Lovelock gravity, string's α' corrections and 2D regular black hole

伍厚文

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arXiv:1906.09650 PRD (Non-SUSY AdS vacua),
arXiv:1909.00830 JHEP (Removing big-bang singularity),
arXiv:1910.05808 JHEP (Construct α' corrected or loop solutions),
arXiv:2012.13312 JHEP (Lovelock gravity from string theory),
and works to appear very soon.

Outlines

Motivation and Background

It is generally known that Lovelock gravity is a natural generalization of general relativity to higher dimensions and only includes second order derivatives. If string theory is the correct candidate of quantum gravity, it must have Lovelock gravity as a descendant somehow.

In 1985, Barton Zwiebach proved that the first-order α' correction of string theory can be transformed to a Gauss-Bonnet term which matches the quadratic term of Lovelock gravity.

There is no further progress during 35 years. People began to believe that these two theories are different.

Difficulties :

- It is difficult to obtain higher order α' corrections of string theory.
- The higher order α' corrections include higher order derivatives of a metric, which breaks a requirement of Lovelock gravity.

Motivation and Background

Consider Einstein's gravity in $D = 4$ dimensions

$$I_{Einstein} = \int d^D x \sqrt{-\tilde{g}} \left(-2\Lambda + \tilde{R} \right),$$

where we set $16\pi G_D = c = 1$ for simplicity and a notation "tilde" indicates the Einstein frame.

For arbitrary D dimensions, if we require a gravitational theory to be ghost free, or equivalently saying its Einstein tensor $G_{\mu\nu}$ satisfies following conditions:

- The tensor is symmetric,
- It is a function of a metric and its first two derivatives (no ghosts),
- It is free of the divergence: $\nabla_\mu G_{\mu\nu} = 0$.

The modified gravitational theory is unique that is constructed by dimensionally extended Euler densities, say Lovelock gravity.

The action of Lovelock gravity is

$$\begin{aligned} I_{Love} &= \int d^D x \sqrt{-\tilde{g}} \sum_{k=0}^{\left[\frac{D-1}{2}\right]} \alpha_k \lambda^{2k-2} \mathcal{L}_k, \\ &= \int d^D x \sqrt{-\tilde{g}} \Big(\alpha_0 \lambda^{-2} + \alpha_1 \tilde{R} + \alpha_2 \lambda^2 (\tilde{R}^2 + \tilde{R}_{\alpha\beta\mu\nu} \tilde{R}^{\alpha\beta\mu\nu} - 4 \tilde{R}_{\mu\nu} \tilde{R}^{\mu\nu}) + \ldots \Big), \\ \mathcal{L}_k &\equiv \frac{1}{2^k} \delta^{\mu_1 \cdots \mu_k \nu_1 \cdots \nu_k}_{\rho_1 \cdots \rho_k \sigma_1 \cdots \sigma_k} \tilde{R}_{\mu_1 \nu_1}{}^{\rho_1 \sigma_1} \cdots \tilde{R}_{\mu_k \nu_k}{}^{\rho_k \sigma_k}, \end{aligned}$$

where $[(D-1)/2]$ denotes the integer part of $(D-1)/2$. α_k are dimensionless and λ has a length scale. The action only has a finite number of terms for $k < D/2$. Terms for $k > D/2$ vanish identically, and the term $k = D/2$ is a topological invariant. To match the Einstein-Hilbert action, we have $\alpha_0 \lambda^{-2} = -2\Lambda$ and $\alpha_1 = 1$.

The term of α_2 is the Gauss-Bonnet.

For example

When $D = 2$

$$I_{Love} = \int d^D x \sqrt{-\tilde{g}} \, (-2\Lambda + \underbrace{\tilde{R}}_{\substack{topological \\ invariant}} + \underbrace{\cdots}_{\substack{do\ not\ contribute \\ to\ the\ EOM}}) \, .$$

When $D = 4$

$$I_{Love} = \int d^D x \sqrt{-\tilde{g}} \left(-2\Lambda + \tilde{R} + \underbrace{\alpha_2 \lambda^2 \left(\tilde{R}^2 + \tilde{R}_{\alpha\beta\mu\nu} \tilde{R}^{\alpha\beta\mu\nu} - 4 \tilde{R}_{\mu\nu} \tilde{R}^{\mu\nu} \right)}_{\substack{topological \\ invariant}} + \underbrace{\cdots}_{\substack{do\ not\ contribute \\ to\ the\ EOM}} \right) .$$

To see it more clearly, let us utilize FLRW cosmological background:

$$ds^2 = -dt^2 + \tilde{a}(t)^2 dx_i dx^i,$$

Lovelock action becomes

$$\begin{aligned} I_{Love} &= \int dt \tilde{a}^{D-1} \sum_{k=0}^{\left[\frac{D-1}{2}\right]} \left(-\frac{1}{2k-1}\right) \alpha_k \lambda^{2k-2} \frac{(D-1)!}{(D-2k-1)!} \tilde{H}^{2k} \\ &= \int dt \tilde{a}^{D-1} \Big[\alpha_0 \lambda^{-2} + \alpha_1 (D-1)(D-2) \tilde{H}^2 \\ &\quad -\frac{1}{3} \alpha_2 \lambda^2 (D-1)(D-2)(D-3)(D-4) \tilde{H}^4 + \dots \Big] \end{aligned}$$

This is why recent work on the 4D Gauss-Bonnet solution can be obtained by the following replacement:

$$\alpha_2 \to \frac{\alpha_2}{D-4}$$

On the other hand, as early as the mid-1980s, it has been speculated that Lovelock theory might be derived from string theory. If string theory is as powerful as claimed, this should be true.

Consider the tree-level low energy effective action of closed string

$$I^{(0)}_{string} = \int d^{d+1}x\sqrt{-g}e^{-2\phi}(R + 4(\partial\phi)^2 - \frac{1}{12}\mathcal{H}^2),$$

with massless string fields:

$$g_{\mu\nu}, \qquad b_{\mu\nu}, \qquad \phi,$$

is valid only in the perturbative regime:

$$g_s = e^{2\phi} << 1 \quad \text{and} \quad |R|\alpha' << 1$$

- The first condition $g_s = e^{2\phi} << 1$ concerns quantum/loop/topology corrections.
- Since $\alpha' \sim \ell^2_{\text{string}}$, the second condition $|R|\alpha' << 1$ concerns the classical stringy correction. This means we have not really included "string" effects!

Motivation and Background

Beyond the perturbative regime, the tree level string effective action receives two kinds of corrections:

- *Classical* stringy effects, namely the higher-derivative expansion, controlled by α'.
- *Quantum* loop corrections, controlled by the string coupling $g_s = e^{2\phi}$.

Ignoring matter sources, the most general perturbative form of the string effective action has the following structure

$$
\begin{aligned}
I_{string} \quad &= \quad \int d^{d+1}x\sqrt{-g}e^{-2\phi}\Big\{ \\
&\quad \Big[(R + 4(\partial\phi)^2 - \frac{1}{12}\mathcal{H}^2) + \frac{\alpha'}{4}(R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} + \cdots) + \mathcal{O}(\alpha'^2)\Big] \\
&+ \quad e^{2\phi}\Big[(c^1_R R + c^1_\phi(\partial\phi)^2 + c^1_{\mathcal{H}}\mathcal{H}^2) + \alpha'(c^1_{\alpha' R}R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} + \cdots) + \mathcal{O}(\alpha'^2)\Big] \\
&+ \quad e^{4\phi}\Big[(c^2_R R + c^2_\phi(\partial\phi)^2 + c^2_{\mathcal{H}}\mathcal{H}^2) + \alpha'(c^2_{\alpha' R}R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} + \cdots) + \mathcal{O}(\alpha'^2)\Big] \\
&+ \quad \cdots\Big\},
\end{aligned}
$$

with unknown $c^i_{[\cdots]}$.

Here, we only consider the closed string's low energy effective action with first-order α'

$$I_{string} = \int d^D x \sqrt{-g} e^{-2\phi} \left[\left(R + 4 (\partial\phi)^2 \right) + \alpha' \lambda_0 \left(R_{\mu\nu\sigma\rho} R^{\mu\nu\sigma\rho} \right) + \mathcal{O}\left(\alpha'^2\right) \right].$$

By using field redefinitions $\phi \to \phi + \alpha'^k \delta\phi$, $g_{\mu\nu} \to g_{\mu\nu} + \alpha'^k \delta g_{\mu\nu}$, the action becomes

$$\begin{aligned} I_{string} \quad = \quad & \int d^D x \sqrt{-g} e^{-2\phi} \left[\left(R + 4 (\partial\phi)^2 \right) \right. \\ & + \alpha' \lambda_0 \left(R_{\mu\nu\sigma\rho} R^{\mu\nu\sigma\rho} + a_1 R_{\mu\nu} R^{\mu\nu} + a_2 R^2 + a_3 R^{\mu\nu} \partial_\mu\phi \partial_\nu\phi + a_4 R (\partial\phi)^2 \right. \\ & \left. \left. + a_5 R \Box\phi + a_6 (\Box\phi)^2 + a_7 \Box\phi (\partial\phi)^2 + a_8 (\partial\phi)^4 \right) + \mathcal{O}\left(\alpha'^2\right) \right]. \end{aligned}$$

- Unambiguous coefficients λ_0: Independent of the field redefinitions.
- Ambiguous coefficients a_i: Transform under the field redefinitions. These coefficients do not affect the S-matrix.
- When the curvature grows, we need to consider the α' corrections to all orders in a non-perturbative way. Therefore, the ghost problem exists in higher-order.

However, it is difficult to obtain the higher-order Lovelock gravity from string theory due to following reasons:

- The higher order α' corrections are obtained by the Sigma model Weyl anomaly coefficients or "β-functions", it is difficult to obtain results beyond two-loops.
- The higher order α' corrections include higher order derivatives of the metric.

Thanks to recent developments on classification of α' corrections, the higher-order α' corrections only include first two derivatives of the FLRW metric, which meets the requirements of Lovelock gravity. It is therefore possible to compare these two kinds of theories.

Classification of α' corrections

Zeroth order

Hohm-Zwiebach action

It is well known that for the time-dependent background, the tree level action can be recast in an $O(d,d)$ covariant form [Veneziano 1991]. To this end, it is convenient to set $b_{ij} = 0$ and write the fields in the form

$$g_{\mu\nu} = \begin{pmatrix} -n(t)^2 & 0 \\ 0 & G_{ij}(t) \end{pmatrix}.$$

The action can be rewritten as

$$I^{(0)}_{string} = \int dtne^{-\Phi}\left[-(\mathcal{D}\Phi)^2 - \frac{1}{8}\mathrm{Tr}\left((\mathcal{D}\mathcal{S}_0)^2\right)\right], \qquad \mathcal{D} \equiv \frac{1}{n(t)}\frac{\partial}{\partial t},$$

where $\mathcal{S}_0$ is the standard form of $O(d,d)$ matrix

$$\mathcal{S}_0 = \begin{pmatrix} B_0 G_0^{-1} & G_0 - B_0 G_0^{-1} B_0 \\ G_0^{-1} & -G_0^{-1} B_0 \end{pmatrix}, \qquad \sqrt{g}e^{-2\phi} = e^{-\Phi},$$

This action is manifestly invariant under the $O(d,d)$ transformations

$$\Phi \to \Phi, \qquad S_0 = \begin{pmatrix} 0 & G_0 \\ G_0^{-1} & 0 \end{pmatrix} \to \tilde{S}_0 = \begin{pmatrix} 0 & G_0^{-1} \\ G_0 & 0 \end{pmatrix},$$

or equivalently,

$$\Phi \to \Phi, \qquad G_0 \to G_0^{-1}.$$

Hohm-Zwiebach action

T-duality	Scale-factor duality
World-sheet theory	Low energy effective theory
Compactified background	Non-compactified background
Discrete $O(d,d;Z)$ group	Continuous $O(d,d;R)$ group
$R \longleftrightarrow \alpha'/R$	$a^2 \longleftrightarrow 1/a^2$

First order

Hohm-Zwiebach action

Let us recall the first order α' correction:

$$\begin{aligned} I_{string} &= \int d^D x \sqrt{-g} e^{-2\phi} \Big[R + 4 (\partial\phi)^2 \\ &+ \frac{1}{4}\alpha' \left(-R_{GB}^2 + 16 \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) \partial_\mu \phi \partial_\nu \phi \right. \\ &\left. -16 \Box\phi (\partial\phi)^2 + 16 (\partial\phi)^4 \right) \Big], \end{aligned}$$

which can be rewritten as

$$\begin{aligned} I_{string} &= \int dtn e^{-\Phi} \Big[- (\mathcal{D}\Phi)^2 - \frac{1}{8} \mathrm{Tr} \left((\mathcal{D}\mathcal{S}_0)^2 \right) \\ & -\frac{1}{4}\alpha' \left(\frac{1}{16} \mathrm{Tr} (\mathcal{D}\mathcal{S}_0)^4 - \frac{1}{64} \left(\mathrm{Tr} (\mathcal{D}\mathcal{S}_0)^2 \right)^2 \right. \\ & \left. -\frac{1}{4} (\mathcal{D}\Phi)^2 \mathrm{Tr} (\mathcal{D}\mathcal{S}_0)^2 - \frac{1}{3} (\mathcal{D}\Phi)^4 + \mathcal{F}(G_0, \Phi) \right) \Big] \end{aligned}$$

$\mathcal{F}(G_0, \Phi)$ does not belong to the $O(d,d)$ invariants.

Hohm-Zwiebach action

$$I_{string} = \int dtne^{-\Phi}\left[-(\mathcal{D}\Phi)^2 - \frac{1}{8}\mathrm{Tr}\left((\mathcal{D}\mathcal{S}_0)^2\right)\right.$$
$$-\frac{1}{4}\alpha'\left(\frac{1}{16}\mathrm{Tr}(\mathcal{D}\mathcal{S}_0)^4 - \frac{1}{64}\left(\mathrm{Tr}(\mathcal{D}\mathcal{S}_0)^2\right)^2\right.$$
$$\left.\left.-\frac{1}{4}(\mathcal{D}\Phi)^2\mathrm{Tr}(\mathcal{D}\mathcal{S}_0)^2 - \frac{1}{3}(\mathcal{D}\Phi)^4 + \mathcal{F}(G_0,\Phi)\right)\right]$$

Meissner proved that there existed a field redefinition [Meissner,1997]:

$$\underbrace{\begin{pmatrix} BG^{-1} & G-BG^{-1}B \\ G^{-1} & -G^{-1}B \end{pmatrix}}_{\mathcal{S}} = \underbrace{\begin{pmatrix} B_0G_0^{-1} & G_0-B_0G_0^{-1}B_0 \\ G_0^{-1} & -G_0^{-1}B_0 \end{pmatrix}}_{\mathcal{S}_0} + \alpha'\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

The action becomes $O(d,d)$ invariant

$$I_{string} = \int dtne^{-\Phi}\left[-(\mathcal{D}\Phi)^2 - \frac{1}{8}\mathrm{Tr}\left((\mathcal{D}\mathcal{S})^2\right)\right.$$
$$-\frac{1}{4}\alpha'\left(\frac{1}{16}\mathrm{Tr}(\mathcal{D}\mathcal{S})^4 - \frac{1}{64}\left(\mathrm{Tr}(\mathcal{D}\mathcal{S})^2\right)^2\right.$$
$$\left.\left.-\frac{1}{4}(\mathcal{D}\Phi)^2\mathrm{Tr}(\mathcal{D}\mathcal{S})^2 - \frac{1}{3}(\mathcal{D}\Phi)^4\right)\right]$$

All orders in α'

Sen proved that, to all orders in α', for configurations independent of m coordinates, the action possesses an $O(m,m)$ symmetry [Sen, 1991,1992]。 In particular for a time-dependent metric which depends on t only, the symmetry is $O(d,d)$.

First Blood

Based on Sen's proof, Hohm and Zwiebach assumed that the terms of action which break the $O(d,d)$ invariance (cannot be written as $\mathcal{S}$) can be absorbed by field redefinitions to all orders in α':

$$\mathcal{S} = \begin{pmatrix} BG^{-1} & G - BG^{-1}B \\ G^{-1} & -G^{-1}B \end{pmatrix} + \alpha' \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} + \mathcal{O}(\alpha'^2),$$

Therefore, the higher-order α' corrections can be constructed by the $O(d,d)$ invariant terms: $\mathcal{S}$, Φ and their higher-order derivatives. It would be described as

$$I^{(k)}_{string} = \alpha'^k \int dtne^{-\Phi} X\left(\{\mathcal{D}\Phi\}, \{\mathcal{S}\}\right),$$

where X is a function of $\mathcal{D}\Phi$, $\mathcal{D}^2\Phi$,... and $\mathcal{S}$, $\mathcal{D}\mathcal{S}$, $\mathcal{D}^2\mathcal{S}$,... as well as their miexd terms.

Hohm-Zwiebach action

The aim here is to simplify the action further:

$$I^{(k)}_{string} \quad = \quad \alpha'^k \int dtne^{-\Phi} X\left(\{\mathcal{D}\Phi\}, \{\mathcal{S}\}\right),$$

Double Kill

Based on the definition of $\mathcal{S}$, we can prove:

$$\mathrm{Tr}\left(\mathcal{S}\right) = \mathrm{Tr}\left(\mathcal{D}\mathcal{S}\right) = \mathrm{Tr}\left(\mathcal{D}^2\mathcal{S}\right) = \cdots = 0,$$
$$\mathrm{Tr}\left(\left(\mathcal{D}\mathcal{S}\right)^{2k+1}\right) = 0, \quad \mathrm{Tr}\left(\mathcal{S}\left(\mathcal{D}\mathcal{S}\right)^k\right) = 0.$$

Triple Kill

Using a series of field redefintions:

$$\Phi \quad \to \quad \Phi + \alpha'^k \delta\Phi,$$
$$g_{ij} \quad \to \quad g_{ij} + \alpha'^k \delta g_{ij}.$$

This step removes any function of Φ, and higher-derivatives of $\mathcal{D}\mathcal{S}$.

Hohm-Zwiebach action

The result gives

$$I^{(k)}_{string} \quad = \quad \alpha'^{k} \int dtne^{-\Phi} X\left(\mathcal{DS}\right).$$

In other words, the most general action takes the following form

$$I_{string} = \int dtne^{-\Phi} \left(L_0 + \alpha' L_1 + \left(\alpha'\right)^2 L_2 + \left(\alpha'\right)^3 L_3 + \cdots \right),$$

where

$$\begin{aligned}
L_1 \quad &= \quad a_1 \mathrm{Tr}\left(\mathcal{DS}\right)^4 + a_2 \left[\mathrm{Tr}\left(\mathcal{DS}\right)^2\right]^2, \\
L_2 \quad &= \quad b_1 \mathrm{Tr}\left(\mathcal{DS}\right)^6 + b_2 \mathrm{Tr}\left(\mathcal{DS}\right)^4 \mathrm{Tr}\left(\mathcal{DS}\right)^2 + b_3 \left[\mathrm{Tr}\left(\mathcal{DS}\right)^2\right]^3, \\
L_3 \quad &= \quad c_1 \mathrm{Tr}\left(\mathcal{DS}\right)^8 + c_2 \left[\mathrm{Tr}\left(\mathcal{DS}\right)^4\right]^2 + c_3 \mathrm{Tr}\left(\mathcal{DS}\right)^6 \mathrm{Tr}\left(\mathcal{DS}\right)^2 \\
&\quad + c_4 \mathrm{Tr}\left(\mathcal{DS}\right)^4 \left[\mathrm{Tr}\left(\mathcal{DS}\right)^2\right]^2 + c_5 \left[\mathrm{Tr}\left(\mathcal{DS}\right)^2\right]^4.
\end{aligned}$$

Hohm-Zwiebach action

Quadra Kill

Using lapse redefinitions:

$$n \to n + \alpha'^k \delta n, \qquad \delta n = n\beta X_{2k}\left(\mathcal{D}\mathcal{S}\right),$$

where β is an undetermined constant, and X_{2k} denotes a term with $2k$ derivatives which is constructed by the products of traces of powers of $\mathcal{D}\mathcal{S}$. It implies any term with a $\mathrm{Tr}\left(\mathcal{D}\mathcal{S}\right)^2$ can be set to 0,

Then, the action becomes

$$I_{string} = \int dtne^{-\Phi}\left(L_0 + \alpha' L_1 + \left(\alpha'\right)^2 L_2 + \left(\alpha'\right)^3 L_3 + \cdots\right),$$

where

$$\begin{aligned} L_1 &= a_1 \mathrm{Tr}\left(\mathcal{D}\mathcal{S}\right)^4, \\ L_2 &= b_1 \mathrm{Tr}\left(\mathcal{D}\mathcal{S}\right)^6, \\ L_3 &= c_1 \mathrm{Tr}\left(\mathcal{D}\mathcal{S}\right)^8 + c_2\left[\mathrm{Tr}\left(\mathcal{D}\mathcal{S}\right)^4\right]^2. \end{aligned}$$

Hohm-Zwiebach action

Penta Kill

Utilizing FLRW metric

$$ds^2 = -n(t)^2\, dt^2 + a(t)^2\, dx_i dx^i.$$

The action for the higher-order correction takes a form

$$L \propto (-1)^k\, 2^{2k+1} c_k d H^{2k}(t) + (-1)^k\, c_{k,l} 2^{2k+1} 2 d^2 H^{2k}(t)\,,$$

where $H(t) \equiv \frac{\dot{a}(t)}{a(t)}$ is Hubble parameter, and we can redefine the coefficients:

$$c_k \rightarrow c_k + 2 d c_{k,l}.$$

Finally, we will get

Hohm-Zwiebach action

$$I_{HZ} = \int dt e^{-\Phi} \left(-\frac{1}{n}\dot{\Phi}^2 - d \sum_{k=1}^{\infty} \frac{(-\alpha')^{k-1}}{n^{2k-1}} 2^{2k+1} c_k H^{2k} \right),$$

where $c_1 = -\frac{1}{8}$, $c_2 = \frac{1}{64}$ and $c_k \geq 3$ are undetermined constants for the bosonic string theory. It is worth noting that this action is non-perturbative in α', since we do not require $\alpha' \rightarrow 0$.

From Hohm-Zwiebach action to Lovelock action

To compare Hohm-Zwiebach action with Lovelock action, we need to transform Hohm-Zwiebach action into the Einstein frame at first:

$$g_{\mu\nu} = \exp\left(\frac{4(\phi-\phi_0)}{D-2}\right)\tilde{g}_{\mu\nu}, \quad \tilde{\phi} = \phi - \phi_0,$$

Setting $\tilde{n} = 1$ and $\tilde{\phi} = 0$, Hohm-Zwiebach action becomes

Hohm-Zwiebach action

$$\tilde{I}_{HZ} = e^{-2\phi_0}\int dt\tilde{a}^{D-1}\sum_{k=1}^{\infty}(-1)^k\, 2^{2k+1}\,(D-1)\,c_k\left(\sqrt{\alpha'}\right)^{2k-2}\tilde{H}^{2k}.$$

Lovelock action in the FLRW metric gives

Lovelock action

$$I_{Love} = \int dt\tilde{a}^{D-1}\sum_{k=0}^{\left[\frac{D-1}{2}\right]}\left(-\frac{1}{2k-1}\right)\frac{(D-1)!}{(D-2k-1)!}\alpha_k\lambda^{2k-2}\tilde{H}^{2k}.$$

However, a conceptual mismatch exists: for a particular dimension $D = d + 1$, Lovelock gravity has finite terms but α' corrections are infinitely many. Lovelock theory is an unique ghost free gravitational theory.

Let us recall the low energy effective actions:

$$I_{string} = \int d^D x \sqrt{-g} e^{-2\phi} \left[\left(R + 4 (\partial\phi)^2 \right) + \frac{\alpha'}{4} \left(R_{\mu\nu\sigma\rho} R^{\mu\nu\sigma\rho} \right) + \ldots \right].$$

$$I_{string} = \int d^D x \sqrt{-g} e^{-2\phi} \left[\left(R + 4 (\partial\phi)^2 \right) + \frac{\alpha'}{4} \left(R^2 + R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} - 4 R_{\mu\nu} R^{\mu\nu} \right) + .$$

The mismatch comes from the field redifitions!

Now, let us try to get the field redefinition $\delta_{k=2}I^{(0)}_{string}$ at first-order. Recall the low energy effective action and Hohm-Zwiebach action at first order. There exists a series of field redefinitions, the results are given by:

$$\begin{aligned} I^{(1)}_{string} &= \alpha' \int d^D x \sqrt{-g} e^{-2\phi} \left(-R^2_{GB} + 16 \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) \partial_\mu \phi \partial_\nu \phi \right. \\ &\quad \left. -16 \Box \phi \left(\partial \phi \right)^2 + 16 \left(\partial \phi \right)^4 \right) \Big] \\ &= \alpha' e^{-\phi_0} \int dt \tilde{a}^{D-1} \left[2^5 c_2 \left(D-1 \right) \left(-\frac{1}{6} \left(D-4 \right) \left(D-3 \right) \left(D-2 \right) \right) \tilde{H}^4 \right], \\ I^{(1)}_{HZ} &= \alpha' e^{-2\phi_0} \int dt \tilde{a}^{D-1} \left[2^5 c_2 \left(D-1 \right) \tilde{H}^4 \right] \end{aligned}$$

Therefore,

$$\begin{aligned} \delta_{k=2} I^{(0)}_{string} &= I^{(1)}_{HZ} - I^{(1)}_{string} \\ &= -\alpha' e^{-2\phi_0} \int dt \tilde{a}^{D-1} 2^5 c_2 \left(D-1 \right) \times \\ &\quad \left(-\frac{1}{6} \left(D-4 \right) \left(D-3 \right) \left(D-2 \right) - 1 \right) \tilde{H}^4. \end{aligned}$$

$$\delta^I_{k=2}\tilde{I}^{(0)}_{string} = \left(-\alpha'\right)^{2-1} e^{-2\phi_0}\int dt\tilde{a}^{D-1}2^5 c_2\,(D-1)\times \left(-\frac{1}{6}(D-4)(D-3)(D-2)-1\right)\tilde{H}^4.$$

$$\downarrow \frac{(D-2)!}{(D-2k-1)!} = (D-2k)\dots(D-3)(D-2)$$

$$\delta^I_{k\geq 2}\tilde{I}^{(0)}_{string} = \left(-\alpha'\right)^{k-1} e^{-2\phi_0}\int dt\tilde{a}^{D-1}2^{2k+1} c_k\,(D-1)\times \left((-)^{k-1}\frac{1}{(k+1)!}\frac{(D-2)!}{(D-2k-1)!}-1\right)\tilde{H}^{2k}.$$

From Hohm-Zwiebach action to Lovelock action

Finally, we will get the suitable Hohm-Zwiebach action by the field redefinitions:

$$I'_{HZ} = I_{HZ} - \sum_{k=2}^{\infty} \delta_{k\geq 2} I^{(0)}_{string}.$$

Hohm-Zwiebach action

$$I'_{HZ} = \int dt \tilde{a}^{D-1} \sum_{k=1}^{\left[\frac{D-1}{2}\right]} \left(-\frac{1}{2k-1}\right) \beta_k \left(\sqrt{\alpha'}\right)^{2k-2} \frac{(D-1)!}{(D-2k-1)!} \tilde{H}^{2k},$$

Lovelock action

$$I_{Love} = \int dt \tilde{a}^{D-1} \sum_{k=0}^{\left[\frac{D-1}{2}\right]} \left(-\frac{1}{2k-1}\right) \alpha_k \lambda^{2k-2} \frac{(D-1)!}{(D-2k-1)!} \tilde{H}^{2k}.$$

where $\beta_k = e^{-2\phi_0} \frac{2k-1}{(k+1)!} 2^{2k+1} c_k$ and $c_1 \equiv -2\,(c_1) = \frac{1}{4}$.

These field redefinitions are not trivial due to the following reasons:

- We cannot absorb (k)-th order into $(k-1)$-th order by a field redefinition. It is because Hohm-Zwiebach action is not perturbative in α' and the modification of c_k and c_{k-1} will change the results of S-matrix.
- One might think the cosmological background is very special and the higher-order effects can be absorbed into the tree-level action by the field redefinitions. Let us look at a counterexample. Considering Witten's 2D black hole ($ds^2 = -dx^2 + a\,(x)^2\,dt^2$), which can be obtained from the beta function to the lowest order. On the other hand, Witten's solution is the leading term of the α'-corrected solutions, say DVV's black hole solution. However, it is impossible to obtain the DVV's black hole solution from the lowest order beta function by the field redefinitions [Grumiller,2005]. It was also clarified by Tseytlin in the three-loop approximation [Tseytlin,1991].[1]

[1] We thank H. Lü for raising this question.

From Hohm-Zwiebach action to Lovelock action

Then, the relation is very clear

$$\alpha_k = \frac{2k-1}{(k+1)!} 2^{2k+1} c_k .$$

where we set $\phi_0 = 0$ for satisfying $\alpha_1 = 1$. Moreover, we will get $\alpha_2 = 1/4$ which agrees with known results.

- It implies all orders coupling constants α_k of Lovelock gravity are uniquely determined by the coefficients c_k of closed string theory.

Based on our derivation, it is possible to ask whether different kinds of generalizations of gravity could be related to Hohm-Zwiebach action by appropriate field redefinitions.

For example, quasi-topological gravity gives a similar action in the cosmological background:

$$\begin{aligned} I_{Quasi} &= \int dt\tilde{a}^{D-1}\sum_{k=1}^{K}\left(-\frac{1}{2k-1}\right)\mu_k\gamma_k\tilde{H}^{2k} \\ &= \int dt\tilde{a}^{D-1}\left(-\mu_1\gamma_1\tilde{H}^2-\frac{1}{3}\mu_2\gamma_2\tilde{H}^4-\frac{1}{5}\mu_3\gamma_3\tilde{H}^6-\frac{1}{7}\mu_4\gamma_4\tilde{H}^8+\dots\right), \end{aligned}$$

where a dimension of μ_k is l^{2k-2} and:

$$\begin{aligned} \gamma_1 &= (D-1)(D-2), \\ \gamma_2 &= (D-1)(D-2)(D-3)(D-4), \\ \gamma_3 &= \frac{(D-1)(D-2)(D-3)(D-6)\left(3D^2-15D+16\right)}{8(2D-3)^4}, \\ \gamma_4 &= (D-1)^2(D-2)^2(D-3)^2(D-4)(D-8)\times \\ &\quad \left(D^5-20D^4+142D^3-472D^2+743D-436\right). \end{aligned}$$

It looks like we can identify two theories directly:

$$
\begin{aligned}
I_{Quasi} = &\qquad I_{HZ} = \\
\int dt\tilde{a}^{D-1}\times &\qquad e^{-2\phi_0}\int dt\tilde{a}^{D-1}\times \\
\Big(-\bar{\mu}_1\tilde{H}^2 &\qquad \Big(-(D-1)(D-2)\tilde{H}^2 \\
-\frac{1}{3}\bar{\mu}_2 l^2\tilde{H}^4 &\qquad +(D-1)\,2^5 c_2\alpha'\tilde{H}^4 \\
-\frac{1}{5}\bar{\mu}_3 l^4\tilde{H}^6 &\qquad -(D-1)\,2^7 c_3\alpha'^2\tilde{H}^6 \\
+\cdots\Big), &\qquad +\cdots\Big).
\end{aligned}
$$

However, the $O(d,d)$ invariant field redefinitions, only can provide following terms to modify ambiguous coefficients in the low energy effective action:

$$\begin{aligned}\delta_{k\geq 2}\tilde{I}^{(0)}_{string} &= (-)^{k-1}\left(\sqrt{\alpha'}\right)^{2k-2} e^{-2\phi_0}\int d t \tilde{a}^{D-1} 2^{2k+1} c_k\,(D-1)\times \\ &\quad \left((-)^{k-1}\frac{1}{(k+1)!}\frac{(D-2)!}{(D-2k-1)!}-1\right)\tilde{H}^{2k} \\ &= \int dt\tilde{a}^{D-1}\left[\sum_{i=0}^{2k-1} a_i D^i\right]\left(\sqrt{\alpha'}\right)^{2k-2}\tilde{H}^{2k},\end{aligned}$$

where a_i are arbitrary constants and D^i denotes an i-th power of the spacetime dimension D. It is worth noting that a_i do not depend on spacetime.

Let us recall the cubic term of quasi-topological gravity:

$$\begin{aligned}I^{(3)}_{Quasi} &= \int dt\tilde{a}^{D-1}\left(-\frac{1}{5}\frac{(D-1)(D-2)(D-3)(D-6)\left(3D^2-15D+16\right)}{8\,(2D-3)^4}\mu_3\tilde{H}^6\right) \\ &= \int dt\tilde{a}^{D-1}\left[\sum_{i=0}^{5}\frac{a_i}{2D-3}D^i\right]\mu_3\tilde{H}^6.\end{aligned}$$

Therefore, we cannot reach the quasi-topological gravity by field redefinitions from Hohm-Zwiebach action.

2D regular black hole

Let us consider 1+1 dimensional low energy effective action with vanishing Kalb-Ramond field:

$$S = \int d^2x\sqrt{-g}e^{-2\phi}\left(R + 4\left(\nabla\phi\right)^2 + \lambda^2\right),$$

where $\lambda^2 = -\frac{2(D-26)}{3\alpha'}$. The black hole solution is given by [Mandal,Sengupta,Wadia, 1991]:

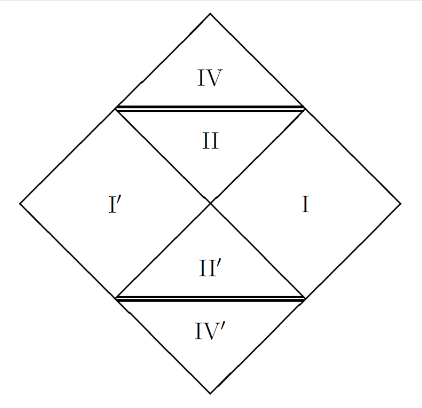
$$\begin{aligned} ds^2 &= -\left(1 - \frac{M}{r}\right)dt^2 + \left(1 - \frac{M}{r}\right)^{-1}\frac{1}{\lambda^2 r^2}dr^2, \\ \phi &= -\frac{1}{2}\ln\left(\frac{2}{M}r\right). \end{aligned}$$

However,this solution is only valid as long as the curvature is small enough. Is there a way to figure out an exact solution of the full action?

- α'-corrected low energy effective action: Unknown
- $SL\left(2,R\right)/U\left(1\right)$ gauged WZW model: ✓

2D regular black hole

In the semiclassical limit $k \to \infty$ ($k \sim 1/\alpha'$ is the Kac–Moody level), Witten find a 2D black hole solution [Witten, 1991].



$$ds^2 = -\tanh^2\left(\frac{\lambda}{2}x\right)dt^2 + dx^2, \text{ Region I}$$

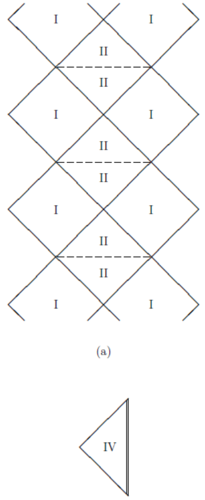
$$ds^2 = -dx^2 + \tan^2\left(\frac{\lambda}{2}x\right)dt^2, \text{ Region II}$$

$$ds^2 = -\coth^2\left(\frac{\lambda}{2}x\right)dt^2 + dx^2, \text{ Region IV}$$

Region I and Region IV are T-dual solutions.

2D regular black hole

For general k, Dijkgraaf, Verlinde and Verlinde discovered the exact 2D string black hole (**DVV's black hole**) [Dijkgraaf, Verlinde, Verlinde, 1992].



(a)

(b)

$$ds^2 = -\frac{\tanh^2\left(\frac{\lambda}{2}x\right)}{1-\frac{\alpha'\lambda^2}{2+\alpha'\lambda^2}\tanh^2\left(\frac{\lambda}{2}x\right)}dt^2+dx^2, \text{ Region I}$$

$$ds^2 = -dx^2+\frac{\tan^2\left(\frac{\lambda}{2}x\right)}{1+\frac{\alpha'\lambda^2}{2+\alpha'\lambda^2}\tan^2\left(\frac{\lambda}{2}x\right)}dt^2, \text{ Region II}$$

It is believed to be valid to all orders in α'.

When $k \to \infty$ or $\alpha' \to 0$, it reduces to the previous Witten's black hole.

Problems of this solution:

- It has been verified that this solution is the perturbative solution of the β-function equations up to three loops [Tseytlin, 1991].
- There still exists a disjoint region which possesses the naked singularity.
- It is difficult to generate new solutions.

Thanks to the recent progress on classifying all orders α' corrections of the low energy effective action, it is possible to re-study the exact 2D string black hole systematically.

Let us recall Witten's 2D black hole solution:

$$ds^2 = -dx^2 + \tan^2\left(\frac{\lambda}{2}x\right)dt^2$$

The ansatz therefore is

$$ds^2 = -dx^2 + a\left(x\right)^2 dt^2.$$

Based on this ansatz, the Hohm-Zwiebach action can be written as

$$\begin{aligned} I_{HZ} &= \int d^2x\sqrt{-g}e^{-2\phi}\Big(R + 4\left(\partial\phi\right)^2 \\ &\quad + \frac{1}{4}\alpha'\left(R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} + \ldots\right) + \alpha'^2\left(\ldots\right) + \ldots\Big), \\ &= \int dx e^{-\Phi}\left(-\dot{\Phi}^2 - \sum_{k=1}^{\infty}\left(-\alpha'\right)^{k-1}2^{2k+1}c_k H^{2k}\right), \end{aligned}$$

where dot denotes as $\dot{f}\left(x\right) \equiv \partial_x f\left(x\right)$, $H\left(x\right) \equiv \frac{\dot{a}(x)}{a(x)}$, $c_1 = -\frac{1}{8}$, $c_2 = \frac{1}{64}$, $c_3 = -\frac{1}{3.2^7}$, $c_4 = \frac{9}{65536} - \frac{1}{2048}\zeta\left(3\right)$ and $c_{k>4}$'s are unknown coefficients for a bosonic case.

The corresponding EOM are given by:

$$\begin{aligned}\ddot{\Phi}+\frac{1}{2}Hf\left(H\right) &= 0,\\ \frac{d}{dx}\left(e^{-\Phi}f\left(H\right)\right) &= 0,\\ \dot{\Phi}^{2}+g\left(H\right)+\lambda^{2} &= 0,\end{aligned}$$

where

$$\begin{aligned}f\left(H\right) &= \sum_{\mathrm{k}=1}^{\infty}\left(-\alpha'\right)^{k-1}2^{2(k+1)}kc_kH^{2k-1}=-2H-\alpha'2H^3+\cdots\,,\\ g\left(H\right) &= \sum_{\mathrm{k}=1}^{\infty}\left(-\alpha'\right)^{k-1}2^{2k+1}\left(2k-1\right)c_kH^{2k}=-H^2-\alpha'\frac{3}{2}H^4+\cdots\,.\end{aligned}$$

Therefore, our aim is to figure out the solutions of these EOM.

To obtain the non-singular solutions of the EOM, two constraints must be respected by such black hole solutions:

a. As $\alpha'\to 0$, the solutions must exactly match the the perturbative solution.
b. The constructed solution is anticipated to be regular everywhere.

To calculate the perturbative solutions we introduce the variable

$$\Omega \equiv e^{-\Phi},$$

and using the new expansions:

$$\begin{aligned} \Omega(x) &= \Omega_0(x) + \alpha' \Omega_1(x) + \alpha'^2 \Omega_2(x) + \ldots, \\ H(x) &= H_0(x) + \alpha' H_1(x) + \alpha'^2 H_2(x) + \ldots, \end{aligned}$$

the EOM can be solved iteratively to arbitrary order in α'.
The perturbative solution is

$$\begin{aligned} H(x) &= \lambda \csc(\lambda x) - \frac{\lambda^2}{4} \frac{\cos(2\lambda x)}{\sin(\lambda x)} \alpha' + \cdots, \\ \Omega(x) &= \sin(\lambda x) + \frac{\lambda^3}{4} \frac{(\cos(2\lambda x) + 4)}{\sin^3(\lambda x)} \alpha' + \cdots, \end{aligned}$$

It is ready to see that the first term of the Hubble parameter or the dilaton matches the Witten's 2D black hole solution.

The general solution is

$$\Phi\left(x\right)=\frac{1}{2}\log\left(\frac{\sum_{\mathrm{k}=1}^{N}\left(\alpha'\lambda^{2}\right)^{k-1}}{\sum_{\mathrm{k}=1}^{N}\sigma_{k}\left(\lambda x,c_{k}\right)\left(\alpha'\lambda^{2}\right)^{k-1}}\right),$$

where σ_k's are functions of λx and c_k. After obtained regular $\Phi\left(x\right)$, the regular solutions $H\left(x\right)$, $f\left(x\right)$ and $g\left(x\right)$ can be simultaneously determined due to the EOM. Moreover, in the perturbative regime $\alpha'\to 0$, the general solution $\Phi\left(x\right)$ is expanded as,

$$\Phi\left(x\right)=-\frac{1}{2}\log\left(\sigma_{1}\right)+\frac{\left(\sigma_{1}-\sigma_{2}\right)}{2\sigma_{1}}\alpha'\lambda^{2}+\frac{\left(\sigma_{1}^{2}-2\sigma_{3}\sigma_{1}+\sigma_{2}^{2}\right)}{4\sigma_{1}^{2}}\left(\alpha'\lambda^{2}\right)^{2}+\cdots.$$

To compare with the perturbative solution,

$$\Phi\left(x\right)=-\log\left(\sin\left(\lambda x\right)\right)-\frac{1}{4}\left(\cot^{2}\left(\lambda x\right)-1\right)\alpha'\lambda^{2}+\cdots.$$

we can fix the functions σ_k's.

2D regular black hole

To cover the first two order corrections, we have $\sigma_1 = \sin^2(\lambda x)$, $\sigma_2 = \frac{1}{2}\sin^2(\lambda x)\left(\cot^2(\lambda x) + 1\right)$ and $N = 2$. The non-perturbative solution is

$$
\begin{aligned}
\Phi(x) &= \log\sqrt{\frac{1+\alpha'\lambda^2}{\sin^2(\lambda x) + \frac{1}{2}\alpha'\lambda^2\sin^2(\lambda x)(\cot^2(\lambda x)+1)}},\\
H(x) &= -\frac{\sqrt{2}\lambda\left(\left(\alpha'\lambda^2+1\right)\cos(2\lambda x)-1\right)}{(\alpha'\lambda^2+1)^{1/2}(\alpha'\lambda^2+1-\cos(2\lambda x))^{3/2}},\\
f(x) &= -2\sqrt{2}\lambda\left(\frac{\alpha'\lambda^2+1}{\alpha'\lambda^2+1-\cos(2\lambda x)}\right)^{1/2},\\
g(x) &= \frac{\lambda^2}{(\alpha'\lambda^2+1-\cos(2\lambda x))^2}\left(-\alpha'\lambda^2\left(\alpha'\lambda^2+2\right)\right.\\
&\quad \left.+2\left(\alpha'\lambda^2+1\right)\cos(2\lambda x)-2\right).
\end{aligned}
$$

2D regular black hole

In other words, the black hole solution is

$$ds^2 = -dx^2 + a(x)^2 \, dt^2,$$

where

$$\begin{aligned} a(x) \quad = \quad & C \exp \sqrt{2} \left[\sqrt{\frac{\alpha' \lambda^2 + 1}{\alpha' \lambda^2}} \mathbb{F} \left(x\lambda \middle| -\frac{2}{\alpha' \lambda^2} \right) \right. \\ & - \sqrt{\frac{\alpha' \lambda^2}{\alpha' \lambda^2 + 1}} \mathbb{E} \left(x\lambda \middle| -\frac{2}{\alpha' \lambda^2} \right) \\ & \left. - \frac{\sin(2\lambda x)}{\sqrt{(\alpha' \lambda^2 + 1)(\alpha' \lambda^2 + 1 - \cos(2\lambda x))}} \right], \end{aligned}$$

$\mathbb{F}(\phi|m)$ and $\mathbb{E}(\phi|m)$ are elliptic integrals of the first and second kinds, and the physical dilaton is given by

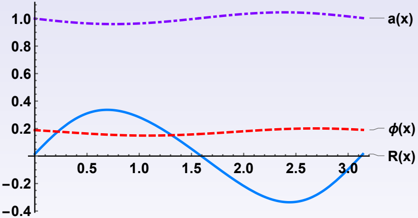
$$\phi(x) = \frac{1}{2}\Phi(x) + \frac{1}{2}\ln a(x).$$

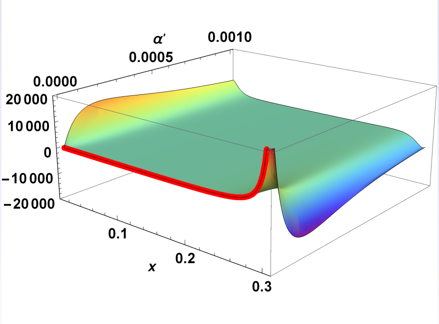
2D regular black hole

To check its curvature singularity, we present Kretschmann scalar in our ansatz

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{1}{2}R_{\mu\nu}R^{\mu\nu} = R^2 = 4\left(\dot{H} + H^2\right)^2 .$$

It is regular when $\alpha'\lambda^2 > 0$ which satisfies $\alpha'\lambda^2 = -\frac{2(D-26)}{3} = 16$.





Future works

- Derive Lovelock gravity from string theory in the black hole spacetime.
- Remove the singularities of D dimensional spherically symmetric black holes and black p-branes.

Thank you!

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