

Wilson Surface Expansion in 6d Supergravity Blocks

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- Yuezhang Tang and Wentao Yu (USTC), in progress

Swampland Program

Landscape

EFTs consistent with quantum gravity
(*finite!*)



Swampland

EFTs that look fine
but have no UV completion
(*vast!*)

No global symmetries

Exact global symmetries are forbidden in quantum gravity,
including higher-form ones.

Completeness

Every allowed gauge charge must be realized as a physical
state or extended object.

...

Why 6d $N=(1,0)$ supergravity?

Landscape

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6d $N=(1,0)$ supergravity is a perfect testing ground:

Highly constrained

Anomaly cancellation + SUSY + unitarity
tightly constrain spectrum and gauge content.

Finiteness proven

Kim–Vafa–Xu '25
the massless spectrum is finite.

Non-perturbative richness

Tensionless BPS strings, SCFTs and LSTs
give non-perturbative tests.

F-theory connection

Many solutions from F-theory on elliptic CY3

BPS strings provide a protected test

Landscape

EFTs consistent with quantum gravity
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A consistent **BPS-string elliptic genus** is **non-perturbative evidence**.

- ▶ **Modular bootstrap:** Anomalies and modularity strongly constrain each string sector.
- ▶ **Localization:** direct calculation for worldsheet SCFT

Outline

1. 6d supergravity theory
2. Wilson surface as a codimension-four defect
3. Wilson-surface expansion

6d supergravity theory

Field content & anomaly cancellation

Gravity multiplet (graviton, gravitino, self-dual 2-form).

T tensor multiplets anti-self-dual 2-forms + scalars.

V vector multiplets gauge algebra $\mathfrak{g}$.

H hypermultiplets matter representations.

1-loop anomalies from chiral fields can be cancelled by the *Green–Schwarz–Sagnotti* mechanism. [Green–Schwarz '84] • [Sagnotti '92]

Anomaly vectors $b^0, b^i \in \mathbb{R}^{1,T}$ live in the tensor-charge lattice; $\Omega^{\alpha\beta}$ is its intersection form.

ANOMALY CANCELLATION CONDITIONS

$$\text{gravitational: } H - V = 273 - 29T,$$

$$(\text{tr } R^2)^2: \quad b^0 \cdot b^0 = 9 - T,$$

$$\text{tr } F_i^4: \quad B_{\text{adj}}^i = \sum_r n_r^i B_r^i,$$

$$\text{tr } R^2 \text{ tr } F_i^2: \quad b^0 \cdot b^i = \frac{\lambda_i}{6} \left(\sum_r n_r^i A_r^i - A_{\text{adj}}^i \right),$$

$$(\text{tr } F_i^2)^2: \quad b^i \cdot b^i = \frac{\lambda_i^2}{3} \left(\sum_r n_r^i C_r^i - C_{\text{adj}}^i \right),$$

$$\text{tr } F_i^2 \text{ tr } F_j^2: \quad b^i \cdot b^j = 2\lambda_i \lambda_j \sum_{r,s} n_{r,s}^{ij} A_r^i A_s^j \quad (i \neq j).$$

$$\text{tr}_r F_i^2 = A_r^i \text{tr } F_i^2, \quad \text{tr}_r F_i^4 = B_r^i \text{tr } F_i^4 + C_r^i (\text{tr } F_i^2)^2.$$

F-theory geometrizes the anomaly data

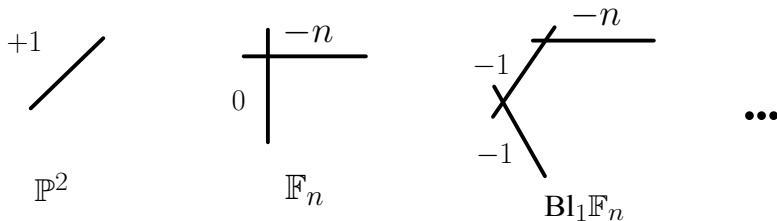
For an elliptic Calabi–Yau threefold $\pi : X \rightarrow B$, field-theory data become curve classes on the base.

6d data		geometry on B
$\Gamma^{1,T}$	$\longleftrightarrow$	$H_2(B, \mathbb{Z})_{\text{free}}$
$\Omega^{\alpha\beta}$	$\longleftrightarrow$	intersection form
b^0	$\longleftrightarrow$	$-K_B$
b^i	$\longleftrightarrow$	gauge divisor $[S_i]$
string charge Q	$\longleftrightarrow$	effective curve $[C]$

$$b^0 \cdot b^i = (-K_B) \cdot S_i, \quad b^i \cdot b^j = S_i \cdot S_j, \quad Q \cdot b^i = C \cdot S_i.$$

Toric data specify the base

For toric B , a finite curve diagram records the lattice and its intersections.



Gauge groups: types of singular fibers over curves in B .

H-strings: Dual to heterotic string.

MINIMAL SUPERGRAVITY COMPLETION

A minimal supergravity completion couples a 6d SCFT on a $(-n)$ -curve to an *H-string*.

6d $SU(N)$ theory on $\mathbb{F}_2$

THE TENSOR-BRANCH THEORY

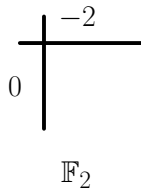
$$\mathcal{N} = (1, 0), \quad G = SU(N), \quad (T, V) = (1, N^2 - 1).$$

MATTER CONTENT

$$N_f = 2N \text{ (N)}, \quad n_{\text{adj}} = n_{\Lambda^2} = 0.$$

ANOMALY CONSTRAINTS

$$\begin{aligned} H_{\text{ch}} &= N_f \dim \mathbf{N} = 2N^2, \\ H_0 + H_{\text{ch}} - V + 29T &= 273, \\ H_0 = 243 - N^2 \geq 0 &\rightarrow \boxed{N \leq 15}. \end{aligned}$$



Elliptic genera

The string-charge lattice has rank $T + 1$, while the tensor-moduli space is only T -dimensional.

TENSOR MODULI SPACE

$$\mathcal{M}_{\text{tens}}^{\text{loc}} = \frac{SO^+(1, T)}{SO(T)}, \quad \dim \mathcal{M}_{\text{tens}} = T.$$

$$\mathcal{J}^\alpha, \quad \alpha = 0, \dots, T, \\ \mathcal{J}^0 > 0, \quad \mathcal{J} \cdot \mathcal{J} = 1.$$

One norm constraint leaves T scalar moduli;
BPS positivity selects the physical chamber
inside this local coset.

$$\mathbf{n} \in \Gamma_{\text{eff}}, \quad \frac{T_{\mathbf{n}}}{M_{\text{Pl}}^2} \sim \mathcal{J} \cdot \mathbf{n} \geq 0.$$

TENSOR-CHARGE EXPANSION

$$Z_{\text{BPS}}(\mathbf{Q}) = Z_{\text{pert}} \left[1 + \sum_{\mathbf{n} \neq 0} \mathbf{Q}^{\mathbf{n}} Z_{\mathbf{n}}(\tau, \epsilon_{1,2}, m, a) \right],$$

$$\mathbf{Q}^{\mathbf{n}} = \prod_{\alpha=0}^T Q_{\alpha}^{n_{\alpha}}, \quad Q_{\alpha} = e^{-t_{\alpha}}.$$

$Z_{\mathbf{n}}$ is the protected elliptic genus at fixed string
charge $\mathbf{n}$.

Elliptic genera

A PROTECTED RR-SECTOR TRACE

$$\mathcal{I}_{r,k}(\tau, \mathbf{z}) = \text{Tr}_{\mathcal{H}_{\text{RR}}^{(r,k)}} \left[(-1)^F q^{L_0 - c_L/24} \bar{q}^{\bar{L}_0 - c_R/24} e^{2\pi i \mathbf{z} \cdot \mathbf{J}} \right], \quad q = e^{2\pi i \tau}.$$

$$e^{2\pi i \mathbf{z} \cdot \mathbf{J}} = \exp \left\{ 2\pi i \left[\epsilon_- J_l + \epsilon_+ (J_r + J_R) + \mathbf{m} \cdot \mathbf{J}_F + \mathbf{a} \cdot \mathbf{J}_G \right] \right\}, \quad \epsilon_{\pm} = \frac{\epsilon_1 \pm \epsilon_2}{2}.$$

THREE NON-VACUUM STRING-CHARGE SECTORS

$$\mathcal{Z}_{\text{string}} = 1 + \mathcal{Z}_H + \mathcal{Z}_{\text{SCFT}} + \mathcal{Z}_{\text{mix}}.$$

H-STRING

$$\mathcal{Z}_H = \sum_{r \geq 1} Q_H^r \mathcal{I}_{r,0}.$$

Pure H-string charge: ($r > 0$, $k = 0$).

$$\mathcal{I}_{1,0} \big|_{\mathbf{m}=0, \epsilon_+=0} = \frac{2E_4 E_6}{\eta^{24} \phi_{-2,1}(\tau, \epsilon_-)}.$$

6d SCFT BPS STRINGS

$$\mathcal{Z}_{\text{SCFT}} = \sum_{k \geq 1} Q_S^k \mathcal{I}_{0,k}.$$

Pure SCFT-string charge: ($r = 0$, $k > 0$).

MIXED / BOUND STATES

$$\mathcal{Z}_{\text{mix}} = \sum_{r,k \geq 1} Q_H^r Q_S^k \mathcal{I}_{r,k}.$$

Both charges are nonzero: ($r > 0$, $k > 0$).

Wilson surface

Wilson surface

PARTICLE / ONE-FORM

$$W_R(C) = \text{Tr}_R P \exp \left(i \oint_C A \right).$$

C : closed worldline



STRING / TWO-FORM

$$W_q(\Sigma) = \exp \left(i q \int_{\Sigma} B \right).$$

Σ : closed world sheet, $B \mapsto B + d\Lambda$

Wilson surface as defect

TYPE IIA BRANE ARRAY

	$\mathbb{T}^2$		$\mathbb{R}^4_{\epsilon_1, \epsilon_2}$							
IIA	0	1	2	3	4	5	6	7	8	9
2 NS5	•	•	•	•	•	•				
k D2	•	•					•			
n D4'	•	•						•	•	•
N D6	•	•	•	•	•	•	•			

MATTER CONTENT

OPEN STRINGS $2d$ MULTIPLLET / REPRESENTATION

D2-D2	vector + adjoint hyper of $U(k)$
D2-D6 _{6d}	fundamental hyper in $(\mathbf{k}, \bar{\mathbf{N}}_{6d})$
D2-D6 _{L,R}	Fermi in $\mathbf{k} \otimes (\bar{\mathbf{N}}_L \oplus \bar{\mathbf{N}}_R)$
D2-D4'	twisted hyper + Fermi in $(\mathbf{k}, \bar{\mathbf{n}})$
D6-D4'	Fermi ρ in $(\mathbf{N}_{6d}, \bar{\mathbf{n}})$, $U(k)$ -neutral

WORLD-SHEET SYMMETRY GAUGE

$$G_{\text{gauge}}^{2d} = U(k).$$

FLAVOR

$$G_{\text{flavor}}^{2d} = SU(N)_{6d} \times SU(N)_L \times SU(N)_R \\ \times U(n)_{\text{def}}.$$

Although gauged in $6d$, $SU(N)_{6d}$ is a background flavor symmetry on the string.

$SU(N)_{L,R}$: semi-infinite D6 stacks

$U(n)_{\text{def}}$: added D4' branes

Wilson surface as defect

D6–D4' fixes the prefactor; D2 modes generate the $Q^{k>0}$ sectors.

$$Z_{6d+\text{def}}^{(n)} = Z_{6d,\text{pert}} \underbrace{Z_{\text{D6–D4}'}^{(n)}}_{Z_{\text{pert}}^{\text{def}}} \left[1 + \sum_{k \geq 1} Q^k \underbrace{Z_k^{\text{D2+def}}}_{\text{D2-coupled}} \right].$$

D6–D4' · k -INDEPENDENT

The $U(k)$ -neutral Fermi ρ gives one theta function per D6–D4' pair.

$$Z_{\text{D6–D4}'}^{(n)} = \prod_{\ell=1}^n \prod_{\alpha=1}^N \frac{\theta_1(\tau | z_\ell - a_\alpha)}{i \eta(\tau)}.$$

SINGLE D4': $n = 1$

$$Z_{\text{pert}}^{\text{def}, n=1} = \prod_{\alpha=1}^N \frac{\theta_1(\tau | z - a_\alpha)}{i \eta(\tau)}.$$

(SU(N)) convention: $\sum_{\alpha=1}^N a_\alpha = 0$.

D2–D4' · STRING-SECTOR DRESSING

For $k > 0$, integrate the D2 holonomies u_s by Jeffrey–Kirwan residues.

$$Z_k^{\text{D2+def}} = \frac{1}{k!} \oint_{\text{JK}} \prod_{s=1}^k \frac{du_s}{2\pi i} Z_{k,\text{bulk}}^{\text{D2}}(\mathbf{u}) \prod_{s=1}^k \prod_{\ell=1}^n \mathcal{W}(u_s - z_\ell),$$

$$\mathcal{W}(u - z) = \frac{\theta_1(\tau | u - z + \epsilon_-) \theta_1(\tau | u - z - \epsilon_-)}{\theta_1(\tau | u - z + \epsilon_+) \theta_1(\tau | u - z - \epsilon_+)}.$$

$Z_{k,\text{bulk}}^{\text{D2}}$ contains the D2–D2 and D2–D6 determinants; $\mathcal{W}$ is the additional D2–D4' factor.

Wilson surface as defect

$$Z_{\text{D6-D4}'}^{(1)}(\tau, z, \mathbf{a}) = \prod_{\alpha=1}^N \frac{\theta_1(\tau \mid z - a_\alpha)}{i\eta(\tau)}, \quad \sum_{\alpha=1}^N a_\alpha = 0,$$

$$= (-1)^N \sum_{\mu=0}^{N-1} \mathcal{X}_\mu^{A_{N-1}}(\tau, \mathbf{a}) \Theta_\mu^{(N)}(\tau, z).$$

THE N -DIMENSIONAL THETA BASIS

THE A_{N-1} LATTICE CHARACTER

$$\Theta_\mu^{(N)}(\tau, z) := \frac{1}{[i\eta(\tau)]^N} \vartheta \left[\begin{matrix} \frac{1}{2} + \frac{\mu}{N} \\ \frac{N}{2} \end{matrix} \right] (Nz \mid N\tau).$$

$$L = A_{N-1} = \{\ell \in \mathbb{Z}^N : \ell \cdot \mathbf{1} = 0\},$$

$$\omega_\mu = (\underbrace{1, \dots, 1}_\mu, 0, \dots, 0) - \frac{\mu}{N} \mathbf{1},$$

$$\mathcal{X}_\mu^{A_{N-1}}(\tau, \mathbf{a}) := \sum_{\lambda \in L + \omega_\mu} q_\tau^{\lambda^2/2} e^{-2\pi i \lambda \cdot \mathbf{a}}, \quad q_\tau = e^{2\pi i \tau}.$$

FULL COEFFICIENT: $\langle W_\mu \rangle := \mathcal{X}_\mu^{A_{N-1}} + \sum_{k \geq 1} Q^k W_{k, \mu}$

$$\mathcal{Z}_{\text{def}}(z) := \frac{Z_{6d+\text{def}}^{(1)}(z)}{Z_{6d}} = (-1)^N \sum_{\mu=0}^{N-1} \Theta_\mu^{(N)}(\tau, z) \langle W_\mu \rangle.$$

Discrete theta projection

$$\langle W_\mu \rangle = \frac{(-1)^N}{N \Theta_\mu^{(N)}(\tau, z_\star)} \sum_{j=0}^{N-1} (-1)^j e^{-2\pi i \mu j / N} \mathcal{Z}_{\text{def}} \left(z_\star + \frac{j}{N} \right), \quad \mu \in \mathbb{Z}_N.$$

$\mathbf{W} = (\langle W_0 \rangle, \dots, \langle W_{N-1} \rangle)^\top$ is a **vector-valued Jacobi form** in the dual theta representation.

SCALAR JACOBI FORM

$$T: \quad \phi_{k,m}(\tau + 1, u) = \phi_{k,m}(\tau, u),$$

$$S: \quad \phi_{k,m} \left(-\frac{1}{\tau}, \frac{u}{\tau} \right) = \tau^k e^{2\pi i m u^2 / \tau} \phi_{k,m}(\tau, u).$$

VECTOR-VALUED JACOBI FORM

$$\vartheta = (\theta_1, \theta_2, \theta_3, \theta_4)^\top, \quad \theta_r = \theta_r(\tau \mid u).$$

$$T: \quad \vartheta(\tau + 1, u) = (e^{\pi i / 4} \theta_1, e^{\pi i / 4} \theta_2, \theta_4, \theta_3)^\top,$$

$$S: \quad \vartheta \left(-\frac{1}{\tau}, \frac{u}{\tau} \right) = \kappa(\tau, u) (-i\theta_1, \theta_4, \theta_3, \theta_2)^\top, \\ \kappa(\tau, u) = (-i\tau)^{1/2} e^{\pi i u^2 / \tau}.$$

Discrete theta projection

$$\langle W_\mu \rangle = \frac{(-1)^N}{N \Theta_\mu^{(N)}(\tau, z_\star)} \sum_{j=0}^{N-1} (-1)^j e^{-2\pi i \mu j / N} \mathcal{Z}_{\text{def}}\left(z_\star + \frac{j}{N}\right), \quad \mu \in \mathbb{Z}_N.$$

$\mathbf{W} = (\langle W_0 \rangle, \dots, \langle W_{N-1} \rangle)^\top$ is a vector-valued Jacobi form in the dual theta representation.

	$N = 1$	$N = 2$	$N = 3$	$N = 4$
$\mathbf{T}$	(1)	diag(1, i)	diag(1, ω , ω)	diag(1, ξ , -1, ξ)
$\mathbf{S}$	(1)	$\frac{e^{-\pi i/4}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	$-\frac{i}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$	$\frac{e^{-3\pi i/4}}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$

Wilson surface expansion

Wilson surface expansion

$$Z_X^{6d} = Z_{1\text{-loop}}^{6d} [Z_0 + Q_F Z_1 + \mathcal{O}(Q_F^2)]$$

$$Z_1 = Z_0 (\mathbf{H}^{(N)})^\top \mathbf{W}^{(N)} = Z_0 \sum_{\mu=0}^{N-1} H_\mu^{(N)} \langle W_\mu \rangle$$

THE WILSON VECTOR

Original Wilson vector

$$\mathbf{W}^{(N)} = (\langle W_0 \rangle, \dots, \langle W_{N-1} \rangle)^\top,$$

$$\langle W_\mu \rangle = \mathcal{X}_\mu^{A_{N-1}} + \sum_{k \geq 1} Q_E^k W_{k,\mu}.$$

TWO CONSISTENCY TESTS

Modular scalarity

$$\rho_H(\gamma)^\top \rho_W(\gamma) = I, \quad \gamma = S, T.$$

Heterotic/K3 normalization

$$\sum_{\mu=0}^{N-1} H_\mu^{(N)} \Theta_\mu^{(N)}(\tau, \mathbf{0}) \stackrel{!}{=} \mathcal{I}_{1,0}^H$$

$$= \frac{2E_4 E_6}{\eta^{24} \phi_{-2,1}(\tau, \epsilon)}.$$

Wilson surface expansion

$$E_{k,1}^{AN-1}(\tau, \mathbf{a}) = \sum_{\mu=0}^{N-1} \mathcal{E}_{\mu}^{(k,N)}(\tau) \mathcal{X}_{\mu}^{AN-1}(\tau, \mathbf{a}), \quad \mathcal{E}_k^{(N)} := (\mathcal{E}_0^{(k,N)}, \dots, \mathcal{E}_{N-1}^{(k,N)})^{\top}.$$

$$\mathbf{H}^{(N)} = \frac{\mathrm{i}^N}{6\eta^{24-N}\phi_{-2,1}(\tau, \epsilon)} \left(5E_4 \mathcal{E}_6^{(N)} + 7E_6 \mathcal{E}_4^{(N)} \right), \quad 2 \leq N \leq 7.$$

RANK-ONE CONSISTENCY CHECK

ORDINARY EISENSTEIN SERIES

H-STRING NORMALIZATION

$$N = 1 : \quad \mathcal{E}^{(4,1)} = E_4, \quad \mathcal{E}^{(6,1)} = E_6. \quad \Rightarrow \quad \mathcal{I}_{1,0}^H = \frac{2E_4E_6}{\eta^{24}\phi_{-2,1}(\tau, \epsilon)}.$$

Wilson surface expansion

WHAT WAS COMPUTED

$$\frac{Z_X^{5d}}{Z_{\text{pert}}^{5d}} = Z_{\text{inst}, \mathbf{0}}^{5d} + \sum_{r \geq 1} (Q_{\text{heavy}}^{5d})^r \sum_{(\mathbf{h}, \beta, \ell) \in \mathcal{S}_r^{(N)}} P_{r, \mathbf{h}, \beta}^{5d}(x) Q_{\text{inst}}^\beta Z_{\text{inst}, \mathbf{h}}^{5d}$$

N	5d theory	$Q_E^{\max}$	$q_\tau^{\max}$	$d_{\text{Coul}}^{\text{tot}}$	overlapped
1	$SU(2)$, theta-special	5	6	—	3
2	$SU(3)_1$	5	5	$[0, 6]$	167
3	$SU(4)_0$	4	4	$[0, 6]$	118
4	$SU(5)_{-1}$	3	4	$[0, 6]$	87
5	$SU(6)_{-2}$	3	4	$[0, 6]$	86

Here $d_{\text{Coul}}^{\text{tot}} = d_{A_6}$ is the total 5d Coulomb expansion degree.

Conclusion

- **Wilson-surface expansion in minimally coupled 6d supergravity**
 - Circle reduction/degravity maps the 6d $SU(N)$ theory to 5d $SU(N+1)_{3-N}$.
- **Wilson-surface VEVs form a vector-valued Jacobi form**

Future directions

- Higher H-string charges
- Other rank-one 6d theories
- Tests in exotic supergravity models
- Models with $U(1)$ or discrete gauge groups

Thank you!