# Some progress on BPS spectrum of 5d/6d field theories

Strings and Related Physics
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# This talk is based on

[1908.11276] “Instantons from Blow-up”

[2101.00023] “Bootstrapping BPS spectra of 5d/6d field theories”

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# Introduction

In this talk, we discuss $\mathcal{N}$=1 5d / 6d field theories
(gauge theories and non-Lagrangian theories).

In particular, I will talk about how to compute BPS spectrum:
Nekrasov partition function on the Omega background ($R^4$ x $S^1$)

There are many systematic ways of obtain the $Z_{Nek}$ :
ADHM, topological vertex, DIM algebra

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There are many systematic ways of obtain the $Z_{Nek}$ :
ADHM, topological vertex, DIM algebra

They have been successful in computing the partition for some cases,
but each has their own limitations:

- exceptional gauge groups,
- matter in higher dimensional representations,
- higher CS level,
- many 5-brane webs are still unknown.
- orientifold planes and refinements, ….

# Introduction

Today, I will discuss yet another powerful way,
which turns out a very powerful way:

Nakajima-Yoshioka Blowup equation

We devised a complete blowup formalism
which enables one to compute BPS spectrum
of *any* supersymmetric field theory
(of UV completion) in 5d / 6d

# Content

- N=1 SQFTs in 5d and geometric engineering
- Blowup equation
- Main conjecture
- Examples
- Conclusion

# N=1 Supersymmetric QFTs in 5d

(including KK theories)

- 8 supercharges
- $SU(2)_R$ symmetry
- particle content and moduli space
  - vector multiplet $(A_\mu, \phi) \rightarrow$ Coulomb branch (CB)
    G $\rightarrow U(1)^r$
  - hypermultiplet $q^{A=1,2} \rightarrow$ Higgs branch (HB)
- Instanton U(1) topological symmetry
- non-renormalizable $\rightarrow$ CFT, UV fixed point

M-theory

$\downarrow$ CY3

5d SCFTs

$\downarrow$ RG

5d gauge theories

# Geometric Engineering

M-theory on shrinkable CY3 engineers 5d SCFTs

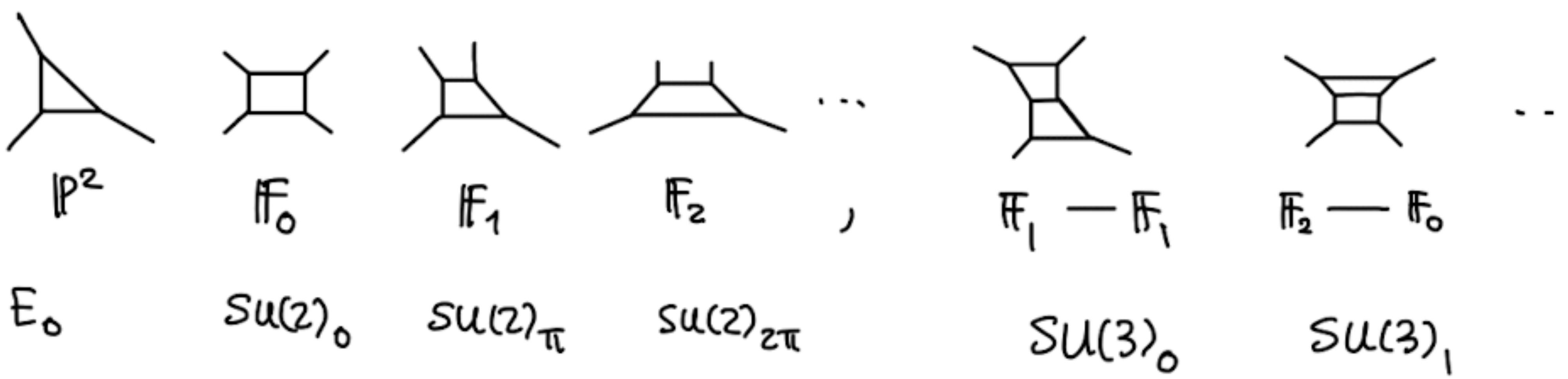
shrinkable means all holomorphic surfaces shrink to a point or non-compact 2-cycles

$\Rightarrow$ singular limit of CY3 $\to$ SCFT at singularity

Hirzebruch surface: $\mathbb{F}_n$ $\mathbb{P}^2$

- building blocks of shrinkable CY3

- glue $\mathbb{F}_n$ and their blowups $\mathbb{F}_n^b$




[Seiberg 96], [Intriligator, Morrison, Seiberg 97]

- (geometric) classifications

[Jefferson, Katz, H.-C. Kim, Vafa 18], [Bhardwaj 19], [Bhardwaj, Zafrir 20], ⋯

| Gauge theory | | Geometry |
| --- | --- | --- |
| SCFT | ⟷ | CY3 |
| Coulomb Branch (CB) | ⟷ | Kahler cone |
| W-bosons | ⟷ | compact 4-cycles |
| Flavors | ⟷ | non-compact 4-cycles |
| BPS states | ⟷ | M2-brane wrapping compact 2-cycles |
| mass of BPS states | ⟷ | volume of 2-cycles |
| BPS charges | ⟷ | Intersection number between 2-cycles and 4 cycles |

# Partition function on the $\Omega$ deformed $\mathbb{R}^4 \times S^1$

Partition function here is the Witten index counting
the BPS states, annihilated by supercharge $Q$ and $Q^\dagger$.

$$Z(\phi, m; \epsilon_1, \epsilon_2) = \mathrm{Tr}\left[(-1)^F e^{-\beta\{Q,Q^\dagger\}} e^{-\epsilon_1(J_1+J_R)} e^{-\epsilon_2(J_2+J_R)} e^{-\phi\cdot\Pi} e^{-m\cdot H}\right]$$

This BPS partition function factorizes

$$Z = Z_{\text{pert}} \cdot Z_{\text{inst}}$$

$$\begin{aligned} Z_{\text{pert}} &= Z_{\text{class}} \cdot Z_{\text{1-loop}} \\ &= e^{\mathcal{E}} \cdot \mathrm{PE}\left[-\frac{1+p_1p_2}{(1-p_1)(1-p_2)}\sum_{e\in\mathbf{R}^+} e^{-e\cdot\phi} + \frac{(p_1p_2)^{1/2}}{(1-p_1)(1-p_2)}\sum_f \sum_{w\in\mathbf{w}_f} e^{-|w\cdot\phi+m_f|}\right] \end{aligned}$$

where $p_{1,2} \equiv e^{-\epsilon_{1,2}}$ and $\mathbf{R}^+$ denotes the positive roots for the gauge group

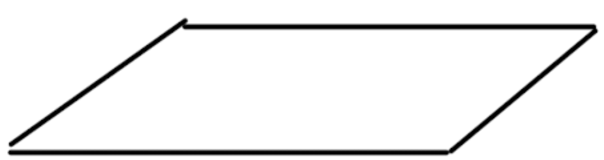
$$\mathrm{PE}\left[f(\mu)\right] \equiv \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} f(n\mu)\right)$$

# Blowup Equation

Recall on the $\Omega$-deformed $\mathbb{R}^4 \times S^1$ ($\mathbb{C}^2 \times S^1$)

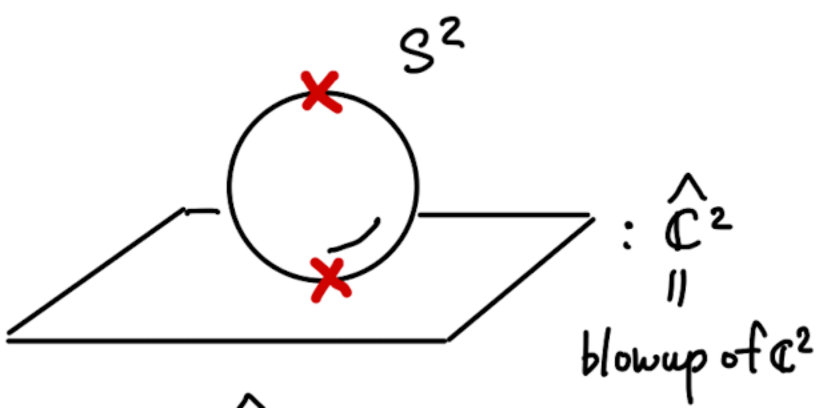
$$Z(\phi, m; \epsilon_1, \epsilon_2) = \mathrm{Tr}\left[(-1)^F e^{-\beta\{Q,Q^\dagger\}} e^{-\epsilon_1(J_1+J_R)} e^{-\epsilon_2(J_2+J_R)} e^{-\phi\cdot\Pi} e^{-m\cdot H}\right]$$

Blowup equation is a functional equation identifying two partition functions on different backgrounds



blowup:

the origin is replaced by an $S^2$ or $\mathbb{P}^1$

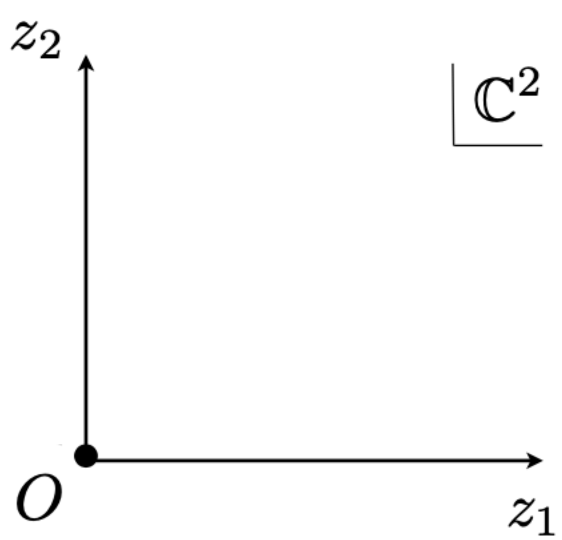


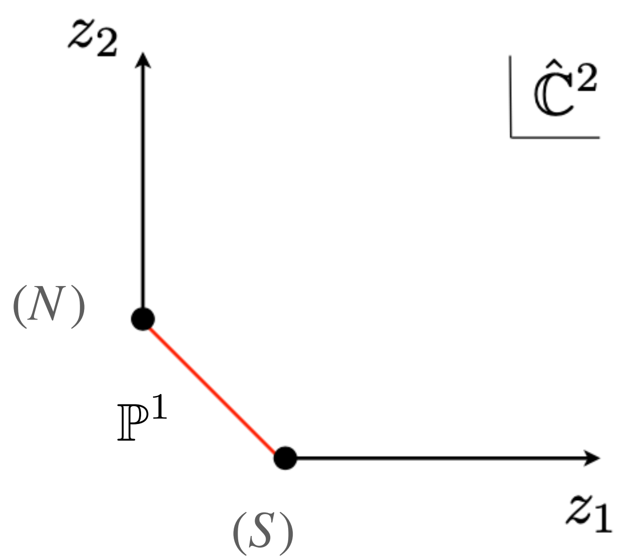

before | blowup | after

$Z$ | | $\hat{Z}^{(N)}$ $\hat{Z}^{(S)}$

summing over fluxes

# (Gottsche-) Nakajima-Yoshioka blowup equation




$$\Lambda(m_j;\epsilon_1,\epsilon_2)\hat{Z}(\phi_i,m_j;\epsilon_1,\epsilon_2)=\sum_{\vec{n}}(-1)^{|\vec{n}|}\hat{Z}^{(N)}(\vec{n},\vec{B})\times\hat{Z}^{(S)}(\vec{n},\vec{B})\,,$$

$$\begin{aligned}\hat{Z}^{(N)}(\vec{n},\vec{B})\ &=\hat{Z}^{(N)}(\phi_i+n_i\epsilon_1,m_j+B_j\epsilon_1;\epsilon_1,\epsilon_2-\epsilon_1)\\ \hat{Z}^{(S)}(\vec{n},\vec{B})\ &=\hat{Z}^{(S)}(\phi_i+n_i\epsilon_2,m_j+B_j\epsilon_2;\epsilon_1-\epsilon_2,\epsilon_2)\end{aligned}$$

$\Lambda \neq \Lambda(\phi)$: unitary / vanishing equation

$(\overrightarrow{n}, \overrightarrow{B})$: magnetic fluxes on $\mathbb{P}^1$ for (gauge, global) symmetries

# Various studies and generalizations:

4d/5d SU(N) [Nakajim, Yoshioka 03, 05, 09], [Gottsche, Nakajim, Yoshioka 06]

exceptional gauge groups / matter [Keller, Song 12] [Kim-SSK-Lee-Lee-Song 19]

local CY3 [Huang, Sun, Wang 17]

elliptic, 6d Gu, Haghighat, Klemm, Sun, Wang 18, 19, 20]

6d (2,0) theories, $\mathcal{N}$=1* [Duan, Lee, Nahmgoong, Wang, 21]

RG flows, dualities, global symmetry [Lee-Sun 21]

surface defects, Painleve [Jeong, Nekrasov 20]

# Our Main conjecture

Partition function Z (on the Omega background) of any *N*=1 theory

$$Z(\phi, m; \epsilon_1, \epsilon_2) = e^{\mathcal{E}(\phi, m; \epsilon_1, \epsilon_2)} Z_{GV}(\phi, m; \epsilon_1, \epsilon_2)$$

can be computed by solving the blowup equation

$$\Lambda(m_j; \epsilon_1, \epsilon_2) \hat{Z}(\phi_i, m_j; \epsilon_1, \epsilon_2) = \sum_{\vec{n}} (-1)^{|\vec{n}|} \hat{Z}^{(N)}(\vec{n}, \vec{B}) \times \hat{Z}^{(S)}(\vec{n}, \vec{B}),$$

with the following inputs:

(i) effective prepotential $\mathcal{E}(\phi, m; \epsilon_1, \epsilon_2)$

(ii) consistent magnetic fluxes $(\vec{n}, \vec{B})$

# Effective Prepotential

In 5d field theories, the effective action consists of

[Witten 96] [Bonetti, Grimm, Hohenegger 13]

- cubic CS term $S_{CS} = \frac{C_{IJK}}{24\pi^2} \int A^I \wedge F^J \wedge F^K$

- mixed gauge/gravitational CS term $S_{\rm grav} = -\frac{1}{48} \int C_i^G A^i \wedge p_1(T)$

$p_1$ : Pontryagin class of tangent bundle of 5d spacetime

- mixed gauge/$SU(2)_R$ CS terms $S_R = \frac{1}{2} \int C_i^R A^i \wedge c_2(R)$

$c_2$ : 2nd Chern class of $SU(2)_R$ symm bundle

Together, we define the effective prepotential:

$$\mathcal{E}(\phi, m; \epsilon_1, \epsilon_2) = \frac{1}{\epsilon_1 \epsilon_2} \left[ \mathcal{F}(\phi, m) + \frac{1}{48} C_i^G \phi^i (\epsilon_1^2 + \epsilon_2^2) + \frac{1}{2} C_i^R \phi^i \epsilon_+^2 \right]$$

the cubic prepotential (or IMS prepotential) [Intriligator, Morrison, Seiberg 97]

$$\mathcal{F} = \sum_a \left( \frac{m_a}{2} K_{ij}^a \phi_i^a \phi_j^a + \frac{\kappa_a}{6} d_{ijk}^a \phi_i^a \phi_j^a \phi_k^a \right) + \frac{1}{12} \left( \sum_{e \in \mathbf{R}} |e \cdot \phi|^3 - \sum_f \sum_{w \in \mathbf{w}_f} |w \cdot \phi + m_f|^3 \right)$$

$m_a$ = gauge coupling

$\kappa_a$ = CS-level

$K_{ij}^a = \mathrm{Tr}(T_i^a T_j^a)$

$d_{ijk}^a = \mathrm{Tr}(T_i^a \{T_j^a, T_k^a\})$

Other coefficients are given by

$$C_i^G = -\partial_i \left( \sum_{e \in \mathbf{R}} |e \cdot \phi| - \sum_f \sum_{w \in \mathbf{w}_f} |w \cdot \phi + m_f| \right) \qquad C_i^R = \frac{1}{2} \partial_i \sum_{e \in \mathbf{R}} |e \cdot \phi|$$

In geometry, $X$ = CY3 $D_I$ = divisors $S_i$ = compact divisors

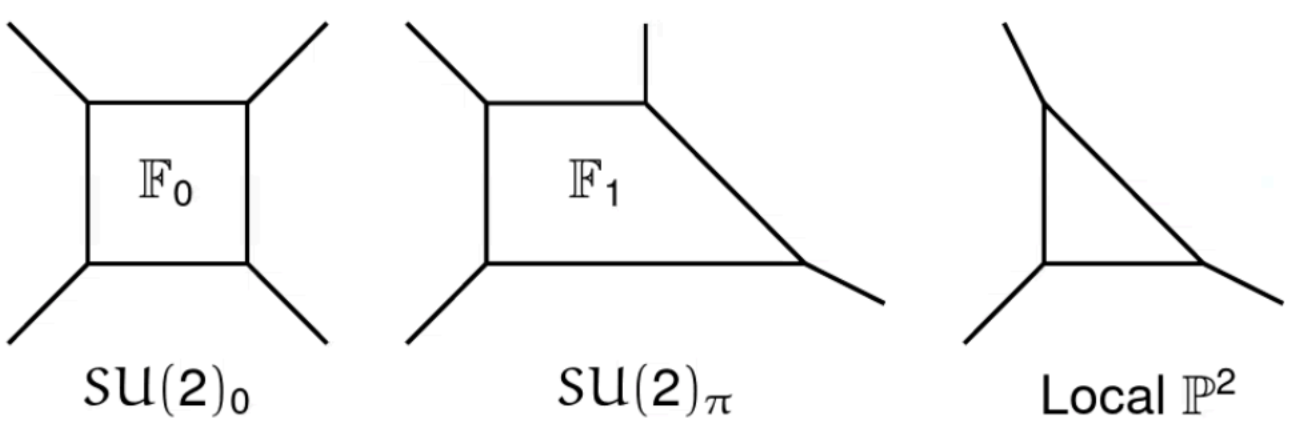
$C_{IJK} = D_I \cdot D_J \cdot D_K$ triple intersection number

$C_i^G = c_2(X) \cdot S_i$ $\qquad c_2(X) \cdot \mathbb{P}^2 = -6, \qquad c_2(X) \cdot \mathbb{F}_n^b = -4 + 2b$ .

$C_i^R = 2$

For instance, pure SU(2) $\quad 6\mathcal{F} = 6m\phi^2 + 8\phi^3, \quad C_i^G = -4, \quad C_i^R = 2$

local $\mathbb{P}^2$ (non-Lag.) $\quad 6\mathcal{F} = 9\phi^3, \quad C_i^G = -6, \quad C_i^R = 2$




Either gauge theory or geometric description is known, we can compute

# Consistent magnetic flux

Magnetic flux $(\vec{n}, \vec{B})$ on $\mathbb{P}^1$ cannot be arbitrary.
It satisfies the quantization condition.

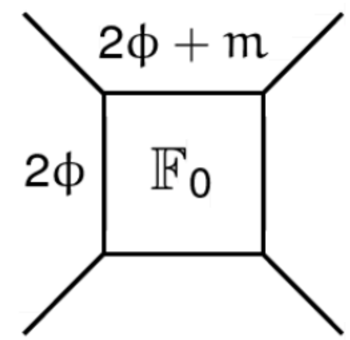
Suppose M2-brane wrapping a 2-cycle $C$ of the charge $(j_l, j_r)$ couples with the flux F

F $\cdot$ $C$ is integral / half-integral when $C^2$ is even / odd

F $\cdot$ $C$ is integral / half-integral when 2$(j_l + j_r)$ is odd / even

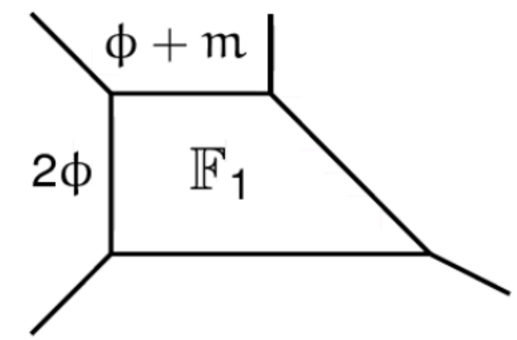
W-bosons (0,1/2): integral flux

hypermultiplet (0,0) : half-integral flux




$SU(2)_0$

$n \in \mathbb{Z}, B_m \in \mathbb{Z}$




$SU(2)_\pi$

$n \in \mathbb{Z}, B_m \in \mathbb{Z} + 1/2$

There exists a chamber such that mass of all BPS particles are non-negative. This further restricts background flux

$n \in \mathbb{Z}, B_m = 0, \pm 1, \pm 2$ $\qquad$ $n \in \mathbb{Z}, B_m = \pm 1/2, \pm 3/2$

# Solving the blowup equation

With two inputs: effective prepotential and magnetic fluxes

$$Z(\phi, m; \epsilon_1, \epsilon_2) = e^{\mathcal{E}(\phi,m;\epsilon_1,\epsilon_2)} Z_{GV}(\phi, m; \epsilon_1, \epsilon_2)\,,$$

$$Z_{GV}(\phi, m; \epsilon_1, \epsilon_2) = \mathrm{PE}\left[\sum_{j_l, j_r, \mathbf{d}} (-1)^{2(j_l+j_r)} N^{\mathbf{d}}_{j_l, j_r} \frac{\chi_{j_l}(p_1/p_2)\,\chi_{j_r}(p_1 p_2)}{(p_1^{1/2} - p_1^{-1/2})(p_2^{1/2} - p_2^{-1/2})} e^{-\mathbf{d}\cdot\mathbf{m}}\right]$$

We recast the blowup equation:

d: degree of 2-cycles; m: vol (2-cycles)

$$\Lambda \hat{Z}_{GV} = \sum_{\vec{n}} (-1)^{|\vec{n}|} e^{-V} \hat{Z}^{(N)}_{GV}(\vec{n}, B) \hat{Z}^{(S)}_{GV}(\vec{n}, B)$$

where $V = \mathcal{E} - \mathcal{E}^{(N)} - \mathcal{E}^{(S)}$

3 different sets of $(\vec{n}, \vec{B})$ gives three linearly independent equations

can solve for three unknowns at each instanton order : $Z_{GV}, Z^{(N)}_{GV}, Z^{(S)}_{GV}$

Less than 3 sets ….

# Strategy for less than 3 sets

Expand the partition function in terms of the Kahler parameters

Solve the blowup equation order by order in the Kahler parameter to determine $N^{\mathbf{d}}_{(j_l,j_r)}$

Even undetermined, those we have determined are another input for higher order equation.

$\Rightarrow$ Solve iteratively $N^{\mathbf{d}}_{(j_l,j_r)}$

# Examples

# SU(2)$_{\theta=0,\pi}$


$$\begin{cases} \mathbb{F}_0: & \mathrm{vol}(f) = 2\phi, \quad \mathrm{vol}(e) = 2\phi + m\,, \\ \mathbb{F}_1: & \mathrm{vol}(f) = 2\phi, \quad \mathrm{vol}(e) = \phi + m\,. \end{cases}$$

Two SU(2) theories are perturbatively indistinguishable (same $\mathcal{E}$), but have different instanton spectra.

→ A good example for having same $\mathcal{E}$ with different fluxes $B$ yielding different results

$$Z_{GV}(\phi, m; \epsilon_1, \epsilon_2) = \mathcal{Z}_{\mathrm{pert}}(\phi; \epsilon_1, \epsilon_2) \cdot \mathcal{Z}_{\mathrm{inst}}(\phi, m; \epsilon_1, \epsilon_2)$$

$$\mathcal{Z}_{\mathrm{inst}}(\phi, m; \epsilon_1, \epsilon_2) = \sum_{k=0}^{\infty} q^k Z_k(\phi; \epsilon_1, \epsilon_2) \qquad \text{with the instanton fugacity } q \equiv e^{-m} \text{ and } Z_0 = 1$$

- Effective prepotential $\quad \mathcal{E} = \frac{1}{\epsilon_1\epsilon_2}\left(\mathcal{F} - \frac{\epsilon_1^2 + \epsilon_2^2}{12}\phi + \epsilon_+^2\phi\right) \qquad 6\mathcal{F} = 6m\phi^2 + 8\phi^3$

  Perturbative part $\quad \mathcal{Z}_{\mathrm{pert}}(\phi; \epsilon_1, \epsilon_2) = \mathrm{PE}\left[-\frac{1 + p_1 p_2}{(1-p_1)(1-p_2)} e^{-2\phi}\right]$

- Magnetic flux $\quad \begin{cases} \mathbb{F}_0: & SU(2)_0 \quad n \in \mathbb{Z}\,, \quad B_m = -2,\ -1,\ 0,\ 1, 2 \\ \mathbb{F}_1: & SU(2)_\pi \quad n \in \mathbb{Z}\,, \quad B_m = -\frac{3}{2},\ -\frac{1}{2},\ \frac{1}{2},\ \frac{3}{2}\,. \end{cases}$

# SU(2)$_{\theta=0,\pi}$

With these inputs, we can expand the blowup equation to the first order in the instanton fugacity $e^{-m}$

$$\Lambda_1 + \hat{Z}_1 = p_1^{B_m}\hat{Z}_1^{(N)} + p_2^{B_m}\hat{Z}_1^{(S)} - \frac{(p_1p_2)^{B_m+1}e^{-2(2+B_m)\phi_1}}{(1-e^{-2\phi})(1-p_1e^{-2\phi})(1-p_2e^{-2\phi})(1-p_1p_2e^{-2\phi})}$$
$$- \frac{(p_1p_2)^{B_m-1}e^{-2(2-B_m)\phi_1}}{(1-e^{-2\phi_1})(1-p_1^{-1}e^{-2\phi_1})(1-p_2^{-1}e^{-2\phi_1})(1-(p_1p_2)^{-1}e^{-2\phi_1})} .$$

Three unknowns: $\hat{Z}_1, \hat{Z}_1^{(N)}, \hat{Z}_1^{(S)}$

There are more than 3 distinct equations coming from fluxes:

$$\begin{cases} \mathbb{F}_0 : & SU(2)_0 \quad n\in\mathbb{Z}\,, \quad B_m = -2,\ -1,\ 0,\ 1, 2 \\ \mathbb{F}_1 : & SU(2)_\pi \quad n\in\mathbb{Z}\,, \quad B_m = -\frac{3}{2},\ -\frac{1}{2},\ \frac{1}{2},\ \frac{3}{2}\ . \end{cases}$$

Solutions:

$$Z_1^{SU(2)_0}(\phi;\epsilon_1,\epsilon_2) = \frac{p_1p_2(1+p_1p_2)e^{-2\phi}}{(1-p_1)(1-p_2)(1-p_1p_2e^{-2\phi})(e^{-2\phi}-p_1p_2)} ,$$
$$Z_1^{SU(2)_\pi}(\phi;\epsilon_1,\epsilon_2) = -\frac{p_1^{3/2}p_2^{3/2}(1+e^{-2\phi})e^{-\phi}}{(1-p_1)(1-p_2)(1-p_1p_2e^{-2\phi})(e^{-2\phi}-p_1p_2)} .$$

# SU(2) again but with one blowup equation.

Recall GV inv. form:

$$Z_{GV}(\phi, m; \epsilon_{1,2}) = \mathrm{PE}\left[\sum_{j_l,j_r}\sum_{d_1,d_2=0}^{\infty}(-1)^{2(j_l+j_r)}N_{j_l,j_r}^{(d_1,d_2)}A_{j_l,j_r}(\epsilon_1,\epsilon_2)e^{-d_1\mathrm{vol}(e)-d_2\mathrm{vol}(f)}\right]$$

$$A_{j_l,j_r}(\epsilon_1,\epsilon_2) \equiv \frac{\chi_{j_l}^{SU(2)}(p_1/p_2)\chi_{j_r}^{SU(2)}(p_1p_2)}{(p_1^{1/2}-p_1^{-1/2})(p_2^{1/2}-p_2^{-1/2})}\ .$$

$(d_1, d_2)$ : degree for 2-cycles e and f, respectively.
e.g., (0,1) means perturbative part

$$\mathcal{Z}_{\mathrm{pert}}(\phi;\epsilon_1,\epsilon_2) = \mathrm{PE}\left[-\frac{1+p_1p_2}{(1-p_1)(1-p_2)}e^{-2\phi}\right] \quad \rightarrow \quad N_{0,\frac{1}{2}}^{(0,1)} = 1 \qquad N_{j_l,j_r}^{(0,d_2)} = 0 \text{ for } d_2 > 1$$

Suppose we found only one flux for SU(2)$_0$ : $B_m = 0$

From the blowup equation at 1st order in the instanton $d_1 = 1$, we then further expand the equation with $d_2$, namely $e^{-2\phi}$

$$\Lambda_1 = \hat{Z}_1^{(N)} + \hat{Z}_1^{(S)} - \hat{Z}_1 - (p_1p_2 + (p_1p_2)^{-1})e^{-4\phi} - \frac{(1+p_1)(1+p_2)(1+p_1^3p_2^3)}{p_1^2p_2^2}e^{-6\phi} + \mathcal{O}(e^{-8\phi})$$

We then solve this equation order by order in $d_2$

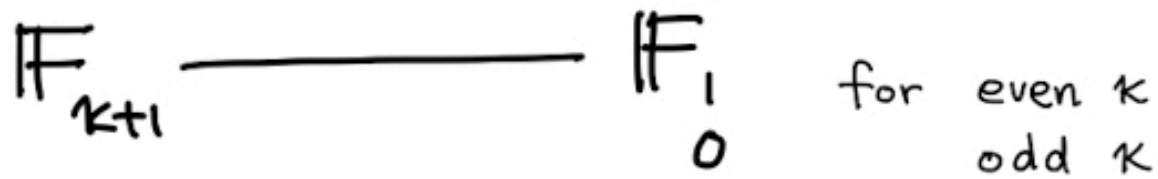
| $\mathbf{d}$ | $\oplus N_{j_l,j_r}^{\mathbf{d}}(j_l,j_r)$ | $\mathbf{d}$ | $\oplus N_{j_l,j_r}^{\mathbf{d}}(j_l,j_r)$ |
|---|---|---|---|
| $(1,0)$ | $N_{0,\frac{1}{2}}^{(1,0)}(0,\frac{1}{2})$ | $(1,1)$ | $N_{0,\frac{1}{2}}^{(1,1)}(0,\frac{1}{2})\oplus(0,\frac{3}{2})$ |
| $(1,2)$ | $N_{0,\frac{1}{2}}^{(1,2)}(0,\frac{1}{2})\oplus(0,\frac{5}{2})$ | $(1,3)$ | $N_{0,\frac{1}{2}}^{(1,3)}(0,\frac{1}{2})\oplus(0,\frac{7}{2})$ |

$N_{0,\frac{1}{2}}^{(1,d_2)}$ is not fixed $\quad \hat{Z}_1^{(N)} + \hat{Z}_1^{(S)} - \hat{Z}_1 \rightarrow 0$

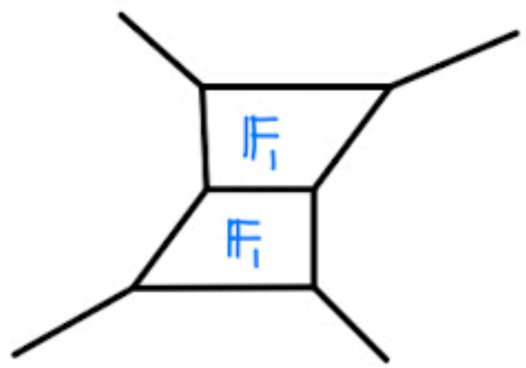
But at higher orders in $d_1$, it is fixed to 1 for $d_2$=0, 0 otherwise

# Rank2: $SU(3)_\kappa\ ,\quad \kappa \leq 7$

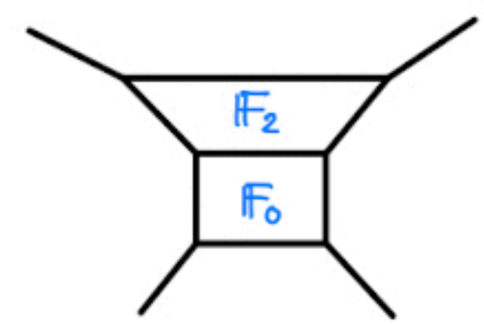
Geometrically, they are engineered by glueing two Hirzebruch surfaces.




$SU(3)_0$




• $SU(3)_1$




$$\mathrm{vol}(f_1) = 2\phi_1 - \phi_2, \quad \mathrm{vol}(f_2) = -\phi_1 + 2\phi_2,$$

$$\mathrm{vol}(e_2) = \begin{cases} \left(1 - \frac{\kappa}{2}\right)\phi_1 + \phi_2 + m & \text{for even } \kappa \\ \frac{1-\kappa}{2}\phi_1 + 2\phi_2 + m & \text{for odd } \kappa \end{cases}$$

# Rank2: $SU(3)_\kappa\ ,\quad \kappa \le 7$

Input:

- Effective prepotential

$$\mathcal{E} = \frac{1}{\epsilon_1\epsilon_2}\left(\mathcal{F} - \frac{\epsilon_1^2+\epsilon_2^2}{12}(\phi_1+\phi_2) + \epsilon_+^2(\phi_1+\phi_2)\right),$$

$$6\,\mathcal{F} = 6m(\phi_1^2 - \phi_1\phi_2 + \phi_2^2) + 3\kappa\phi_1\phi_2(\phi_1-\phi_2) + 8\phi_1^3 - 3\phi_1^2\phi_2 - 3\phi_1\phi_2^2 + 8\phi_2^3\,,$$

- Magnetic flux

$$n_1, n_2 \in \mathbb{Z} \quad \text{and} \quad B_m = \begin{cases} -\frac{1}{2}, \frac{1}{2}, \frac{3}{2} & \kappa = 0,2,4,6 \\ -1, 0, 1 & \kappa = 1,3,5,7 \end{cases}$$

- GV form

$$Z_{GV}(\phi_i, m; \epsilon_1, \epsilon_2) = \mathcal{Z}_{\text{pert}}(\phi_i; \epsilon_1, \epsilon_2)\cdot \mathcal{Z}_{\text{inst}}(\phi_i, m; \epsilon_1, \epsilon_2)\ ,$$

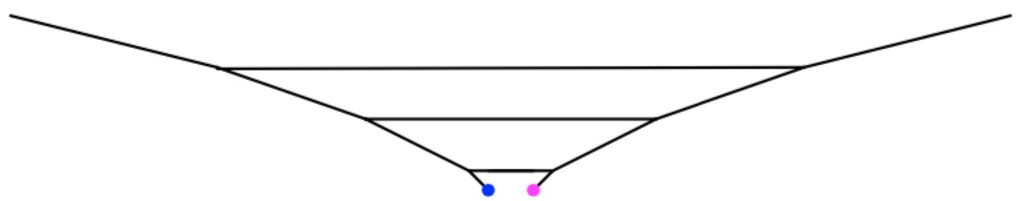
$$\mathcal{Z}_{\text{pert}}(\phi_i; \epsilon_1, \epsilon_2) = \text{PE}\left[-\frac{1+p_1p_2}{(1-p_1)(1-p_2)}\left(e^{-(2\phi_1-\phi_2)} + e^{-(2\phi_2-\phi_1)} + e^{-(\phi_1+\phi_2)}\right)\right]$$

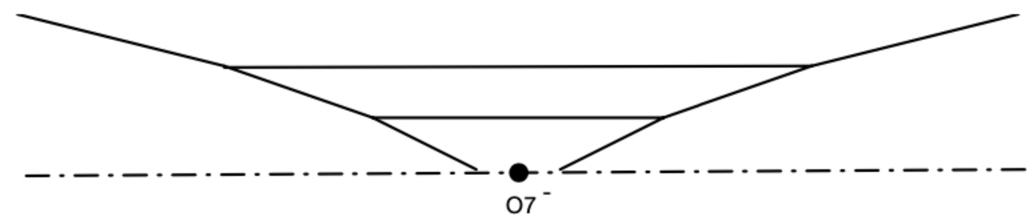
$$\mathcal{Z}_{\text{inst}}(\phi_i, m; \epsilon_1, \epsilon_2) = \sum_{k=0}^{\infty} q^k Z_k(\phi; \epsilon_1, \epsilon_2)\ ,$$

# Rank2: $SU(3)_5$ and $Sp(2)_\pi$

Two theories are known to be UV-dual / fiber-base dual.

[Gaiotto, Kim 15]






Geometrically, $\mathbb{F}_6 \overset{e \qquad\quad h+2f}{\text{———}} \mathbb{F}_0$

SU(3) frame: $\mathrm{vol}(f_1) = 2\phi_1 - \phi_2, \quad \mathrm{vol}(f_2) = -\phi_1 + 2\phi_2, \quad \mathrm{vol}(e_2) = -2\phi_1 + 2\phi_2 + m$

Sp(2) frame: $\mathrm{vol}(f_1) = 2\phi_1 - \phi_2, \quad \mathrm{vol}(f_2) = -2\phi_1 + 2\phi_2, \quad \mathrm{vol}(e_2) = -\phi_1 + 2\phi_2 + m$

Map between two theories

[Hayashi-SSK-Lee-Yagi 15,16]

$$\phi_1^{SU} = \phi_1^{Sp} + \frac{1}{3} m^{Sp} \,, \quad \phi_2^{SU} = \phi_2^{Sp} + \frac{2}{3} m^{Sp} \,, \quad m^{SU} = -\frac{2}{3} m^{Sp}$$

# Rank2: SU(3)$_5$ and Sp(2)$_\pi$

- Effective prepotential

$$\mathcal{E} = \frac{1}{\epsilon_1\epsilon_2}\left(\mathcal{F} - \frac{\epsilon_1^2+\epsilon_2^2}{12}(\phi_1+\phi_2) + \epsilon_+^2(\phi_1+\phi_2)\right) \ ,$$

$$6\mathcal{F} = 6m(2\phi_1^2 - 2\phi_1\phi_2 + \phi_2^2) + 8\phi_1^3 + 12\phi_1^2\phi_2 - 18\phi_1\phi_2^2 + 8\phi_2^3 \ ,$$

- Magnetic flux

$$n_1, n_2 \in \mathbb{Z} \ , \quad B_m = -1, 0, 1$$

Duality under the map

$$SU(3)_5 \Longleftrightarrow Sp(2)_\pi$$

| $\mathbf{d}$ | $\oplus N^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$ | $\mathbf{d}$ | $\oplus N^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$ |
|---|---|---|---|
| $(1,0,0)$ | $(0,\frac{1}{2})$ | $(1,0,1)$ | $(0,\frac{3}{2})$ |
| $(1,0,2)$ | $(0,\frac{5}{2})$ | $(1,1,0)$ | $(0,\frac{1}{2})$ |
| $(1,1,1)$ | $(0,\frac{1}{2})\oplus(0,\frac{3}{2})$ | $(1,1,2)$ | $(0,\frac{3}{2})\oplus(0,\frac{5}{2})$ |
| $(1,2,0)$ | $(0,\frac{1}{2})$ | $(1,2,1)$ | $(0,\frac{1}{2})\oplus(0,\frac{3}{2})$ |
| $(1,2,2)$ | $(0,\frac{1}{2})\oplus(0,\frac{3}{2})\oplus(0,\frac{5}{2})$ | $(2,0,1)$ | $(0,\frac{5}{2})$ |
| $(2,0,2)$ | $(0,\frac{5}{2})\oplus(0,\frac{7}{2})\oplus(\frac{1}{2},4)$ | $(2,1,1)$ | $(0,\frac{3}{2})\oplus(0,\frac{5}{2})$ |
| $(2,1,2)$ | $(0,\frac{3}{2})\oplus 3(0,\frac{5}{2})\oplus 2(0,\frac{7}{2})\oplus (\frac{1}{2},3)\oplus(\frac{1}{2},4)$ | $(2,2,1)$ | $(0,\frac{1}{2})\oplus(0,\frac{3}{2})\oplus(0,\frac{5}{2})$ |
| $(2,2,2)$ | $(0,\frac{1}{2})\oplus 3(0,\frac{3}{2})\oplus 4(0,\frac{5}{2})\oplus 2(0,\frac{7}{2})\oplus(\frac{1}{2},2)\oplus(\frac{1}{2},3)\oplus(\frac{1}{2},4)$ | $(3,0,1)$ | $(0,\frac{7}{2})$ |
| $(3,0,2)$ | $(0,\frac{5}{2})\oplus(0,\frac{7}{2})\oplus 2(0,\frac{9}{2})\oplus (\frac{1}{2},4)\oplus(\frac{1}{2},5)\oplus(1,\frac{11}{2})$ | $(3,1,1)$ | $(0,\frac{5}{2})\oplus(0,\frac{7}{2})$ |
| $(3,1,2)$ | $(0,\frac{3}{2})\oplus 3(0,\frac{5}{2})\oplus 5(0,\frac{7}{2})\oplus 3(0,\frac{9}{2})\oplus(\frac{1}{2},3)\oplus 3(\frac{1}{2},4)\oplus 2(\frac{1}{2},5)\oplus(1,\frac{9}{2})\oplus(1,\frac{11}{2})$ | $(3,2,1)$ | $(0,\frac{3}{2})\oplus(0,\frac{5}{2})\oplus(0,\frac{7}{2})$ |
| $(3,2,2)$ | $(0,\frac{1}{2})\oplus 3(0,\frac{3}{2})\oplus 8(0,\frac{5}{2})\oplus 7(0,\frac{7}{2})\oplus 4(0,\frac{9}{2})\oplus(\frac{1}{2},2)\oplus 3(\frac{1}{2},3)\oplus 4(\frac{1}{2},4)\oplus 2(\frac{1}{2},5)\oplus(1,\frac{7}{2})\oplus(1,\frac{9}{2})\oplus(1,\frac{11}{2})$ | | |

**Table 6**. BPS spectrum of $SU(3)_5$ for $d_1 \leq 3$ and $d_2, d_3 \leq 2$. Here, $\mathbf{d} = (d_1, d_2, d_3)$ labels the BPS states with charge $d_1e_2 + d_2f_1 + d_3f_2$.

# Rank2: $SU(3)_8$

Geometrically $\quad \mathbb{F}_9 \overset{e \qquad\qquad h+3f}{\rule{3cm}{0.4pt}} \mathbb{F}_1$

3-primitive curves : $\mathrm{vol}(f_1) = 2\phi_1 - \phi_2\,, \quad \mathrm{vol}(f_2) = -\phi_1 + 2\phi_2\,, \quad \mathrm{vol}(e_2) = -3\phi_1 + \phi_2 + m\,,$

but non-shrinkable ( cannot be all non-negative , as $m \to 0$ )

$\rightsquigarrow$ Cannot have a UV completion in the geom. phase.

[Jefferson, Katz, Kim, Vafa '18]

it was pointed out that $\exists$ KK theory :

$$SU(3)_{15/2} + 1F \longleftrightarrow G_2 + 1\,\mathrm{Adj}\,.$$

$\rightsquigarrow$ by decoupling 1F

$SU(3)_8$.

[Bhardwaj '19]

Interestingly, No 5-brane webs yet; No ADHM methods.

$\Rightarrow$ The partition function has not been computed.

But Blowup can compute Z.

# Rank2: SU(3)$_8$

- Effective prepotential

$$\mathcal{E} = \frac{1}{\epsilon_1\epsilon_2}\left(\mathcal{F} - \frac{\epsilon_1^2+\epsilon_2^2}{12}(\phi_1+\phi_2) + \epsilon_+^2(\phi_1+\phi_2)\right),$$

$$6\mathcal{F} = 8\phi_1^3 + 21\phi_1^2\phi_2 - 27\phi_1\phi_2^2 + 8\phi_2^3 + 6m(\phi_1^2 - \phi_1\phi_2 + \phi_2^2)$$

- Magnetic fluxes

$$n_1 \in \mathbb{Z}+1/3\,, \quad n_2 \in \mathbb{Z}+2/3\,, \quad B_m = -1/6$$

- BPS spectrum

(1,0,0), (1,0,2), (1,2,0) states:
- negative masses
- hypermultiplets

↓ flop

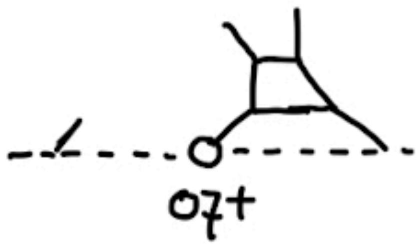
non-trivial phase of CB.

| $\mathbf{d}$ | $\oplus N^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$ | $\mathbf{d}$ | $\oplus N^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$ |
|---|---|---|---|
| $(1,0,0)$ | $(0,0)$ | $(1,0,1)$ | $(0,1)$ |
| $(1,0,2)$ | $(0,2)$ | $(1,0,3)$ | $(0,3)$ |
| $(1,1,0)$ | $(0,0)$ | $(1,1,1)$ | $(0,0)\oplus(0,1)$ |
| $(1,1,2)$ | $(0,1)\oplus(0,2)$ | $(1,1,3)$ | $(0,2)\oplus(0,3)$ |
| $(1,2,0)$ | $(0,0)$ | $(1,2,1)$ | $(0,0)\oplus(0,1)$ |
| $(1,2,2)$ | $(0,0)\oplus(0,1)\oplus(0,2)$ | $(1,2,3)$ | $(0,1)\oplus(0,2)\oplus(0,3)$ |
| $(1,3,0)$ | $(0,0)$ | $(1,3,1)$ | $(0,0)\oplus(0,1)$ |
| $(1,3,2)$ | $(0,0)\oplus(0,1)\oplus(0,2)$ | $(1,3,3)$ | $(0,0)\oplus(0,1)\oplus(0,2)\oplus(0,3)$ |
| $(2,0,2)$ | $(0,\frac{5}{2})$ | $(2,0,3)$ | $(0,\frac{5}{2})\oplus(0,\frac{7}{2})\oplus(\frac{1}{2},4)$ |
| $(2,1,2)$ | $(0,\frac{3}{2})\oplus(0,\frac{5}{2})$ | $(2,1,3)$ | $(0,\frac{3}{2})\oplus 3(0,\frac{5}{2})\oplus 2(0,\frac{7}{2})\oplus(\frac{1}{2},3)\oplus(\frac{1}{2},4)$ |
| $(2,2,1)$ | $(0,\frac{1}{2})$ | $(2,2,2)$ | $(0,\frac{1}{2})\oplus 2(0,\frac{3}{2})\oplus(0,\frac{5}{2})$ |
| $(2,2,3)$ | $(0,\frac{1}{2})\oplus 3(0,\frac{3}{2})\oplus 5(0,\frac{5}{2})\oplus 2(0,\frac{7}{2})\oplus(\frac{1}{2},2)\oplus(\frac{1}{2},3)\oplus(\frac{1}{2},4)$ | $(2,3,1)$ | $2(0,\frac{1}{2})\oplus(\frac{1}{2},0)$ |
| $(2,3,2)$ | $3(0,\frac{1}{2})\oplus 3(0,\frac{3}{2})\oplus(0,\frac{5}{2})\oplus(\frac{1}{2},1)$ | $(2,3,3)$ | $3(0,\frac{1}{2})\oplus 6(0,\frac{3}{2})\oplus 6(0,\frac{5}{2})\oplus 2(0,\frac{7}{2})\oplus(\frac{1}{2},1)\oplus 2(\frac{1}{2},2)\oplus(\frac{1}{2},3)\oplus(\frac{1}{2},4)$ |

**Table 29**. BPS spectrum of $SU(3)_8$ for $d_1 \leq 2$ and $d_2, d_3 \leq 3$. Here, $\mathbf{d}=(d_1,d_2,d_3)$ labels the state wrapping curve $d_1e_2 + d_2f_1 + d_3f_2$.

# KK theory: Rank1

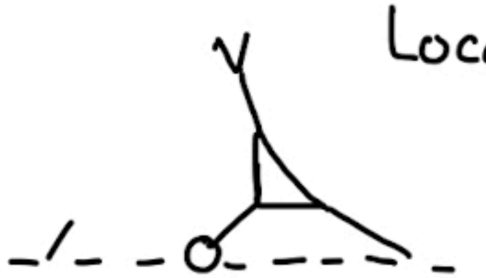
- $SU(2) + 8F$ — 6d E-string on a circle
  ( $\mathcal{E}$, Magnetic fluxes ) ⇒ BPS
- $SU(2)_0 + 1\,Adj$ — 6d M-string on a circle
- $SU(2)_\pi + 1\,Adj$ — 6d $\mathcal{N}=(2,0)$ $A_2$ thy w/ $\mathbb{Z}_2$-twist




like $SU(2)_{0,\pi}$ : two distict sets of fluxes

↓ decoupling "instantonic" hyper.

Local $\mathbb{P}^2 + 1\,Adj$ : non - Lagrangian [Bhardwaj '



# KK theory: Rank2

$Sp(2) + 3\Lambda^2$, $G_2 + 6F$, $Sp(2) + 1\Lambda^2 + 8F$ (rank-2 E-string)

$SU(3)_0 + 10F$, $SU(2)\times SU(2) + 2biF$, $SU(3)_0 + 1Adj$

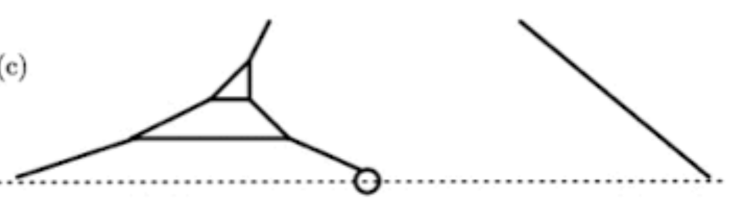
$Sp(2)_0 + 1Adj$, $Sp(2)_\pi + 1Adj$, $SU(3)_0 + 1Sym + 1F$

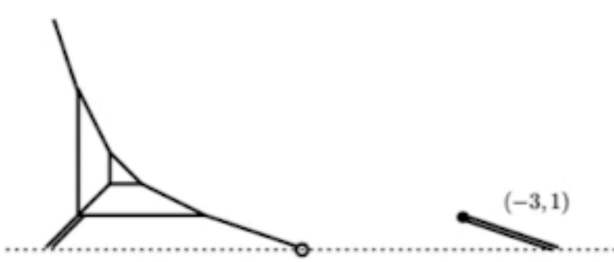
$SU(3)_{\frac{15}{2}} + 1F$ ($G_2 + 1Adj$)

$Sp(2)_\pi + 1Adj \downarrow \mathbb{P}^2 \cup \mathbb{F}_3 + 1Sym$

$SU(3)_0 + 1Sym + 1F \downarrow \mathbb{P}^2 \cup \mathbb{F}_6 + 1Sym$.

(c)





# Conclusion

A systematic bootstrap method for BPS spectra of 5d N = 1 field theories (including KK theories),
based on the Nakajima-Yoshioka's blowup equation

for any theories: either gauge theory description
or geometric description

With inputs: effective prepotential, consistent magnetic fluxes.

Various examples: rank 1 and rank 2, KK theories

Wilson lines: Online talk on July 14 by Minsung KIm

**Strings and QFTs for Eurasian time zone**
External homepage

HEP - theory

| | |
|---|---|
| Audience: | Researchers in the topic |
| Seminar series times: | Wednesday 15:00-16:00, 16:00-17:00 |
| Organizer: | Satoshi Nawata* |