplitudes from general principles



Conformal Bootstrap

Constrain CFT data using symmetry, unitarity, and crossing symmetry



Gauge Bootstrap

Use Wilson lines, positivity or high energy input to constrain gauge theory



Cosmological Bootstrap

Constrain cosmological models using fundamental consistency conditions



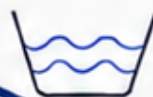
Amplitude Bootstrap

Constrain quantum field theory amplitudes using principles like unitarity and locality



Quantum Mechanics Bootstrap

Constrain quantum systems using consistency of spectra and correlation functions



String Bootstrap

Constrain string theory using consistency conditions and dualities



...



DEVELOPMENT ECONOMICS

Bootstrap Development

Stimulate economic growth through small, initial investments that multiply



MACHINE LEARNING

Bootstrap Aggregating (Bagging)

Train models on bootstrap samples and aggregate predictions



NATURAL LANGUAGE PROCESSING

Bootstrap Finetuning

Improve language models using self-training on model outputs



EDUCATION

Bootstrap Learning

Learn new topics using prior knowledge and iterative self-improvement



PSYCHOLOGY

Bootstrap Confidence

Building confidence through small wins and positive feedback loops

S-matrix principles

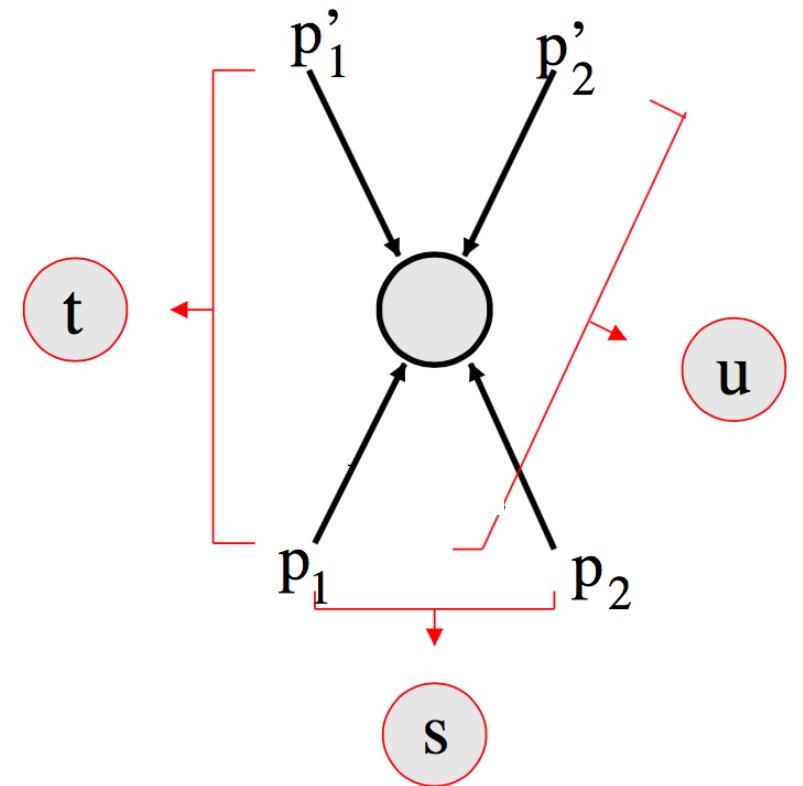
- **Poincare invariance**

$$\mathcal{M}(\{p\}) = \mathcal{M}(s, t), \quad s + t + u = 4m^2$$

- **Locality:** Froissart—Jin—Martin bound

$$\lim_{|s| \rightarrow \infty} \frac{\mathcal{M}(s, t)}{s^2} = 0, \quad t < t_* \quad \text{or} \quad |t| < t_*$$

- **Unitarity:** conservation of probability in quantum theory
- **Causality:** micro-causality $[\mathcal{O}(x), \mathcal{O}(y)] = 0, (x - y)^2 > 0$



Unitarity $S^\dagger S = \mathbf{1}$

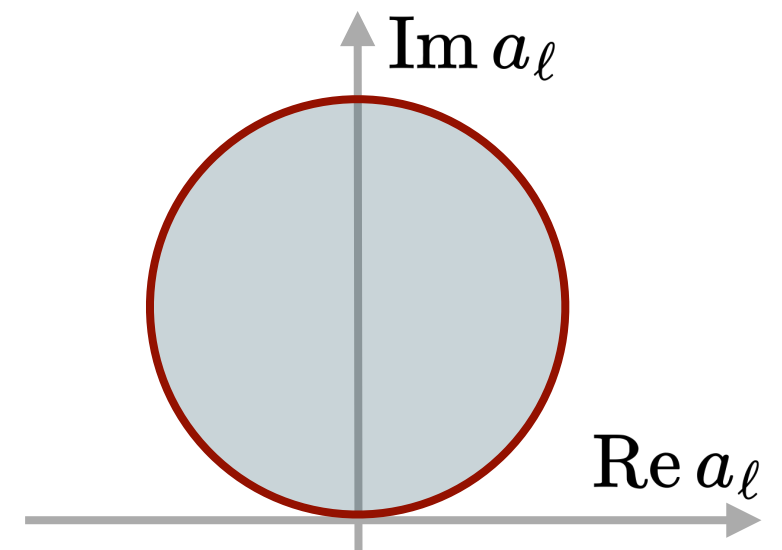
Partial wave expansion

$$\mathcal{M}(s, t) \propto \sum_{\ell \geq 0} (2\ell + 1) a_\ell(s) P_\ell \left(1 + \frac{2t}{s - 4m^2} \right)$$

Partial wave unitarity

$$2 \operatorname{Im} a_\ell \geq (\operatorname{Re} a_\ell)^2 + (\operatorname{Im} a_\ell)^2$$

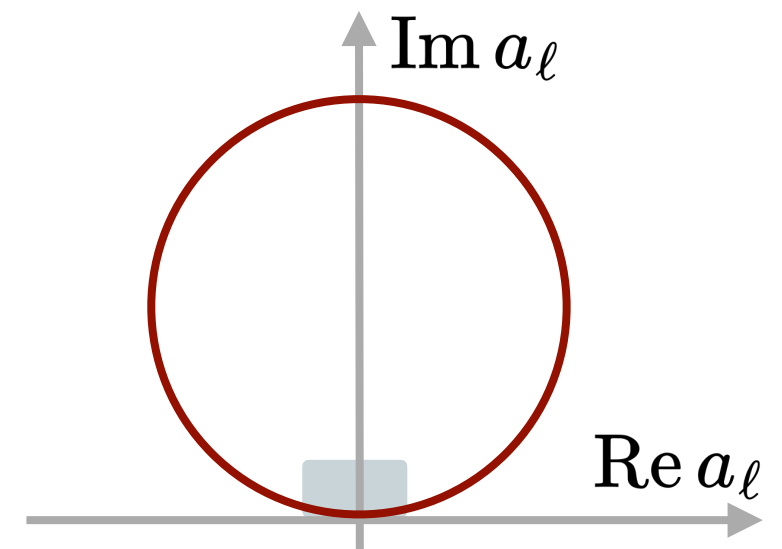
important for strongly coupled theories



For weakly coupled EFTs

$$\operatorname{Im} a_\ell \geq 0 \Rightarrow \text{positivity bounds}$$

already rich bounds on Wilson coefficients



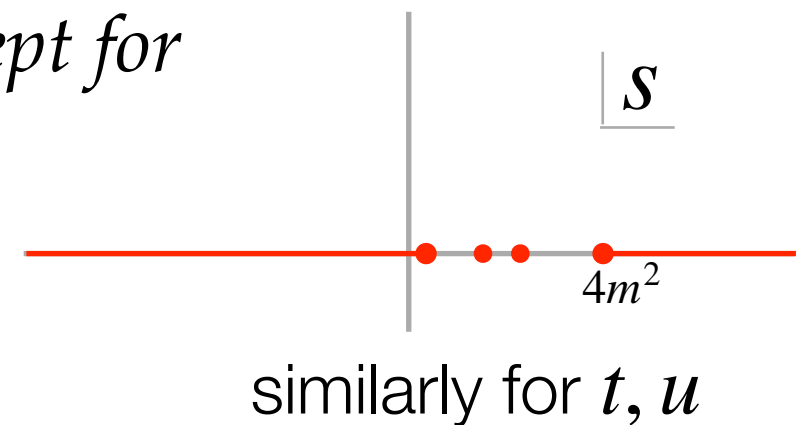
Causality $\Rightarrow$ Analyticity

Maximal analyticity (Mandelstam conjecture)

physical-sheet amplitude is analytic *except for*

$$s \geq 4m^2, \quad t \geq 4m^2, \quad u \geq 4m^2$$

(up to bound state poles)



Martin's **proven** analyticity

physical-sheet amplitude is analytic *within*

$$\text{doubly cut } s \text{ plane, } |t| \leq 4m^2$$

(up to bound state poles)

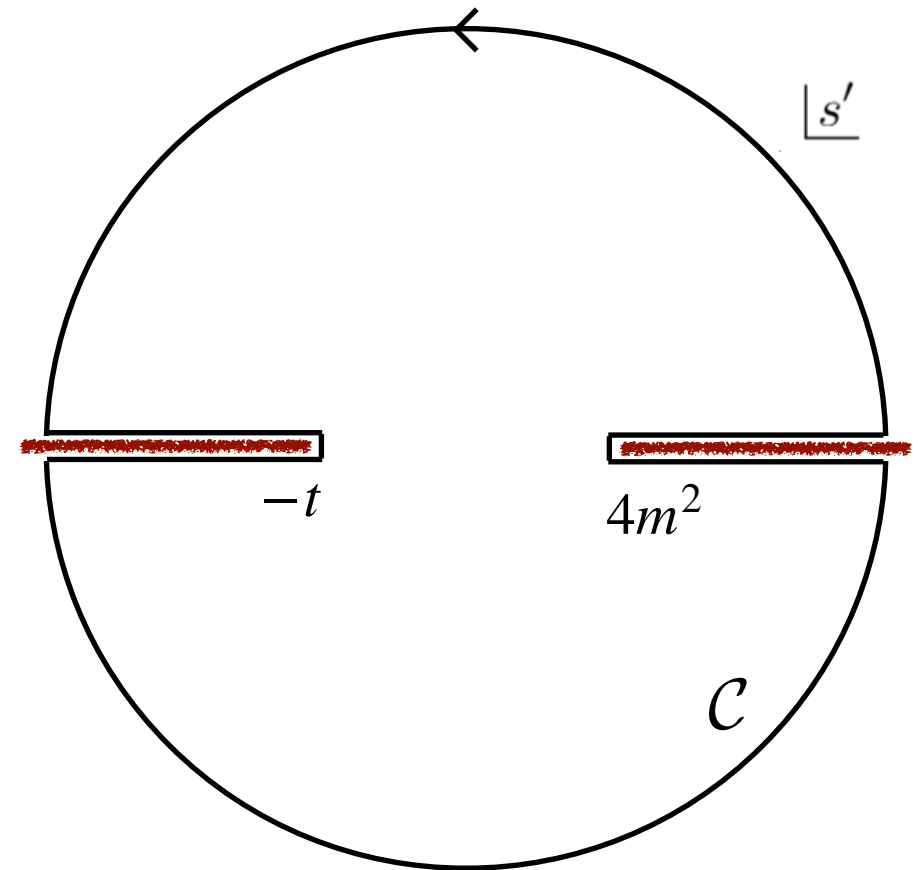
Crossing symmetry: $\mathcal{M}(s, t) = \mathcal{M}(u, t) = \mathcal{M}(t, s)$

Analyticity $\Rightarrow$ Dispersion relation

- Analyticity in complex s plane (fixed t)

$$\mathcal{M}(s, t) = \oint_C \frac{ds'}{2\pi i} \frac{\mathcal{M}(s', t)}{s' - s}$$

- Froissart-Martin bound $\mathcal{M}(|s| \rightarrow \infty, t)/s^2 \rightarrow 0$
- su crossing symmetry $\mathcal{M}(s, t) = \mathcal{M}(s, u)$



Twice-subtracted dispersion relation

$$\mathcal{M}(s, t) = \mathcal{M}(s_0, t_0) + \frac{1}{\pi} \int_{4m^2}^{\infty} d\mu \left(\text{Im} \mathcal{M}(\mu, t) K_{s,t}^{\mu, t_0} + \text{Im} \mathcal{M}(\mu, t_0) K_{t,t_0}^{\mu, s_0} \right)$$

$$-4m^2 < t < 4m^2$$

$$K_{s,t}^{\mu, t_0} = \frac{1}{\mu - s} + \frac{1}{\mu - 4m^2 + s + t} - \frac{1}{\mu - t_0} - \frac{1}{\mu - 4m^2 + t + t_0}$$

analytical function is fixed by singularities and asymptotics!

S-matrix bootstrap: Dual vs Primal

S-matrix bootstrap: use basic principles to constrain theory space

S-matrix theory space

Dual bootstrap

- rule out unviable space
- eliminate amplitudes

EFT setup:
positivity bounds

de Rham, Kundu, Reece, Tolley, SYZ,
Snowmass White Paper, 2203.06805

Primal bootstrap

- rule in viable space
- construct amplitudes

Kruczenski, Penedones, van Rees,
Snowmass White paper, 2203.02421

Duality gap: ideally, it shrinks to zero
Many models worked out in both approaches

Primal S-matrix Bootstrap

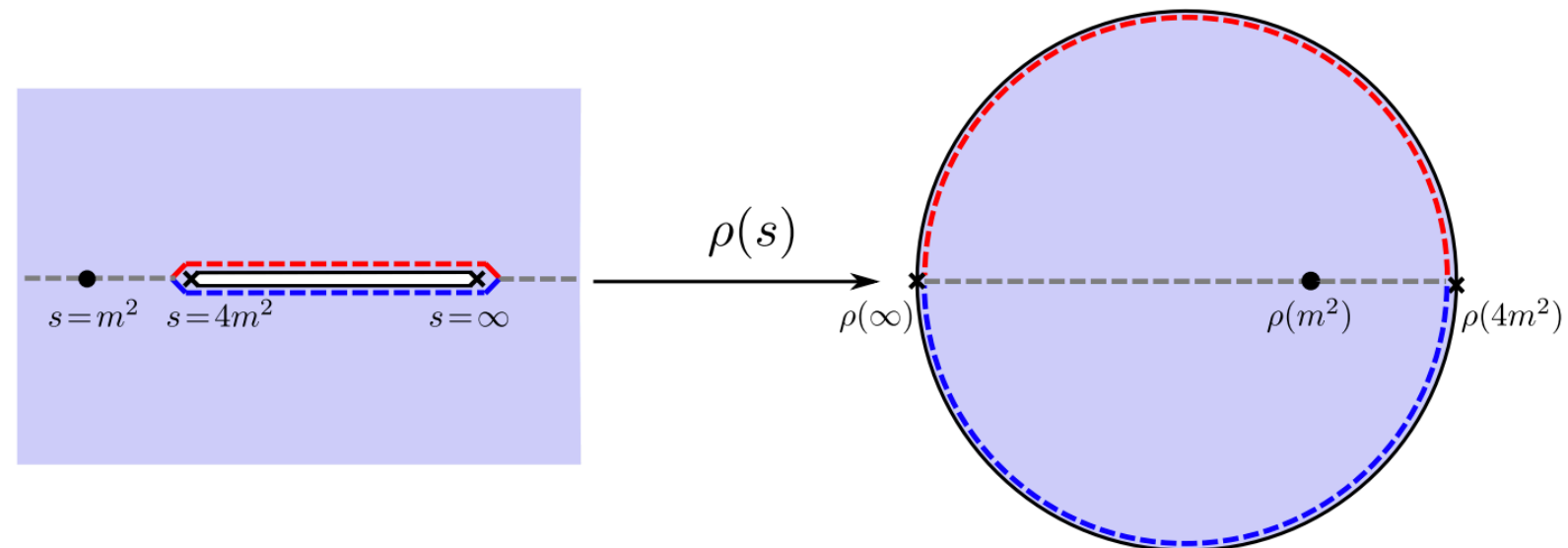
Primal bootstrap method (1)

Paulos, Penedones, Toledo, van Rees, Vieira, 1708.06765

Conformal mapping

$$s \rightarrow \rho(s)$$

$$\rho(x) = \frac{\sqrt{4m^2 - x_0} - \sqrt{4m^2 - x}}{\sqrt{4m^2 - x_0} + \sqrt{4m^2 - x}}$$



Build all possible amplitudes

(crossing symmetric)

$$\mathcal{M}(s, t) = \sum_{a+b+c \leq N_{\max}} d_{abc} \rho^{(a)}(s) \rho^{(b)}(t) \rho^{(c)}(u)$$

parametrize amplitudes

Subject to

$$s + t + u = 4m^2 \rightarrow \text{polynomial constraints on } \rho(s), \rho(t), \rho(u)$$

Primal bootstrap method (2)

Paulos, Penedones, Toledo, van Rees, Vieira, 1708.06765

Partial wave projection

$$a_\ell(s) \sim \int_{-(s-4m^2)}^0 dt P_\ell \left(1 + \frac{2t}{s-4m^2} \right) \mathcal{M}(s, t)$$

Impose partial wave unitarity

$$\begin{array}{ccc} \mathbf{S}_\ell \mathbf{S}_\ell^\dagger = \mathbf{1} & \xrightarrow{\quad} & \mathbf{S}_\ell \mathbf{S}_\ell^\dagger \leq \mathbf{1} \quad \text{or} \quad |1 + iS_m a_\ell(s)| \leq 1 \\ \mathbf{S}_\ell = \begin{pmatrix} S_\ell & \cdots \\ \vdots & \ddots \end{pmatrix} & & \text{physical region } s \geq 4m^2 \\ & & S_m = \sqrt{(s-4m^2)/s} \end{array}$$

$$\xrightarrow{\text{for efficient numerics}} \begin{pmatrix} 1 - \frac{1}{2} S_m \text{Im } a_\ell(s), & S_m^{1/2} \text{Re } a_\ell(s) \\ S_m^{1/2} \text{Re } a_\ell(s), & 2 \text{Im } a_\ell(s) \end{pmatrix} \succeq 0 \quad s \geq 4m^2$$

$a_\ell = (\text{linear in decision variables } d_{abc}) \Rightarrow$ semi-definite program (SDP)
use high precision SDPB

Primal bootstrap results

Bounds on amplitude coefficients

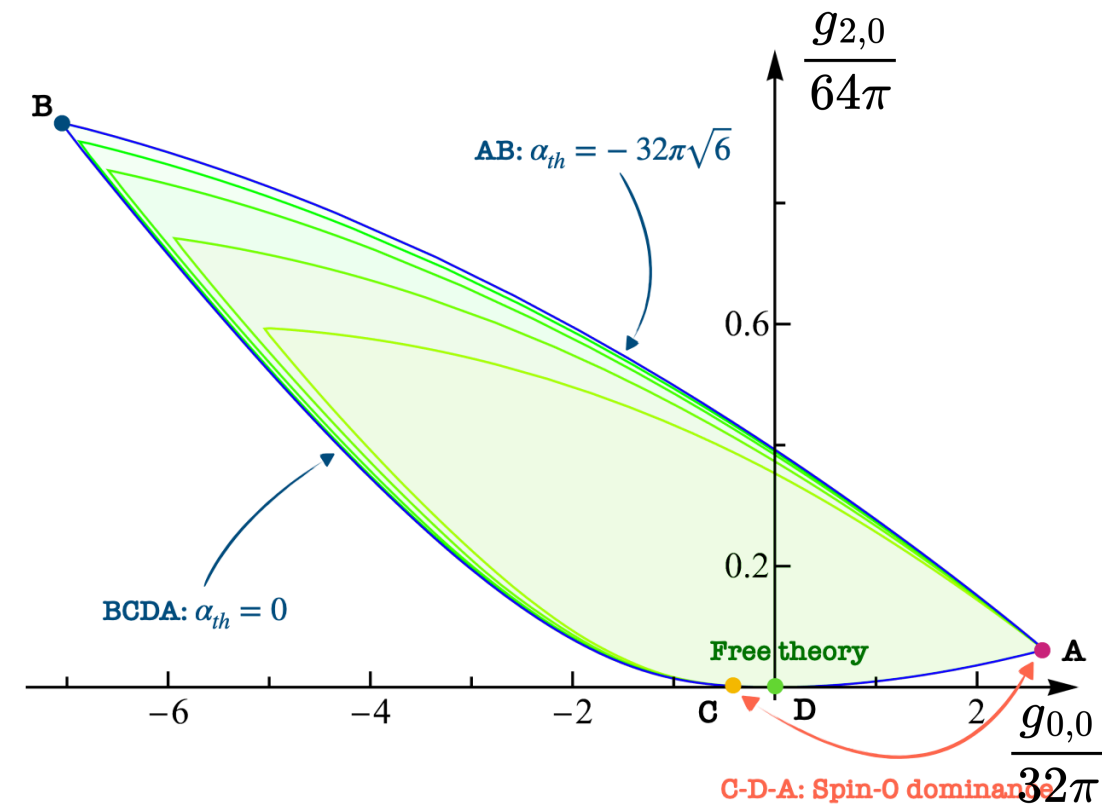
non-perturbative observables

$$\mathcal{M}(s, t) = \sum_{i,j=0} g_{i,j} (s - s_*)^i (t - t_*)^j$$

$$s_* = t_* = 4m^2/3$$

Many other results, for example:

- Effects of symmetries
- Extra UV information
- Massless fields
- Photon EFTs
- QCD bootstrap
- Where is string theory
- Bounds on jets scattering
-



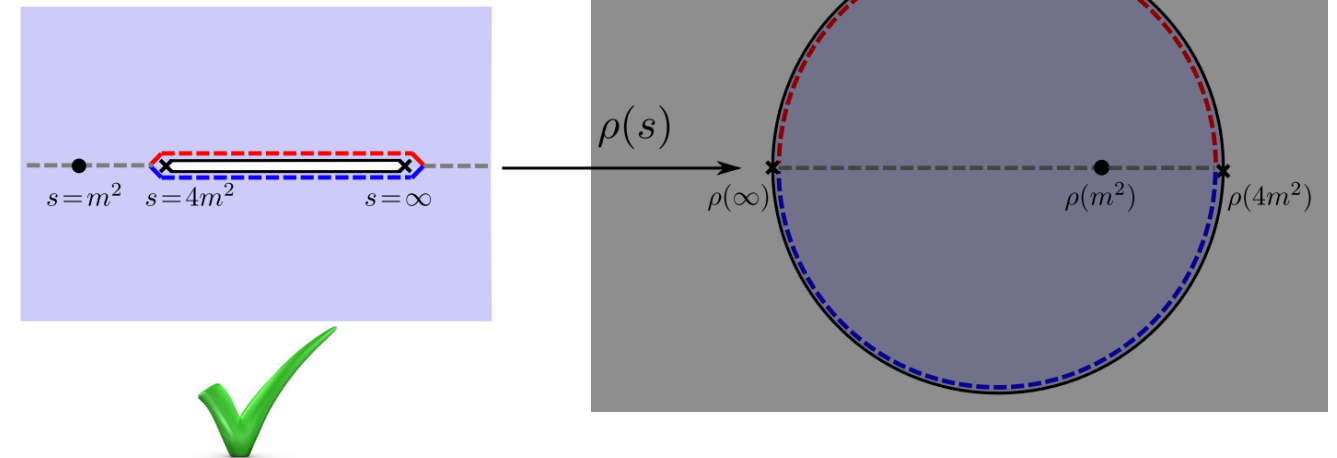
Paulos, Penedones, Toledo, van Rees, Vieira,
1708.06765, 1810.12849
Doroud, Miro, 1804.04376
Guerrieri, Penedones, Vieira, 1810.12849, 2102.02847
Miro, Guerrieri, Hebbar, Penedones, Vieira, 1906.08098
Hebbar, Karateev, Penedones, 2011.1708
Miro, Guerrieri, Gumus, 2210.01502
Haring, Karateev, Meineri, Penedones, 2211.05795
He, Kruczenski, 2309.12402

New primal bootstrap method (1)

de Rham, Tolley, Wang, SYZ, 2506.22546

Use Mandelstam variables
to do primal bootstrap

Via dispersion relation



$$\text{Re}\mathcal{M}(s, t) = \mathcal{M}(s_0, t_0) + \mathcal{P} \int_{4m^2}^{\infty} \frac{d\mu}{\pi} \left(\text{Im}\mathcal{M}(\mu, t) K_{s,t}^{\mu, t_0} + \text{Im}\mathcal{M}(\mu, t_0) K_{t, t_0}^{\mu, s_0} \right)$$

Benefits:

- Only use proven Martin's analyticity
- Parametrize amplitudes in physical region
- Can control UV behavior of amplitude $\mathcal{M}(s \rightarrow \infty, t) \sim s^{1+\eta(t)}$
- Can add spin-2 bound states
- Fast numerical convergence

$$\text{Re}\mathcal{M}(s, t) = \mathcal{M}(s_0, t_0) + \mathcal{P} \int_{4m^2}^{\infty} \frac{d\mu}{\pi} \left(\text{Im}\mathcal{M}(\mu, t) K_{s,t}^{\mu, t_0} + \text{Im}\mathcal{M}(\mu, t_0) K_{t,t_0}^{\mu, s_0} \right)$$

New primal bootstrap method (2)

Partial wave expansion in physical region $s \geq 4m^2$

$$\text{Im}\mathcal{M}(s, t) = 16\pi \sum_{\text{even } \ell \geq 0} (2\ell + 1) \text{Im}a_{\ell}(s) P_{\ell} \left(1 + \frac{2t}{s - 4m^2} \right)$$

Parametrize $\text{Im} a_{\ell}(s)$ in physical region ($s \geq 4m^2$)



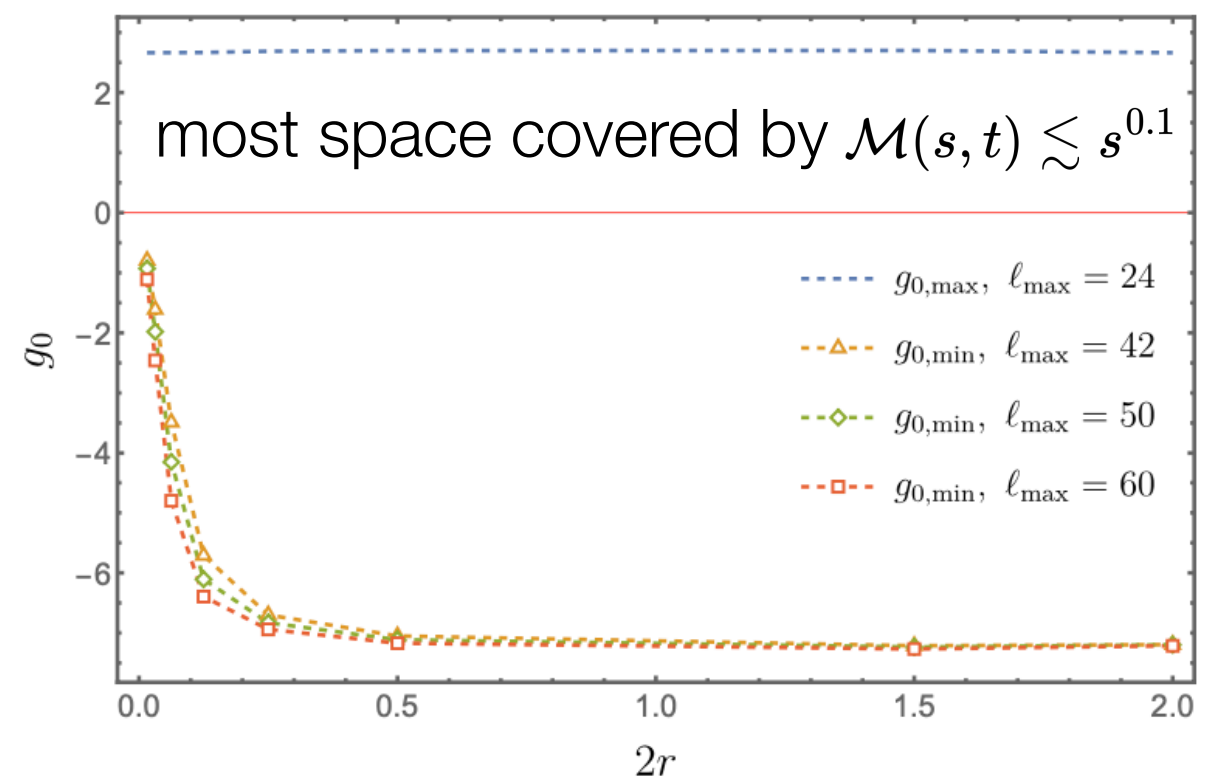
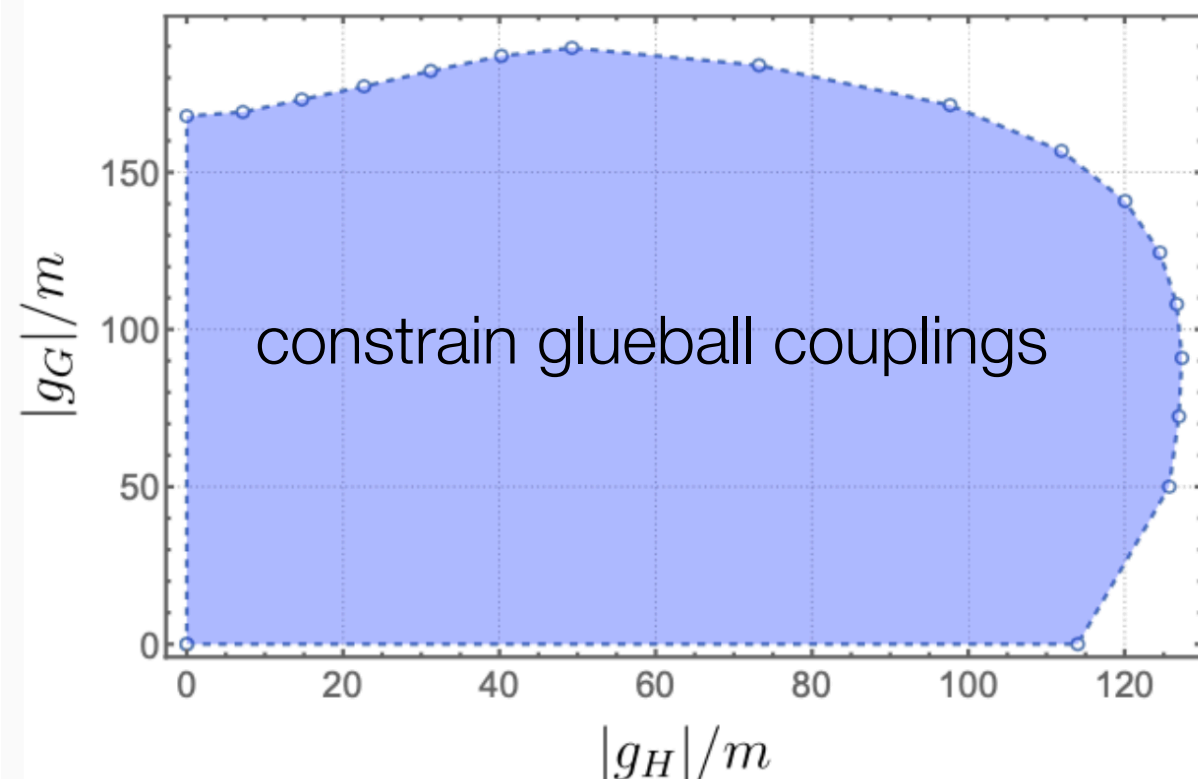
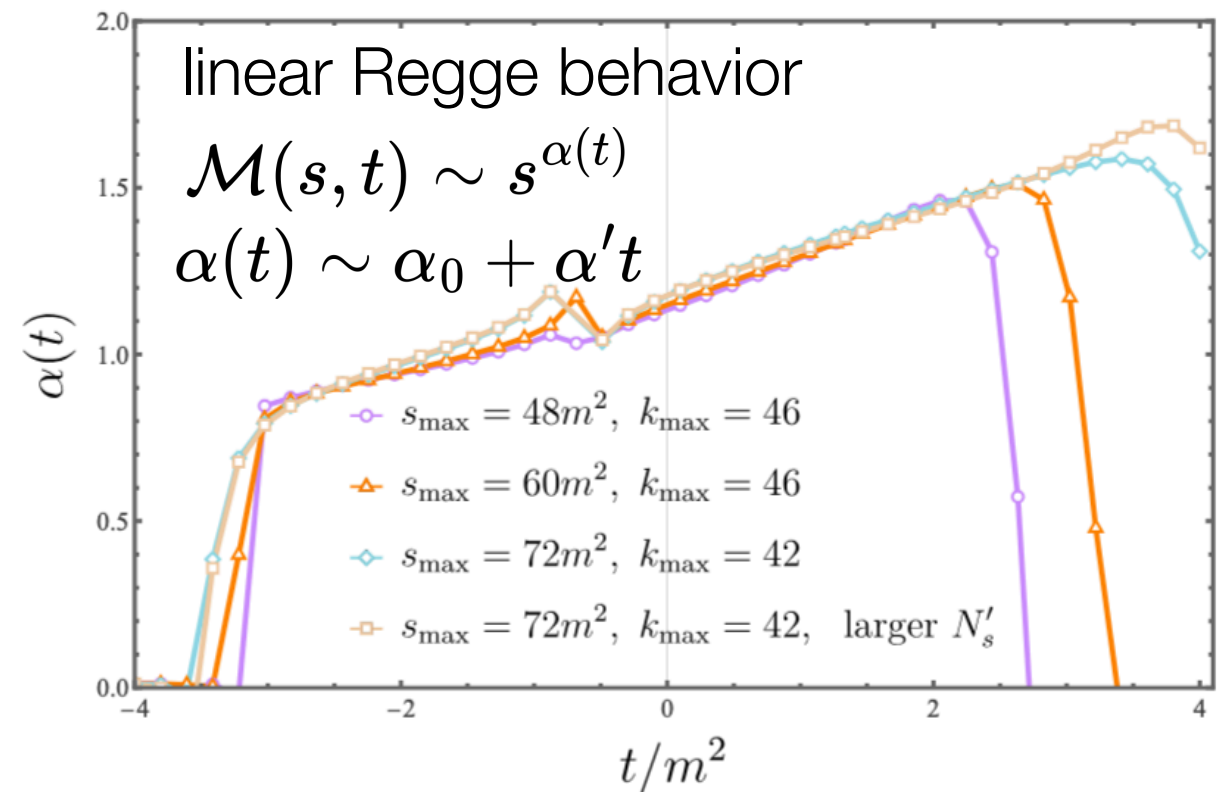
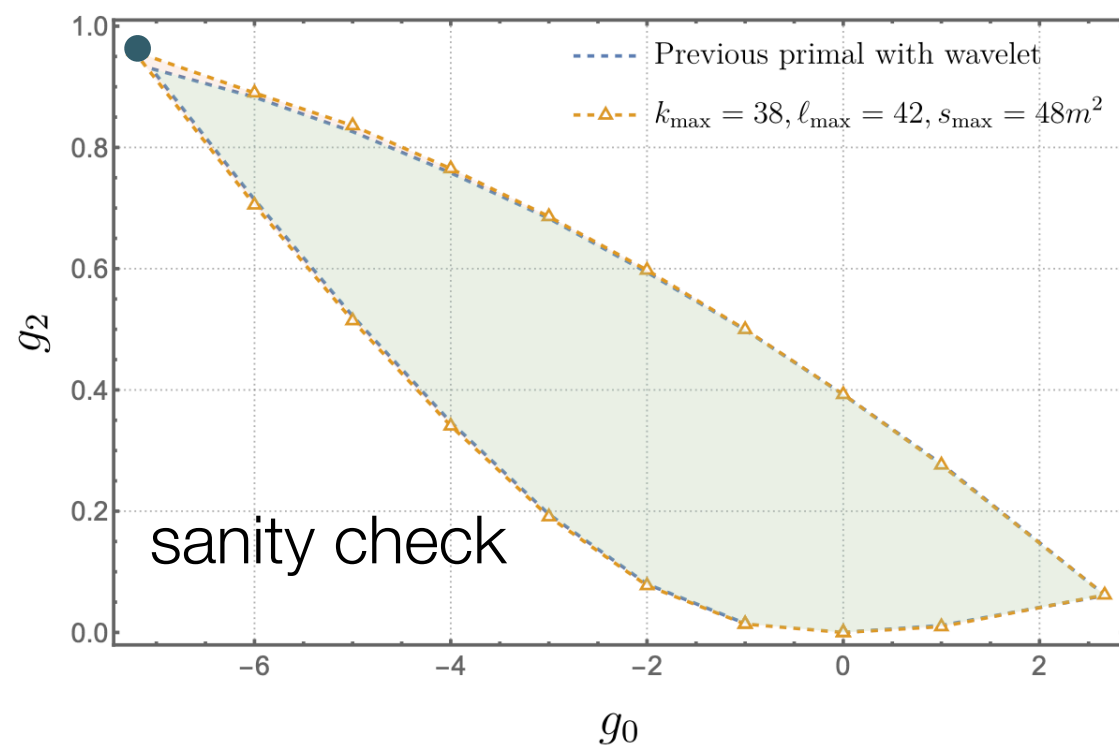
$$\text{Im} a_{\ell}(s) = \left(\frac{s - 4m^2}{s} \right)^{\ell - \frac{1}{2}} \sum_{k=0}^{k_{\max}} c_{\ell,k} P_k \left(8m^2/s - 1 \right)$$

$s \geq 4m^2$ threshold divergence decision variables Legendre polynomials

Solve for $\text{Re} a_{\ell}(s)$ via dispersion relation: discretize t

New primal bootstrap results

de Rham, Tolley, Wang, SYZ, 2506.22546



Properties of extremal theories

CB: Blue shift of resonances

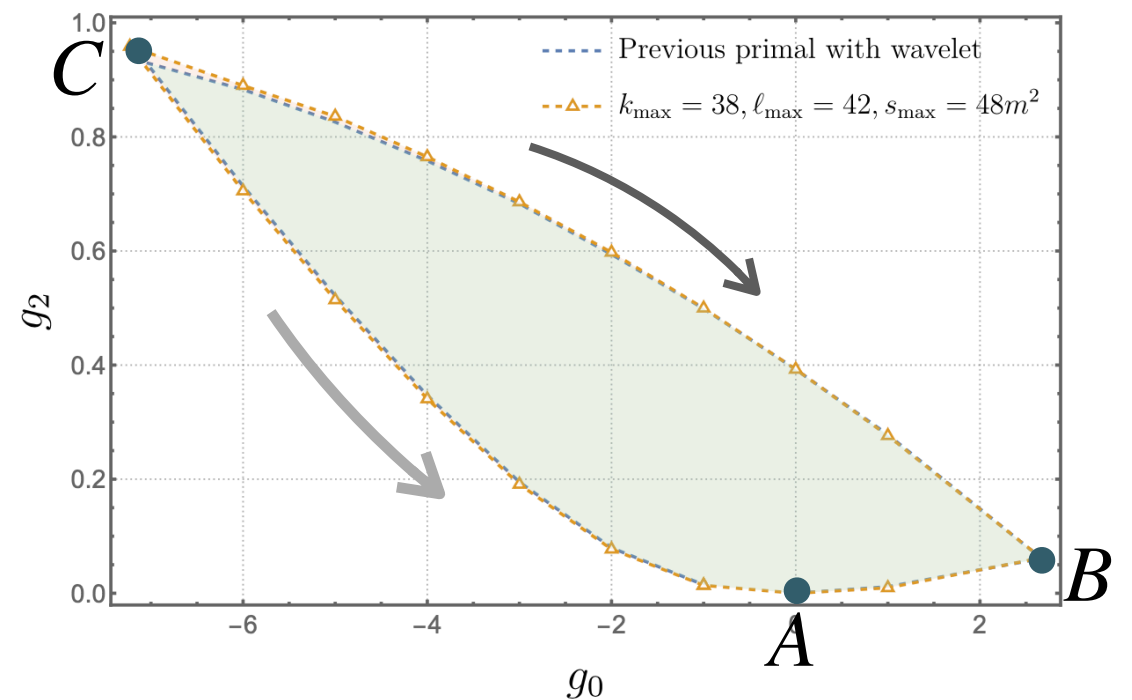
CA: Decrease of couplings

AB: Asymptotic free $-\phi^4$

Correia, Georgoudis, Guerrieri, 2506.04313

de Rham, Tolley, Wang, SYZ, 2506.22546

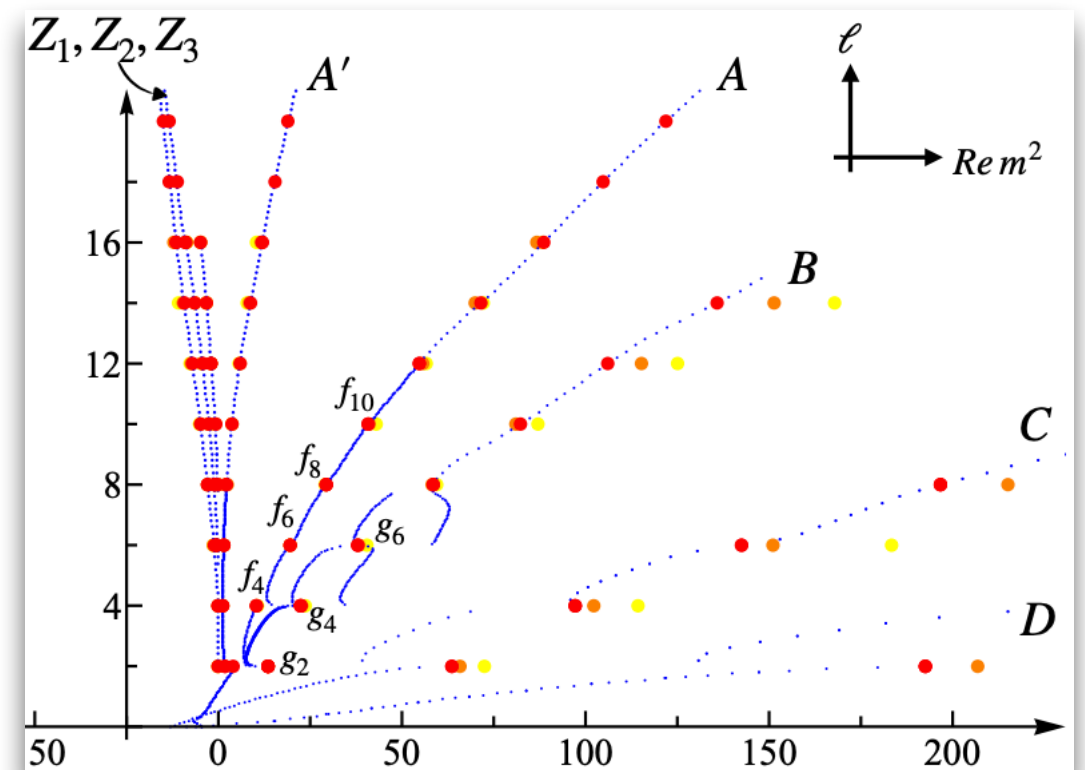
Miro, Guerrieri, Gumus, 2605.06613



Regge phenomenology

- Pomeron trajectory
- Subleading Regge trajectories
- Regge cuts (?)
- Froissart-Martin bound near **C**

$$\mathcal{M} \sim s^{1.7 \sim 1.8}$$



QCD bootstrap

Use high-energy pQCD

$$\chi\text{PT}_{IR} + \text{pQCD}_{UV} \rightarrow S_{\pi\pi}$$

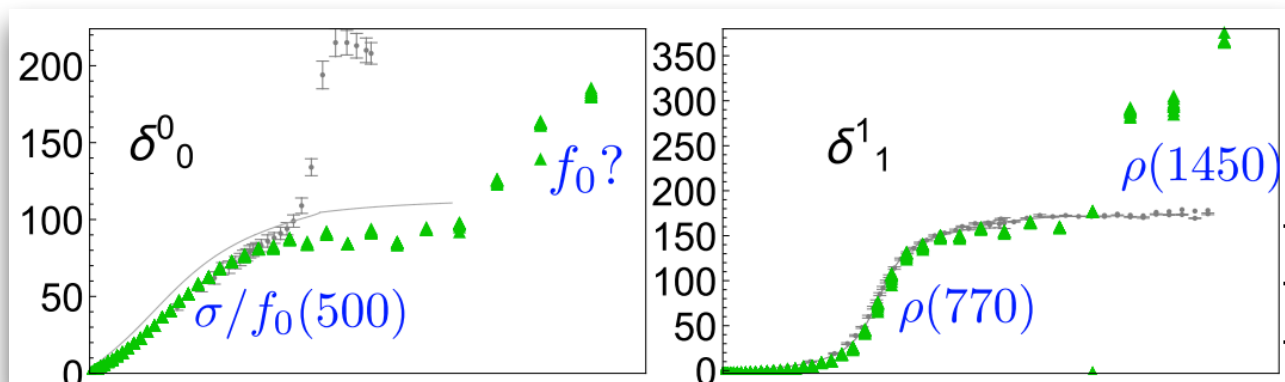
via finite-energy sum rules • Form factors $\mathcal{F}$
 He, Kruczenski, 2023 • Current correlators ρ

Extended unitarity

Karateev, Kuhn, Penedones, 1912.08940

$$\begin{pmatrix} \langle \text{in} |_{P', I, \ell} & | \text{in} \rangle_{P, I, \ell} & | \text{out} \rangle_{P, I, \ell} & \mathcal{O}_{P, I, \ell} | 0 \rangle \\ \langle \text{out} |_{P', I, \ell} & 1 & S_\ell^I(s) & \mathcal{F}_\ell^I(s) \\ \langle 0 | \mathcal{O}_{P', I, \ell}^\dagger & S_\ell^{I*}(s) & 1 & \mathcal{F}_\ell^{I*}(s) \\ & \mathcal{F}_\ell^{I*}(s) & \mathcal{F}_\ell^I(s) & \rho_\ell^I(s) \end{pmatrix} \succeq 0$$

Add Watson's equation $S_\ell^I = \mathcal{F}_\ell^I / \mathcal{F}_\ell^{I*}$

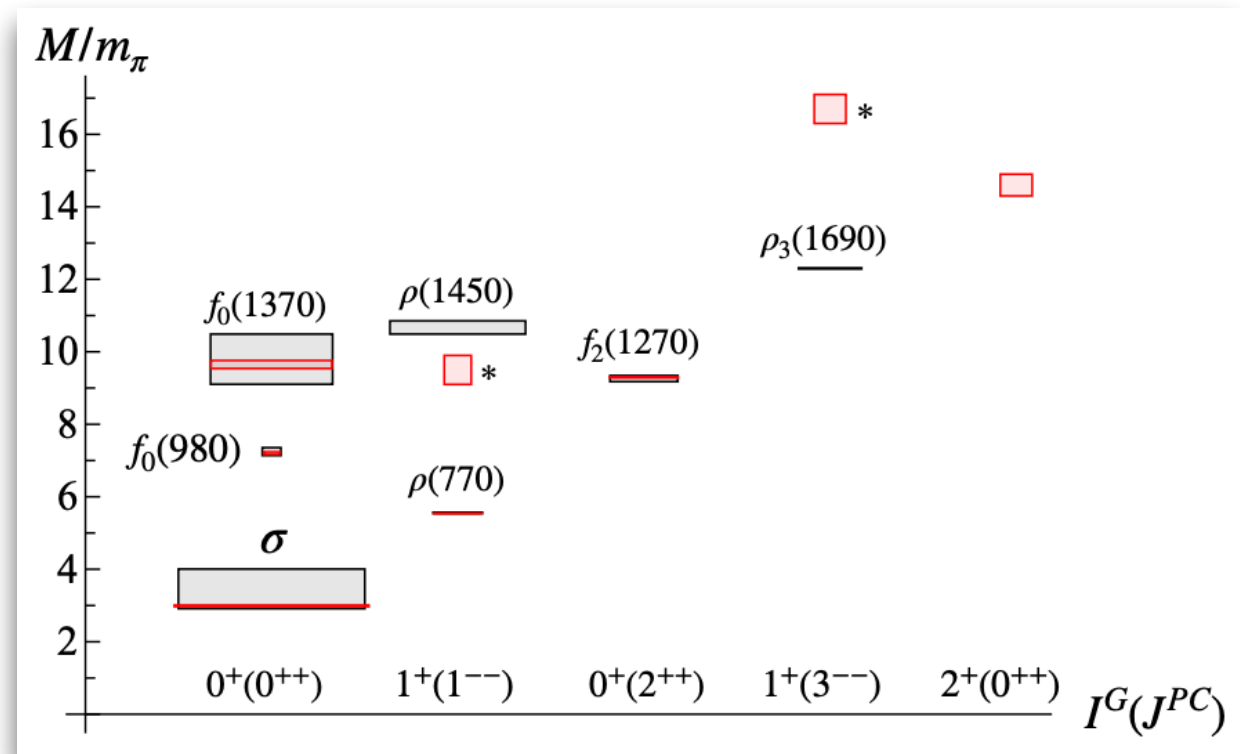


He, Kruczenski, 2505.19332

Use experimental and lattice data

Guerrieri, Haring, Su, 2410.23333

- Phase shifts data of S_0, S_2, P, D_0
- Inelasticity model $|S_\ell(s)| \leq \eta_\ell(s)$
- Existence of chiral zeros
- Existence of some resonances



bootstrap assisted fitting

Mass bounds on self-interacting dark matter

Self-interaction solves DM's core-cusp problem

$$\sigma_{\text{self}} / M \sim 0.1 - 10 \text{ cm}^2/\text{g}$$

Generic unitarity bound on DM's mass

$$M \lesssim 12\text{GeV} \quad (\text{model-independent})$$

Hui, [astro-ph/0102349](#)

Bootstrap bounds on weakly-coupled DM's mass

Chen, Wang, SYZ, [2607.13141](#)

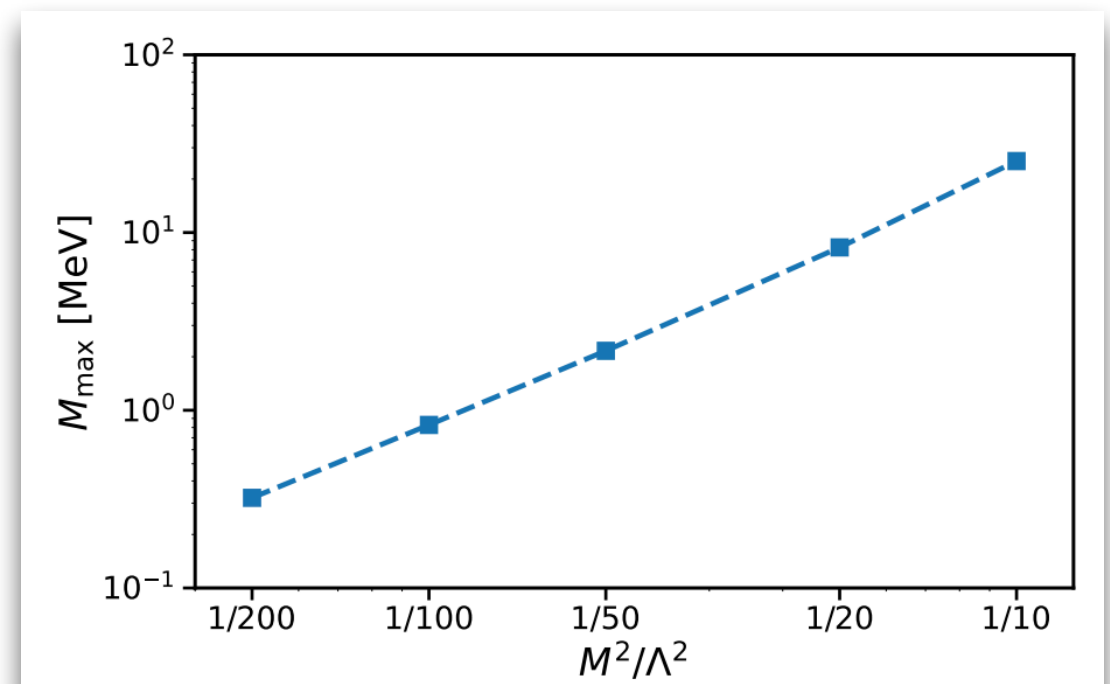
- Generic scalar DM

$$M \lesssim 0.29\text{GeV} \quad (\text{model-independent})$$

- pseudo-Goldstone boson DM

$$M \lesssim 26\text{MeV} \quad (\text{model-independent})$$

(depends on EFT cutoff)



Unitary dual-resonance models

Veneziano amplitude

$$V_{\alpha'}(s, t) = \frac{\Gamma(-\alpha' s) \Gamma(-\alpha' t)}{\Gamma(-\alpha' s - \alpha' t)}$$

positive but not unitary
loop string amplitude is unitary

New unitary dual-resonance amplitude

de Rham, Tolley, Wang, SYZ, 2607.24922

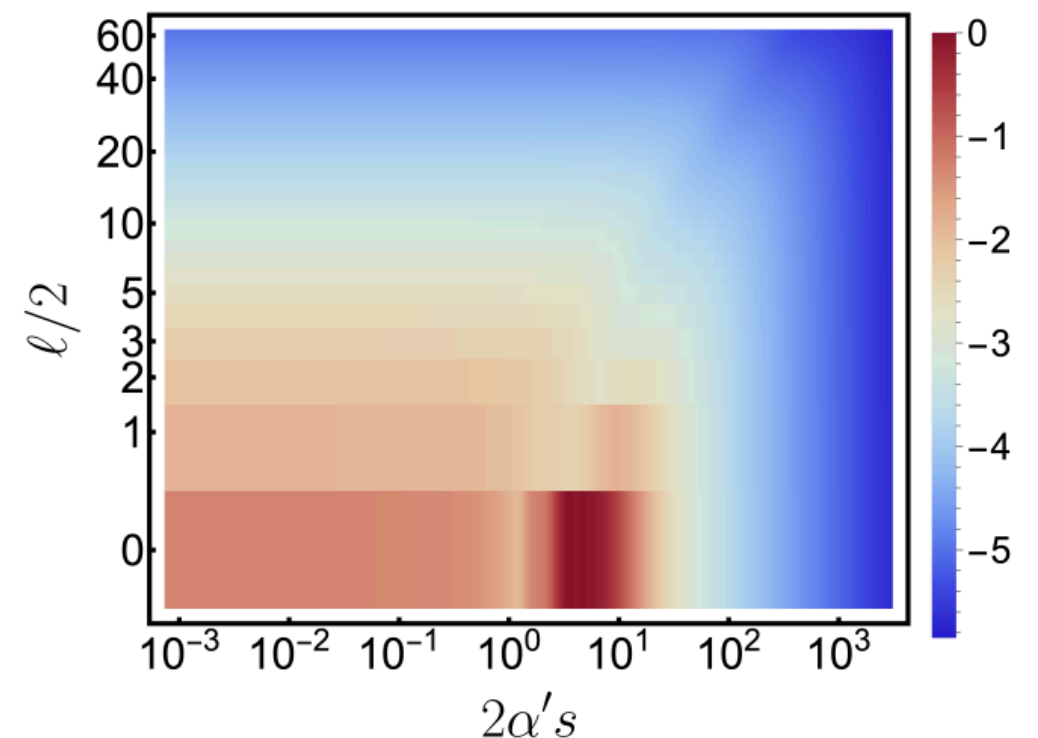
$$\mathcal{M}(s, t) = \int_{\alpha'}^{\infty} dy \phi(y) W_y(s, t, u), \quad W_y(s, t, u) \equiv V_y(s, t) + V_y(u, t) + V_y(s, u)$$

$$\phi(y) = C \alpha'^2 \left(1 - \frac{\alpha'}{y}\right)^{\beta} \left(\frac{y}{\alpha'}\right)^{-p+2} \frac{\mathbb{T}^2}{(y - y_0)^2 + \mathbb{T}^2}$$

$$\text{Regge trajectory: } s_n^{(\ell)} = \frac{n}{y_0 + i\mathbb{T}}$$

Verified:

- Unitarity (numerically + analytically)
- Maximal analyticity (analytically+numerically)
- Locality (analytically)



Dual S-matrix Bootstrap

A diagram of a complex plane with a horizontal axis labeled $\text{Re } a_\ell$ and a vertical axis labeled $\text{Im } a_\ell$. A red circle is centered on the vertical axis, with its bottom point touching the horizontal axis. A small gray rectangle is positioned on the horizontal axis, centered under the bottom of the circle.

- $$A(s, t) = \frac{1}{2\pi i} \oint_{\mathcal{C}} ds' \frac{A(s', t)}{s' - s}$$

-
- The diagram shows a complex plane with two concentric circles. The outer circle is labeled C and the inner circle is labeled C' . The region between the two circles is labeled UV . The region inside the inner circle C' is labeled EFT . The region outside the outer circle C is labeled s' . The horizontal axis is labeled $-\Lambda^2$ on the left and Λ^2 on the right. Two red 'x' marks are located on the inner circle C' .

$$A(s, t) \sim \int_{\Lambda^2} \frac{ds'}{\pi s'^2} \left[\frac{s^2}{s' - s} + \frac{u^2}{s' - u} \right] \text{Im } A(s', t)$$

EFT amplitude

IR—UV connection

UV amplitude

EFT sum rules $\Rightarrow$ Positivity bounds

$$\sum_{i,j} g^{i,j} s^i t^j \sim A(s, t) \sim \int_{\Lambda^2} \frac{ds'}{s'^2} \left[\frac{s^2}{s' - s} + \frac{u^2}{s' - u} \right] \sum_{\ell} P_{\ell} \left(1 + \frac{2t}{s'} \right) \text{Im } a_{\ell}(s')$$

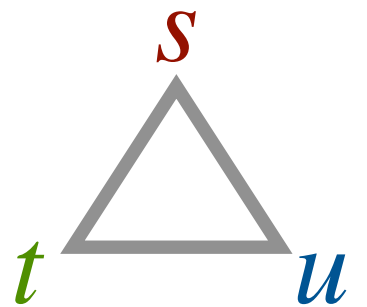
EFT/Wilson coefficients partial wave expansion

Sum rules $g^{i,j} = \sum_{\ell} \int d\mu \frac{G_{i,j}(J^2)}{s'^{i+j}}$ $d\mu \sim ds' \text{Im } a_{\ell}(s')$
 $J^2 = \ell(\ell + 1)$

Full crossing symmetry $A(u, t) = A(s, t) = A(t, s)$

Null constraints

$$\sum_{i,j} \alpha_{i,j} g^{i,j} = 0 \quad \text{or} \quad 0 = \sum_{\ell} \int d\mu \frac{N_{i,j}(J^2)}{s'^{i+j}}$$



Positivity cone for dim-8 for multi-fields

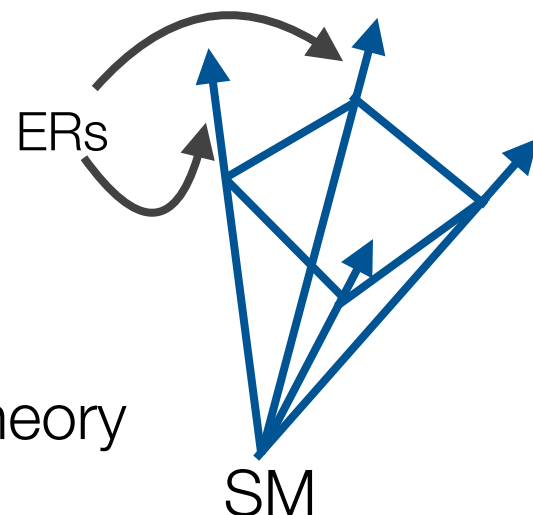
- bounds on s^2 coefficients $\rightarrow$ dim-8
- phenomenologically more relevant

$$A_{ij \rightarrow kl}(s, t = 0) = \frac{c_{ij \rightarrow kl}^{2,0} s^2}{\Lambda^4} + \dots$$

physical $c_{ij \rightarrow kl}^{2,0}$ form (high dimensional) convex cone

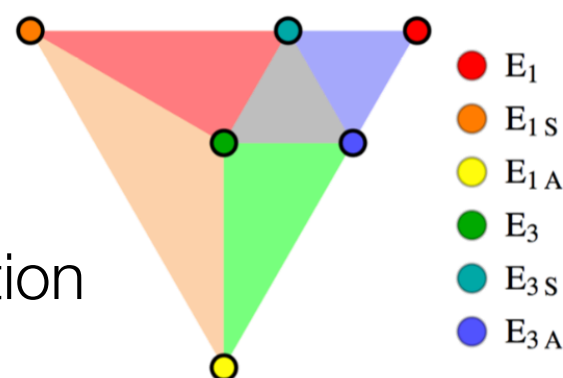
Polyhedral cone

- many symmetries
- finite discrete ERs
- obtained by group theory



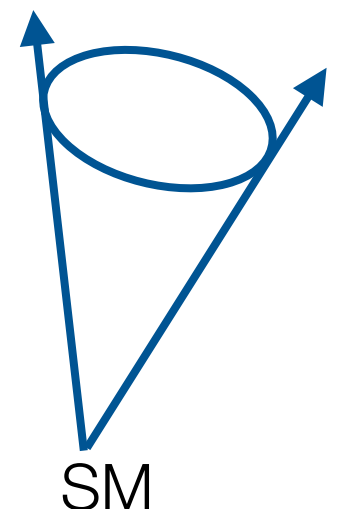
ER $\leftrightarrow$ UV states

- importance of dim-8
- useful for UV reconstruction



Curly cone

- less symmetries
- infinitely many ERs
- obtained numerically by SDP



- switch to dual amplitude cone
- numerically efficient to get optimal bounds

Bounds on Standard Model EFT

aQGC dim-8 operators in SMEFT

(actively constrained by LHC)

$$O_{S,0} = \left[(D_\mu \Phi)^\dagger D_\nu \Phi \right] \times \left[(D^\mu \Phi)^\dagger D^\nu \Phi \right],$$

$$O_{S,1} = \left[(D_\mu \Phi)^\dagger D^\mu \Phi \right] \times \left[(D_\nu \Phi)^\dagger D^\nu \Phi \right],$$

$$O_{S,2} = \left[(D_\mu \Phi)^\dagger D_\nu \Phi \right] \times \left[(D^\mu \Phi)^\dagger D^\nu \Phi \right],$$

$$O_{M,0} = \text{Tr} \left[\hat{W}_{\mu\nu} \hat{W}^{\mu\nu} \right] \times \left[(D_\beta \Phi)^\dagger (D^\beta \Phi) \right],$$

$$O_{M,1} = \text{Tr} \left[\hat{W}_{\mu\nu} \hat{W}^{\nu\beta} \right] \times \left[(D_\beta \Phi)^\dagger D^\mu \Phi \right]$$

$$O_{M,2} = \hat{B}_{\mu\nu} \hat{B}^{\mu\nu} \times \left[(D_\beta \Phi)^\dagger D^\beta \Phi \right],$$

$$O_{M,3} = O_{M,4} = \left[(D_\mu \Phi)^\dagger \hat{W}_{\beta\nu} (D^\mu \Phi) \right] \times \hat{B}^{\beta\nu},$$

$$O_{M,5} = \frac{1}{2} \left[(D_\mu \Phi)^\dagger \hat{W}_{\beta\nu} (D^\nu \Phi) \right] \times \hat{B}^{\beta\mu} + \text{h.c.},$$

$$O_{M,7} = \left[(D_\mu \Phi)^\dagger \hat{W}_{\beta\nu} \hat{W}^{\beta\mu} (D^\nu \Phi) \right],$$

$$O_{M,8} = i \left[D_\mu \Phi^\dagger \tilde{W}_{\nu\rho} D^\nu \Phi \right] \hat{B}^{\mu\rho} + \text{h.c.},$$

$$O_{M,9} = i D^\mu \Phi^\dagger \hat{W}_{\mu\rho} \tilde{W}^{\nu\rho} D_\nu \Phi + \text{h.c.},$$

$$O_{T,0} = \text{Tr} \left[\hat{W}_{\mu\nu} \hat{W}^{\mu\nu} \right] \text{Tr} \left[\hat{W}_{\alpha\beta} \hat{W}^{\alpha\beta} \right],$$

$$O_{T,1} = \text{Tr} \left[\hat{W}_{\alpha\nu} \hat{W}^{\mu\beta} \right] \text{Tr} \left[\hat{W}_{\mu\beta} \hat{W}^{\alpha\nu} \right],$$

$$O_{T,2} = \text{Tr} \left[\hat{W}_{\alpha\mu} \hat{W}^{\mu\beta} \right] \text{Tr} \left[\hat{W}_{\beta\nu} \hat{W}^{\nu\alpha} \right],$$

$$O_{T,3} = \text{Tr} \left[\hat{W}_{\mu\nu} \hat{W}_{\alpha\beta} \right] \text{Tr} \left[\hat{W}^{\alpha\nu} \hat{W}^{\mu\beta} \right],$$

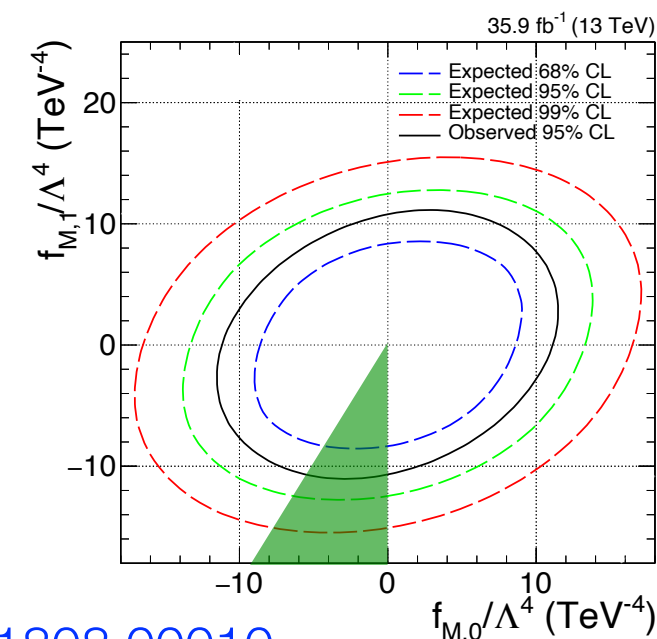
$$O_{T,4} = \text{Tr} \left[\hat{W}_{\mu\nu} \hat{W}_{\alpha\beta} \right] \hat{B}^{\alpha\nu} \hat{B}^{\mu\beta},$$

$$O_{T,6} = \text{Tr} \left[\hat{W}_{\alpha\nu} \hat{W}^{\mu\beta} \right] \hat{B}_{\mu\beta} \hat{B}^{\alpha\nu},$$

$$O_{T,7} = \text{Tr} \left[\hat{W}_{\alpha\mu} \hat{W}^{\mu\beta} \right] \hat{B}_{\beta\nu} \hat{B}^{\nu\alpha},$$

$$O_{T,8} = \hat{B}_{\mu\nu} \hat{B}^{\mu\nu} \hat{B}_{\alpha\beta} \hat{B}^{\alpha\beta},$$

$$O_{T,9} = \hat{B}_{\alpha\mu} \hat{B}^{\mu\beta} \hat{B}_{\beta\nu} \hat{B}^{\nu\alpha}.$$



Zhang, SYZ, 1808.00010

positivity cone
0.05% of total

22D aQGC space

**Most of the naive SMEFT
space is redundant!**

Chang, Chen, SYZ, 2604.00107

EFT sum rules as moments

Forward amplitude as Hausdorff moment

Arkani-Hamed, Huang, Huang, 2017
Bellazzini, Miro, Rattazzi, Riembau, Riva, 2021

$$g^{2n,0} = \int_{\Lambda^2}^{\infty} \frac{2 \, ds'}{\pi s'^{1+2n}} \operatorname{Im} A(s', 0) \quad \xrightarrow{x \equiv \Lambda/s'} \quad g^{2n,0} = \int_0^1 x^{2n} \rho(x) dx$$

General amplitude as

- Bi-variate moment formulation
- Mix-variate moment formulation

Chiang, Huang, Li, Rodina & Weng, 2105.02862

- Multiple DoFs
- Fields with spin

(identified N & S condition)

Wan, SYZ, 2411.11964

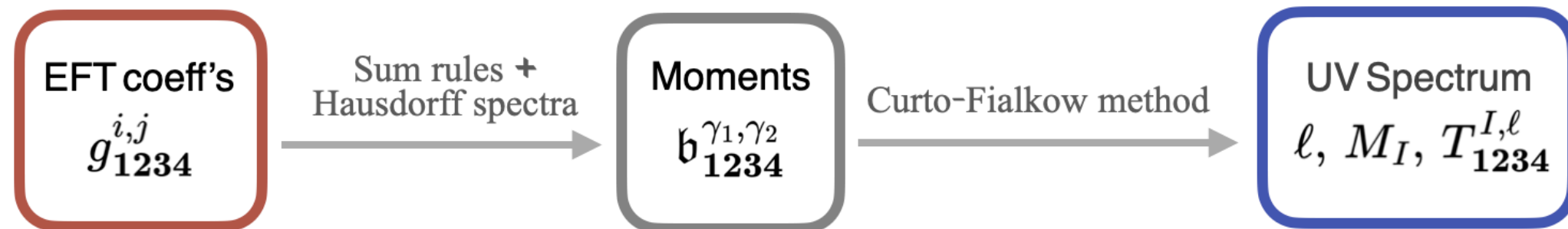
- Series of Hausdorff moment problems

Wan, SYZ, 2605.11084

$$A(s, t) \sim \sum_{p,q} a_{p,q} s^{p-q} t^q \quad a_{p+q,q} = \int_0^1 x^p \rho_q(x) dx$$

Inverse problem in EFT

Moment approach to reconstruct UV spectrum from EFT coefficients



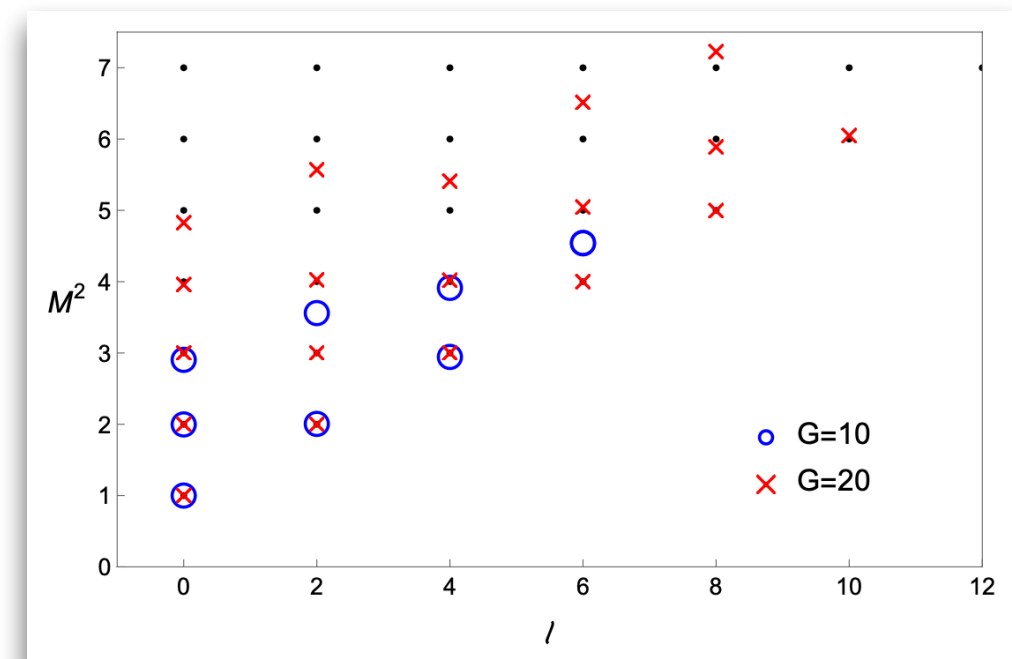
Wan, SYZ, 2411.11964

$$A_{1234}^{(\text{UV})}(s, t) \supset \sum_{I, \ell} \frac{\mathcal{T}_{1234}^{I, \ell} d_{h_{12}, h_{34}}^{\ell}(\cos \theta)}{M_I^2 - s - i\epsilon} + (\text{crossing terms})$$

To also reconstruct zeros

$$Q(s) = \frac{A'(s, 0)}{A(s, 0)} = - \sum_{\text{poles}} \frac{1}{s - \mu} + \sum_{\text{zeros}} \frac{1}{s - \rho} + \dots$$

Closed-string Regge trajectories from Wilson coefficients



Calisto, Cheung, Remmen, Sciotti, Tarquini, 2604.15423

String bootstrap

How unique are perturbative string amplitudes?

work in the $g_s \rightarrow 0$ limit, so tree-level

How much information is need to uniquely determine the (stripped) Veneziano amplitude?

$$\boxed{A_V(s, t) = -\frac{\Gamma(-s)\Gamma(-t)}{\Gamma(1-s-t)}} \quad \alpha' = 1$$

closed-string Virasoro-Shapiro amplitude via KLT-double copy

Recent uniqueness results for Veneziano

Cheung, Hillman, Remmen, 2406.02665

- Level truncation + positivity bounds + UV super-softness

$$A(s, t) = \sum_{n=0}^{\infty} \frac{R(n, t)}{\mu(n) - s} \quad R(n, t) = c(n) \prod_{k=1}^n (t - \xi(k)) \quad \lim_{|s| \rightarrow \infty} A(s, t_0) \sim s^{J(t_0)}, \quad J(t_0 \rightarrow -\infty) \rightarrow -\infty$$

- Level truncation from minimal zeros

Cheung, Remmen, Sciotti, Tarquini, 2508.09246

$$A(s, t) \stackrel{s \rightarrow \infty}{\sim} f(t)(-s)^{\alpha(t)} \quad \text{All zeros in } R(n, t) \text{ are Regge zeros: } \alpha(t < 0) \in \mathbb{Z}$$

- Maximal SUSY + positivity bounds

Berman, Elvang, 2406.03543

Berman, Caron-Huot, Chandra, Elvang et al, 2607.14230

$\mathcal{N} = 4$ or 8 SUSY EFT in 4D: R-symmetry + 5,6-point factorization + parity

- Minimal entanglement in scattering + lowest 2 Wilson coeff's

$$\Delta E = \Delta(1 - \text{tr}_A \rho_A^2) \sim \text{Im } \mathcal{M}(p_1, p_2 \rightarrow p_1, p_2) \quad \text{Bhata, Chowdhury, Saha, Sinha, 2409.18259}$$

- String monodromy condition + positivity bounds

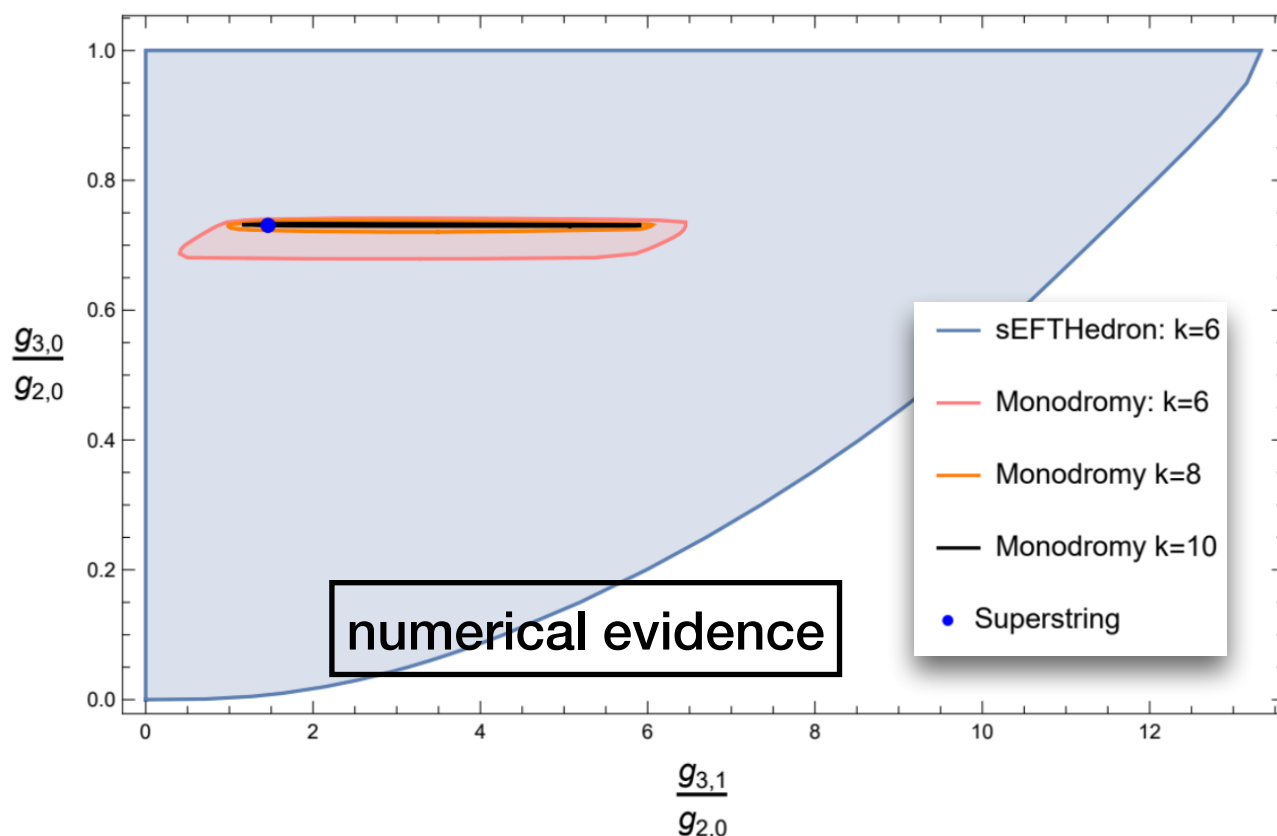
Wan, SYZ, 2605.11084

- Splitting and hidden-zero condition + positivity bounds

Monodromy or Splitting + Positivity $\Rightarrow$ Veneziano

String monodromy condition

$$A(s, t) + e^{i\pi s} A(s, u) + e^{-i\pi t} A(t, u) = 0$$



Yu-tin Huang, Liu, Rodina, Wang, 2008.02293

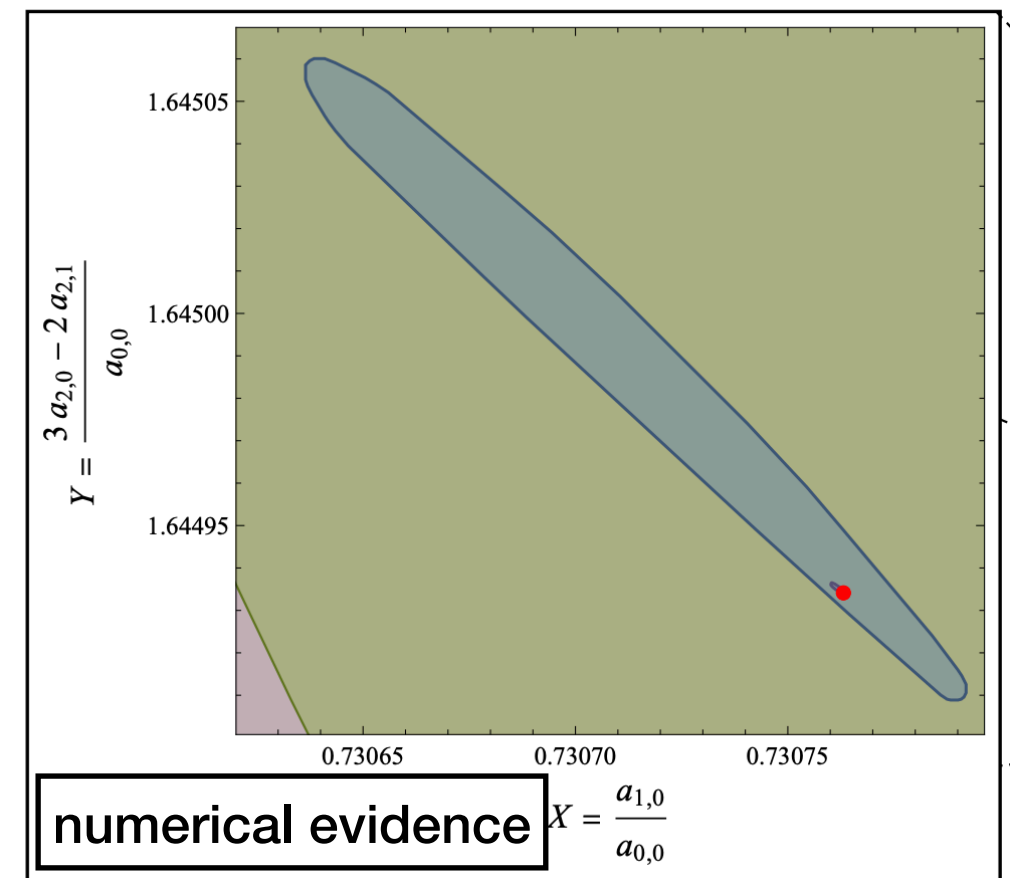
Chiang, Huang, Weng, 2310.10710

Berman, Elvang, Herderschee, 2310.10729

5-point splitting condition

Arkani-Hamed, Cao, He, et al, 23', 24'

$$B(s_1, s_2)B(s_1 + s_2, s_3) = B(s_1, s_2 + s_3)B(s_2, s_3)$$



Berman, Elvang, Figueiredo, 2506.22538

We analytically prove: either of them determines Veneziano uniquely

Wan, SYZ, 2605.11084

String monodromy condition $\Rightarrow$ Veneziano amplitude

$$A(s, t) + e^{i\pi s} A(s, u) + e^{-i\pi t} A(t, u) = 0$$



Positivity bounds

$$A(s, t) = -\frac{\Gamma(-s)\Gamma(-t)}{\Gamma(1-s-t)}$$

Explicitly: For $A(s, t) = -\frac{1}{st} + \sum_{p \geq q \geq 0} a_{p,q} s^{p-q} t^q$

Prove $a_{p,q} = a_{p,q}^V = \zeta(p - q + 2, \{1\}^q)$

Multiple zeta values $\zeta(p_1, \dots, p_d) \equiv \sum_{\infty > j_1 > \dots > j_d \geq 1} \frac{1}{j_1^{p_1} \cdots j_d^{p_d}}$

Leading order ($q = 0$)

$$A(s, t) = -\frac{1}{st} + \sum_{p \geq q \geq 0} a_{p,q} s^{p-q} t^q$$

$t \rightarrow 0$ limit of monodromy condition $\Rightarrow$ even coeff's

$$a_{2p,0} = \zeta(2p+2) = \sum_{j=1}^{\infty} \frac{1}{j^2} \left(\frac{1}{j}\right)^{2p} \quad \text{but } a_{2p+1,0} \text{ are unfixed}$$

Key point: use dispersive sum rule to get $a_{2p+1,0}$

$$a_{2p,0} = \int_0^1 x^{2p} \rho_0(x) dx \quad \xrightarrow{\text{determinacy of Hausdorff moments}} \quad a_{2p+1,0} = \int_0^1 x^{2p+1} \rho_0(x) dx$$

Explicitly: $x = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{k_1} (-1)^{k_1+k_2} \binom{1/2}{k_1} \binom{k_1}{k_2} x^{2k_2} \quad x \in [0, 1]$

Thus, odd coeff's are also of Veneziano

$$a_{2p+1,0} = \sum_{j \geq 1} \frac{1}{j^2} \left(\frac{1}{j}\right)^{2p+1} = \zeta(2p+3)$$

Next leading order ($q = 1$)

$$A(s, t) = -\frac{1}{st} + \sum_{p \geq q \geq 0} a_{p,q} s^{p-q} t^q$$

Monodromy condition $\Rightarrow$

$$A(s, t) = A(t, s) \quad \Rightarrow \quad a_{p,q} = a_{p,q-p}$$

∂_t of monodromy condition $\Rightarrow$ odd coeff's

$$a_{2p+1,1} = (p+1)a_{2p+1,0} - \sum_{k=1}^p \zeta(2k) a_{2p-2k+1,0} \xrightarrow[\text{Euler's formula}]{q=0 \text{ results}} a_{2p+1,1} = \zeta(2p+2, 1)$$

Next, again, use dispersive sum rule

$$a_{2p+1,1} = \int_0^1 x^{2p+1} \rho_1(x) dx \xrightarrow[\text{determinacy of Hausdorff moments}]{\text{determinacy of Hausdorff moments}} a_{2p+2,1} = \int_0^1 x^{2p+2} \rho_1(x) dx$$

$$\xrightarrow{\quad} a_{2p+2,1} = \sum_{j=2}^{\infty} \left(\frac{1}{j^2} \sum_{j_1=1}^{j-1} \frac{1}{j_1} \right) \left(\frac{1}{j} \right)^{2p+1} = \zeta(2p+3, 1)$$

All orders: Parametrization

Parametrization that fully resolves monodromy condition

$$A(s, t) = \underset{\substack{\uparrow \\ \text{Veneziano}}}{A_V(s, t)} \cdot \underset{\substack{\uparrow \\ \text{stu-triple symmetric}}}{C(s, t, u)}$$

Re-writing

$$1 - C = \sum_{n=0}^{\infty} (1 + stA_V)^n (stA - stA_V)$$

Define $\Delta_{p,q} \equiv a_{p,q} - a_{p,q}^V$ and $C(s, t, u) = 1 - st \sum_{p \geq q \geq 0} c_{p,q} s^{p-q} t^q$



$$c_{p,q} = \Delta_{p,q} + \sum_{p',q'} \mathcal{M}_{p,q;p',q'} \Delta_{p',q'}$$

- $\mathcal{M}$ is strictly triangular
- Null constraints $\mathcal{T}c = 0$

All orders: Proof by induction

Null constraints

$$\mathcal{T} = \mathcal{T}_0 \oplus \mathcal{T}_1 \oplus \cdots \oplus \mathcal{T}_p \oplus \cdots \quad \mathcal{T}_p = (\mathcal{T}_p^{st}, \mathcal{T}_p^{tu})^T$$

combine $\tilde{\mathcal{T}}_p^{tu} \sim \mathcal{T}_p^{tu} (\mathcal{I}_p - \mathcal{T}_p^{st})$ to get $\tilde{\mathcal{T}}^{tu} c = 0$

Triangular recursion relation

$$\Delta_{p,q} \equiv a_{p,q} - a_{p,q}^V$$

$$\Delta_{p,q} = - \sum_{p',q'} \tilde{\mathcal{M}}_{p,q;p',q'} \Delta_{p',q'} \quad \text{for } p - q = \text{even}$$

for all orders, monodromy fixes half of coeff's

Use dispersive sum rules/Hausdorff moments to get the rest

$$a_{p,q} = \sum_{j > j_1 > \cdots > j_q \geq 1} \frac{1}{j^2 j_1 \cdots j_q} \left(\frac{1}{j} \right)^p = a_{p+q,q}^V$$

Full proof: proceed by mathematical induction

Summary

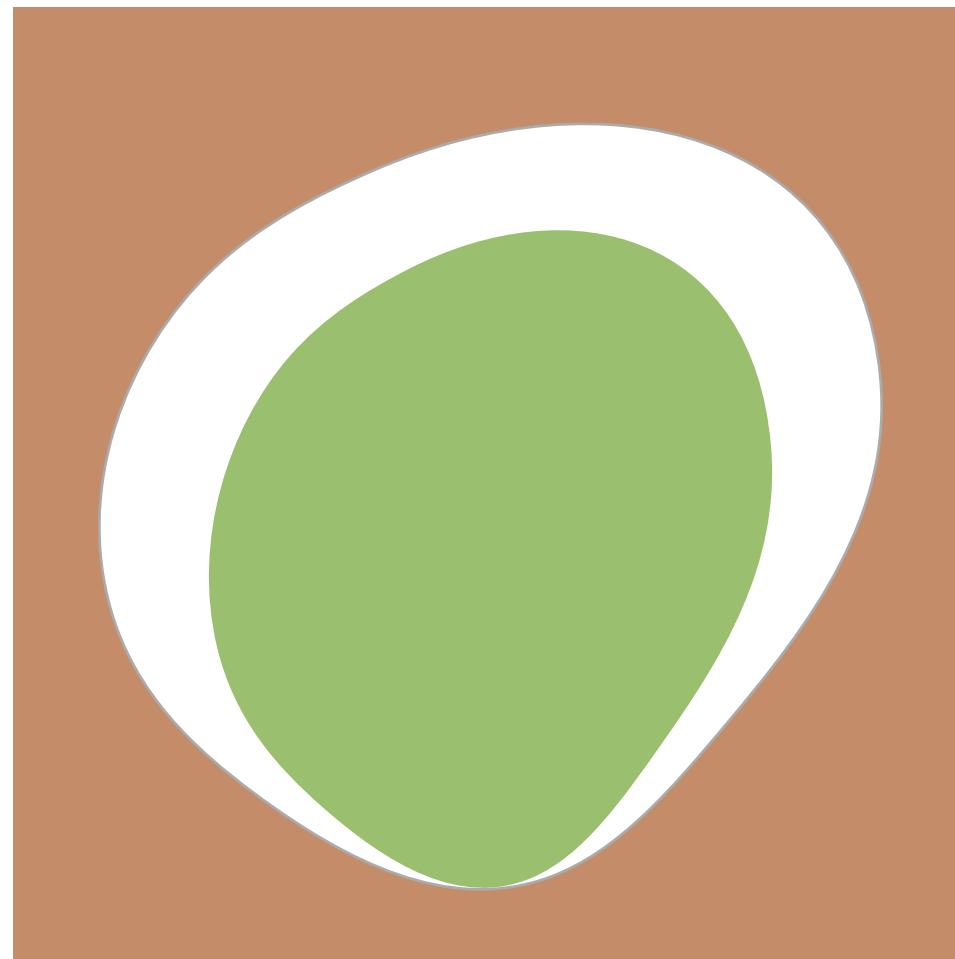
S-matrix bootstrap: use basic principles to constrain theory space, sometimes to uniquely determine a theory

S-matrix theory space

Dual bootstrap

- rule out unviable space
- eliminate amplitudes

EFT setup:
positivity bounds



Primal bootstrap

- rule in viable space
- construct amplitudes

Thank you!