

Non-abelian BF Theory as continuous Symmetry TFT (SymTFT)

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Background and Motivation

Generalized Symmetries

[Gaiotto, Kapustin, Seiberg, Willett, 14]

In recent years, it was advocated that symmetry can be understood in terms of topological operators:

- ▶ The generators are codimensional-1 operators satisfying the fusion rule

$$\begin{array}{c} \uparrow \\ g \end{array} \begin{array}{c} \uparrow \\ h \end{array} = \begin{array}{c} \uparrow \\ gh \end{array}$$

- ▶ The symmetry action is implemented by wrapping the symmetry generator around the local operators

$$g \circ \mathcal{O}^a = \rho(g)^a_b \mathcal{O}^b$$

Gauging (sub)symmetries and SymTFT

[Bhardwaj, Tachikawa, 17] [Tachikawa, 17]

Given a theory $\mathfrak{T}(\mathcal{C})$ with the (generalized) finite symmetry $\mathcal{C}$

- ▶ If $\mathcal{C}$ is anomaly-free, we can gauge $\mathcal{C}$
- ▶ Even if $\mathcal{C}$ is anomalous, we can still gauge an anomaly-free sub-symmetry $\mathcal{H} \in \mathcal{C}$

One might ask the following two questions:

- ▶ What's the resulting theory $\mathfrak{T}/\mathcal{C}$ (or $\mathfrak{T}/\mathcal{H}$)
- ▶ What's the dual symmetry $\mathcal{C}^\vee$ after gauging?

SymTFT is a useful tool encoding various gauging procedures in a unified picture. [Witten, 98] [Kong, Lan, Wen, Zhang, Zheng, 20] [Gaiotto, Kulp, 20] [Apruzzi, Bonetti, Garcia Etxebarria, Hosseini, Schafer-Nameki, 21] [Kaidi, Ohmori, Zheng, 22]

Our goal is to generalize the SymTFT picture to continuous and infinite symmetries, such as Lie groups G .

Some motivations

- ▶ Continuous symmetries, like G -flavor symmetries, play a fundamental role in physics.
- ▶ *Flat gauging* continuous symmetry can lead to new CFTs in 2D, like “R-W” CFT
[Runkel,Watts,01] [Roggenkamp,Wendland,04] [Gaberdiel,Suchanek,11]
- ▶ It is a TFT with uncountable infinite number of topological operators, which is not yet mathematically well-studied
- ▶ It provides an interesting model generalizing the modular structure for finite 3D TFT
- ▶ Boundary CFT and 3D gravities [Maloney, Witten,20][Afkhami-Jeddi,Cohn,Hartman,Tajdini,20][Yu,26]

Content

This talk is a summary of our recent study on continuous SymTFT. After a brief review of SymTFT, we shall focus on

- ▶ Non-abelian BF theory as a SymTFT for Lie group
[Jia,Tian,Luo,Wang,Zhang,25]
- ▶ Modular properties and gapped boundaries [Jia,Ma,Tian,26]

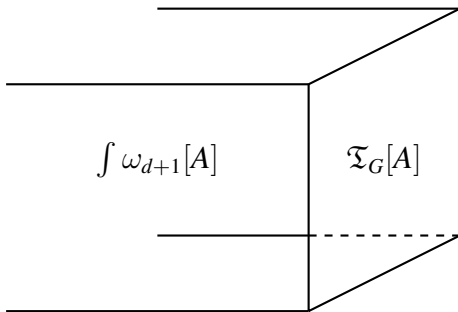
then we will conclude and briefly mention other aspects that are not covered, and on-going works

Review of SymTFT

Basic idea

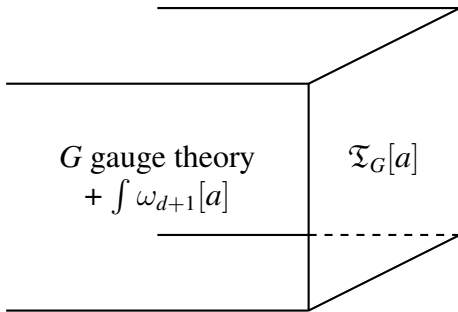
Suppose we have a d -dim theory $\mathfrak{T}_G[A]$ with global symmetry G and background A .

First, we couple it with a $(d + 1)$ -dim TFT $S_{\text{inflow}} = \int \omega_{d+1}[A]$, which cancels the anomaly by inflow

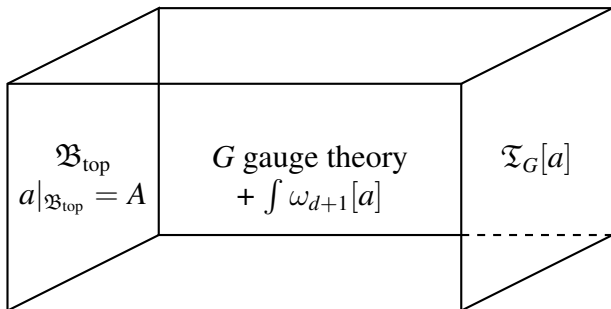


If G is anomaly-free, then S_{inflow} is a trivial TFT

In the spirit of holography, we promote the bulk G symmetry to a gauge symmetry [Dijkgraaf, Witten, 90]



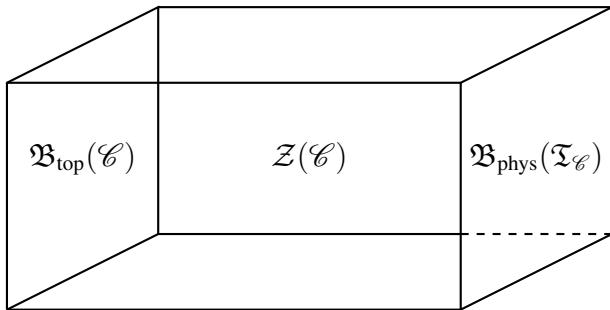
To recover the d -dim theory $\mathfrak{T}_G[A]$, we add a topological boundary $\mathfrak{B}_{\text{top}}$, where the gauge field a satisfies the Dirichlet boundary condition $a|_{\mathfrak{B}_{\text{top}}} = A$.



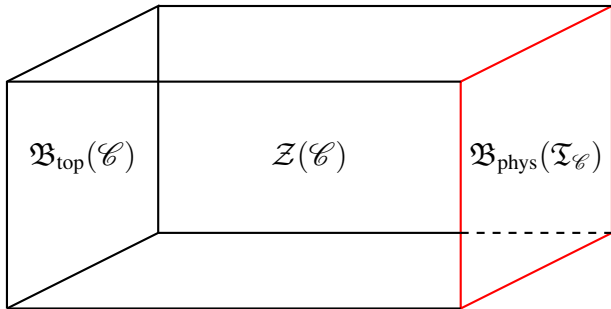
Since the bulk is a TFT, we can shrink the interval and recover $\mathfrak{T}_G[A]$.

What is a SymTFT

Given a d -dimensional theory $\mathfrak{T}_{\mathcal{C}}$ with the global symmetry $\mathcal{C}$ living on M_d , the SymTFT is a $(d + 1)$ -dimensional TFT $\mathcal{Z}(\mathcal{C})$ living on $[0, 1] \times M_d$ with two boundaries, which separates the symmetry data from the dynamics

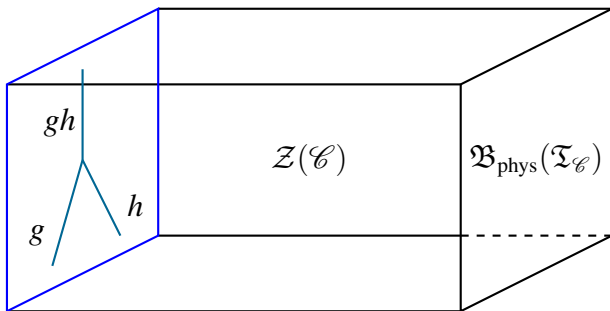


Physical boundary $\mathfrak{B}_{\text{phys}}$



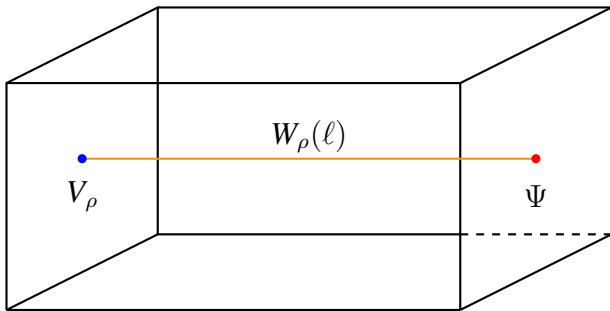
Physical (dynamical) boundary $\mathcal{B}_{\text{phys}} = \{1\} \times M_d$ encodes the dynamical data of the theory $\mathfrak{T}_{\mathcal{C}}$

Symmetry (topological) boundary $\mathfrak{B}_{\text{top}}$



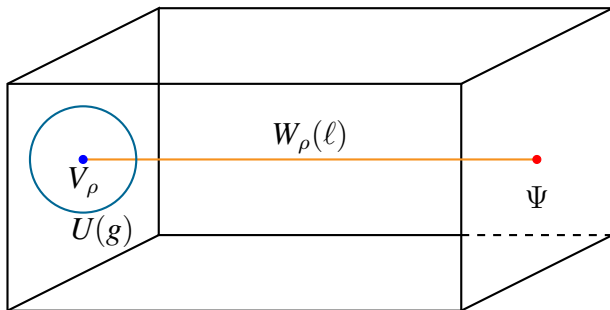
The symmetry (topological) boundary $\mathcal{B}_{\text{top}} = \{0\} \times M_d$ is topological, and the symmetry generators live here. It is purely kinematic.

Charged operators



The charged operators ψ_ρ in $\mathfrak{T}_\mathcal{C}$ is represented as certain topological line operators $W_\rho(\ell)$, labeled by ρ , stretching between the two boundaries.

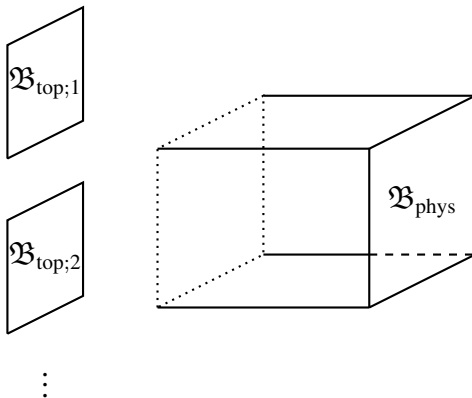
Symmetry transformation



The endpoint V_ρ on $\mathfrak{B}_{\text{top}}$ is a vector that encodes the components of ψ_ρ . Under the symmetry action $U(g)$ on $\mathfrak{B}_{\text{top}}$, the junction V_ρ is rotated by

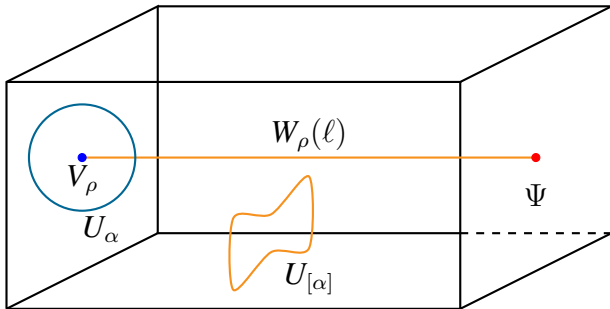
$$U(g)V_\rho = \sum_b \rho(g)_a{}^b (V_\rho)_b.$$

Gauging as changing boundary



Gauging in $\mathcal{T}_{\mathcal{C}}$ is implemented by changing different topological boundaries $\mathfrak{B}_{\text{top}}$ while fixing physical boundary $\mathfrak{B}_{\text{phys}}$.

Non-abelian BF theory as SymTFT for Lie group symmetry



SymTFT action

[Brennan,Sun,24] [Antinucci,Benini,24] [Bonetti,Del Zotto,Minasian,24]
[Jia,Tian,Luo,Wang,Zhang,25]

The action of non-abelian BF theory is

[Horowitz,89] [Blau,Thompson,91]

$$S_G = \int_{M_{d+1}} \text{Tr} B \wedge F + k \text{CS}[A] ,$$

where B is a $\mathfrak{g}$ -valued $(d-1)$ -form, $F = dA - iA \wedge A$ is the field strength, and $k \text{CS}[A]$ encodes the anomaly of G .

- ▶ B is a Lagrange multiplier imposing $F = 0$.
- ▶ EOM of A gives $D_A B = dB - i[A, B] = 0$
- ▶ There are two kinds of gauge transformation

$$\begin{aligned} A &\rightarrow gAg^{-1} + igdg^{-1}, & B &\rightarrow gBg^{-1} \\ B &\rightarrow B + D_A K, \end{aligned}$$

with g a G -valued function and K a $\mathfrak{g}$ -valued $(d-2)$ -form.

Extended operators

First of all, we have the Wilson loop operators

$$W_\rho(\ell) = \text{Tr} \mathcal{P} \exp \left(i \oint_\ell A_\rho \right) .$$

Labeled by the representation ρ of G . They are topological given the flatness condition $F = 0$.

We should also have Wilson operators for the B -field. Let us consider the following operators

$$U_\alpha(\Sigma_{d-1}) = \exp \left(i \int_{\Sigma_{d-1}} \text{Tr} \alpha B \right) \equiv \exp \left(i \int_{\Sigma_{d-1}} (\alpha, B) \right) ,$$

where α is a ($\mathfrak{g}$ -valued) covariantly constant section of $\text{ad}(P)$, satisfying $D_A \alpha = 0$ for given flat A , and $(\cdot, \cdot)$ is the inner product defined by trace.

Observation

- ▶ $U_\alpha(\Sigma_{d-1})$ is topological under small deformations due to

$$\int_{\Sigma_{d-1}-\Sigma'_{d-1}} (\alpha, B) = \int_{\partial\Sigma_d} (\alpha, B) = \int_{\Sigma_d} d(\alpha, B) = 0,$$

given $D_A\alpha = D_AB = 0$.

- ▶ $U_\alpha(\Sigma_{d-1})$ is invariant under $B \rightarrow B + D_AK$ since

$$\int_{\Sigma_{d-1}} (\alpha, D_AK) = - \int_{\Sigma_{d-1}} (D_A\alpha, K) = 0$$

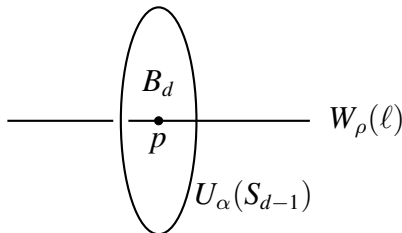
- ▶ $U_\alpha(\Sigma_{d-1})$ is also invariant under

$$A \rightarrow gAg^{-1} + igdg^{-1}, \quad B \rightarrow gBg^{-1},$$

since $\alpha \rightarrow g\alpha g^{-1}$ as a section, and $(\text{Ad}_g\alpha, \text{Ad}_gB) = (\alpha, B)$

Action of $U_\alpha(S_{d-1})$ on $W_\rho(\ell)$

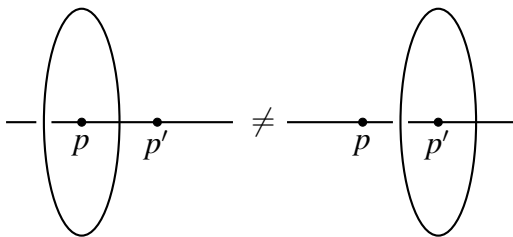
Let us focus on the case $k = 0$, and consider the action of $U_\alpha(S_{d-1})$ on $W_\rho(\ell)$ when S_{d-1} links ℓ in M_{d+1}



which gives

$$\begin{aligned}
 & \langle U_\alpha(S_{d-1}) W_\rho(\ell) \rangle \\
 &= \int \mathcal{D}B \mathcal{D}A e^{i \int_{M_{d+1}} \text{Tr}(B \wedge F)} \exp \left(i \int_{S_{d-1}} (\alpha, B) \right) \mathcal{P} \exp \left(i \int_\ell A_\rho \right) \\
 &= \left\langle \mathcal{P} \int^p \exp \left(i \int_\ell A_\rho \right) \times e^{-i \alpha_\rho(p)} \times \mathcal{P} \int_p \exp \left(i \int_\ell A_\rho \right) \right\rangle
 \end{aligned}$$

The action depends on the point p where they are linked



and it is not topological in the presence of Wilson lines.

To build a genuine topological operator, we need to get rid of the point dependence. Hence, we consider

$$U_{[\alpha]}(S_{d-1}) = \int dg \exp \left(i \int_{\Sigma_{d-1}} (\alpha_g, B) \right) ,$$

where α_g is defined by picking any point $\text{pt} \in S_{d-1}$, perform the conjugation $g\alpha(\text{pt})g^{-1}$, and then parallel transport $g\alpha(\text{pt})g^{-1}$ to other points of S_{d-1} .

Then the linking between $\tilde{U}$ and W reads

$$\begin{aligned} & \langle U_{[\alpha]}(\Sigma) W_{\rho}(\ell) \rangle \\ = & \langle \mathcal{P} \int^P \exp \left(i \int_{\ell} A_{\rho} \right) \times \left(\int dg g e^{-i\alpha_{\rho}(p)} g^{-1} \right) \times \mathcal{P} \int_p \exp \left(i \int_{\ell} A_{\rho} \right) . \end{aligned}$$

The middle operator commutes with any group element

$$\begin{aligned} & h \left(\int dg g e^{-i\alpha_{\rho}(p)} g^{-1} \right) \\ = & \int dg h g e^{-i\alpha_{\rho}(p)} (hg)^{-1} h \\ = & \left(\int dg g e^{-i\alpha_{\rho}(p)} g^{-1} \right) h , \end{aligned}$$

where we choose the Haar measure dg to be left-invariant.

By Schur's lemma, one has

$$\int dg \, g e^{-i\alpha_\rho(p)} g^{-1} = \frac{\chi_\rho(e^{-i\alpha(p)})}{\dim \rho} \times \mathbb{I}_{\dim \rho} ,$$

where χ_ρ is the group character of representation ρ , and it is independent of p since α at different points are related by parallel transport.

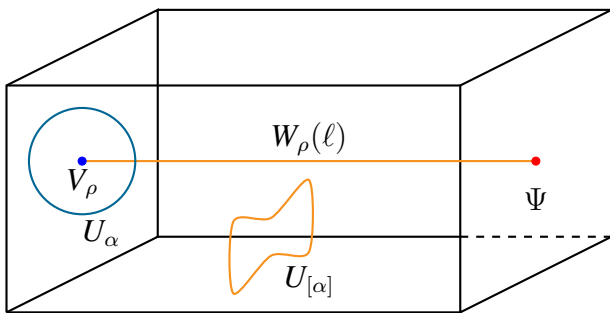
Thus, we have [Cattaneo,Rossi,02] [Jia,Tian,Luo,Wang,Zhang,25]

$$\langle U_{[\alpha]}(S_{d-1}) \, W_\rho(\ell) \rangle = \frac{\chi_\rho(e^{-i\alpha(p)})}{\dim \rho} W_\rho(\ell) ,$$

which depends only on the conjugacy class $[g] = [e^{-i\alpha}]$, rather than on a specific group element.

How does SymTFT work [Jia,Tian,Luo,Wang,Zhang,25]

Although $U_\alpha(\Sigma)$ is not genuinely topological in the bulk, it is perfectly well-behaved at the boundary.

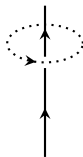
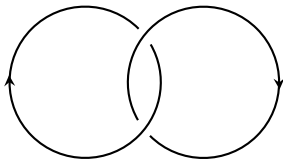


It acts at the endpoint of Wilson line according to

$$\langle U_\alpha(\Sigma) \Psi W_\rho(\ell) V_\rho \rangle = \langle \Psi \mathcal{P} \int_p \exp \left(i \int_\ell A_\rho \right) e^{-i\alpha_\rho V_\rho} \rangle$$

so that it rotates $V_\rho \rightarrow e^{-i\alpha_\rho} V_\rho$ as generators of G -symmetry.

Modular S/T-Kernels



General line operators in 3D

Let us focus on $3d$ non-abelian BF theory, where both W_ρ and $U_{[g]}$ are line operators, and we can put them together. What do the general line operators look like?

- ▶ From our construction, the $U_{[g]}$ operators can be understood as the collection

$$U_{[g]} = \int_G dx U_{\text{Ad}_x(g)} = \int_{[g]} d\mu(g) U_g .$$

- ▶ For a single U_g , it is invariant under the gauge transformation of the centralizer group $C_G(g) \in G$

$$C_G(g) = \{h \in G | hg = gh\} .$$

Therefore we can further assign to U_g a representation of $C_G(g)$.

Since $C_G(g)$ for all $g \in [g]$ are isomorphic, we then conclude that a general line operator $U_{[g],\rho}$ should be labeled by the pair $([g], \rho)$ such that

- ▶ $[g]$ is the conjugacy class of G
- ▶ ρ is an irreducible representation $C_G(g)$

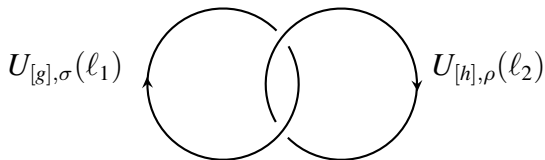
and formally, we have

$$U_{[g],\rho}(\ell) \sim \int dg \exp \left(i \int_{\ell} (\alpha_g, B) \right) \times \mathcal{P} \exp \left(i \int_{\ell} A_{\rho}^{C_G(g)} \right),$$

where $A_{\rho}^{C_G(g)}$ is the gauge field restricted to the Lie algebra of $C_G(g)$.

S-kernel [Jia, Ma, Tian, 2026]

We are interested in the Hopf link between two general line operators



And it gives the S -kernel element of the TQFT

$$\begin{aligned} S_{([g],\sigma),([h],\rho)}^{(k)} &= \int \mathcal{D}B \mathcal{D}A \, e^{i \int \text{tr} BF + \frac{k}{4\pi} \text{CS}[A]} U_{[g],\sigma}(\ell_1) U_{[h],\rho}(\ell_2) , \\ &= \int \mathcal{D}B \mathcal{D}A \, e^{i \int_{S^3} \text{tr} BF + \frac{k}{4\pi} \text{CS}[A]} \\ &\quad \times \int_{G \times G} dx dy \, \text{tr}_\sigma \mathcal{P} e^{i \int_{\ell_1} A} U_{\text{Ad}_x g}(\ell_1) \, \text{tr}_\rho \mathcal{P} e^{i \int_{\ell_2} A} U_{\text{Ad}_y h}(\ell_2) , \end{aligned}$$

We give a semi-classical expression.

We make the following approximation and assumption

- We integrate B , which imposes the sources of A

$$F = -\alpha_x \delta_{\ell_1}^{(2)} - \beta_y \delta_{\ell_2}^{(2)}, \quad (e^{-i\alpha_x} = \text{Ad}_x(g), e^{-i\alpha_y} = \text{Ad}_y(h))$$

which imposes the boundary condition

$$\text{Hol}_A(\ell_2) = \text{Ad}_x g, \quad \text{Hol}_A(\ell_1) = \text{Ad}_y h.$$

- We assume the integral $\int_{G \times G} dx dy$ is effectively restricted to the moduli space of flat connection with induced measure

$$\mathcal{M} := \{x, y \in G \times G | \text{Hol}_A(\ell_1) \text{Hol}_A(\ell_2) = \text{Hol}_A(\ell_2) \text{Hol}_A(\ell_1)\}.$$

- We ignore loop corrections

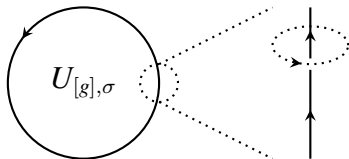
After that, we obtain

$$S_{([g], \sigma), ([h], \rho)}^{(k)} = \sum_{w \in W} e^{-\frac{ik}{2\pi} \text{tr}(\alpha \text{Ad}_w \beta)} \chi_{\sigma}^* (\text{Ad}_w h) \chi_{\rho}^* (\text{Ad}_{w^{-1}} g),$$

where $g = e^{-i\alpha}$, $h = e^{-i\beta}$ are regular elements of G , and W is the Weyl group.

T-kernel

To evaluate the T-kernel, consider a line operator $U_{[g],\sigma}$, whose two endpoints are glued by a 2π -twist.



Schematically, one is led to

$$T_{([g],\sigma),([h],\rho)}^{(k)} \propto \delta_{[g],[h]} \delta_{\sigma,\rho} \int \mathcal{D}B \mathcal{D}A \, e^{i \int_{S^3} BF + \frac{k}{4\pi} \text{CS}} U_{[g],\sigma}(\ell) .$$

Follow the same approximation, we have

$$T_{([g],\sigma),([h],\rho)}^{(k)} = \delta_{[g],[h]} \delta_{\sigma,\rho} e^{\frac{ik}{4\pi} \text{tr } \alpha^2} \frac{\chi_{\sigma}(g)}{d_{\sigma}} .$$

Modular properties

We have also checked the S/T -kernel (regular case) we obtained satisfy several desirable properties

$$SS^\dagger = TT^\dagger = 1, \quad S^2 = (ST)^3 = C.$$

- ▶ We check them for $G = SU(N)$ with arbitrary k .
- ▶ The S -kernel with singular elements do not contribute, since they have measure zero.

Example : $U(1)^{(k)}$

The simplest example is $G = U(1)$, the group element is $e^{i\theta}$ for $\theta \in [0, 2\pi)$. The Weyl group is trivial, and the centralizer $C_{U(1)}(g) = U(1)$ for all $g \in U(1)$.

The S -kernel reads

$$S_{(g,n),(h,m)}^{(k)} = e^{-i\frac{2k}{2\pi}\theta_x\theta_y} e^{-im\theta_x} e^{-in\theta_y} = e^{-i(m\theta_x+n\theta_y+\frac{2k}{2\pi}\theta_x\theta_y)} ,$$

where $g = e^{i\theta_x}$, $h = e^{i\theta_y}$, $n, m \in \mathbb{Z}$. Similarly, the T -kernel is evaluated as

$$T_{(g,n),(h,m)}^{(k)} = \delta_{2\pi}(\theta_x - \theta_y) \delta_{n,m} \exp \left(i \left(n\theta_x + \frac{k\theta_x^2}{2\pi} \right) \right) .$$

Example : $SU(2)^{(k)}$

One can parametrize the maximal torus T as

$$\exp(i\alpha H) = \begin{pmatrix} e^{\frac{i\alpha}{2}} & \\ & e^{-\frac{i\alpha}{2}} \end{pmatrix}, \quad \text{with} \quad H = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $\alpha \sim \alpha + 4\pi$.

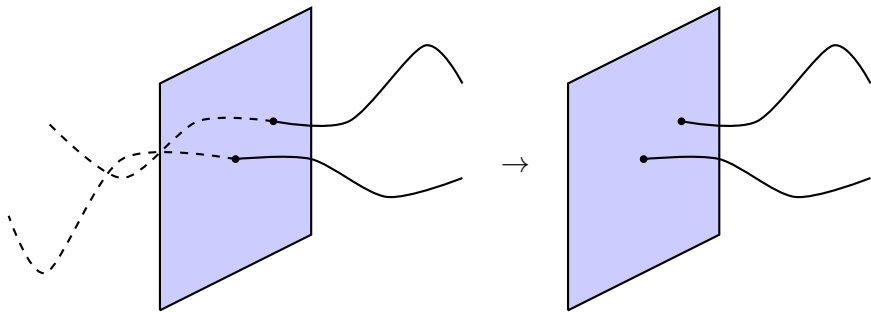
- ▶ The Weyl reflection $S_2 = \mathbb{Z}_2$ flips the sign of α .
- ▶ We can restrict $\alpha \in (0, 2\pi)$ as a representative of the conjugacy class $g = e^{i\alpha} \in [g]$.
- ▶ The centralizer is $U(1)$, whose representations are labeled by integers.

The S/T kernels are separately given by

$$\begin{aligned} S_{([g],n),([h],m)}^{(k)} &= e^{-i\frac{k}{4\pi}\alpha\beta} e^{-\frac{i}{2}m\alpha - \frac{i}{2}n\beta} + e^{i\frac{k}{4\pi}\alpha\beta} e^{\frac{i}{2}m\alpha + \frac{i}{2}n\beta} \\ &= 2 \cos \frac{1}{2} \left(m\alpha + n\beta + \frac{k}{2\pi}\alpha\beta \right). \end{aligned}$$

$$T_{([g],n),([h],m)}^{(k)} = e^{i\frac{k}{8\pi}\alpha^2} e^{\frac{i}{2}n\alpha} \delta_{4\pi}(\alpha - \beta) \delta_{n,m}.$$

Gapped boundaries and Lagrangian algebras



Lagrangian algebra

The collection of bulk line operators L_i that can end on the topological boundary $\mathfrak{B}_{\text{top}}$ are encoded into the algebraic object

$$\mathcal{L} = \bigoplus_i n_i U_i, \quad n_i \in \mathbb{Z}_{\geq 0}$$

with n_i the dimension of the topological junction. Denote the vector $V_{\mathcal{L}}$ as

$$V_{\mathcal{L}} = (n_0, n_1, \dots, n_N),$$

they are the invariant vectors of S/T

$$S \cdot V_{\mathcal{L}} = V_{\mathcal{L}}, \quad T \cdot V_{\mathcal{L}} = V_{\mathcal{L}}.$$

For continuous SymTFT, both S/T and $V_{\mathcal{L}}$ are infinite dimensional.

Lagrangian algebras for $SU(2)^{(0)}$ [Jia, Ma, Tian, 2026]

We work out several Lagrangian algebra for $SU(2)^{(0)}$ and check the invariance under the S/T action.

► Dirichlet type

$$\mathcal{L}_{\text{Dir}} = \bigoplus_{j \geq 0} (j+1) U_{(0,j)} ,$$

► Neumann type

$$\mathcal{L}_{\text{Neu}} = \bigoplus_{\alpha \in [0, 2\pi]} U_{(\alpha, 0)} .$$

► $SO(3)$ type

$$\mathcal{L}_{\text{SO}(3)} = \bigoplus_{j \geq 0} \left((2j+1) U_{(0, 2j)} \oplus (2j+1) U_{(2\pi, 2j)} \right) .$$

Finally, we have the A_q -type candidate data $\mathcal{L}_{A_q}$

$$\mathcal{L}_{A_{q,\text{odd}}} = \left(\bigoplus_{j \geq 0} d_q(j) U_{(0,j)} \right) \oplus \left(\bigoplus_m \bigoplus_{n=1}^{\left[\frac{q}{2}\right]} 2U_{\left(\frac{4\pi}{q}n, qm\right)} \right),$$

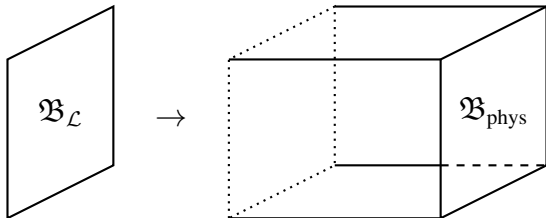
for odd q and

$$\begin{aligned} \mathcal{L}_{A_{q,\text{even}}} = & \left(\bigoplus_{j \geq 0} d_q(j) U_{(0,j)} \right) \oplus \left(\bigoplus_m \bigoplus_{n=1}^{\left[\frac{q}{2}\right]-1} 2U_{\left(\frac{4\pi}{q}n, qm\right)} \right) \\ & \oplus \left(\bigoplus_{j \geq 0} d_q(j) U_{(2\pi,j)} \right), \end{aligned}$$

for even q , where $d_q(j)$ is defined as

$$d_q(j) = \frac{1}{q} \sum_{n=0}^{q-1} \chi_j \left(\frac{4\pi}{q}n \right).$$

Topological boundaries



Changing different topological boundaries gives different theories related by (flat) gauging

$\mathcal{L}_{\text{Dir}}$	Original theory
$\mathcal{L}_{\text{Neu}}$	(Flat) Gauge $SU(2)$
$\mathcal{L}_{\text{SO}(3)} (= \mathcal{L}_{A_2})$	Gauge $\mathbb{Z}_2 \in SU(2)$
$\mathcal{L}_{A_N}$	Gauge $\mathbb{Z}_N \in SU(2)$

Conclusion

- ▶ We review the non-abelian BF theory as a SymTFT of continuous symmetry
- ▶ We give a semi-classical computation of the modular S/T -kernels, and check some Lagrangian algebras in $SU(2)$ case
- ▶ Open questions : D/E -type boundaries, Verlinde formula, dual symmetries $\dots$

Other directions

- ▶ We propose a categorical construction for the Lie group symmetry, as the category of unitary representation of the Hopf C^* -algebra $C_0(G)$. [Jia,Tian,Luo,Wang,Zhang,25]
- ▶ We also study the Hilbert space and the action of line operators in the non-abelian BF theory [Jia,Tian,2026]
- ▶ (On-going) non-abelian BF and $\lim_{k \rightarrow \infty} \hat{G}_k \times \hat{G}_{-k}$

Thank you