Emergent Einstein Equation in p-adic CFT Tensor Networks

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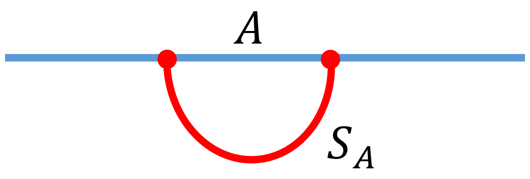
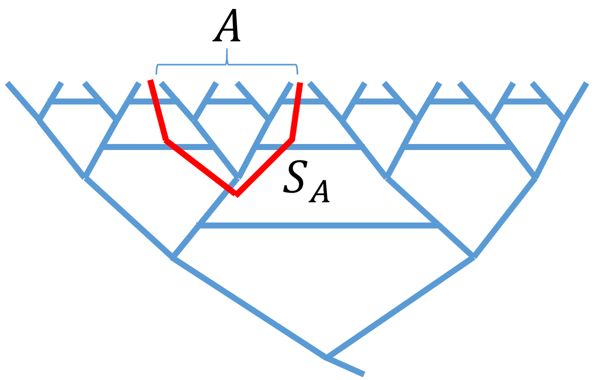
Why p-adic CFT tensor networks?

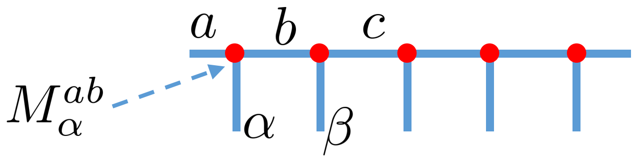
- In AdS/CFT, the Ryu-Takayanagi formula relates entanglement entropy in CFT with areas of minimal surfaces in AdS. [Ryu and Takayanagi 06]



$$S_A \sim \log L_A$$



- Tensor networks are widely used to construct many-body wave functions. They are also geometrization of patterns of entanglement of many-body wave-functions.



$$|\Psi\rangle = \sum_{\alpha,\beta,\ldots} M_\alpha^{ab} M_\beta^{bc} \ldots |\alpha,\beta,\ldots\rangle$$

- Tensor networks capture the microscopic mechanism behind the AdS/CFT correspondence. [B. Swingle 09]

Why p-adic CFT tensor networks?

- So we should reconstruct gravitational dynamics using tensor networks. But it remains a tremendous challenge.
- Tensor networks describing a general CFT are unclear. While the tensor networks describing the p-adic CFT are known and simple.
- Our work shows that a **unique** Einstein equation naturally emerges from the p-adic CFT tensor networks!

The difficulty of the tensor network program

One has to make guesses of

- the tensor network describing the p-adic CFT
- the "metric" extracted out of the tensor network
- the bulk matter dynamics encoded in the tensor network
- Einstein equation relating the guessed metric and bulk matter

Outline

I. The p-adic number and p-adic CFTs
II. The p-adic AdS/CFT
III. The p-adic CFT Tensor Networks
IV. Distances and graph curvature from the TN
V. The graph Einstein equation
VI. Solving the Einstein constraints
VII. Summary

I. The p-adic number and p-adic CFTs

The **p-adic number field $\boldsymbol{Q_p}$** is the field extension of the rational numbers $\mathbb{Q}$. For $x \in Q_p$:

$$x = p^v \sum_{i=0}^{\infty} a_i p^i, \quad (v \in \mathbb{Z},\ a_i \in \mathbb{F}_p = \{0, 1, \ldots, p-1\},\ a_0 \neq 0)$$

The p-adic norm is $|x|_p = p^{-v}$. And p is a prime number.

The **p-adic CFT** is a field theory living in the p-adic number field Q_p, i.e. its coordinates $x \in Q_p$. Two pieces of algebraic data: [Melzer 1989]

(a) the spectrum of primary operators O_a with conformal dimensions Δ_a, transformation rule:

$$x \quad \to \quad x' = \frac{ax+b}{cx+d}, \quad (a, b, c, d \in Q_p), \quad \mathcal{O}_a(x) \to \tilde{\mathcal{O}}_a(x') = \left|\frac{ad-bc}{(cx+d)^2}\right|_p^{-\Delta_a} \mathcal{O}_a(x).$$

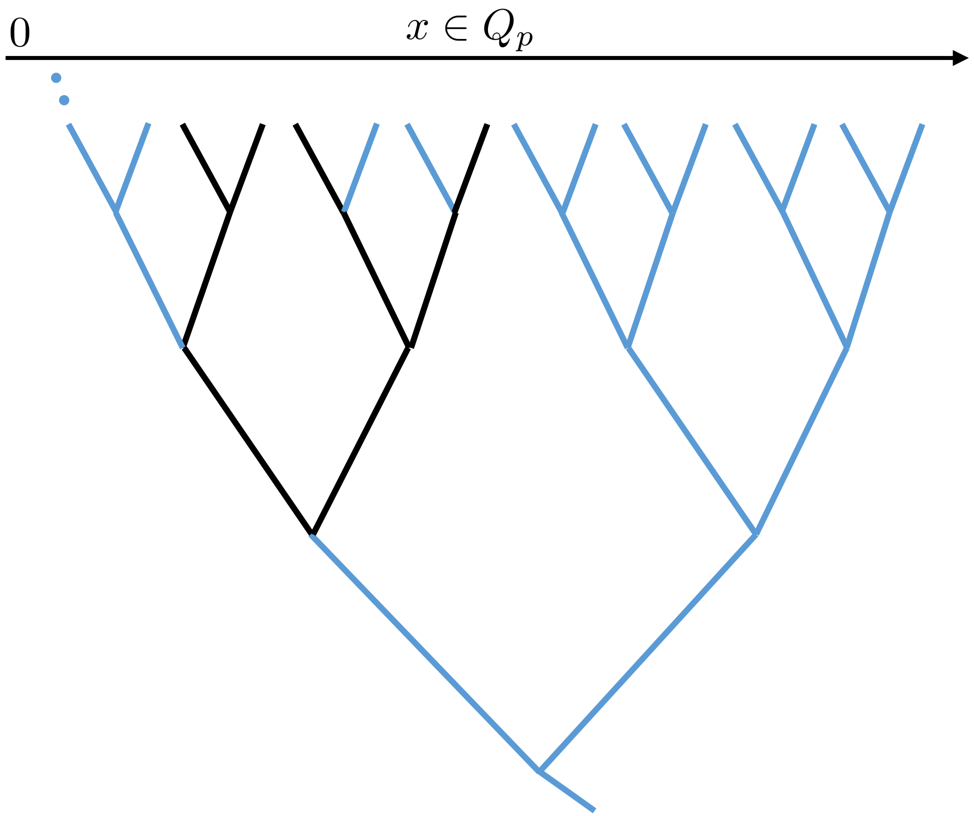
(b) OPE coefficients C^{abc}.

$$\langle \mathcal{O}_a(x_1) \mathcal{O}_b(x_2) \rangle = \frac{\delta^{ab}}{|x_1 - x_2|_p^{2\Delta_a}}.$$

$$\langle \mathcal{O}_a(x_1) \mathcal{O}_b(x_2) \mathcal{O}_c(x_3) \rangle = \frac{C^{abc}}{|x_{12}|_p^{\Delta_a+\Delta_b-\Delta_c} |x_{23}|_p^{\Delta_b+\Delta_c-\Delta_a} |x_{31}|_p^{\Delta_c+\Delta_a-\Delta_b}}.$$

II. The p-adic AdS/CFT [Gubser et.al. 16, Heydeman et.al. 16]

The bulk space time for p-adic CFT is the **Bruhat-Tits tree**. It is an infinite p+1 valent tree.



p=2 case

Bulk scalar field ϕ_a with mass m_a

$\Updownarrow$

CFT operator O_a with dimension Δ_a

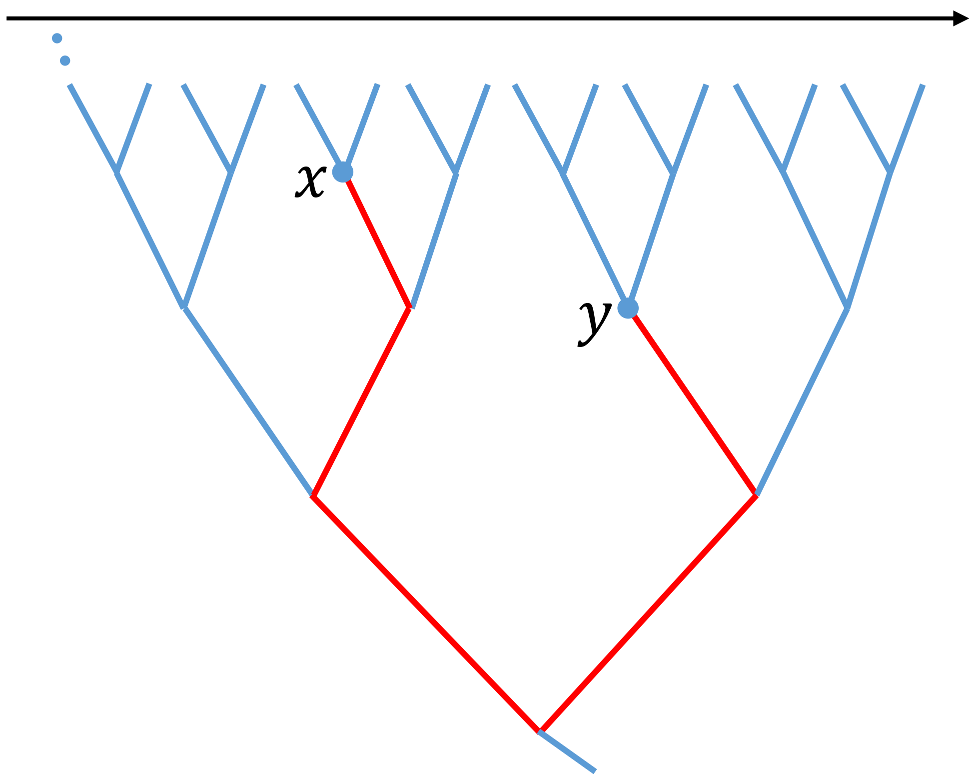
$$m_a^2 = -\frac{1}{\zeta_p(\Delta_a - 1)\zeta_p(-\Delta_a)} \ , \quad \zeta_p(s) \equiv \frac{1}{1-p^{-s}}$$

The Witten diagram of the bulk scalar fields on the BT tree can recover the correlation function of p-adic CFT.

Action of the bulk quantum field theory on BT tree: [Gubser et.al. 16]

$$S_m = \sum_{\langle xy\rangle} \frac{1}{2}(\phi_x^a - \phi_y^a)^2 + \sum_x \frac{1}{2} m_a^2 (\phi_x^a)^2 + \mathcal{O}(\phi^3),$$

x denotes the vertices. $\langle xy\rangle$ denotes the edges.



p=2 case

The propagator is

$$\langle \phi^a(x)\phi^b(y)\rangle \sim \delta^{ab} p^{-\Delta_a d(x,y)}$$

$d(x,y)$ counts the edges between x and y.

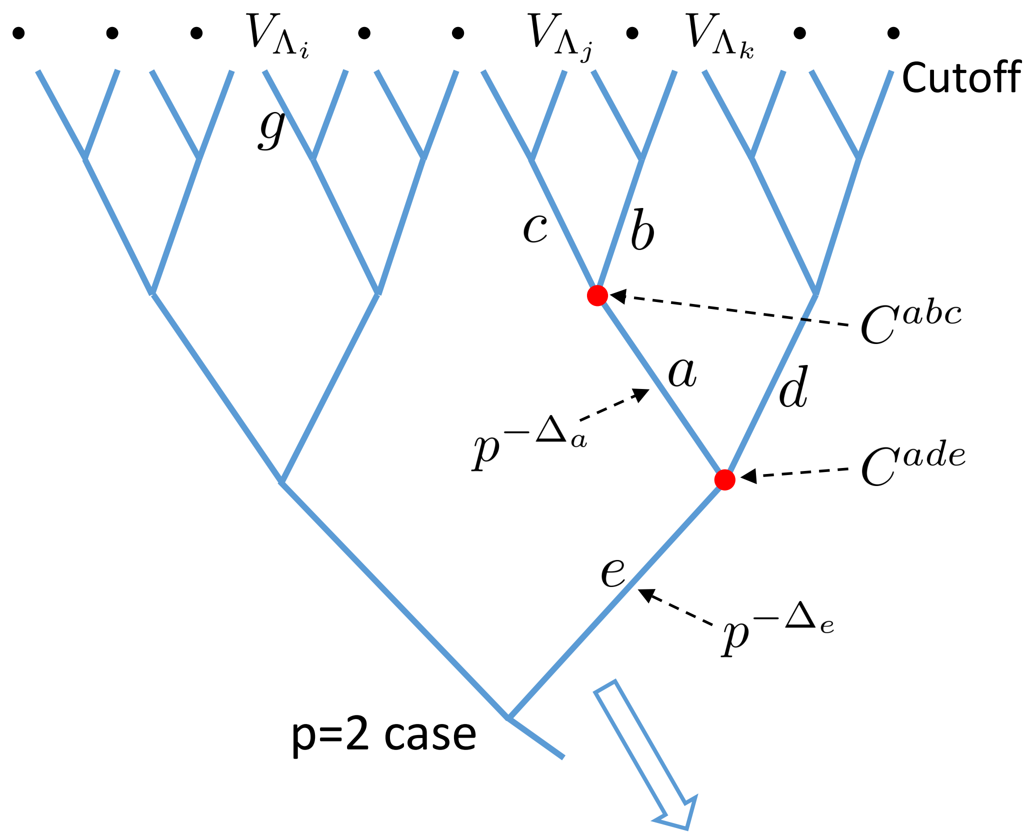
$$\Downarrow$$

Can recover 2-pt function in p-adic CFT:

$$\langle \mathcal{O}_a(x_1)\mathcal{O}_b(x_2)\rangle = \frac{\delta^{ab}}{|x_1 - x_2|_p^{2\Delta_a}}.$$

III. The p-adic CFT Tensor Networks [L.-Y. Hung et.al. 19]

Interpret the BT tree as a tensor network:



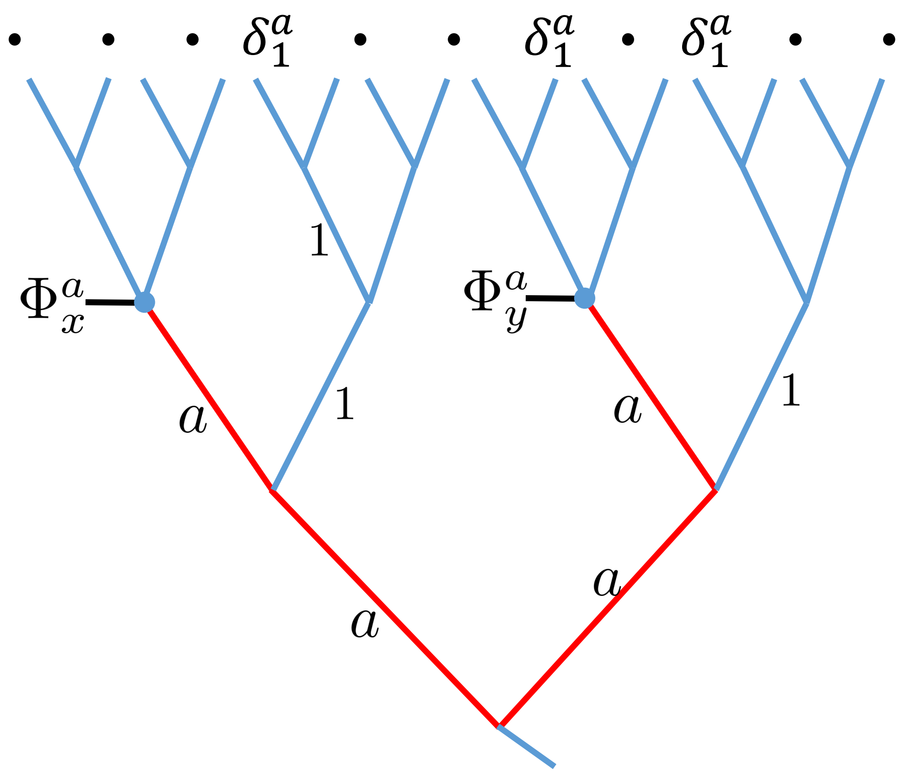
① Put a tensor at every vertex. When p=2, the tensor is C^{abc}. It is designed to recover the OPE coefficients in the correlation function.

② When an edge is labeled by a, it will contribute $p^{-\Delta_a}$. It is designed to recover the $p^{-\Delta_a}$ behavior in the correlation function.

③ Boundary condition. The dangling legs of the tensors are contracted with a reference vector $V^a_{\Lambda_i}$.

④ Operator insertion Φ^a_x: fusing an extra leg with label a to the vertex x .

$$Z = \sum_{\dots,a,b,c,d,e,\dots,g,\dots} \dots p^{-\Delta_b} p^{-\Delta_c} C^{abc} p^{-\Delta_a} C^{ade} p^{-\Delta_d} p^{-\Delta_e} \dots p^{-\Delta_g} V^g_{\Lambda_i} \dots$$



Insertion of two bulk operators:

$$\Longrightarrow \quad \langle \Phi_x^a \Phi_y^b \rangle = \delta^{ab} p^{-\Delta_a d(x,y)}$$

Recall:

$$\langle \phi^a(x)\phi^b(y)\rangle \sim \delta^{ab} p^{-\Delta_a d(x,y)}$$

$$\langle \mathcal{O}_a(x_1)\mathcal{O}_b(x_2)\rangle = \frac{\delta^{ab}}{|x_1 - x_2|_p^{2\Delta_a}}.$$

Tensor network can recover these 2-pt functions.

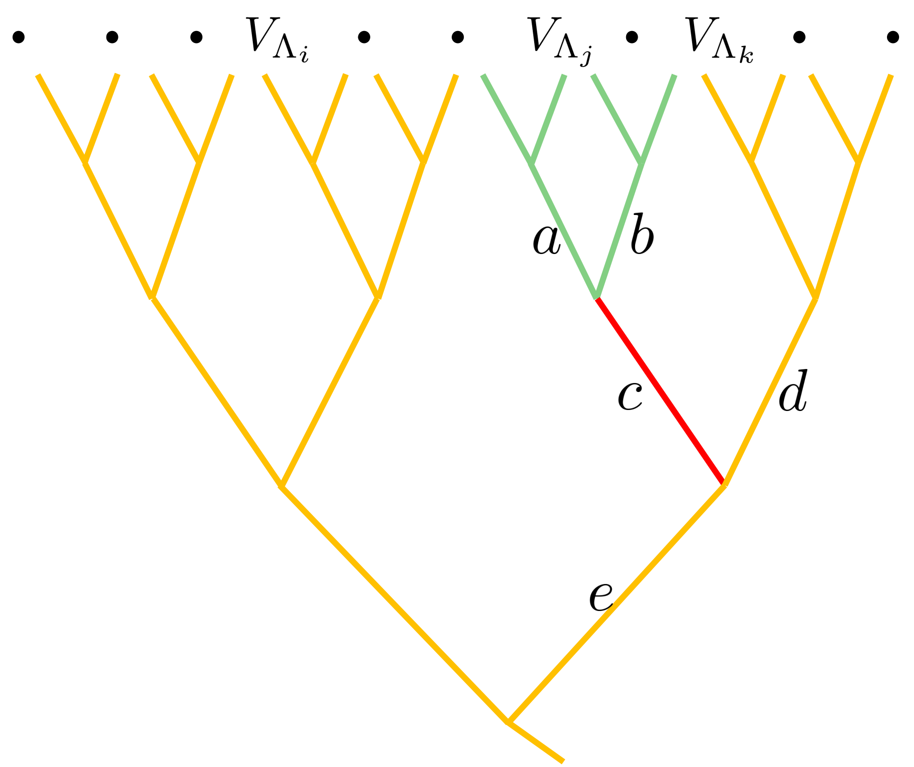
Choosing all $V_{\Lambda_i}^a = \delta_1^a$ (1 is the identity operator), inserting n operators $\Phi_{x_i}^{a_i}$ at the boundary, the tensor network result can recover n-point function of the p-adic CFT.

This tensor network can exactly recover the p-adic CFT!

IV. Distances and graph curvature from the TN

Perturbing the p-adic CFT a bit is achieved by changing the boundary condition:

$$V^a_{\Lambda_i} = \delta^a_1 \quad \Rightarrow \quad V^a_{\Lambda_i} = \delta^a_1 + \lambda v^a_{\Lambda_i}.$$ (λ a small parameter)



The edge length can be mesured by the guys living on the edge. The mesurement they can do is to observe the correlation functions on this edge, which only cares about the final results of contracting the green edges or the yellow edges. $\boldsymbol{V^a}$ **and** $\boldsymbol{\tilde{V}^a}$ **determine the edge distance!**

$$V^c \equiv \sum_{\dots,a,b} \dots p^{-\Delta_a} p^{-\Delta_b} C^{abc} \qquad \tilde{V}^c \equiv \sum_{d,e,\dots} C^{cde} p^{-\Delta_d} p^{-\Delta_e} \dots$$

$$\begin{aligned} V^a &= \delta^a_1 + w^a = \delta^a_1 + \lambda^a + \eta^a + \mathcal{O}(\lambda^3), \\ \tilde{V}^a &= \delta^a_1 + \tilde{w}^a = \delta^a_1 + \tilde{\lambda}^a + \tilde{\eta}^a + \mathcal{O}(\lambda^3). \end{aligned}$$

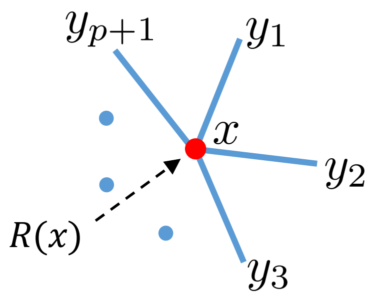
The edge distance d admits an expansion in $\omega^a, \tilde{\omega}^a$:

$$d = 1 + j$$

$$j = A^a(\omega^a + \tilde{\omega}^a) + B^{ab}(\omega^a\omega^b + \tilde{\omega}^a\tilde{\omega}^b) + C^{ab}\omega^a\tilde{\omega}^b + \mathcal{O}(\omega^3)$$

A^a, B^{ab}, C^{ab} are unknowns now.

Graph curvature and the Einstein Hilbert action



Locality means that graph curvature $R(x) = R(d_{xy_1}, d_{xy_2}, \ldots, d_{xy_{p+1}})$,
Isometry means $R(d_{xy_1}, d_{xy_2}, \ldots, d_{xy_{p+1}})$ is a symmetric function of the lengths d_{xy_i} of edges.
Using $d = 1 + j$, $j \ll 1$, we can expand $R(x)$ as

$$R(d_{xy_1}, d_{xy_2}, \ldots, d_{xy_{p+1}}) \quad = \quad a_0 + a_1 \sum_i j_{xy_i} + b \sum_i j_{xy_i}^2 + c \sum_{i \neq k} j_{xy_i} j_{xy_k} + \mathcal{O}(j^3).$$

Imposed symmetric dependence of $\{j_{xy_i}\}$. a_0, a_1, b, c are some unknown universal constant.

On Riemann manifold the Einstein-Hilbert action takes the form:

$$S_{EH}[g_{\mu\nu}] \quad = \quad \int d^d x \sqrt{-g}(R + \Lambda)$$

The **Einstein-Hilbert action** on the BT tree should take the general form:

$$S_{EH} \quad = \quad \sum_x R(d_{xy_1}, d_{xy_2}, \ldots, d_{xy_{p+1}}) + \sum_{\langle xy \rangle} d_{xy} \Lambda$$

where the first sum runs over all bulk vertices x, and $\sum_{\langle xy \rangle}$ indicates a sum over edges.

V. The graph Einstein equation

The **Einstein-Hilbert action** on the BT tree should take the general form:

$$
\begin{aligned}
S_{EH} &= \sum_x R(d_{xy_1}, d_{xy_2}, \ldots, d_{xy_{p+1}}) + \sum_{\langle xy \rangle} d_{xy}\Lambda \\
R(d_{xy_1}, d_{xy_2}, \ldots, d_{xy_{p+1}}) &= a_0 + a_1 \sum_i j_{xy_i} + b \sum_i j_{xy_i}^2 + c \sum_{i \neq k} j_{xy_i} j_{xy_k} + \mathcal{O}(j^3).
\end{aligned}
$$

The general ansatz for **the matter action**, coupling edge lengths to the matter fields：

$$
S_m^{cov} = \sum_{\langle xy \rangle} d_{xy}^k \frac{\zeta_p(2\Delta_a)}{2p^{\Delta_a}} (\tilde{\phi}_x^a - \tilde{\phi}_y^a)^2 + \sum_{\langle xy \rangle} d_{xy} \frac{\zeta_p(2\Delta_a)}{2p^{\Delta_a}} \frac{1}{p+1} m_a^2 ((\tilde{\phi}_x^a)^2 + (\tilde{\phi}_y^a)^2) + \mathcal{O}(\tilde{\phi}^3)
$$

Here k is some unknown universal constant. And $m_a^2 = -\dfrac{1}{\zeta_p(\Delta_a - 1)\zeta_p(-\Delta_a)}$.

Graph Einstein equation by varying $S_{EH} + S_m^{cov}$ with respect to the edge lengths: $\boldsymbol{G + T = 0}$

$$
G \equiv \frac{\delta S_{EH}}{\delta d_{xy}} = \Lambda + 2a_1 + 4b j_{xy} + c\Big(\sum_{\substack{i \\ (y_i \neq y)}} j_{xy_i} + \sum_{\substack{i \\ (x_i \neq x)}} j_{x_i y} \Big) + \mathcal{O}(j^2),
$$

$$
\begin{aligned}
j = {} & A^a(\omega^a + \tilde{\omega}^a) + B^{ab}(\omega^a\omega^b + \tilde{\omega}^a\tilde{\omega}^b) \\
& + C^{ab}\omega^a\tilde{\omega}^b + \mathcal{O}(\omega^3)
\end{aligned}
$$

$$
T \equiv \frac{\delta S_m^{cov}}{\delta d_{xy}} = k \frac{\zeta_p(2\Delta_a)}{2p^{\Delta_a}} (\tilde{\phi}_x^a - \tilde{\phi}_y^a)^2 + \frac{\zeta_p(2\Delta_a)}{2p^{\Delta_a}} \frac{1}{p+1} m_a^2 ((\tilde{\phi}_x^a)^2 + (\tilde{\phi}_y^a)^2) + \mathcal{O}(\tilde{\phi}^3).
$$

In classical Einstein equation, $\tilde{\phi}_x^a = \langle \Phi_x^a \rangle$

The graph Einstein equation should hold for any perturbed boundary condition $V_{\Lambda_i}^a = \delta_1^a + \lambda v_{\Lambda_i}^a$, then **it becomes a constraint for the unknowns:** $\boldsymbol{A^a, B^{ab}, C^{ab}; a_1, b, c; k}$.

VI. Solving the Einstein constraints

There may no consistent solution for the unknowns: $A^a, B^{ab}, C^{ab}; a_1, b, c; k$.
Then it means that reconstructing gravitational dynamics using tensor networks maybe impossible since we have written down a very general ansatz for gravitational dynamics.

Luckily a solution exists:

$$\Lambda + 2a_1 = 0, \quad A^a = 0$$

$$C^{ab} = \frac{2B^{ab}(1 + p^{-\Delta_a - \Delta_b})}{p^{-\Delta_a} + p^{-\Delta_b}}, \quad B^{ab} = \frac{\delta^{ab}}{2c(p+1)\left(1 - p^{2\Delta_a}\right)}$$

$$\frac{2b}{c} = -p, \qquad k = 1$$

The **only** undetermined unknown is c. And c will cancel each other in the Einstein equation. **So a unique Einstein equation emerges from the tensor network!**

What's more, **our G matches with the mathematics literature for G (perturbatively).**

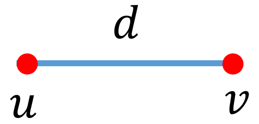
[Y. Lin and Yau 11, Gubser et.al. 16]

$$G \equiv \frac{\delta S_{EH}}{\delta d_{xy}} = 4b j_{xy} + c\Big(\sum_{\substack{i \\ (y_i \neq y)}} j_{xy_i} + \sum_{\substack{i \\ (x_i \neq x)}} j_{x_i y} \Big) = -c \Box j_{xy}$$

$$\Box j_e \equiv \sum_{f \sim e} (j_e - j_f)$$

Here $\sum_{f \sim e}$ denotes the sum over all edges f that share a vertex with a fixed edge e.

Fisher metric and the edge lengths



The edge distance $d = 1 + j$ can be written as Fisher information metric:

$$d = 1 - \langle u|v\rangle$$

$$|u\rangle = \sum_a \mathcal{N}_a \tilde{\phi}^a_u |a\rangle, \quad |v\rangle = \sum_a \mathcal{N}_a \tilde{\phi}^a_v |a\rangle, \quad \mathcal{N}_a = \sqrt{\frac{p^{\Delta_a}}{2c(p+1)(p^{2\Delta_a - 1})}}$$

The state $|u\rangle$ is obtained by sticking a dangling leg to the vertex u of the tensor network.
So the edge distance is the Fisher information distance between two states each with a leg stuck at one of the two ends of the edge!

VII. Summary

- At the beginning, we know nothing about how to assign edge distance and graph curvature to the tensor network. Instead of guessing, we write down the general form for edge distance, graph curvature and the total action. The cost is introducing many unknowns $A^a, B^{ab}, C^{ab}; a_1, b, c; k$.
- Then the Einstein equation becomes a constraint for the unknowns. They can be solved except c which means assigning edge distance and graph curvature to the tensor network is not arbitrary.
- We find that there exists a unique emergent graph Einstein equation. And we obtain the same graph Einstein tensor in the mathematics literature (perturbatively).
- The assigned edge distance corresponds to some Fisher metric.

This is perhaps a first quantitative demonstration that a concrete Einstein equation can be extracted directly from the tensor network.