SOUTHEAST UNIVERSITY

SEU Quantum Field Theory Group

Quantum Field Theory Methods for Classical Binary Dynamics

刘正文

东南大学物理学院与丘成桐中心

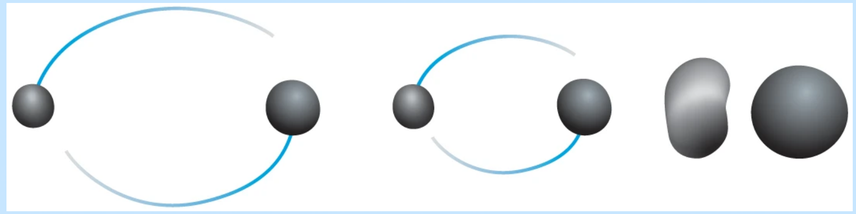
第七届全国“场论与弦论”学术研讨会

彭桓武高能基础理论研究中心

Peng Huanwu Center for Fundamental Theory

Outline

- Two-body problem: scattering vs bound



- Modern Effective Field Theory Methods

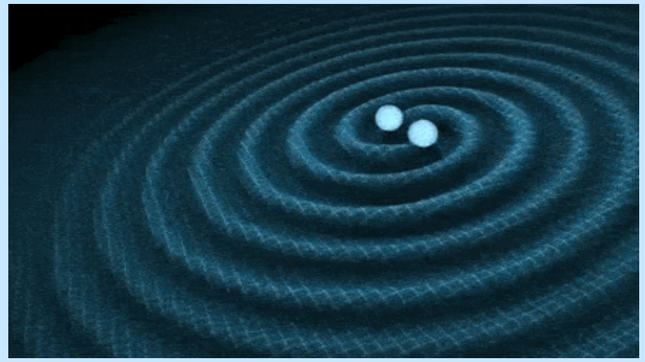
$e^{i\mathcal{S}_{\rm eff}} = \int \mathcal{D}h e^{i(\mathcal{S}_{\rm EH}+\mathcal{S}_{\rm WL})}$ Feynman's path integral → Feynman diagrams → $\mathrm{Li}_3(z) = \int_0^z \frac{\mathrm{d}x\,\mathrm{Li}_2(x)}{x}$ Feynman loop integrals → GW Observables

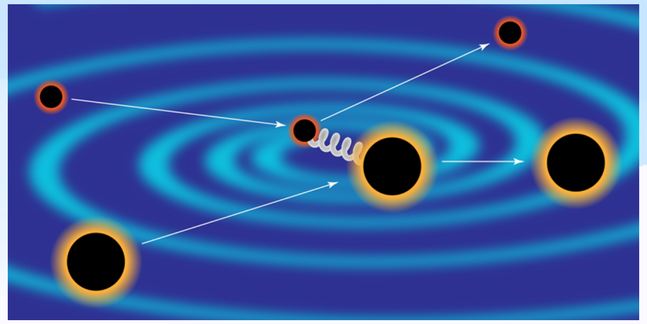
- QFT toolbox: IBP, DE & iterated integrals

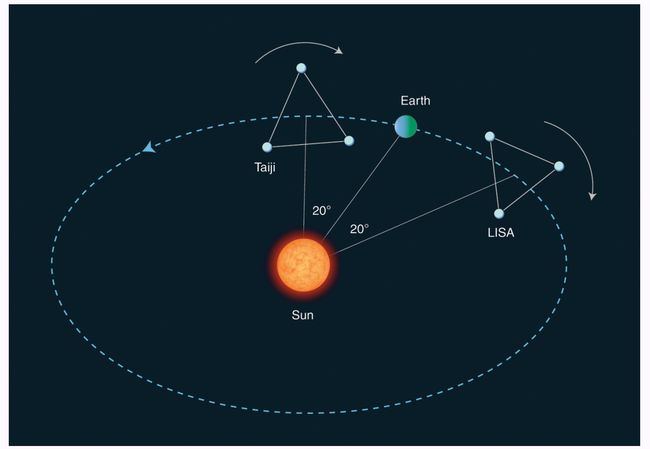
$\int d^d\ell \cdots \frac{\partial}{\partial \ell^\mu} \frac{v^\mu}{P(\ell,\ldots)} = 0$, $\boldsymbol{f}(x,\epsilon) = \{I_1, I_2, \ldots, I_N\}$ → $\frac{d\boldsymbol{f}(x,\epsilon)}{dx} = \boldsymbol{A}(x,\epsilon)\,\boldsymbol{f}(x,\epsilon)$ → $\boldsymbol{g} = \boldsymbol{T}\,\boldsymbol{f}$ → $\frac{d\boldsymbol{g}(x,\epsilon)}{dx} = \epsilon\,\mathcal{M}(x)\boldsymbol{g}(x,\epsilon)$ → $\boldsymbol{g}(x,\epsilon) = \sum_n \epsilon^n \boldsymbol{g}^{(n)}(x)$, $\boldsymbol{g}^{(n)}(x) = \int_{x_0}^{x} \mathcal{M}(t)\boldsymbol{g}^{(n-1)}(t)\,dt + \boldsymbol{g}_0^{(n)}$

- Precision Frontier: two-body dynamics to NNNNLO

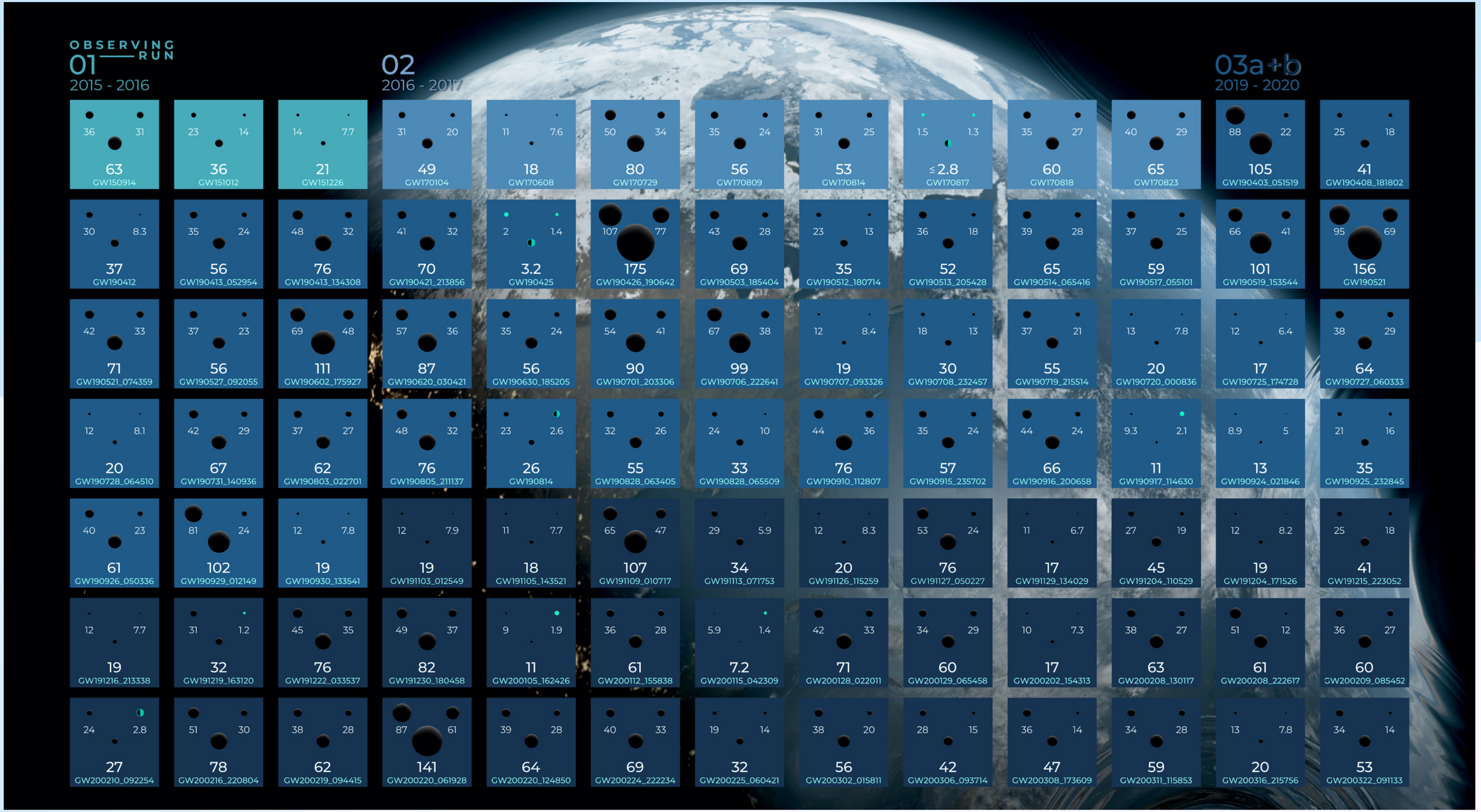
$$\alpha_1 G + \alpha_2 G^2 + \alpha_3 G^3 + \alpha_4 G^4 + \alpha_5 G^5 + \cdots$$

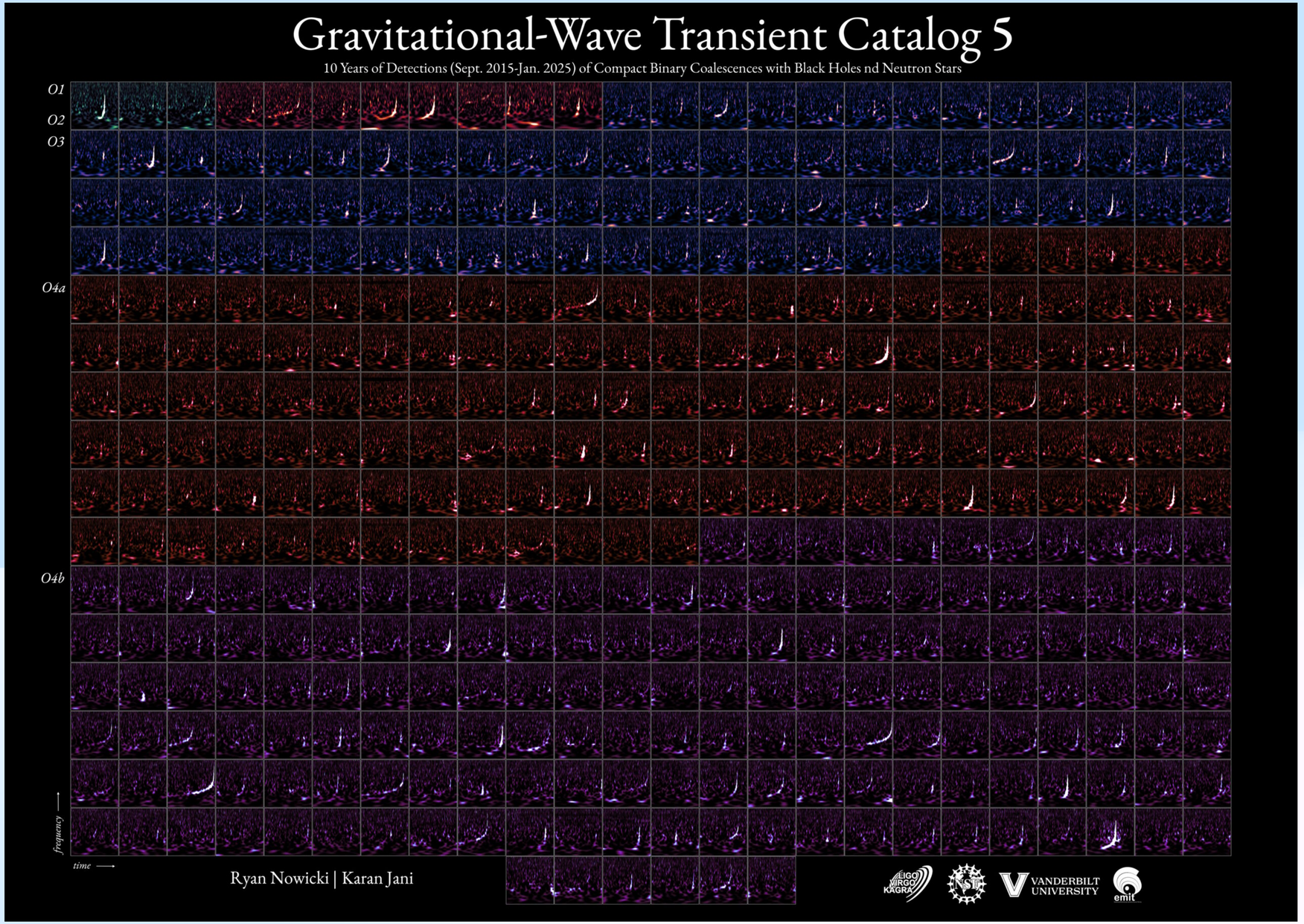






Gravitational-wave physics is now a precision science!





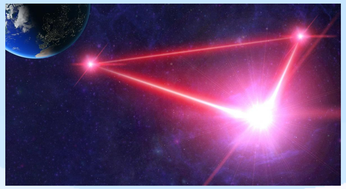
Binary Black Hole: 381

Binary Neutron Star: 2

Binary BH-NS: 7

https://gwosc.org/eventapi/html/GWTC-5.0/

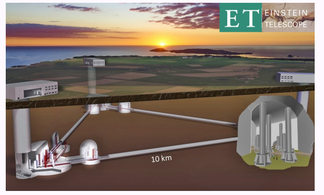










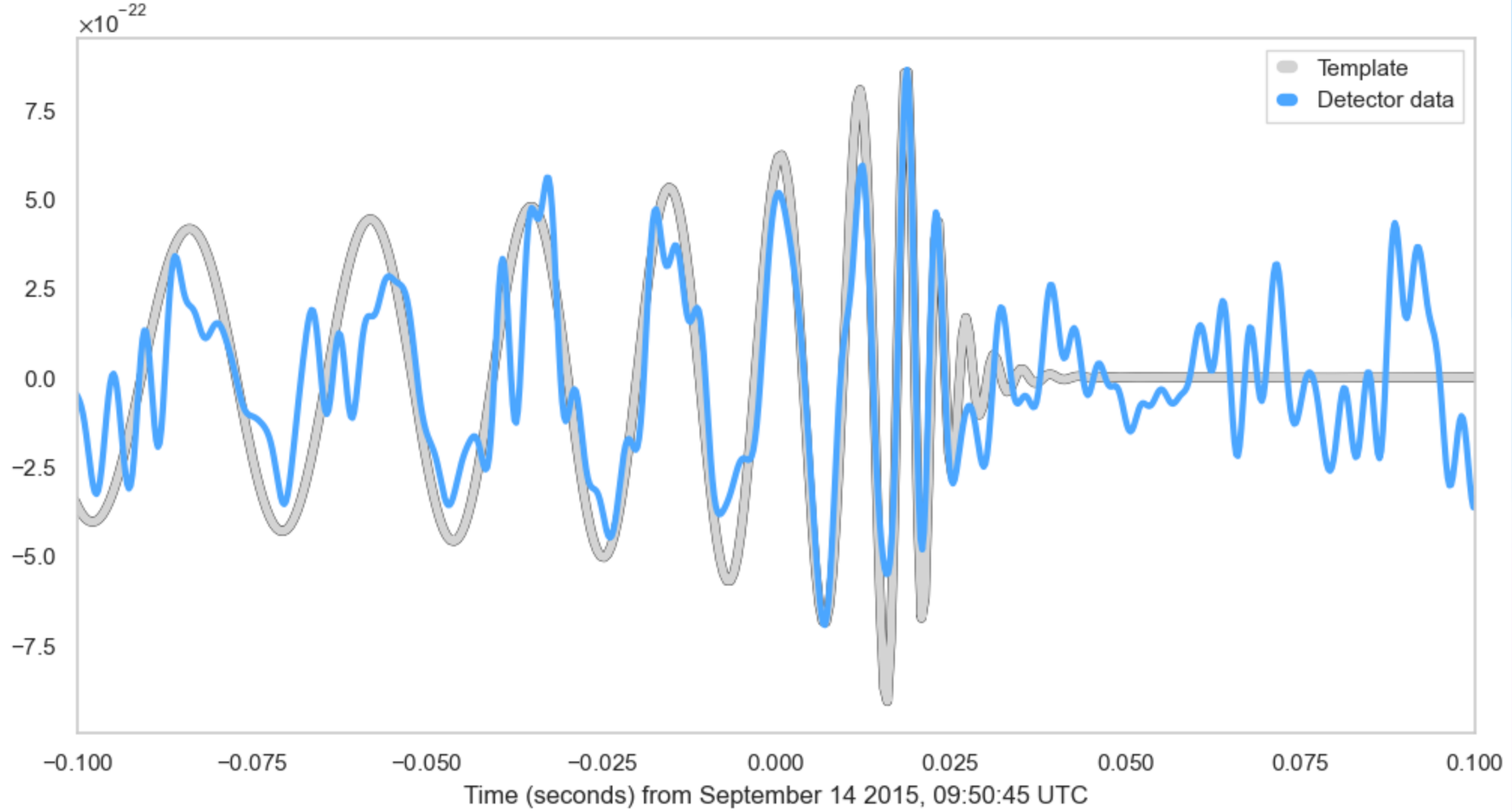




Recording a gravitational wave has never been a big motivation for LIGO, the motivation has always been to open a new window on the universe.

GW waveforms

Discovery Potential = Precise Theoretical Predictions

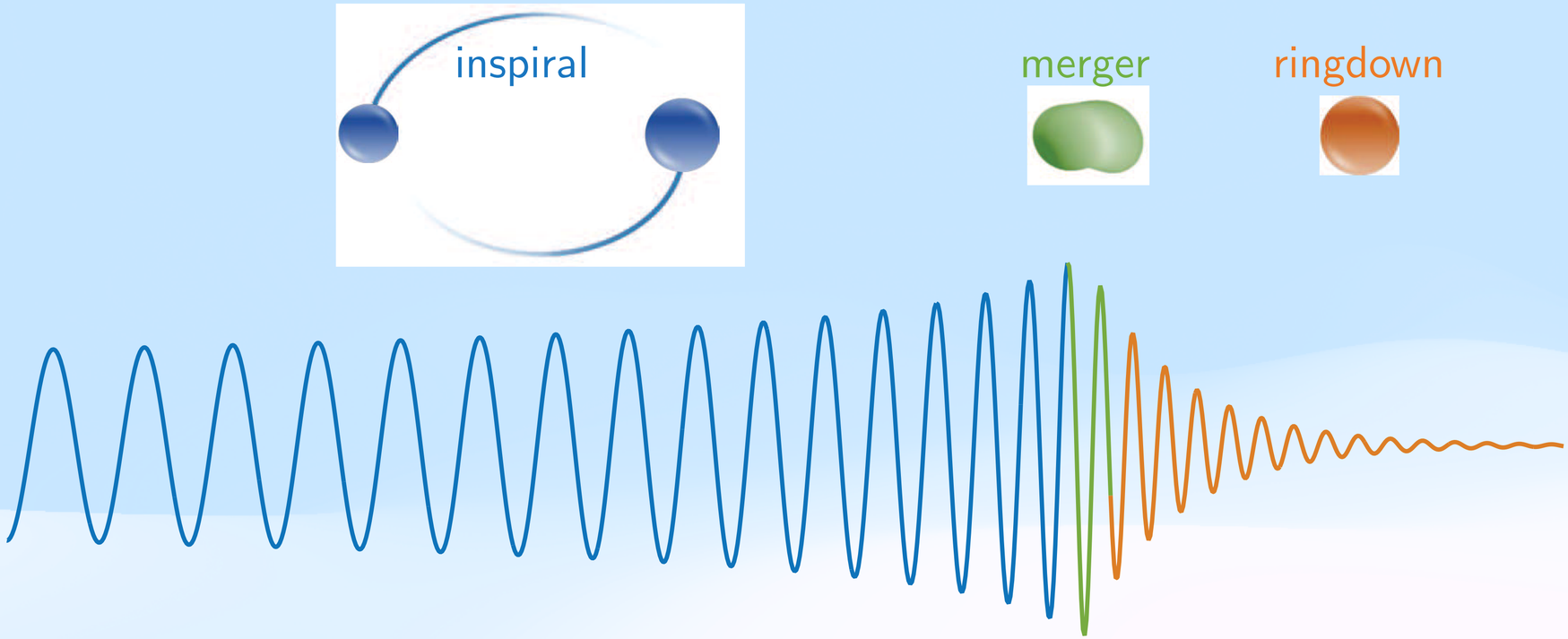




K. Thorne: Waveforms will be far more complex and carry more information than expected.

Improved modeling is crucial for extracting the GW's information

Gravitational two-body problem



Merger: Numerical Relativity

Ringdown: black hole perturbation theory

Inspiral: the interaction between two bodies is weak

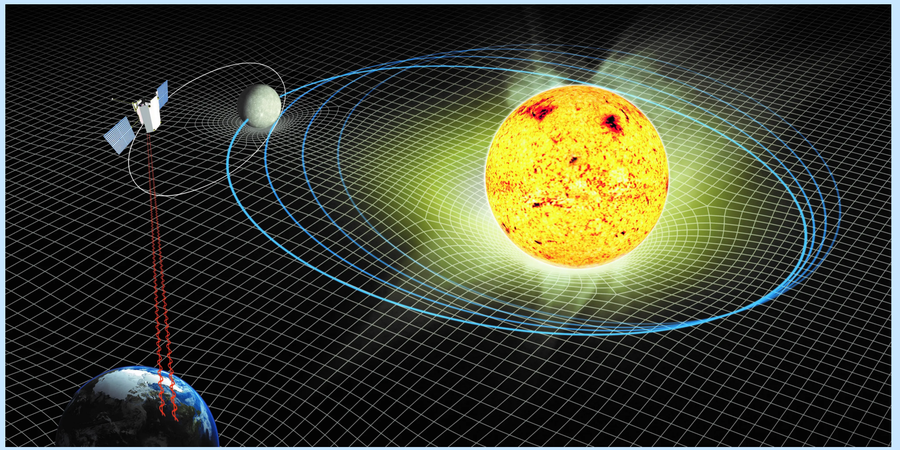
- Numerical Relativity: accurately, but computationally expensive
- Analytic methods: corrections in v or G are perturbatively calculable

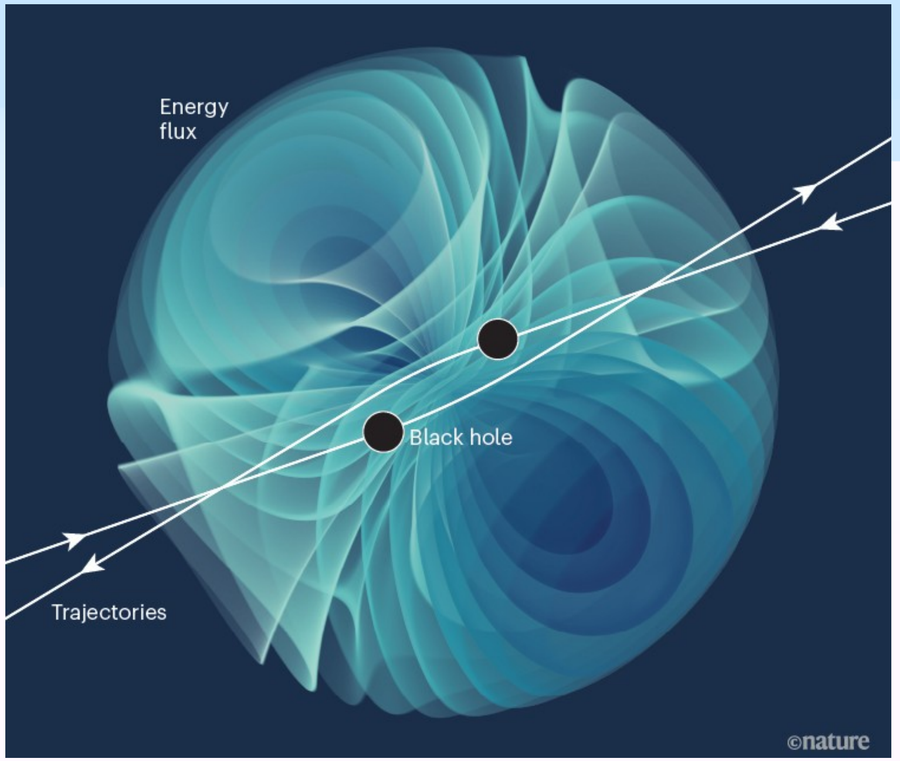
low velocity: $v^2 \sim \frac{GM}{r} \ll 1$

Scattering vs bound

Radial momentum

$$p_r^2(r, \mathscr{E}, J) = \mathbf{p}^2(r, \mathscr{E}) - \frac{J^2}{r^2}$$





Scattering vs bound

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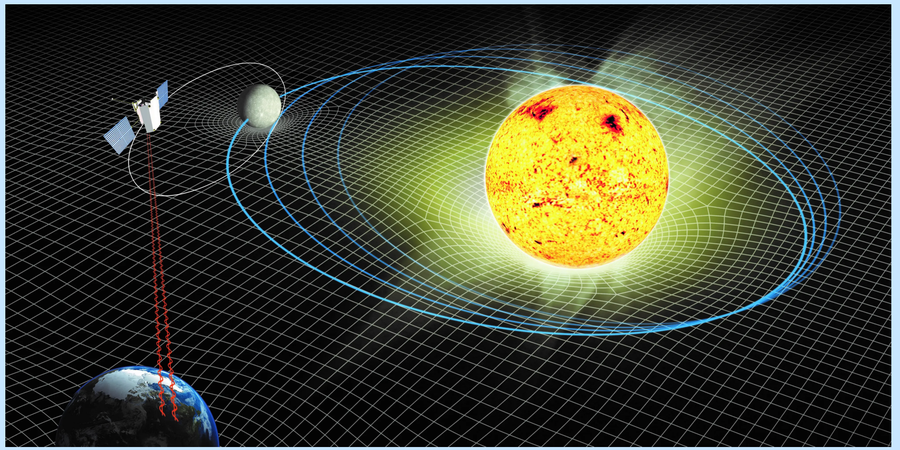
$$\frac{d\theta}{dr} = -\frac{\partial p_r}{\partial J} = \frac{J}{r^2\sqrt{\mathbf{p}^2(r, \mathscr{E}) - J^2/r^2}}$$

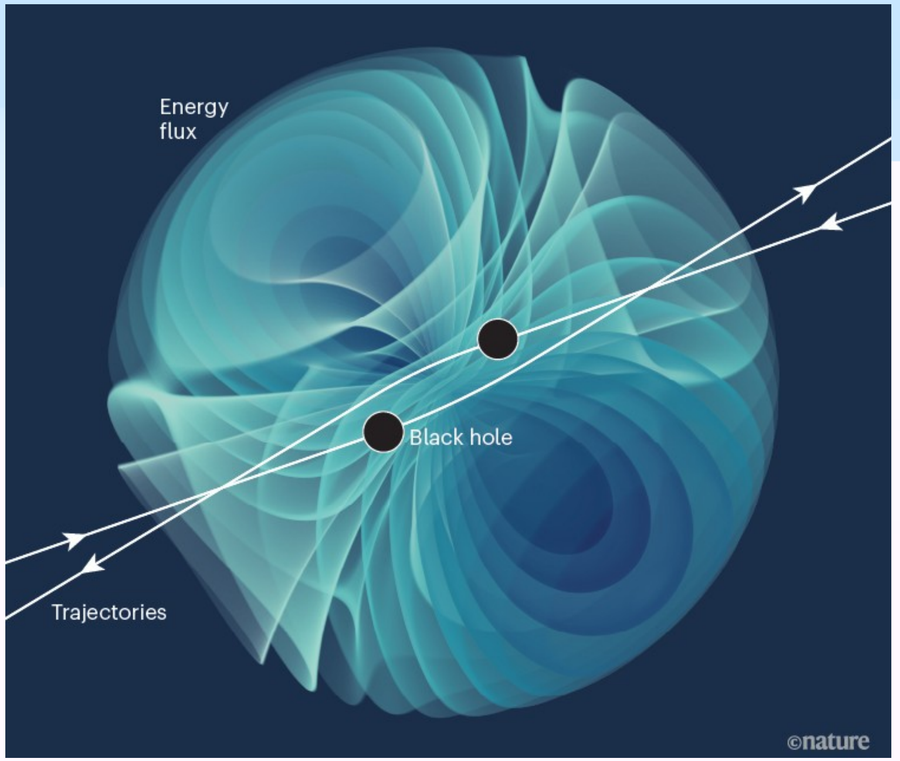
Bound orbits: the periastron advance per radial period

$$\Delta\Phi + 2\pi = 2J\int_{r_-}^{r_+} \frac{dr}{r^2\sqrt{\mathbf{p}^2(r, \mathscr{E}) - J^2/r^2}} \qquad \mathscr{E} < 0$$

Scattering angle

$$\chi + \pi = 2J\int_{r_{\min}}^{\infty} \frac{dr}{r^2\sqrt{\mathbf{p}^2(r, \mathscr{E}) - J^2/r^2}} \qquad \mathscr{E} > 0$$





Scattering vs bound

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$$p_r^2(r,\mathscr{E},J) = \mathbf{p}^2(r,\mathscr{E}) - \frac{J^2}{r^2}$$

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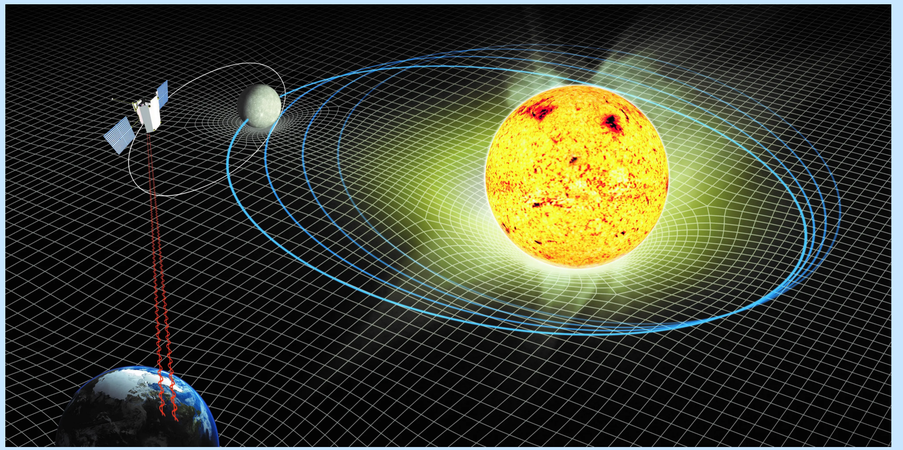
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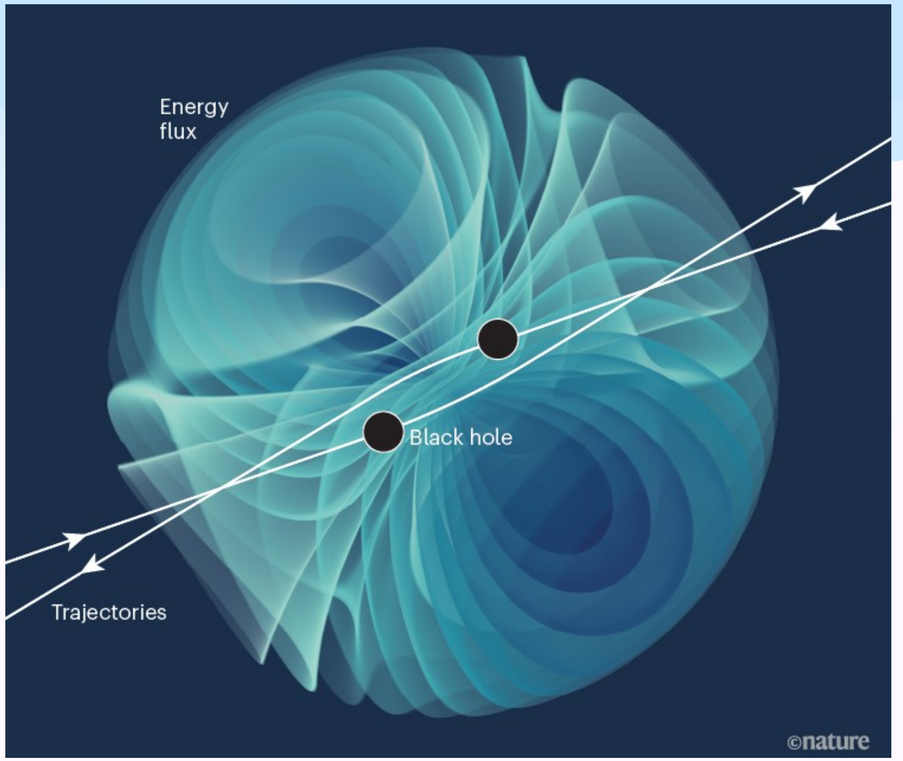
Scattering angle

$$\chi + \pi = 2J\int_{r_{\rm min}}^{\infty} \frac{dr}{r^2\sqrt{\mathbf{p}^2(r,\mathscr{E}) - J^2/r^2}} \qquad \mathscr{E} > 0$$

Analytic continuation:

$$r_{\rm min}(\mathscr{E}<0, J) = r_- \qquad r_{\rm min}(\mathscr{E}<0, -J) = r_+$$





Kälin-Porto JHEP 01 (2020) 072

Scattering vs bound

Analytical continuation of scattering angle

$$\chi(J) + \pi = 2J \int_{r_{\min}(J)}^{\infty} \frac{dr}{r^2 \sqrt{\mathbf{p}^2(r, \mathscr{E}) - J^2/r^2}}$$

Scattering vs bound

Analytical continuation of scattering angle

$$2J\int_{r_{\min}(J)}^{\infty}\frac{dr}{r^2\sqrt{\mathbf{p}^2(r,\mathscr{E})-J^2/r^2}}-2J\int_{r_{\min}(-J)}^{\infty}\frac{dr}{r^2\sqrt{\mathbf{p}^2(r,\mathscr{E})-J^2/r^2}}\quad\mathscr{E}<0$$

Scattering vs bound

Analytical continuation of scattering angle

$$\chi(J) + \chi(-J) + 2\pi = 2J\int_{r_{\min}(J)}^{\infty} \frac{dr}{r^2\sqrt{\mathbf{p}^2(r,\mathscr{E}) - J^2/r^2}} - 2J\int_{r_{\min}(-J)}^{\infty} \frac{dr}{r^2\sqrt{\mathbf{p}^2(r,\mathscr{E}) - J^2/r^2}} \quad \mathscr{E} < 0$$

$$= 2J\int_{r_-}^{r_+} \frac{dr}{r^2\sqrt{\mathbf{p}^2(r,\mathscr{E}) - J^2/r^2}} = \Delta\Phi + 2\pi$$

Periastron advance

Scattering vs bound

orbital motion = scattering + anti-scattering

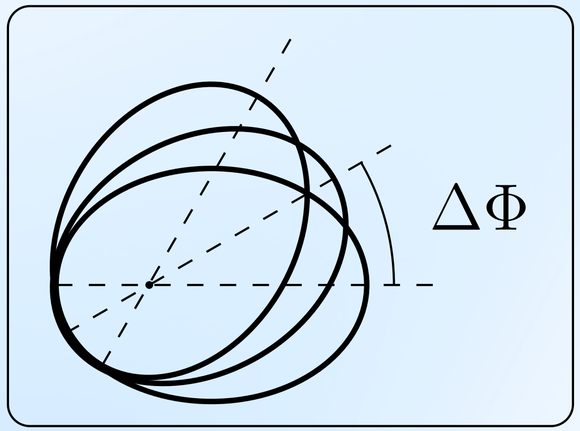
Analytical continuation of scattering angle

Kälin-Porto JHEP 01 (2020) 072; JHEP 02 (2020) 120

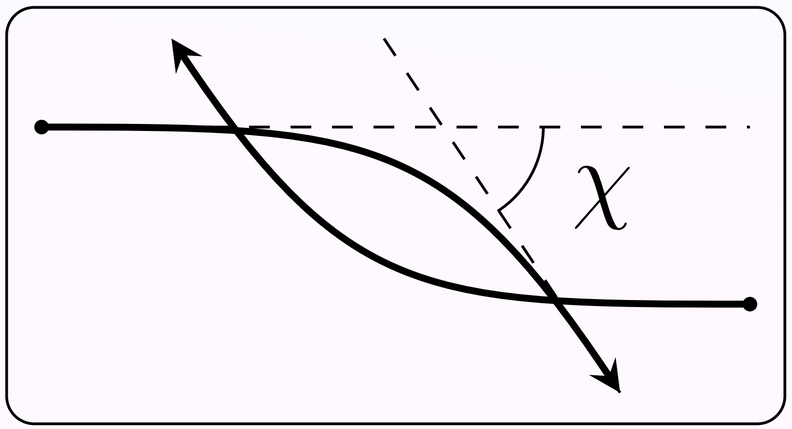
$$\chi(J) + \chi(-J) + 2\pi = 2J\int_{r_{\min}(J)}^{\infty} \frac{dr}{r^2\sqrt{\mathbf{p}^2(r,\mathscr{E}) - J^2/r^2}} - 2J\int_{r_{\min}(-J)}^{\infty} \frac{dr}{r^2\sqrt{\mathbf{p}^2(r,\mathscr{E}) - J^2/r^2}} \qquad \mathscr{E} < 0$$

$$= 2J\int_{r_-}^{r_+} \frac{dr}{r^2\sqrt{\mathbf{p}^2(r,\mathscr{E}) - J^2/r^2}} = \Delta\Phi + 2\pi$$

Periastron advance



$$\Delta\Phi = \chi(J) + \chi(-J)$$



$$I_r^{(b)}(\mathscr{E}, J) = I_r^{(s)}(\mathscr{E}, J) - I_r^{(s)}(\mathscr{E}, -J)$$

The two-body problem

Two compact objects

$$S_{\rm pp} = -\sum_{i=1}^{2} \frac{m_i}{2} \int d\tau_i \left(g_{\mu\nu} \dot{x}_i^\mu \dot{x}_i^\nu + 1 \right) + \cdots$$

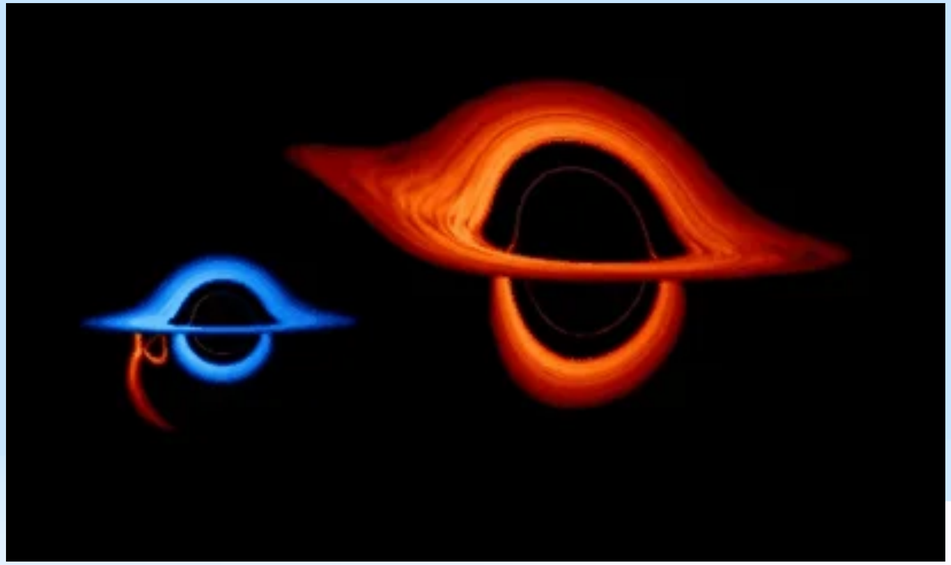
Gravitational interactions are governed by General Relativity

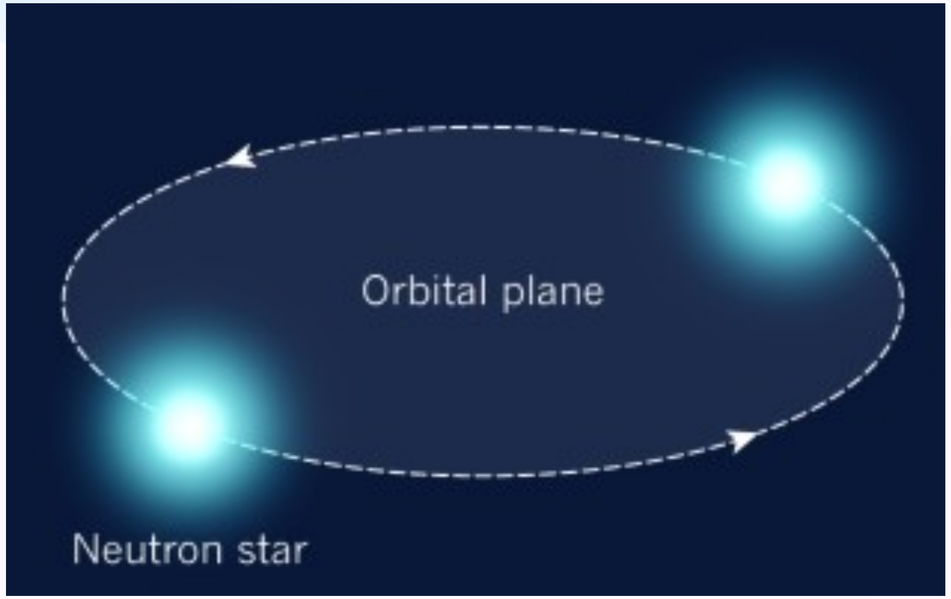
$$S_{\rm HE} = -\frac{1}{16\pi G} \int d^4x \sqrt{-g}\, R$$

Equations of motion

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \frac{8\pi G}{c^4} T_{\mu\nu} = 0$$

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0$$





Effective Field Theory

Effective action for a two-body system

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad e^{iS_{\text{eff}}[x_a]} = \int \mathscr{D}h_{\mu\nu}\, e^{iS_{\text{pp}}[x_a,h]+iS_{\text{EH}}[h]+iS_{\text{GF}}[h]}$$

Perturbative expansion:

$$L_{\text{eff}} = L_0 + L_1 + L_2 + L_3 + \cdots \qquad L_0 = -\sum_i \frac{1}{2} m_i \dot{x}_i^\mu \dot{x}_i^\nu \eta_{\mu\nu}$$

Equations of motion:

$$m_i \ddot{x}_i^\mu = -\eta^{\mu\nu} \sum_{n=1}^{\infty} \left(\frac{\partial L_n}{\partial x_i^\nu} - \frac{d}{d\tau_i} \frac{\partial L_n}{\partial \dot{x}_i^\nu} \right)$$

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saddle-point approximation

PURELY CLASSICAL

Perturbative expansion:

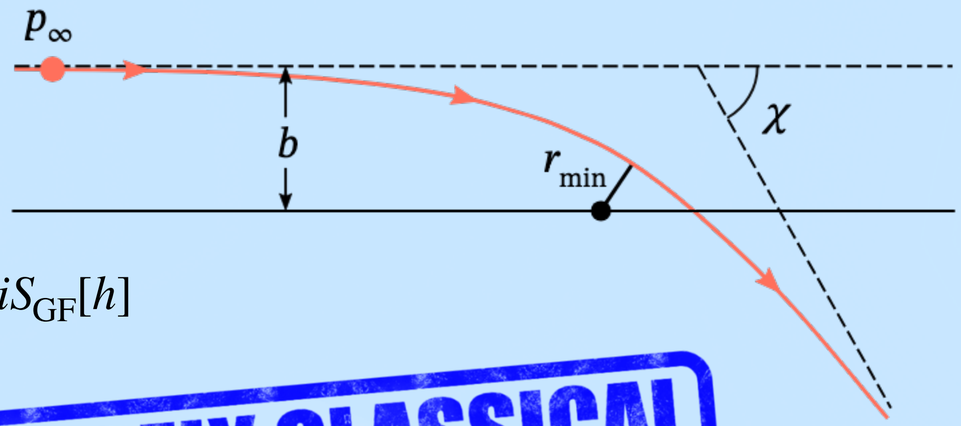
$$L_{\rm eff} = L_0 + L_1 + L_2 + L_3 + \cdots \qquad L_0 = -\sum_i \frac{1}{2} m_i \dot{x}_i^\mu \dot{x}_i^\nu \eta_{\mu\nu}$$

classical sources

Equations of motion:

$$m_i \ddot{x}_i^\mu = -\eta^{\mu\nu} \sum_{n=1}^{\infty} \left(\frac{\partial L_n}{\partial x_i^\nu} - \frac{d}{d\tau_i} \frac{\partial L_n}{\partial \dot{x}_i^\nu} \right)$$

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PURELY CLASSICAL

Perturbative expansion:

saddle-point approximation

$$L_{\rm eff} = L_0 + L_1 + L_2 + L_3 + \cdots \qquad L_0 = -\sum_i \frac{1}{2} m_i \dot{x}_i^\mu \dot{x}_i^\nu \eta_{\mu\nu}$$

Equations of motion:

classical sources

$$m_i \ddot{x}_i^\mu = -\eta^{\mu\nu} \sum_{n=1}^{\infty} \left(\frac{\partial L_n}{\partial x_i^\nu} - \frac{d}{d\tau_i} \frac{\partial L_n}{\partial \dot{x}_i^\nu} \right) \qquad x_i^\mu(\tau_i) = b_i^\mu + u_i^\mu \tau_i + \delta_1 x_i^\mu + \delta_2 x_i^\mu + \delta_3 x_i^\mu + \cdots$$

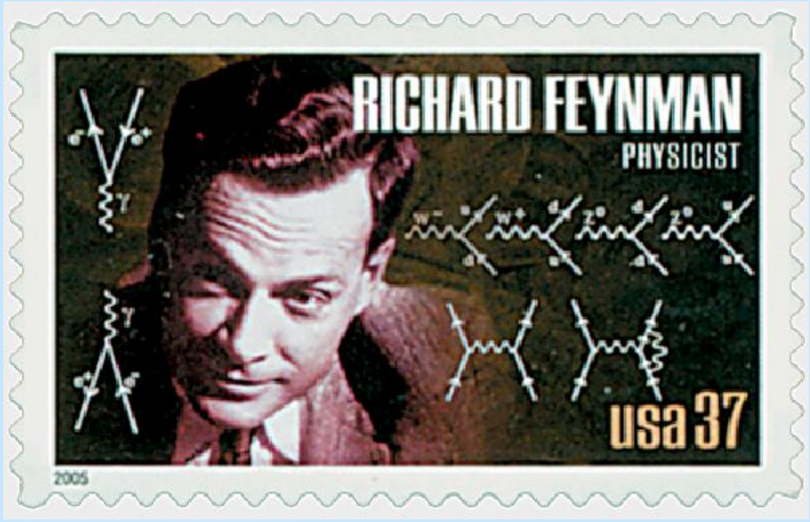
Impulse:

$$\Delta p_i^\mu = \int_{-\infty}^{\infty} d\tau_i\, m_i \ddot{x}_i^\mu = -\eta^{\mu\nu} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\tau_i \left(\frac{\partial L_n}{\partial x_i^\nu} \right)$$

Scattering angle

$$2 \sin\left(\frac{\chi}{2}\right) = \frac{|\Delta p_i|}{p_\infty}$$

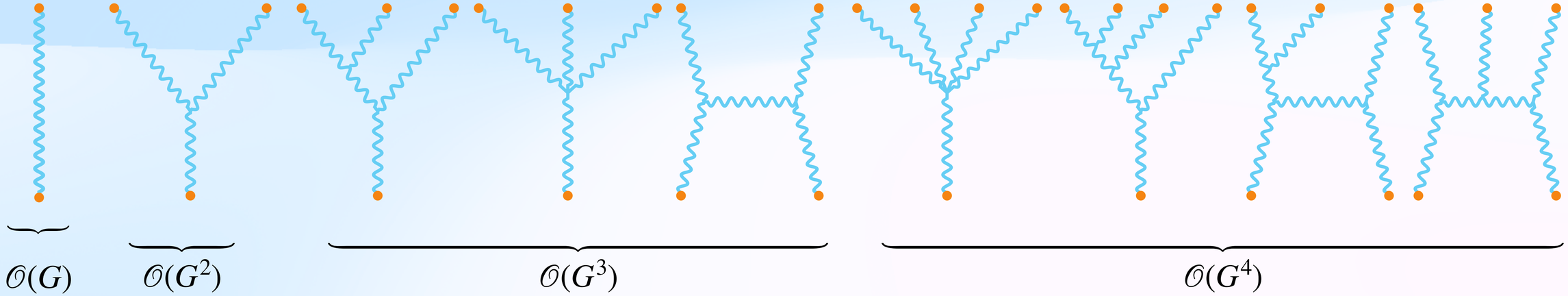
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$$e^{iS_{\rm eff}[x_a]} = \int \mathscr{D}h_{\mu\nu}\, e^{iS_{\rm pp}[x_a,h]+iS_{\rm GR}[h]}$$

Perturbative expansion: only tree graviton graphs & the sources are purely classical.



Classical physics: enjoy the advantages of QFT methods and classical physics:

powerful and systematic & purely classical at all steps (simplicity)

Closed-time path integral

Scattering amplitude: the transition amplitude from an initial "***in***" state to a final "***out***" state

$$\langle f \text{ out} \,|\, i \text{ in} \rangle = \langle f \,|\, \mathcal{S} \,|\, i \rangle = \mathcal{S}_{fi} \qquad \mathcal{S} = U(+\infty, -\infty) = \mathcal{T} \exp\left[-i \int_{-\infty}^{+\infty} \mathrm{d}t \, H_{\text{int}}(t) \right]$$

Real-time dynamics: an initial-value problem (*no particular final state is specified*)

$$\langle O(t) \rangle = \langle \text{in} \,|\, O_H(t) \,|\, \text{in} \rangle$$

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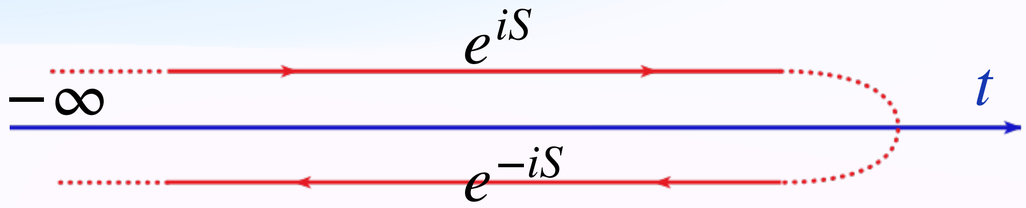
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$$= \langle \text{in} \,|\, U(-\infty, t)\, O_S\, U(t, -\infty) \,|\, \text{in} \rangle$$

a closed time contour



Closed-time path integral

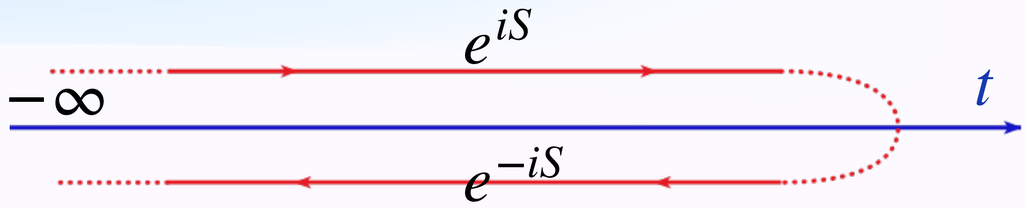
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$$= \langle \text{in} \,|\, U(-\infty, t)\, O_S\, U(t, -\infty) \,|\, \text{in} \rangle$$



a closed time contour

$$= \sum\nolimits_X \langle \text{in} \,|\, U^\dagger(+\infty, -\infty) \,|\, \text{out}_X \rangle \langle \text{out}_X \,|\, U(+\infty, t)\, O_S\, U(t, -\infty) \,|\, \text{in} \rangle$$

all final state X contributes

Inclusive: summing over unobserved radiation generates a causal, dissipative self-force.

Schwinger 1961 Feynman & Vernon 1963 Keldysh 1964

Closed-time path integral

EQUILIBRIUM AND NONEQUILIBRIUM FORMALISMS MADE UNIFIED

Kuang-chao CHOU, Zhao-bin SU,* Bai-lin HAO and Lu YU

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Received 5 June 1984

Physics Reports

Volume 118, Issues 1–2, February 1985, Pages 1-131

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周光召



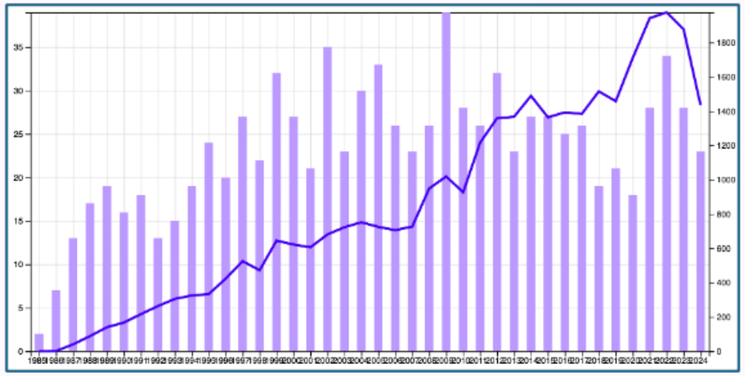
郝柏林



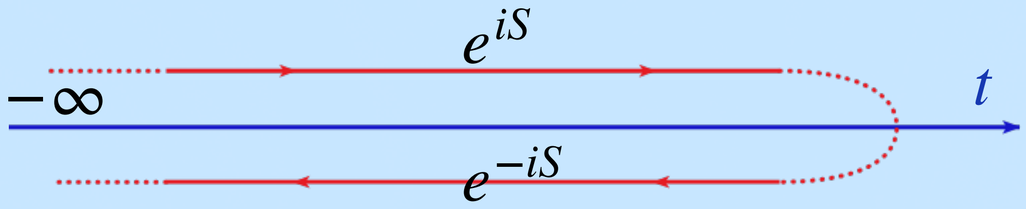
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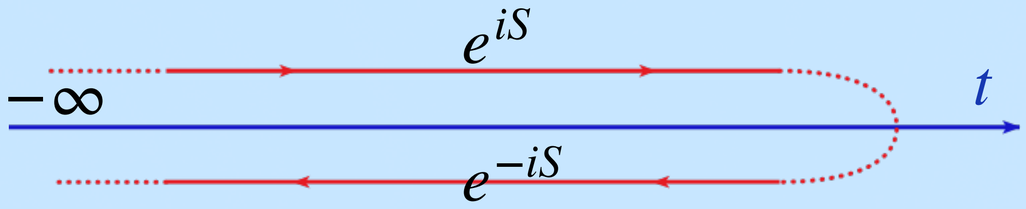
CTP: two-body system



Effective action

$$e^{iS_{\rm eff}[x_{a,1},x_{a,2}]} = \int \mathscr{D}h_1 \mathscr{D}h_2\, e^{i\left(S_{\rm GR}[h_1]+S_{\rm pp}[h_1,x_{a,1}]-S_{\rm GR}[h_2]-S_{\rm pp}[h_2,x_{a,2}]\right)}$$

CTP: two-body system



Effective action

$$e^{iS_{\rm eff}[x_{a,1},x_{a,2}]}=\int \mathscr{D}h_1\mathscr{D}h_2\, e^{i\left(S_{\rm GR}[h_1]+S_{\rm pp}[h_1,x_{a,1}]-S_{\rm GR}[h_2]-S_{\rm pp}[h_2,x_{a,2}]\right)}$$

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Keldysh basis:

$$x^\mu_{a,+}=\frac{1}{2}(x^\mu_{a,1}+x^\mu_{a,2}) \qquad h^+_{\mu\nu}=\frac{1}{2}(h_{1,\mu\nu}+h_{2,\mu\nu})$$

$$x^\mu_{a,-}=x^\mu_{a,1}-x^\mu_{a,2} \qquad h^-_{\mu\nu}=h_{1,\mu\nu}-h_{2,\mu\nu}$$

$$\begin{pmatrix} 0 & -G_{\rm adv} \\ G_{\rm ret} & 0 \end{pmatrix}$$

The doubled CTP histories are auxiliary and coincide only after variation: $x_{i,-}\to 0,\ x_{i,+}\to x_i$

Only the retarded response contributes to the equation of motion, ensuring causal dissipation.

Equations of motion and observables:

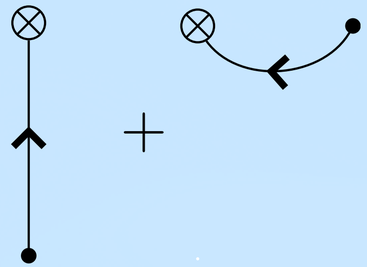
$$m_i\ddot{x}^\mu_i(\tau)=-\eta^{\mu\nu}\frac{\delta S_{\rm int}[x_\pm]}{\delta x^\nu_{i,-}(\tau)}\bigg|_{\rm Phys\ Limit} \qquad \Delta p^\mu_i=\int_{-\infty}^{\infty}d\tau\, m\ddot{x}^\mu_i$$

CTP: two-body system

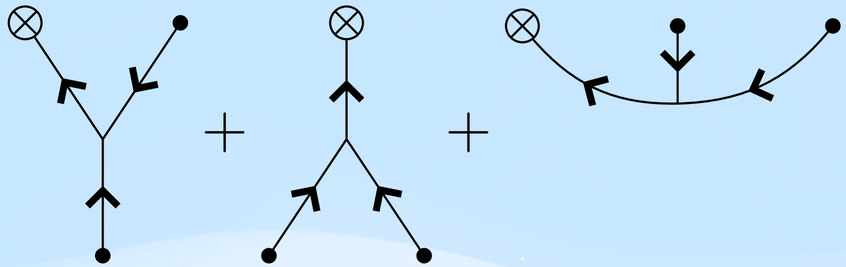
Diagrammatic construction of the effective action:

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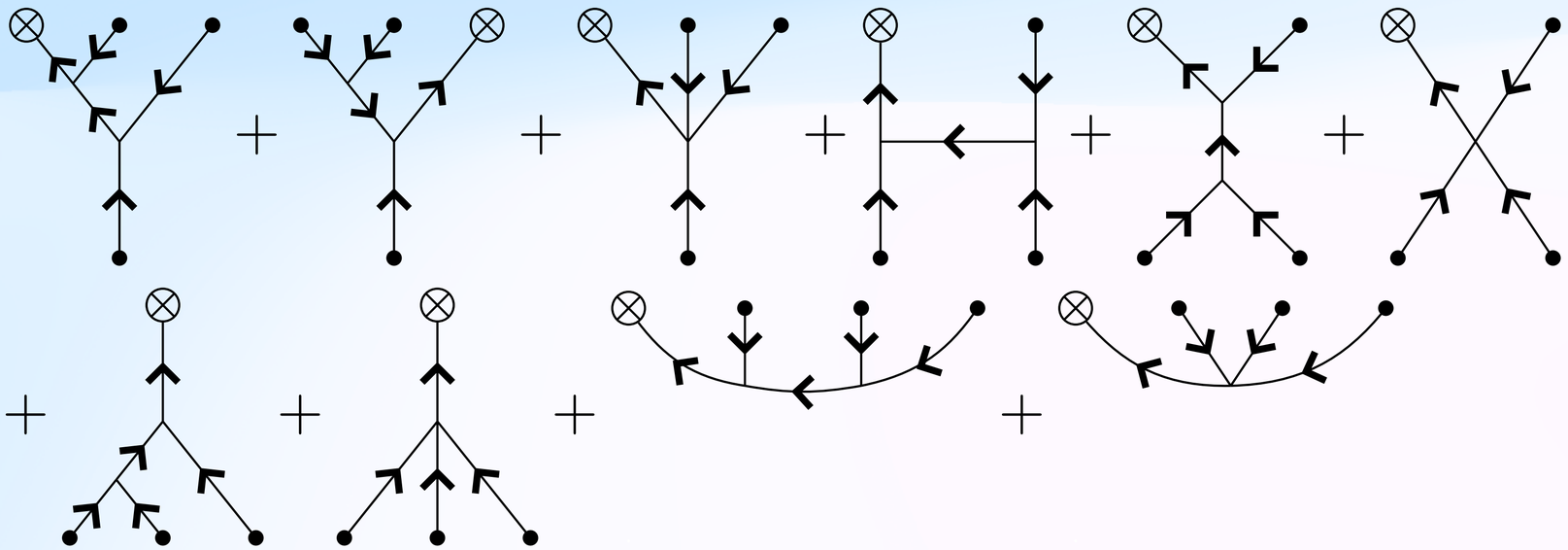
$$\left.\frac{\delta S_{\rm eff}[x_\pm]}{\delta x^\alpha_{1-}}\right|^{\mathcal{O}(G)}_{\rm PL} =$$



$$\left.\frac{\delta S_{\rm eff}[x_\pm]}{\delta x^\alpha_{1-}}\right|^{\mathcal{O}(G^2)}_{\rm PL} =$$



$$\left.\frac{\delta S_{\rm eff}[x_\pm]}{\delta x^\alpha_{1-}}\right|^{\mathcal{O}(G^3)}_{\rm PL} =$$



Effective Field Theory

Observables (scattering impulse) at $\mathcal{O}(G^N)$

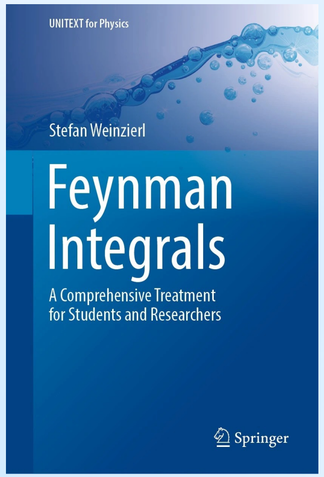
$$\Delta p_i^\mu \sim \underbrace{\int d^D q \frac{e^{iq\cdot b}\,\delta(q\cdot u_1)\,\delta(q\cdot u_2)}{|\,q^2\,|^\sharp}}_{\text{Fourier integral}} \underbrace{\int \left(\prod_{i=1}^{N-1} d^D \ell_i \frac{\delta(\ell_j\cdot u_a)}{(\ell_i\cdot u_b - i0)^{\nu_i}}\right) \frac{\mathcal{N}^\mu(q,u_a)}{D_1 D_2 D_3 \cdots}}_{(N-1)\text{–loop integral}}$$

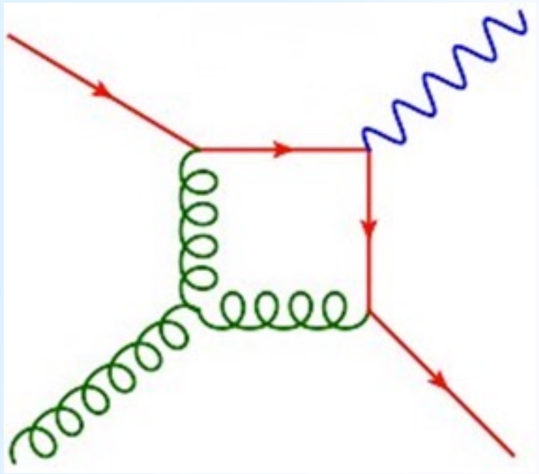
Effective Field Theory

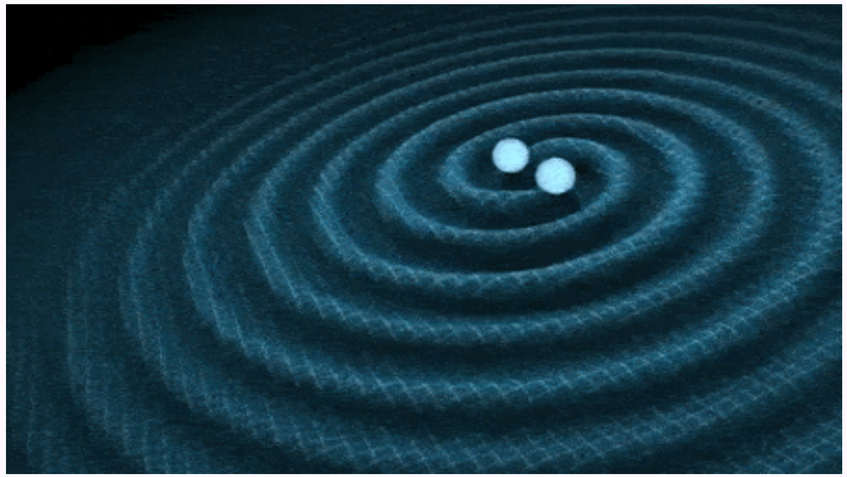
Observables (scattering impulse) at $\mathcal{O}(G^N)$

$$\Delta p_i^\mu \sim \underbrace{\int d^D q \frac{e^{iq\cdot b}\,\delta(q\cdot u_1)\,\delta(q\cdot u_2)}{|\,q^2\,|^\sharp}}_{\text{Fourier integral}} \underbrace{\int \left(\prod_{i=1}^{N-1} d^D \ell_i \frac{\delta(\ell_j\cdot u_a)}{(\ell_i\cdot u_b - i0)^{\nu_i}}\right) \frac{\mathcal{N}^\mu(q,u_a)}{D_1 D_2 D_3 \cdots}}_{(N-1)\text{–loop integral}}$$

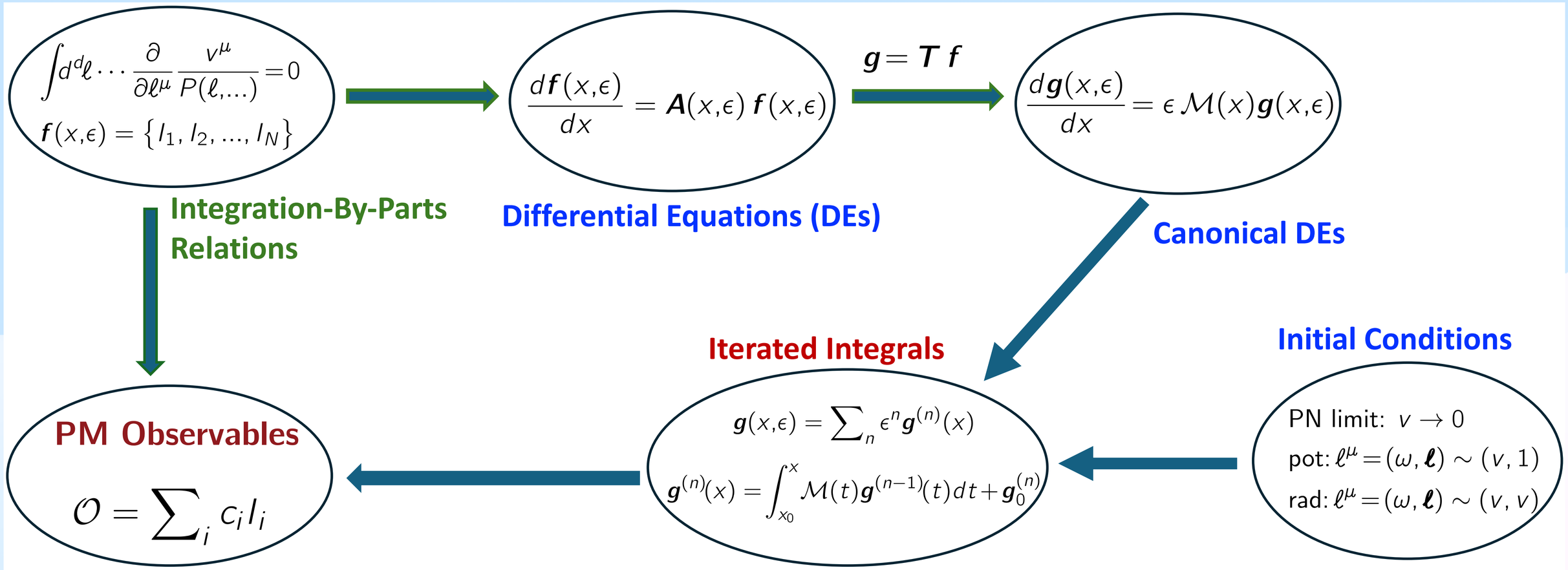
Modern Feynman integral techniques can be used to solve classical gravitational problems!







Quantum Field Theory toolbox



Post-Minkowskian physics can be bootstrapped from Post-Newtonian data using Differential Equations!

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LO: 1PM

Observable from the 1PM diagram

$$\Delta p_1^\mu = \int_{-\infty}^{\infty} d\tau_1\, m_1 \ddot{x}_1^\mu(\tau_1) = -4\pi G m_1 m_2 \int d\tau_1 \int d\tau_2 \int \frac{d^D q}{(2\pi)^D} \frac{i q^\mu\, e^{iq\cdot(x_1 - x_2)}}{-q^2 - i0} \left(2(\dot{x}_1\cdot\dot{x}_2)^2 - (\dot{x}_1^2)(\dot{x}_2^2)\right)$$

Leading order (LO): using $x_i^\mu = b_i^\mu + u_i^\mu \tau_i \qquad u_1^2 = u_2^2 = 1 \quad u_1\cdot u_2 = \gamma$

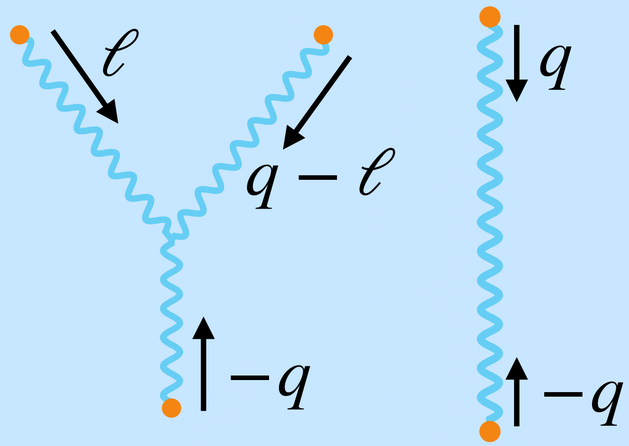
$$\Delta p_1^\mu\Big|_{\rm LO} = -4\pi G m_1 m_2 \int d\tau_1 \int d\tau_2 \int \frac{d^D q}{(2\pi)^D} \frac{i q^\mu\, e^{iq\cdot(b + u_1\tau_1 - u_2\tau_2)}}{-q^2 - i0} \left(2\gamma^2 - 1\right)$$

$$= -4\pi G m_1 m_2 (2\gamma^2 - 1) \int \frac{d^D q}{(2\pi)^D} \frac{i q^\mu\, (2\pi)^2\, \delta(q\cdot u_1)\, \delta(q\cdot u_2)\, e^{iq\cdot b}}{-q^2 - i0}$$

$$= -\frac{\left(2\gamma^2 - 1\right)}{\sqrt{\gamma^2 - 1}} \frac{2 G m_1 m_2\, b^\mu}{|b|^2}$$

$$\int \frac{d^d p}{(2\pi)^d} \frac{e^{ip\cdot x}}{(p^2)^a} = \frac{1}{4^a \pi^{d/2}} \frac{\Gamma(d/2 - a)}{\Gamma(a)} \frac{1}{(x^2)^{d/2 - a}}$$

NLO: 2PM



The NLO impulse has this structure:

$$G^2\int\frac{d^Dq}{(2\pi)^D}\frac{\delta(q\cdot u_1)\,\delta(q\cdot u_2)\,e^{iq\cdot b}\mathcal{N}^\mu}{[-q^2-i0]^\sharp}\int\frac{d^D\ell\,\delta(\ell\cdot u_2)}{[\ell\cdot u_1-i0]\,[-\ell^2-i0][-(\ell-q)^2-i0]}$$

Analytic expression

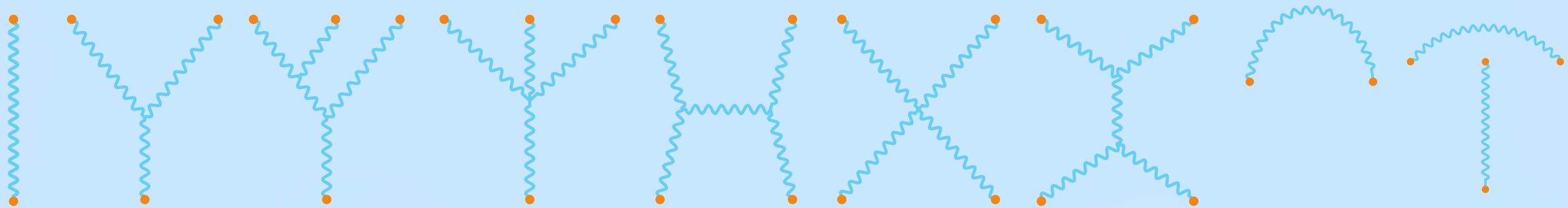
$$\Delta p_1^\mu\Big|_{G^2}=-\frac{3\pi G^2m_1m_2(m_1+m_2)}{4}\frac{5\gamma^2-1}{\sqrt{\gamma^2-1}}\frac{b^\mu}{|\mathbf{b}|^3}\qquad \gamma=u_1\cdot u_2$$

$$+2\frac{m_1m_2(2\gamma^2-1)^2}{(\gamma^2-1)^2}\frac{G^2}{|\mathbf{b}|^2}\left((\gamma m_2+m_1)u_2^\mu-(\gamma m_1+m_2)u_1^\mu\right)$$

Spin effects can also be systematically incorporated in EFT framework. **ZL**-Porto-Yang JHEP 2021

NNLO: 3PM

Kälin-**ZL**-Porto PRL 2020



- Two-loop integrals

$$I_{a_1\ldots a_7} = \int \frac{d^D\ell_1\, d^D\ell_2}{\pi^D} \frac{\delta(\ell_1\cdot u_1)\,\delta(\ell_2\cdot u_2)}{[\ell_1\cdot u_2]^{a_1}\,[\pm\ell_2\cdot u_1]^{a_2}} \frac{[-(\ell_1-q)^2]^{-a_6}\,[-(\ell_2-q)^2]^{-a_7}}{[-\ell_1^2]^{a_3}[-\ell_2^2]^{a_4}[-(\ell_1+\ell_2-q)^2]^{a_5}}$$

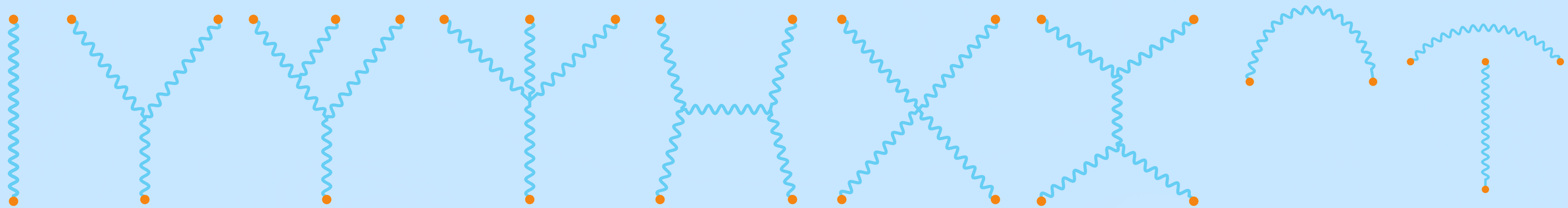
- Solving IBP relations, all integrals can be reduced to a set of 8 master integrals $\vec{f} = \{I_1, I_2, \cdots\}$
- Canonical differential equations:

$$\gamma = \frac{x^2+x}{2x}$$

$$\vec{g} = T\cdot\vec{f} \qquad \frac{d\,\vec{g}(x,\epsilon)}{dx} = \epsilon\left(\frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}\right)\vec{g}(x,\epsilon)$$

NNLO: 3PM

Kälin-**ZL**-Porto PRL 2020 PRD 2020



- Scattering impulse at $\mathcal{O}(G^3)$:

$$\Delta p_1^\mu = \frac{G^3 b^\mu}{|b^2|^2}\left(\frac{8m_1^2m_2^2(4\gamma^4-12\gamma^2-3)}{(\gamma^2-1)}\operatorname{arccosh}(\gamma)\right.$$

$$\left.-\frac{4m_1^2m_2^2\gamma(20\gamma^6-90\gamma^4+120\gamma^2-53)}{3(\gamma^2-1)^{5/2}}-\frac{2m_1m_2(m_1^2+m_2^2)(16\gamma^6-32\gamma^4+16\gamma^2-1)}{(\gamma^2-1)^{5/2}}\right)$$

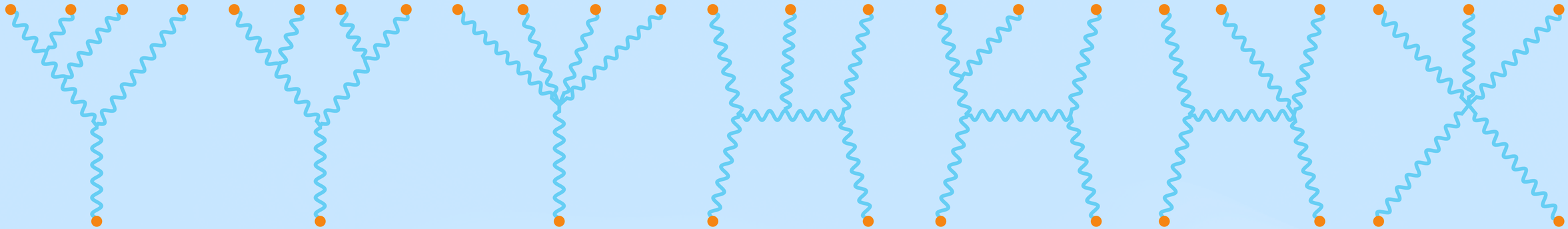
$$+\frac{3\pi}{2}\frac{(2\gamma^2-1)(5\gamma^2-1)}{(\gamma^2-1)^2}\frac{G^3m_1m_2(m_1+m_2)}{|b^2|^{3/2}}\left((m_1+\gamma m_2)u_2^\mu-(m_2+\gamma m_1)u_1^\mu\right)$$

- This is the first confirmation for the amplitude's derivation. Bern-Cheung-Roiban-Shen-Solon-Zeng (BCRSSZ) 2019
- We also obtained the *quadrupolar* and *octupolar* tidal corrections at $\mathcal{O}(G^3)$. Kälin-**ZL**-Porto PRD 2020
- Radiation reaction Kälin-**ZL**-Porto Kälin-Neef-Porto 2022 Jakobsen *et al.* 2022 Herrmann *et al.* 2021

NNNLO: 4PM

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Dlapa-Kälin-**ZL**-Neef-Porto PRL 2023



Three-loop integrals

$$\int d^D q\, \delta(q\cdot u_1)\, \delta(q\cdot u_2)\, e^{iq\cdot b} \int \frac{d^D\ell_1 d^D\ell_2 d^D\ell_3\, \delta(\ell_1\cdot u_1)\, \delta(\ell_2\cdot u_1)\, \delta(\ell_3\cdot u_2)}{[-\ell_1^2]\,[-(\ell_2-q)^2]\cdots}$$

IBP reduction:

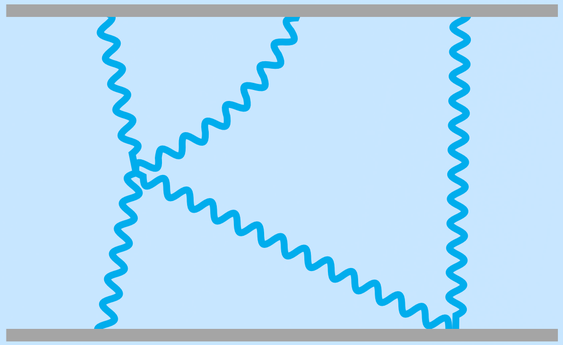
conservative (in-out): $\mathcal{O}(10^2)$ master integrals full (in-in): $\mathcal{O}(10^3)$ master integrals

Differential equations:

$$\frac{d\vec{f}(x,\epsilon)}{dx} = \epsilon\, \mathscr{M}(x)\, \vec{f}(x,\epsilon)$$

- The majority can be solved in terms of multiple polylogarithms.
- The analytical structure reveals elliptic integrals for the first time in PM gravity.

NNNLO: 4PM



- An elliptic example:

$$\frac{d}{dx}\begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix} = \begin{pmatrix} \frac{1-x^2}{2x(1+x^2)} & \frac{1+x^2}{4x(1-x^2)} & \frac{3x}{(1-x^2)(1+x^2)} \\ -\frac{1-x^2}{x(1+x^2)} & \frac{3(1+x^2)}{2x(1-x^2)} & -\frac{6x}{(1-x^2)(1+x^2)} \\ \frac{1-x^2}{x(1+x^2)} & -\frac{1+x^2}{2x(1-x^2)} & -\frac{1-4x^2+x^4}{x(1-x^2)(1+x^2)} \end{pmatrix} \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix} + \mathcal{O}(\epsilon)$$

It can then be written as a third-order differential equation:

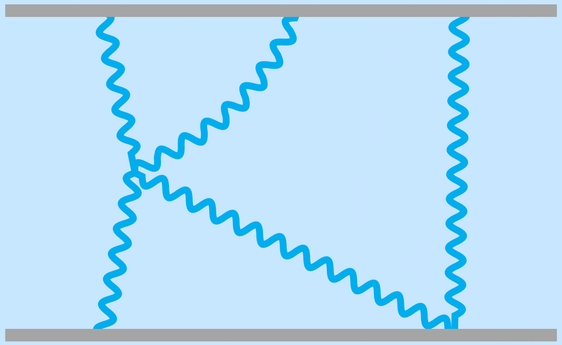
$$\left[\frac{d^3}{dx^3} - \frac{6x}{1-x^2}\frac{d^2}{dx^2} - \frac{1-4x^2+7x^4}{x^2(1-x^2)^2}\frac{d}{dx} - \frac{1+x^2}{x^3(1-x^2)}\right] f_1(x) = 0$$

It is easy to find the three solutions:

$$x\,\mathrm{K}^2(1-x^2), \qquad x\,\mathrm{K}(1-x^2)\mathrm{K}(x^2), \qquad x\,\mathrm{K}^2(x^2)$$

Complete elliptic integrals: $\mathrm{K}(x) \equiv \int_0^1 \frac{dt}{\sqrt{(1-t^2)\,(1-xt^2)}}$

NNNLO: 4PM



With the knowledge of leading-ϵ solutions, one may transform the elliptic diagonal block into

$$\frac{d}{dx}\vec{g}(x,\epsilon) = \epsilon\, M(x)\, \vec{g}(x,\epsilon) + \ldots$$

with

$$M = \begin{pmatrix} -\frac{4(1+x^2)}{3x(1-x^2)} & \frac{\pi^2}{x(1-x^2)\mathrm{K}^2(1-x^2)} & 0 \\ \frac{2(1+110x^2+x^4)\mathrm{K}^2(1-x^2)}{3\pi^2 x(1-x^2)} & -\frac{4(1+x^2)}{3x(1-x^2)} & \frac{\pi^2}{x(1-x^2)\mathrm{K}^2(1-x^2)} \\ \frac{16(1+x^2)(1-18x+x^2)(1+18x+x^2)\mathrm{K}^4(1-x^2)}{27\pi^2 x(1-x^2)} & \frac{2(1+110x^2+x^4)\mathrm{K}^2(1-x^2)}{3\pi^2 x(1-x^2)} & -\frac{4(1+x^2)}{3x(1-x^2)} \end{pmatrix}$$

- Elliptic integrals appear in the transformation matrix: automated by INITIAL
- Higer $\mathcal{O}(\epsilon)$: Iterated integrals involving elliptic kernels.

NNNLO: 4PM

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The full impulse at $\mathcal{O}(G^4)$:

$$\Delta p_1^\mu\Big|_{\rm NNNLO} = \frac{G^4}{|b|^4}\left(c_b\,\frac{b^\mu}{|b|} + c_1\,\frac{\gamma u_2^\mu - u_1^\mu}{\gamma^2-1} + c_2\,\frac{\gamma u_1^\mu - u_2^\mu}{\gamma^2-1}\right)$$

$$\frac{c_b}{\pi} = -\frac{3h_{34}m_2m_1(m_1^3+m_2^3)}{64v_\infty^5} + \frac{m_1^2m_{12}m_2^2}{4}\left[\frac{3h_6\mathrm{K}^2(w_2)}{4v_\infty^3} - \frac{3h_8\mathrm{K}(w_2)\mathrm{E}(w_2)}{4v_\infty^3} + \frac{21h_5w_3\mathrm{E}^2(w_2)}{8v_\infty^3} - \frac{\pi^2h_{16}v_\infty}{4(\gamma+1)} + \frac{3\gamma h_{10}(\mathrm{Li}_2(w_2)-4\mathrm{Li}_2(\sqrt{w_2}))}{w_3v_\infty^2} + \log(v_\infty)\left(\frac{h_{32}}{2v_\infty^3} - \frac{3h_{14}\log(\frac{w_3}{2})}{v_\infty} - \frac{3\gamma h_{22}\log(w_1)}{2v_\infty^4}\right)\right] + m_2^2m_1^3\left[\frac{h_{52}}{48v_\infty^6} - \frac{h_{63}}{768\gamma^9w_3v_\infty^5} - \frac{3v_\infty(h_{40}\mathrm{Li}_2(w_2)+2w_3h_{33}\mathrm{Li}_2(-w_2))}{64w_3} + \frac{3h_{14}\log(\frac{w_3}{2})\log(w_3)}{4v_\infty} + \frac{\gamma h_{39}\log(w_1)}{8w_3^3v_\infty^2} + \frac{3\gamma h_{22}\log(w_3)\log(w_1)-h_{35}\log(\frac{w_3}{2})}{8v_\infty^4} - \frac{\gamma h_{51}\log(w_1)}{16v_\infty^7} + \frac{h_{56}\log(2)-h_{57}\log(w_3)+2\gamma h_{55}\log(\gamma)}{32v_\infty^5}\right] + m_1^2m_2^3\left[\frac{h_{58}}{192\gamma^7v_\infty^5} + \frac{h_{53}}{48v_\infty^6} + \frac{\gamma h_{49}\log(w_1)}{16v_\infty^6} - \frac{2\gamma h_{50}\log(w_1)+3\gamma^2h_{13}\log^2(w_1)}{32v_\infty^7} - \frac{h_{41}\log(\frac{w_3}{2})}{8v_\infty^4} - \frac{h_{36}\log(w_3)}{4v_\infty^3} + \frac{3\gamma\log(w_1)\big(5h_{26}\log(2)+8h_{12}\log(w_3)\big)}{8v_\infty^4} + \frac{\gamma h_{30}\log(\gamma)}{2v_\infty^3} + \frac{h_{37}\log(2)}{8v_\infty^3} + \frac{3(h_{17}w_3\mathrm{Li}_2(w_2)-2h_{23}\mathrm{Li}_2(-w_2)+h_{15}\log^2(w_3)-h_9\log^2(2))}{8v_\infty} - \frac{3h_7\log(2)\log(w_3)}{v_\infty}\right]$$

$$c_1 = m_1m_2^2\left(\frac{2h_{46}m_{12s}}{v_\infty^6} + \frac{9\pi^2h_1m_{12}^2}{32v_\infty^2}\right) + m_1^2m_2^3\left(\frac{4\gamma h_{47}}{3v_\infty^6} - \frac{8\gamma h_2\log(w_1)}{v_\infty^6} + \frac{16h_{25}\log(w_1)}{v_\infty^3} - \frac{8h_3}{3v_\infty^5}\right)$$

$$c_2 = -m_1^4m_2\left(\frac{9\pi^2h_1}{32v_\infty^2} + \frac{2h_{46}}{v_\infty^6}\right) + m_2^2m_1^3\left[+\frac{h_{60}}{705600\gamma^8v_\infty^5} - \frac{4\gamma h_{48}}{3v_\infty^6} + \frac{3h_{38}(\mathrm{Li}_2(w_2)-4\mathrm{Li}_2(\sqrt{w_2}))-\gamma h_{21}(\mathrm{Li}_2(-w_1^2)+2\log(\gamma)\log(w_1))}{16v_\infty^4} + \frac{3\gamma h_{31}(2\mathrm{Li}_2(-w_1)+\log(w_1)\log(w_3))}{8v_\infty^4} + \frac{h_{62}\log(w_1)}{6720\gamma^9v_\infty^6} + \frac{32\gamma^2h_{44}\log^2(w_1)}{v_\infty^7} + \frac{8\gamma\big(2h_4\log(2)-h_{27}\log(w_1)\big)\log(w_1)}{v_\infty^4} - \frac{32h_{29}\log(w_1)}{3v_\infty^3} + \frac{\pi^2h_{42}}{192v_\infty^4}\right] + m_2^3m_1^2\left[\frac{h_{59}}{1440\gamma^7v_\infty^5} - \frac{h_{19}(\mathrm{Li}_2(-w_1^2)+2\log(\gamma)\log(w_1))}{8v_\infty^4} - \frac{2h_{54}}{45v_\infty^6} + \frac{h_{43}(\mathrm{Li}_2(w_2)-4\mathrm{Li}_2(\sqrt{w_2}))}{32v_\infty^4} - \frac{h_{20}(2\mathrm{Li}_2(-w_1)+\log(w_1)\log(w_3))}{4v_\infty^4} - \frac{h_{61}\log(w_1)}{480\gamma^8v_\infty^6} - \frac{16\gamma^2h_{11}\log^2(w_1)}{v_\infty^4} - \frac{32\gamma h_{45}\log^2(w_1)}{v_\infty^7} + \frac{16\gamma h_{28}\log(w_1)}{5v_\infty^3} - \frac{32h_{24}\log(2)\log(w_1)}{v_\infty^4} - \frac{\pi^2h_{18}}{48v_\infty^4}\right]$$

with $\gamma \equiv u_1\cdot u_2$, $v_\infty = \sqrt{\gamma^2-1}$, $w_1 = \gamma - v_\infty$, $w_2 = (\gamma-1)/(\gamma+1)$, $w_3 = \gamma+1$, h_i = polynomial in γ.

$$\mathrm{Li}_2(z) \equiv \int_0^z \frac{dx\log(1-x)}{-x}$$

$$K(z) \equiv \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-zx^2)}}$$

$$E(k) \equiv \int_0^1 dx\,\frac{\sqrt{1-zx^2}}{\sqrt{1-x^2}}$$

NNNLO: 4PM

Dlapa-Kälin-**ZL**-Porto PRL 2022 PLB 2022 JHEP 2023

Dlapa-Kälin-**ZL**-Neef-Porto PRL 2023

The full impulse at $\mathcal{O}(G^4)$:

$$\Delta p_1^\mu\Big|_{\rm NNNLO} = \frac{G^4}{|b|^4}\left(c_b\,\frac{b^\mu}{|b|} + c_1\,\frac{\gamma u_2^\mu - u_1^\mu}{\gamma^2-1} + c_2\,\frac{\gamma u_1^\mu - u_2^\mu}{\gamma^2-1}\right)$$

- The first analytic result of the full dynamics of scattering impulse at $\mathcal{O}(G^4)$.
- Conservative part agrees perfectly with Amplitudes' derivations.

  Bern-Parra-Martinez-Roiban-Ruf-Shen-Solon-Zeng 2021

- Perfect agreement with the state-of-the-art PN computations

  Cho-Dandapat-Gopakumar 2021 Cho 2022 Bini-Geralico 2021 2022 Bini-Damour 2022

- Later, two new calculations confirmed our results.

  Damgaard-Hansen-Planté-Vanhove 2023 (exponentiation of amplitudes)

  Jakobsen-Mogull-Plefka-Sauer-Xu 2023 (worldline QFT)

NNNNLO: 5PM



NNNNLO: 5PM

$$
\begin{array}{lccccccccccc}
 & & \text{0PN} & & \text{1PN} & & \text{2PN} & & \text{3PN} & & \text{4PN} & \cdots \\
\text{1PM} & \left(\frac{GM}{r}\right) \times (& 1 & + & v^2 & + & v^4 & + & v^6 & + & v^8 & + \cdots) \\
\text{2PM} & \left(\frac{GM}{r}\right)^2 \times (& 1 & + & v^2 & + & v^4 & + & v^6 & + & v^8 & + \cdots) \\
\text{3PM} & \left(\frac{GM}{r}\right)^3 \times (& 1 & + & v^2 & + & v^4 & + & v^6 & + & v^8 & + \cdots) \\
\text{4PM} & \left(\frac{GM}{r}\right)^4 \times (& 1 & + & v^2 & + & v^4 & + & v^6 & + & \cdots) \\
\text{5PM} & \left(\frac{GM}{r}\right)^5 \times (& 1 & + & v^2 & + & v^4 & + & \cdots) \\
\text{6PM} & \left(\frac{GM}{r}\right)^6 \times (& 1 & + & v^2 & + & \cdots) \\
\text{7PM} & \left(\frac{GM}{r}\right)^7 \times (& 1 & + & \cdots)
\end{array}
$$

$$
\begin{array}{lccccccc}
 & & \text{0SF} & & \text{1SF} & & \text{2SF} & \text{3SF} \cdots \\
= & \left(\frac{GM}{r}\right) \times (& 1 & & & & &) \\
= & \left(\frac{GM}{r}\right)^2 \times (& 1 & & & & &) \\
= & \left(\frac{GM}{r}\right)^3 \times (& 1 & + & \nu & & &) \\
= & \left(\frac{GM}{r}\right)^4 \times (& 1 & + & \nu & & &) \\
= & \left(\frac{GM}{r}\right)^5 \times (& 1 & + & \nu & + & \nu^2 &) \\
= & \left(\frac{GM}{r}\right)^6 \times (& 1 & + & \nu & + & \nu^2 &) \\
= & \left(\frac{GM}{r}\right)^7 \times (& 1 & + & \nu & + & \nu^2 & + \nu^3)
\end{array}
$$

$$\nu \equiv \frac{m_1 m_2}{(m_1+m_2)^2}$$

Virial theorem

$$v^2 \sim \frac{GM}{r}$$

NNNNLO: 5PM

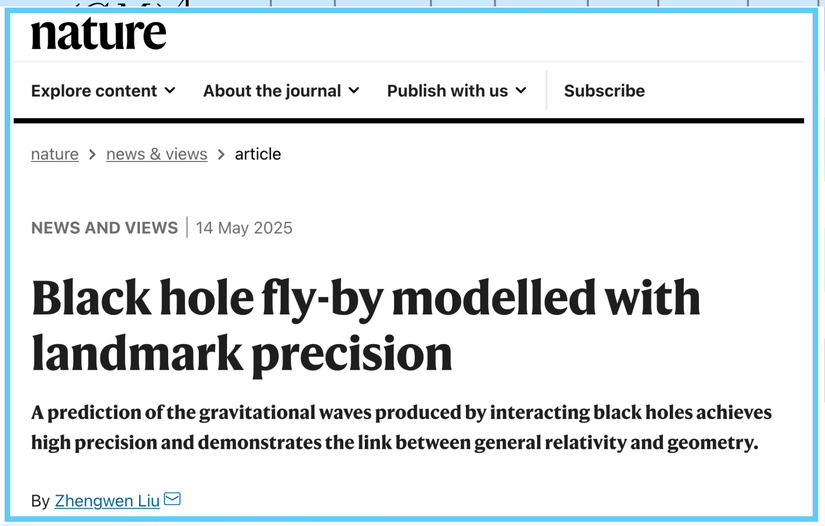
$$\begin{array}{lccccccccccc}
 & \text{0PN} & & \text{1PN} & & \text{2PN} & & \text{3PN} & & \text{4PN} & & \text{5PN} & & \text{6PN} & \cdots \\
\text{1PM} & \left(\frac{GM}{r}\right) \times \Big(& \boxed{1} & + & v^2 & + & v^4 & + & v^6 & + & v^8 & + & v^{10} & + & v^{12} & + \cdots \Big) \\
\text{2PM} & \left(\frac{GM}{r}\right)^2 \times \Big(& 1 & + & v^2 & + & v^4 & + & v^6 & + & v^8 & + & v^{10} & + & \cdots \Big) \\
\text{3PM} & \left(\frac{GM}{r}\right)^3 \times \Big(& 1 & + & v^2 & + & v^4 & + & v^6 & + & v^8 & + & \cdots \Big) \\
\text{4PM} & & & & & & & & & & & & & & \\
\text{5PM} & & & & & & & & & & & & & & \\
\text{6PM} & & & & & & & & & & & & & & \\
\text{7PM} & & & & & & & & & & & & & &
\end{array}$$

$$\begin{array}{lcccccccc}
 & & & & & & & & \\
\text{1PM} & + \cdots \Big) & = & \left(\frac{GM}{r}\right) & \times & \Big(& 1 & & \\
\text{2PM} & + \cdots \Big) & = & \left(\frac{GM}{r}\right)^2 & \times & \Big(& 1 & & \\
\text{3PM} & + \cdots \Big) & = & \left(\frac{GM}{r}\right)^3 & \times & \Big(& 1 & + & \nu \\
\text{4PM} & + \cdots \Big) & = & \left(\frac{GM}{r}\right)^4 & \times & \Big(& 1 & + & \nu \\
\text{5PM} & + \cdots \Big) & = & \left(\frac{GM}{r}\right)^5 & \times & \Big(& 1 & + & \nu & + & \nu^2 \\
\text{6PM} & + \cdots \Big) & = & \left(\frac{GM}{r}\right)^6 & \times & \Big(& 1 & + & \nu & + & \nu^2 \\
\text{7PM} & + \cdots \Big) & = & \left(\frac{GM}{r}\right)^7 & \times & \Big(& 1 & + & \nu & + & \nu^2 & + & \nu^3
\end{array}$$

$$\begin{array}{cccc} \text{0SF} & \text{1SF} & \text{2SF} & \text{3SF} \cdots \end{array}$$

$$\nu \equiv \frac{m_1 m_2}{(m_1+m_2)^2}$$





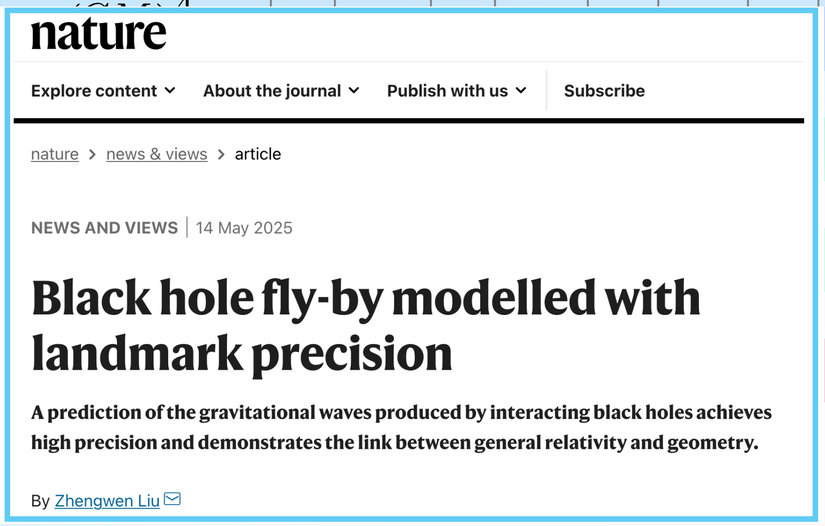
1SF: Driesse-Jakobsen-Mogull-Plefka-Sauer-Usovitsch PRL 2025 Driesse *et al.* Nature 2025

NNNNLO: 5PM

$$\begin{array}{lccccccccccc}
 & \text{0PN} & \text{1PN} & \text{2PN} & \text{3PN} & \text{4PN} & \text{5PN} & \text{6PN} & \cdots & & & \\
\text{1PM} & \left(\frac{GM}{r}\right)\times\Big(& 1 & + & v^2 & + & v^4 & + & v^6 & + & v^8 & + & v^{10} & + & v^{12} & + \cdots\Big) & = \left(\frac{GM}{r}\right)\times\big(1\big) \\
\text{2PM} & \left(\frac{GM}{r}\right)^2\times\Big(& 1 & + & v^2 & + & v^4 & + & v^6 & + & v^8 & + & v^{10} & + & \cdots\Big) & = \left(\frac{GM}{r}\right)^2\times\big(1\big) \\
\text{3PM} & \left(\frac{GM}{r}\right)^3\times\Big(& 1 & + & v^2 & + & v^4 & + & v^6 & + & v^8 & + & \cdots\Big) & & & = \left(\frac{GM}{r}\right)^3\times\big(1+\nu\big) \\
\text{4PM} & & & & & & & & & & & & & & & \cdots\Big) = \left(\frac{GM}{r}\right)^4\times\big(1+\nu\big) \\
\text{5PM} & & & & & & & & & & & & & & & \cdots\Big) = \left(\frac{GM}{r}\right)^5\times\big(1+\nu+\nu^2\big) \\
\text{6PM} & & & & & & & & & & & & & & & \cdots\Big) = \left(\frac{GM}{r}\right)^6\times\big(1+\nu+\nu^2\big) \\
\text{7PM} & & & & & & & & & & & & & & & \cdots\Big) = \left(\frac{GM}{r}\right)^7\times\big(1+\nu+\nu^2+\nu^3\big) \\
 & & & & & & & & & & & & & & & & \text{0SF} \quad \text{1SF} \quad \text{2SF} \quad \text{3SF} \cdots
\end{array}$$

$$\nu \equiv \frac{m_1 m_2}{(m_1+m_2)^2}$$




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NEWS AND VIEWS | 14 May 2025

Black hole fly-by modelled with landmark precision

A prediction of the gravitational waves produced by interacting black holes achieves high precision and demonstrates the link between general relativity and geometry.

By Zhengwen Liu

1SF: Driesse-Jakobsen-Mogull-Plefka-Sauer-Usovitsch PRL 2025 Driesse *et al.* Nature 2025

2SF: Driesse *et al*. PRL 2026 (conservative) Bern *et al*. PRL 2026 (potential) Dlapa-Kälin-**ZL**-Porto WIP

Local-in-time dynamics

Time-nonlocal dynamics arises from hereditary tail interactions that couple the binary's motion at different times. The nonlocal-in-time radial action takes Damour-Jaranowski-Schäfer 2014

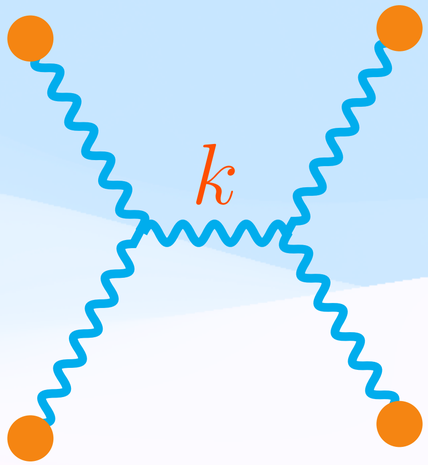
$$\mathcal{S}_r^{(\mathrm{nloc})} \sim -\frac{GE}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{dE_{\mathrm{src}}}{d\omega}\log(GM\omega)^2 \qquad \frac{\chi}{2\pi} \sim -\frac{\partial \mathcal{S}_r}{\partial J}$$

Since scattering-to-bound analytic continuation applies only to the local radial action, the nonlocal tail contribution must first be isolated and subtracted.

Local-in-time dynamics

Time-nonlocal dynamics arises from hereditary tail interactions that couple the binary's motion at different times. The nonlocal-in-time radial action takes Damour-Jaranowski-Schäfer 2014

$$\mathcal{S}_r^{(\mathrm{nloc})} \sim -\frac{GE}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{dE_{\mathrm{src}}}{d\omega}\log(GM\omega)^2 \qquad \frac{\chi}{2\pi} \sim -\frac{\partial \mathcal{S}_r}{\partial J}$$



Since scattering-to-bound analytic continuation applies only to the local radial action, the nonlocal tail contribution must first be isolated and subtracted.

$$\omega = k \cdot u_{\mathrm{com}} \quad u_{\mathrm{com}} = (m_1 u_1 + m_2 u_2)/M$$

Typical Feynman integrals:

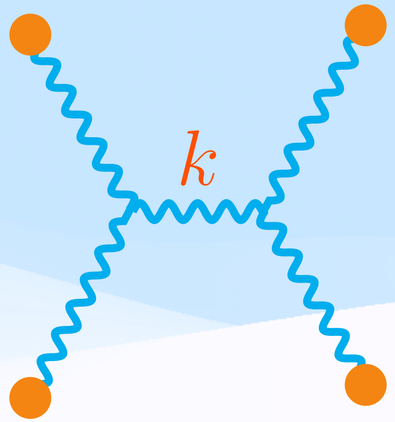
$$\int d^D\ell_1 d^D\ell_2 \frac{\delta(\ell_1\cdot u_1)\delta(\ell_2\cdot u_2)}{[\ell_1\cdot u_2][\ell_2\cdot u_1]}\frac{\log(\omega^2)}{[\ell_1^2][\ell_2^2][(\ell_1+\ell_2-q)^2][(\ell_1-q)^2][(\ell_2-q)^2]}$$

Local-in-time dynamics

Typical Feynman integrals

$$\omega = k \cdot u_{\rm com} \quad u_{\rm com} = (m_1 u_1 + m_2 u_2)/M$$

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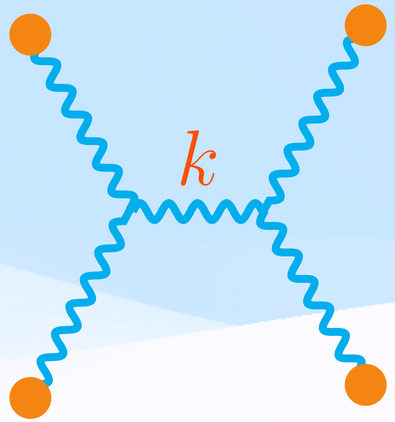


Local-in-time dynamics

Typical Feynman integrals

$$\omega = k \cdot u_{\rm com} \quad u_{\rm com} = (m_1 u_1 + m_2 u_2)/M$$

$$\int d^D\ell_1 d^D\ell_2 \frac{\delta(\ell_1 \cdot u_1)\delta(\ell_2 \cdot u_2)}{[\ell_1 \cdot u_2][\ell_2 \cdot u_1]} \frac{\log(\omega^2)}{[\ell_1^2][\ell_2^2][(\ell_1+\ell_2-q)^2][(\ell_1-q)^2][(\ell_2-q)^2]}$$



Strategy I:

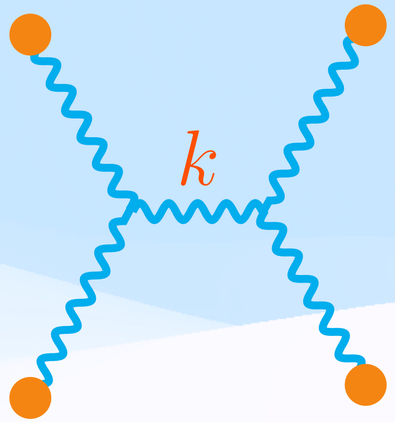
$$\frac{1}{[\omega^2]^\eta} = 1 - \eta \log \omega^2 + \cdots$$

Local-in-time dynamics

Typical Feynman integrals

$$\omega = k \cdot u_{\rm com} \quad u_{\rm com} = (m_1 u_1 + m_2 u_2)/M$$

$$\int d^D\ell_1 d^D\ell_2 \frac{\delta(\ell_1\cdot u_1)\delta(\ell_2\cdot u_2)}{[\ell_1\cdot u_2][\ell_2\cdot u_1]} \frac{\log(\omega^2)}{[\ell_1^2][\ell_2^2][(\ell_1+\ell_2-q)^2][(\ell_1-q)^2][(\ell_2-q)^2]}$$



Strategy I:

$$\frac{1}{[\omega^2]^\eta} = 1 - \eta \log\omega^2 + \cdots$$

Strategy II:

$$Q = m_2/m_1$$

$$\log\omega^2 = \log\left(\left(\frac{k\cdot u_1 + Qk\cdot u_2}{1+Q}\right)^2\right) = \log\left(\left(\ell_2\cdot u_1 + Q\ell_1\cdot u_2\right)^2\right) - 2\log(1+Q)$$

$$\log\left(\left(\ell_2\cdot u_1 + Q\ell_1\cdot u_2\right)^2\right) = \log\left(\left(\ell_2\cdot u_1\right)^2\right) - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{\ell_1\cdot u_2}{\ell_2\cdot u_1}Q\right)^n$$

Local-in-time dynamics

- Local-in-time conservative binary dynamics at 4PM

Dlapa-Kälin-**ZL**-Porto PRL 2024

$$\frac{1}{\pi\Gamma}\chi_{b(\mathrm{nloc})}^{(4)(n\mathrm{SF})} = \frac{\nu}{\left(\gamma^2-1\right)^2}\left\{h_1 + \frac{\pi^2 h_2}{\sqrt{\gamma^2-1}} + h_3 \log\left(\frac{\gamma+1}{2}\right) + \frac{h_4 \operatorname{arccosh}(\gamma)}{\sqrt{\gamma^2-1}} + h_5 \log\left(\frac{\gamma-1}{8}\right) + h_6 \log^2\left(\frac{\gamma+1}{2}\right)\right.$$
$$\left. + h_7 \operatorname{arccosh}(\gamma)^2 + \frac{h_8 \log(2)\operatorname{arccosh}(\gamma)}{\sqrt{\gamma^2-1}} + h_9 \log\left(\frac{\gamma-1}{8}\right)\log\left(\frac{\gamma+1}{2}\right) + \cdots + h_{11}\mathrm{Li}_2\left(\frac{\gamma-1}{\gamma+1}\right)\right\}$$

Local-in-time dynamics

- Local-in-time conservative binary dynamics at 4PM — Dlapa-Kälin-**ZL**-Porto PRL 2024

$$\frac{1}{\pi\Gamma}\chi^{(4)(n\mathrm{SF})}_{b(\mathrm{nloc})} = \frac{\nu}{\left(\gamma^2-1\right)^2}\left\{h_1 + \frac{\pi^2 h_2}{\sqrt{\gamma^2-1}} + h_3\log\left(\frac{\gamma+1}{2}\right) + \frac{h_4\,\mathrm{arccosh}(\gamma)}{\sqrt{\gamma^2-1}} + h_5\log\left(\frac{\gamma-1}{8}\right) + h_6\log^2\left(\frac{\gamma+1}{2}\right)\right.$$
$$\left. + h_7\,\mathrm{arccosh}(\gamma)^2 + \frac{h_8\log(2)\mathrm{arccosh}(\gamma)}{\sqrt{\gamma^2-1}} + h_9\log\left(\frac{\gamma-1}{8}\right)\log\left(\frac{\gamma+1}{2}\right) + \cdots + h_{11}\mathrm{Li}_2\left(\frac{\gamma-1}{\gamma+1}\right)\right\}$$

- Local-in-time conservative binary dynamics at 5PM (1SF) — Dlapa-Kälin-**ZL**-Porto PRL 2025

$$\frac{\chi^{(5)(1\mathrm{SF})}_{(\mathrm{nloc})}}{\Gamma\nu} = \frac{\mathcal{G}_1(x)h_{13}(x)}{25200(x-1)^6x^3(x+1)^6\left(x^2+1\right)^7} + \frac{\mathcal{G}_2(x)h_{14}(x)}{90(x-1)^6x^3(x+1)^6\left(x^2+1\right)^7} + \frac{\mathcal{G}_3(x)h_{15}(x)}{12600(x-1)^7x^3(x+1)^7\left(x^2+1\right)^8}$$
$$+\frac{\mathcal{G}_4(x)h_{16}(x)}{1680(x-1)^7x^5(x+1)^7} + \frac{\mathcal{G}_5(x)h_{17}(x)}{30(x-1)^7x^3(x+1)^7\left(x^2+1\right)^8} + \frac{\mathcal{G}_6(x)h_{18}(x)}{420(x-1)^8x^3(x+1)^8} + \cdots \qquad \gamma = \frac{x^2+x}{2x}$$

Local-in-time dynamics

- Local-in-time conservative binary dynamics at 4PM — Dlapa-Kälin-**ZL**-Porto PRL 2024

$$\frac{1}{\pi\Gamma}\chi^{(4)(n\mathrm{SF})}_{b(\mathrm{nloc})} = \frac{\nu}{\left(\gamma^2-1\right)^2}\left\{h_1 + \frac{\pi^2 h_2}{\sqrt{\gamma^2-1}} + h_3\log\left(\frac{\gamma+1}{2}\right) + \frac{h_4\,\mathrm{arccosh}(\gamma)}{\sqrt{\gamma^2-1}} + h_5\log\left(\frac{\gamma-1}{8}\right) + h_6\log^2\left(\frac{\gamma+1}{2}\right)\right.$$
$$\left. + h_7\,\mathrm{arccosh}(\gamma)^2 + \frac{h_8\log(2)\mathrm{arccosh}(\gamma)}{\sqrt{\gamma^2-1}} + h_9\log\left(\frac{\gamma-1}{8}\right)\log\left(\frac{\gamma+1}{2}\right) + \cdots + h_{11}\mathrm{Li}_2\left(\frac{\gamma-1}{\gamma+1}\right)\right\}$$

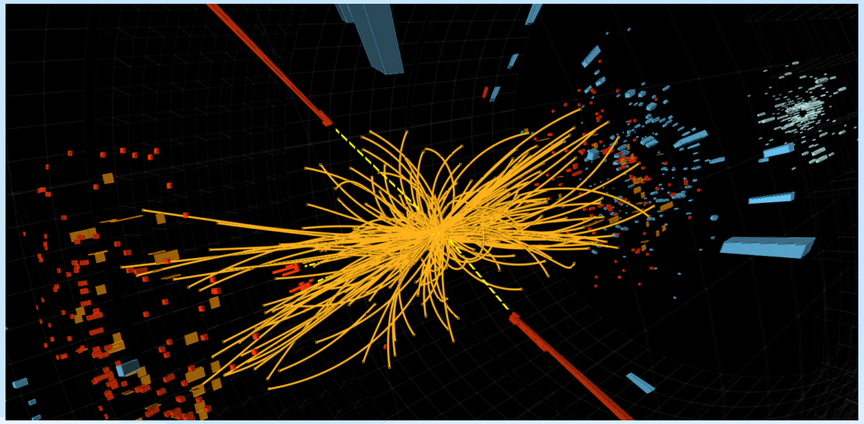
- Local-in-time conservative binary dynamics at 5PM (1SF) — Dlapa-Kälin-**ZL**-Porto PRL 2025

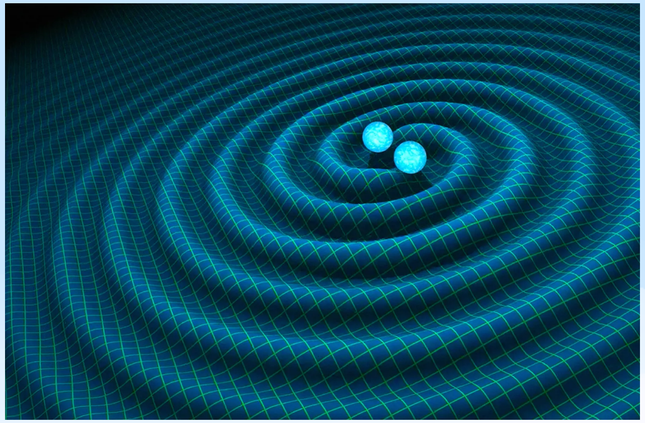
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- Local-in-time conservative binary dynamics at 5PM (10SF) — Dlapa-Kälin-**ZL**-Porto PRD 2026 & WIP

Conclusion

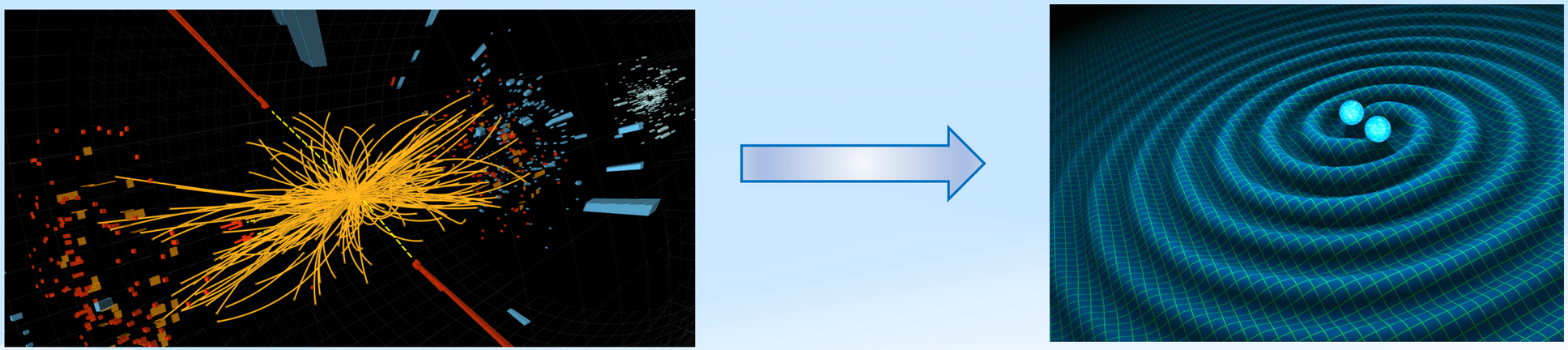
Modern techniques from Quantum Field Theory have already proven useful to improve theoretical predictions for gravitational-wave observables.





Conclusion

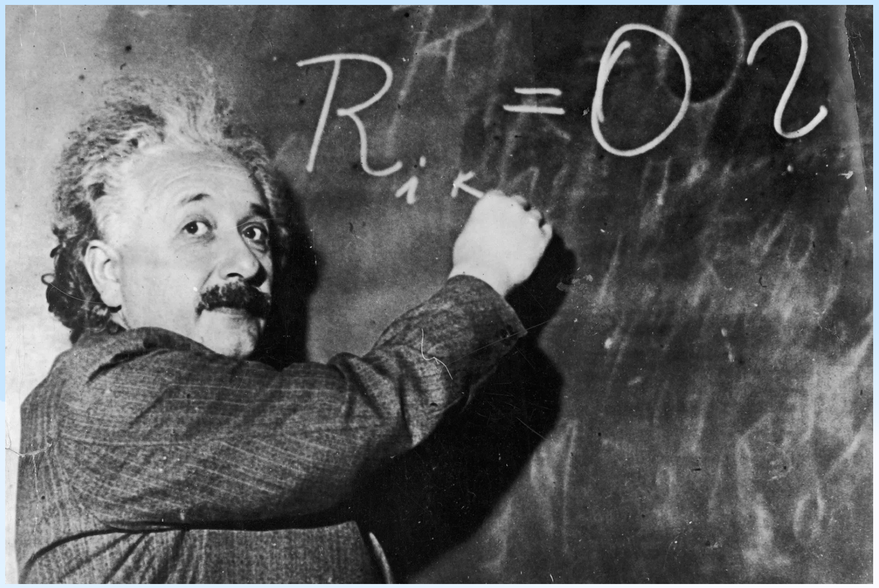
Modern techniques from Quantum Field Theory have already proven useful to improve theoretical predictions for gravitational-wave observables.

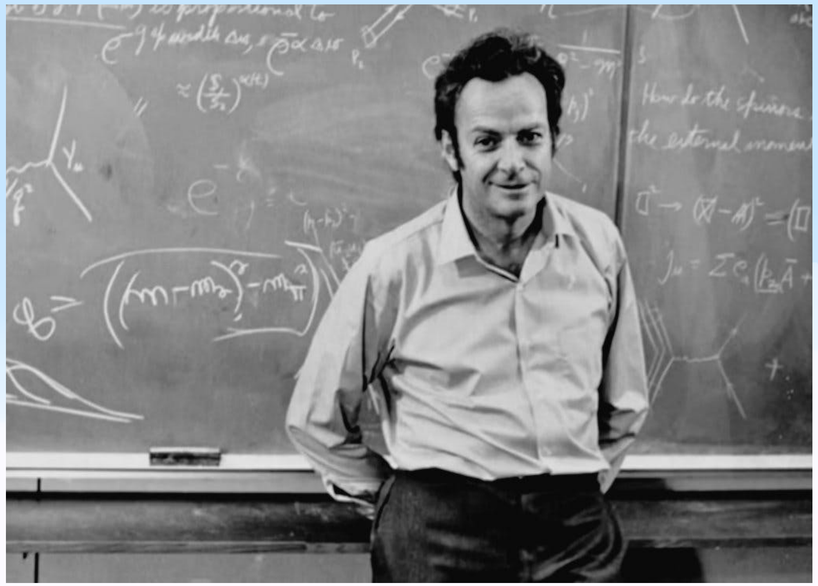


We developed an efficient framework for binary dynamics and have made breakthroughs to NNNNLO.

- Conservative spin & tidal effects at NLO JHEP 06 (2021) 012 PRD 102 (2020) 124025
- Two-body dynamics at NNLO PRL 125 (2020) 261103
- Two-body dynamics at NNNLO PLB 822 (2021) 136698 PRL 128 (2022) 161104 PRL 130 (2023) 101401
- Local- & nonlocal-in-time separation PRL 132 (2024) 221401 PRL 135 (2025) 251401 PRD 114 (2026) 024029
- Novel techniques to evaluate loop integrals in classical gravity JHEP 07 (2023) 181 JHEP 08 (2023) 109
- Beyond Einstein’s general relativity 2602.08876

Feynman's integrals solve Einstein's equations!





谢谢！