Gravitational Waves in Gauge Theory Gravity

许剑飞 (SEU)

@彭桓武高能基础理论研究中心

“弦理论与相关物理”研讨会

2021.07.13

References:

[1] C. J. L. Doran and A. N. Lasenby, "Geometric Algebra for Physicists," Cambridge University Press (2003).

[2] A. N. Lasenby, "Geometric Algebra, Gravity and Gravitational Waves," Adv. Appl. Clifford Algebras 29, 79 (2019) [arXiv:1912.05960].

[3] JX, "Gravitational waves in gauge theory gravity with a negative cosmological constant" [arXiv:2012.15001]

Content:

- Introduction
- Gauge principle for gravity
- Gravitational waves with $\Lambda=0$ and $\Lambda<0$
- Polarizations
- Velocity memory effects

• Introduction

General relativity is the most well know gravity theory under two principles:

1. Equivalence principle:

   ——strong version and weak version;

2. General principle of relativity:

   ——the physical laws take the same forms among all reference frames.

The absolute positions and orientations of fields are irrelevant for writing down physical laws (redundant gauge degrees of freedom)

- The traditional metric approach:

$$\{x^\mu \,, \mu = 0, 1, 2, 3\} \qquad ds^2 = g_{\mu\nu} dx^\mu dx^\nu \qquad R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}$$

Shortcoming:

(1) The metric tensor contains redundant frame information;

(2) It is not apparent which quantities can be treat as small when doing perturbation;

(3) The equations of gauge invariant quantities should be derived from the Einstein equation.

- The gauge theory approach:

  - Gauge theory is built on a flat background spacetime;

  - Gauge fields are introduced to maintain the position and orientation invariant of physical laws.

Position gauge field: $\mathrm{h}(a)$

Rotation gauge field: $\Omega(a)$

- Geometric algebra:

  - Geometric algebra is a covariant language for physics and geometry

  - For two vectors a and b, the inner and outer products are defined in terms of geometric product:

$$a \cdot b = \frac{1}{2}(ab + ba) \qquad a \wedge b = \frac{1}{2}(ab - ba)$$

  - The geometric product for two vectors can be written as

$$ab = a \cdot b + a \wedge b$$

- Axiomatic development of the geometric algebra:

  - In a vector space with any dimension, the geometric product obey three axioms:

    (1) Associative: $a(bc) = (ab)c = abc$

    (2) Distributive over addition: $a(b + c) = ab + ac, \quad (b + c)a = ba + ca$

    (3) The square of any vector is a real scalar: $a^2 \in \mathcal{R}$

- Axiomatic development of the geometric algebra:

  - By successively multiplying vectors together we can generate the complete algebra: $\mathcal{G}_n$

  - The elements of this algebra are called multivectors, which are linear combinations of geometric products of vectors,

$$A = \alpha(abc\cdots) + \beta(ef\cdots) + \cdots ,$$

- The outer product for vectors $a_1, a_2, \cdots, a_r$ is a grade-r multivector:

$$a_1 \wedge a_2 \wedge \cdots \wedge a_r = \frac{1}{r!} \sum (-1)^{\epsilon} a_{k_1} a_{k_2} \cdots a_{k_r}$$

- An arbitrary multivector A can be decomposed into a sum fixed grade terms:

$$A = \langle A \rangle_0 + \langle A \rangle_1 + \cdots = \sum_r \langle A \rangle_r$$

- The geometric product of a vector a and a grade-r multivector A:

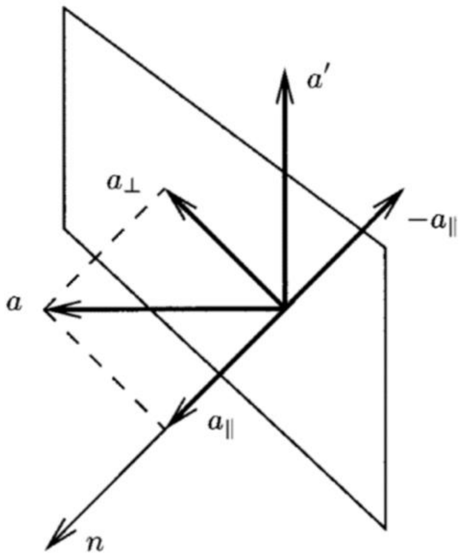
$$aA_r = a \cdot A_r + a \wedge A_r$$

- Reflections and rotations:

  - The reflection of a vector a along a unit vector n with $n^2 = 1$

$$a = n^2 a = n(n \cdot a + n \wedge a) = a_{||} + a_{\perp} \qquad a_{||} = n n \cdot a, \quad a_{\perp} = n n \wedge a$$

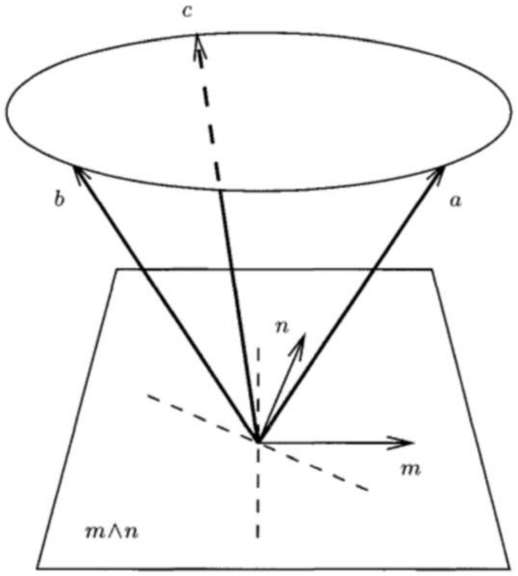
$$\begin{aligned} a' = a_{\perp} - a_{||} &= n n \wedge a - n n \cdot a \\ &= -n \cdot a n - n \wedge a n \\ &= -n a n \,, \end{aligned}$$



- Reflections and rotations:

  - A rotation can be achieved by successive reflection in the hyperplane perpendicular to two vectors:

$$a \mapsto c = nmamn$$



  - In geometric algebra, a rotation is generated by a rotor R:

$$R = nm = e^{-\hat{B}\theta/2}$$

$$\cos(\theta/2) = m \cdot n, \qquad \hat{B} = \frac{m \wedge n}{\sin(\theta/2)}$$

  - Under rotation, any vector a transforms in the following way

$$a \mapsto a' = RaR^{\dagger}$$

- Linear functions:

  - In geometric algebra, we use a frame independent and index free linear function instead of the tensors to describe the physical fields.

  - A linear function F is a quantity which maps vectors to vectors linearly in the same space:

$$\mathrm{F}(\lambda a + \mu b) = \lambda \mathrm{F}(a) + \mu \mathrm{F}(b)$$

$$\mathrm{F}(a \wedge b \wedge \cdots \wedge c) = \mathrm{F}(a) \wedge \mathrm{F}(b) \wedge \cdots \wedge \mathrm{F}(c)$$

$$a \cdot \bar{\mathrm{F}}(b) = \mathrm{F}(a) \cdot b$$

- Spacetime algebra:

  - The spacetime algebra is generated by four frame vectors $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$, satisfying the following algebraic relations

$$\gamma_0^2 = 1, \quad \gamma_0 \cdot \gamma_i = 0, \quad \gamma_i \cdot \gamma_j = -\delta_{ij}$$

  - In terms of geometric product, the frame vectors of the spacetime algebra satisfy

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu}, \qquad \eta_{\mu\nu} = \mathrm{diag}\{1, -1, -1, -1\} \qquad E_4 = \gamma_0\gamma_1\gamma_2\gamma_3$$

  - The frame $\{\gamma_\mu\}$ establish an basis for the spacetime algebra $\mathcal{G}(1,3)$:

$1,$	$\{\gamma_\mu\},$	$\{\gamma_\mu \wedge \gamma_\nu\},$	$\{E_4\gamma_\mu\},$	$E_4.$
1 scalar,	4 vectors,	6 bivectors,	4 trivectors,	1 volume element.

• Gauge principle for gravity

➢ Due to the universality of physics laws, the establishment of all physical relations should be completely independent of where we choose x to place:

$$\Psi'(x) = J'(x) \quad \Longleftrightarrow \quad \Psi(x') = J(x')$$

➢ The orientation irrelevance requires that if we rotate fields in Ψ and J, we will have

$$\Psi'(x) = J'(x) \quad \Longleftrightarrow \quad R\Psi(x)R^{\dagger} = RJ(x)R^{\dagger}$$

- The position gauge fields:

  - The physical quantity changes covariantly under displacement:

$$\phi'(x) = \phi(x') \qquad x' = f(x)$$

  - The derivative has the following transformation under displacement

$$\begin{aligned} a \cdot \nabla_x \phi'(x) &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\phi(f(x + \epsilon a)) - \phi(f(x)) \right) \\ &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\phi(x' + \epsilon \mathrm{f}(a)) - \phi(x') \right) \\ &= \mathrm{f}(a; x) \cdot \nabla_{x'} \phi(x') \,, \end{aligned}$$

$$\Rightarrow \quad \nabla_x \phi'(x) = \bar{\mathrm{f}}(\nabla_{x'}) \phi(x')$$

$$\mathrm{f}(a; x) = a \cdot \nabla_x f(x)$$

- The position gauge fields:

  - The position gauge field $\bar{\mathrm{h}}(a;x)$ is introduced to make the derivative operator covariant under displacement:

$$x \mapsto x' = f(x)$$

$$\bar{\mathrm{h}}(a;x) \mapsto \bar{\mathrm{h}}'(a;x) = \bar{\mathrm{h}}(\bar{\mathrm{f}}^{-1}(a); f(x))$$

$$\bar{\mathrm{h}}(\nabla_x;x) \mapsto \bar{\mathrm{h}}(\bar{\mathrm{f}}^{-1}(\nabla_x); f(x)) = \bar{\mathrm{h}}(\nabla_{x'};x')$$

$$A(x) = \bar{\mathrm{h}}(\nabla\phi(x))$$

$$A(x) \;\mapsto\; A'(x) \;=\; A(x')$$

- The rotation gauge fields:

  - The partial derivative has the following transformation under local rotation

$$\partial_\mu(R\mathcal{J}R^\dagger) = R\partial_\mu\mathcal{J}R^\dagger + 2(\partial_\mu RR^\dagger) \times (R\mathcal{J}R^\dagger)$$

  - The covariant derivative is introduced to make the derivative operator covariant under rotation:

$$\mathcal{D}'_\mu(R\mathcal{J}R^\dagger) = R\mathcal{D}_\mu\mathcal{J}R^\dagger \qquad \mathcal{D}_\mu = \partial_\mu + \Omega(e_\mu)\times$$

$$\Omega(a;x) \mapsto \Omega(a;x)' = R\Omega(a;x)R^\dagger - 2a\cdot\nabla RR^\dagger$$

$$\mathcal{D}\mathcal{J} = \bar{\mathrm{h}}(e^\mu)\mathcal{D}_\mu\mathcal{J}$$

- The gravitational field equations:

  - In gauge theory gravity, the dynamical gravitational fields $\mathrm{h}(a)$ and $\Omega(a)$ are introduced through gauge covariance.

  - Similar to the electromagnetism, the field equations can be constructed from an action made by the field strength

$$[\mathcal{D}_a,\ \mathcal{D}_b]M = R(a \wedge b)M$$

$$R(a \wedge b) = a \cdot \nabla\Omega(b) - b \cdot \nabla\Omega(a) + \Omega(a) \times \Omega(b)$$

- The gravitational field equations:

  - The field strength transforms according to

$$R(B;x) \mapsto R'(B;x) = R(\mathrm{f}(B);x')$$

$$R(B) \mapsto R'(B) = RR(B)R^{\dagger}$$

  - A covariant field strength therefore can be constructed:

$$\mathcal{R}(B;x) = R(\mathrm{h}(B);x)$$

$$\text{Displacements}: \quad \mathcal{R}'(B;x) = \mathcal{R}(B;x'),$$

$$\text{Rotations}: \quad \mathcal{R}'(B) = R\mathcal{R}(R^{\dagger}BR)R^{\dagger}$$

- The gravitational field equations:
  - The analogue of Ricci tensor and scalar, and Einstein tensor can be formulated with vector derivative inner product:

$$\begin{aligned} \text{Ricci Tansor}: &\qquad \mathcal{R}(b) = \partial_a \cdot \mathcal{R}(a \wedge b)\,, \\ \text{Ricci Scalar}: &\qquad \mathcal{R} = \partial_b \cdot \mathcal{R}(b)\,, \\ \text{Einstein Tensor}: &\qquad \mathcal{G}(a) = \mathcal{R}(a) - \frac{1}{2} a\mathcal{R}\,. \end{aligned}$$

  - The Ricci scalar is a good candidate for a Lagrangian density of the gravitational gauge fields, since it is now displacement covariant and rotational invariant:

$$S = \int d^4x \det(\mathrm{h}^{-1}) \left(\frac{1}{2}\mathcal{R} + \Lambda - \kappa \mathcal{L}_m \right)$$

- The gravitational field equations:

  - ➢ The variational principle leads to "Einstein equation" and "torsion equation":

$$\partial_{\bar{\mathrm{h}}(a)}\left(\det(\mathrm{h}^{-1})\left(\frac{\mathcal{R}}{2}+\Lambda-\kappa\mathcal{L}_m\right)\right)=0 \quad \Longrightarrow \quad \mathcal{G}(a)-\Lambda a=\kappa\mathcal{T}(a)$$

$$\partial_{\Omega(a)}\mathcal{R}-\det(\mathrm{h})\partial_b\cdot\nabla(\partial_{\Omega(a),b}\mathcal{R}\det(\mathrm{h}^{-1}))=2\kappa\partial_{\Omega(a)}\mathcal{L}_m$$

$$\Downarrow$$

$$\mathcal{D}\wedge\bar{\mathrm{h}}(a)=\kappa\left(S(a)+\frac{1}{2}(\mathrm{h}^{-1}(\partial_b)\cdot S(b))\wedge\bar{\mathrm{h}}(a)\right) \qquad S(a)=\partial_{\Omega(a)}\mathcal{L}_m$$

- Black holes:

  - The spherical symmetrical configuration in the absence of matter:

$$\bar{\mathsf{h}}(e^t) = f_1 e^t$$
$$\bar{\mathsf{h}}(e^r) = g_1 e^r + g_2 e^t$$
$$\bar{\mathsf{h}}(e^\theta) = e^\theta$$
$$\bar{\mathsf{h}}(e^\phi) = e^\phi$$

$$\Omega(\mathrm{h}(e_t)) = G e_r e_t$$
$$\Omega(\mathrm{h}(e_r)) = F e_r e_t$$
$$\Omega(\mathrm{h}(\hat{\theta})) = g_2/r\hat{\theta}e_t + (g_1 - 1)/r e_r\hat{\theta}$$
$$\Omega(\mathrm{h}(\hat{\phi})) = g_2/r\hat{\phi}e_t + (g_1 - 1)/r e_r\hat{\phi}$$

$$M = \tfrac{1}{2} r({g_2}^2 - {g_1}^2 + 1)$$
$$f_1 = \exp\{\textstyle\int^r -G/g_1\, ds\}$$
$$L_t g_1 = G g_2 \qquad L_t = e_t \cdot \bar{\mathsf{h}}(\nabla)$$
$$L_r g_2 = F g_1 \qquad L_r = e_r \cdot \bar{\mathsf{h}}(\nabla)$$

$$\mathcal{R}(B) = -\frac{M}{2r^3}(B + 3\sigma_r B \sigma_r)$$

$$\sigma_r = e_r e_t$$

- Black holes:

  - One specific gauge choice:

$$g_1 = 1 \qquad g_2 = -\sqrt{2M/r} \qquad f_1 = 1 \qquad G = 0 \qquad F = -\frac{M}{g_2 r^2} = \left(\frac{M}{2r^3}\right)^{1/2}$$

$$\bar{\mathsf{h}}(a) = a - \sqrt{2M/r}\, a \cdot e_r\, e_t$$

  - Compere to the metric approach, the metric tensor can be recovered from the position gauge field:

$$g_{\mu\nu} = g_\mu \cdot g_\nu = \mathsf{h}^{-1}(e_\mu) \cdot \mathsf{h}^{-1}(e_\nu)$$

$$ds^2 = dt^2 - \left(dr + \left(\frac{2M}{r}\right)^{1/2} dt\right)^2 - r^2(d\theta^2 + \sin^2(\theta)\, d\phi^2)$$

Painlevé
Gullstrand

- Black holes:

  - Point particle trajectories:

$$v = \mathsf{h}^{-1}(\dot{x}) = \dot{t}\, e_t + (\dot{t}\sqrt{2M/r} + \dot{r})e_r + \dot{\theta} e_\theta + \dot{\phi} e_\phi$$

$$v^2 = 1 \quad + \quad v \cdot \mathcal{D}v = 0 \quad \Rightarrow$$

$$\frac{\alpha^2 - 1}{2} = \frac{\dot{r}^2}{2} + V_{\mathit{eff}} \qquad V_{\mathit{eff}} = -\frac{M}{r} + \frac{J^2}{2r^2}\left(1 - \frac{2M}{r}\right)$$

  - For out going photons:

$$k = \mathsf{h}^{-1}(\dot{x}) \qquad k^2 = 0 \qquad k = \nu(e_t + e_r)$$

$$\dot{x} = \mathsf{h}(v) = \nu\big(e_t + (1 - \sqrt{(2M/r)})e_r\big)$$

$$\frac{dr}{dt} = 1 - \sqrt{(2M/r)}$$

- **Gravitational waves with Λ=0**

➢ In four dimensional space, denote the frame as $\{e_t, e_x, e_y, e_z\}$, the gravitational wave propagating in the z direction takes the form [A. Lasenby]

$$\bar{\mathrm{h}}(a) = a - \frac{1}{2} H a \cdot e_+ e_+ \qquad e_+ = e_t + e_z$$

$$H(t, x, y, z) = G(t - z) f(x, y)$$

➢ The wave solution in flat spacetime contains two modes:

$$\mathcal{G}(a) = -\frac{1}{2}(e_+ \cdot a) e_+ \nabla^2 H = 0 \quad \Rightarrow \quad \partial_x^2 H + \partial_y^2 H = 0$$

$$f(x, y) = c_1(x^2 - y^2) + 2c_2 xy$$

- **Gravitational waves with Λ=0**

  - The polarizations of the wave can be found in the Weyl tensor:

$$\mathcal{W}(a\wedge b) = \mathcal{R}(a\wedge b) - \tfrac{1}{2}\left(\mathcal{R}(a)\wedge b + a\wedge\mathcal{R}(b)\right) + \tfrac{1}{6}a\wedge b\,\mathcal{R}$$

$$= -\frac{1}{8}e_+\nabla((a\wedge b)\nabla H)e_+$$

$$= -\frac{1}{4}G(t-z)\{c_1\mathcal{W}^+(a\wedge b) + c_2\mathcal{W}^\times(a\wedge b)\}$$

$$\mathcal{W}^+(B) = e_+\left(e_x B e_x - e_y B e_y\right)e_+$$

$$\mathcal{W}^\times(B) = e_+\left(e_x B e_y + e_y B e_x\right)e_+$$

• Gravitational waves with Λ=0

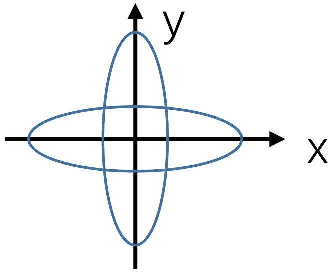
➢ The way of polarization is dictated in the "geodesic deviation":

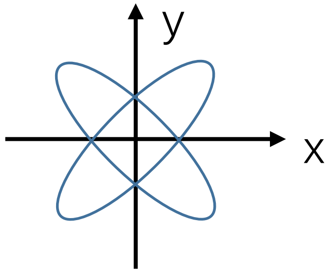
$$
\begin{aligned}
a &= v\cdot\mathcal{D}(v\cdot\mathcal{D}n) \\
&= \mathcal{R}(v\wedge n)\cdot v \\
&= \left(\mathcal{W}(v\wedge n) + \frac{1}{2}(\mathcal{R}(v)\wedge n + v\wedge\mathcal{R}(n)) - \frac{1}{6}v\wedge n\mathcal{R}\right)\cdot v \\
&= \mathcal{W}(v\wedge n)\cdot v
\end{aligned}
$$

$$
v\cdot\mathcal{D}v = 0\,, \qquad [v,\ n] = 0\,.
$$

$$
\mathcal{W}^{+}(e_t\wedge e_x)\cdot e_t \propto e_x \qquad \mathcal{W}^{\times}(e_t\wedge e_x)\cdot e_t \propto e_y
$$

$$
\mathcal{W}^{+}(e_t\wedge e_y)\cdot e_t \propto e_y \qquad \mathcal{W}^{\times}(e_t\wedge e_y)\cdot e_t \propto e_x
$$





- **Gravitational waves with Λ<0**

  - For Λ<0, the position gauge field takes the form [JX]

$$\bar{\mathrm{h}}(a) = \frac{x}{\ell}a - \frac{x}{2\ell}H(t-z,x,y)a\cdot e_+e_+$$

  - The Einstein tensor in this case can be figured out:

$$\mathcal{G}(a) = -\frac{3}{\ell^2}a - \frac{x^2}{2\ell^2}a\cdot e_+\left(\nabla^2 H + \frac{2}{x}e_x\cdot\nabla H\right)e_+$$

$$\mathcal{G}(a) - \Lambda a = 0 \quad \Rightarrow \quad \Lambda = -3/\ell^2 \qquad \nabla^2 H + \frac{2}{x}e_x\cdot\nabla H = 0$$

- **Gravitational waves with Λ<0**

  - The H equation also appears in Siklos spacetime when studying gravitational pp wave in AdS. The solutions take the form

$$H = x^2 \frac{\partial}{\partial x}\left(\frac{\xi(t-z, x+iy)+\bar{\xi}(t-z, x-iy)}{x}\right)$$

  - This wave solution is of Petrov type-N since $\mathcal{W}^2(a\wedge b)=0$. The Weyl tensor is given by

$$\mathcal{W}(a\wedge b) = -\frac{x^2}{8\ell^2} e_+ \nabla((a\wedge b)\nabla H)e_+$$

- **Polarizations**

➢ The deviation acceleration now has contributions both from the Weyl tensor and the cosmological constant:

$$\begin{aligned} a &= v \cdot \mathcal{D}(v \cdot \mathcal{D}n) \\ &= \left(\mathcal{W}(v \wedge n) + \frac{1}{2}(\mathcal{R}(v) \wedge n + v \wedge \mathcal{R}(n)) - \frac{1}{6} v \wedge n \mathcal{R} \right) \cdot v \\ &= \mathcal{W}(v \wedge n) \cdot v - \frac{1}{\ell^2} n \end{aligned}$$

➢ Some explicit solutions and the corresponding Weyl tensors:

$$H_1 = c_1(t-z)(x^2+y^2) \quad \Rightarrow \quad \mathcal{W}_1 = -c_1 \frac{x^2}{4\ell^2} e_+(e_x B e_x + e_y B e_y) e_+$$

$$H_2 = c_2(t-z)x^3 \quad \Rightarrow \quad \mathcal{W}_2 = -c_2 \frac{3x^2}{4\ell^2} e_+ e_x B e_x e_+$$

- **Polarizations**

$$H_3 = c_3(t-z)(3x^4 - 6x^2y^2 - y^4)$$

$$\Rightarrow$$

$$\begin{aligned}\mathcal{W}_3 = & -c_3\frac{3x^2}{2\ell^2}e_+[(3x^2-y^2)e_xBe_x - (x^2+y^2)e_yBe_y]e_+ \\ & + c_3\frac{3x^3y}{\ell^2}e_+(e_xBe_y + e_yBe_x)e_+\end{aligned}$$

$$H_4 = c_4(t-z)x^3(x^2-5y^2)$$

$$\Rightarrow$$

$$\begin{aligned}\mathcal{W}_4 = & -c_4\frac{5x^3}{4\ell^2}e_+[(2x^2-3y^2)e_xBe_x - x^2e_yBe_y]e_+ \\ & + c_4\frac{15x^4y}{4\ell^2}e_+(e_xBe_y + e_yBe_x)e_+\end{aligned}$$

- **Polarizations**

➢ To the first case,

$$\mathcal{W}_1 = -c_1 \frac{x^2}{4\ell^2} e_+ (e_x B e_x + e_y B e_y) e_+ \qquad \mathcal{W}_1 (e_t \wedge e_x) \cdot e_t = \mathcal{W}_1 (e_t \wedge e_x) \cdot e_t = 0$$

➢ To the second case,

$$\mathcal{W}_2 = -c_2 \frac{3x^2}{4\ell^2} e_+ e_x B e_x e_+ \qquad \mathcal{W}_2 (e_t \wedge e_x) \cdot e_t \propto e_x \,, \quad \mathcal{W}_2 (e_t \wedge e_y) \cdot e_t \propto e_y$$

➢ To the third case,

$$\mathcal{W}_3 = -c_3 \frac{3x^2}{2\ell^2} e_+ [(3x^2 - y^2) e_x B e_x - (x^2 + y^2) e_y B e_y] e_+ + c_3 \frac{3x^3 y}{\ell^2} e_+ (e_x B e_y + e_y B e_x) e_+ \qquad \mathcal{W}_3 = \mathcal{W}_3^+ + \mathcal{W}_3^\times$$

- **Velocity memory effects**

  - When the gravitational wave pass through a free falling particle, the change in the velocity of that particle will record the wave information.

  - Consider a massive particle free falling in the background of the gravitational gauge field. The motion of the particle is governed by

$$v \cdot \mathcal{D} v = 0$$

  - The covariant velocity is given by the inverse of the position gauge field:

$$v = \mathrm{h}^{-1}(\dot{x}) = \frac{\ell}{r} + \frac{\ell}{2r} H(t - z, r, y)(\gamma_+ \cdot \dot{x})\gamma_+$$

- **Velocity memory effects**

  - Explicitly, the position vector x satisfies

$$\ddot{x} = \frac{r}{\ell^2}\gamma_1 + 2\frac{\dot{r}}{r}\dot{x} + \frac{1}{2}(\gamma_+ \cdot \dot{x})^2 \nabla H - (\gamma_+ \cdot \dot{x})(\nabla H \cdot \dot{x})\gamma_+$$

  - In components

$$\ddot{u} = 2\frac{\dot{r}}{r}\dot{u}\,, \quad \Rightarrow \quad \dot{u} = kr^2 \qquad\qquad u = t - z$$

$$\ddot{v} = 2\frac{\dot{r}}{r}\dot{v} + \frac{1}{2}\dot{u}^2\partial_u H - \frac{1}{2}\dot{u}(\nabla H \cdot \dot{x})\,, \qquad\qquad v = t + z$$

$$\ddot{r} = \frac{r}{\ell^2} + 2\frac{\dot{r}^2}{r} - \frac{1}{4}\dot{u}^2\partial_r H\,,$$

$$\ddot{y} = 2\frac{\dot{r}}{r}\dot{y} - \frac{1}{4}\dot{u}^2\partial_y H\,,$$

- **Velocity memory effects**

➢ In terms of u, the equation of r's and y's components take the form

$$\frac{d^2 r}{du^2} = \frac{1}{k^2 \ell^2 r^3} - \frac{1}{4}\partial_r H(u, r, y)$$

$$\frac{d^2 y}{du^2} = -\frac{1}{4}\partial_y H(u, r, y)$$

➢ Consider the following impulsive gravitational wave

$$H = r^3 F(u) \qquad F(u) = \begin{cases} Ae^{-bu}, & u > 0 \\ Ae^{bu}, & u < 0 \end{cases}$$

- **Velocity memory effects**

➢ When A is small, the r-equation can be solved perturbatively:

$$r(u) = r_0(u) - \frac{3}{4} A r_1(u) + \mathcal{O}(A^2)$$

$$r_0(u) = \sqrt{u^2 + 1/(k^2\ell^2)}$$

$$r_1(u) = \frac{\sqrt{u^2+1/(k^2\ell^2)}}{4/(k\ell)} \left(\frac{u - i/(k\ell)}{u + i/(k\ell)}\right) \left(C_0 - i \int_0^u e^{-bs}(s + i/(k\ell))^2 \sqrt{s^2 + 1/(k^2\ell^2)} ds\right) + \text{hermitian conjugate}\,, \quad 2)$$

$$C_0 = -i \int_{u_0}^0 e^{bs}(s + i/(k\ell))^2 \sqrt{s^2 + 1/(k^2\ell^2)} ds$$

- **Velocity memory effects**

➢ Long after the impulsive wave left, the change in velocity along r-direction can be written as

$$\Delta\left(\frac{dr}{du}\right) = \frac{\sqrt{u_0^2 + 1/(k^2\ell^2)} - u_0}{\sqrt{u_0^2 + 1/(k^2\ell^2)}}$$
$$-\frac{3k\ell A}{8}\mathrm{Re}\left(C_0 - i\int_0^{+\infty} e^{-bs}(s + i/(k\ell))^2\sqrt{s^2 + 1/(k^2\ell^2)}ds\right)$$

where the particle is initially placed at $r(u_0) = \sqrt{u_0^2 + 1/(k^2\ell^2)}$ with velocity

$$\frac{dr}{du}|_{u=u_0} = \frac{u_0}{\sqrt{u_0^2 + 1/(k^2\ell^2)}}$$

In progress……

Thank you!