“弦理论及相关物理”研讨会, 7.10-13, 2021

New progress of color-kinematics duality

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Based on arXiv: 2106.05280 and in progress
with Guanda Lin (林冠达) and Siyuan Zhang (张思源)

Quantum field theory

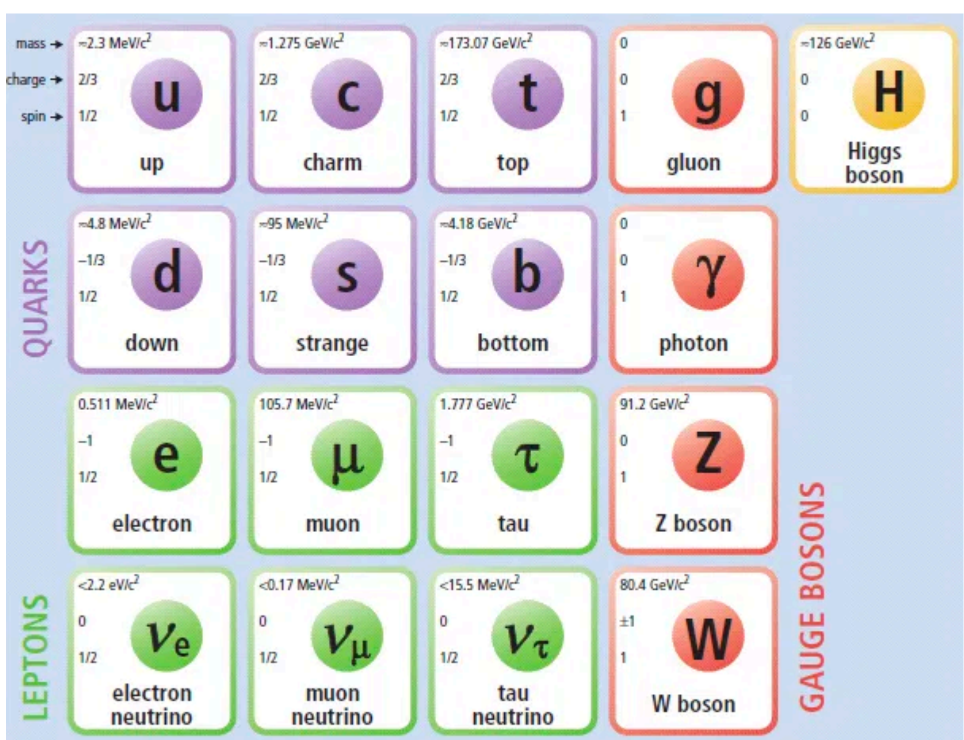
Quantum mechanics

Special relativity

→ Quantum field theory

A unified framework for strong, weak and electromagnetic forces

Standard model of particle physics



Higgs particle finally discovered in 2012

Physics 2013

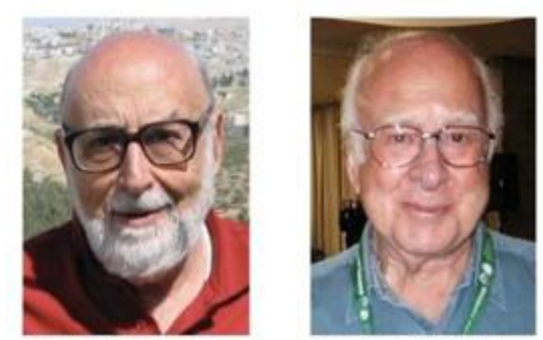


Photo: Pnicolet via Wikimedia Commons
François Englert

Photo: G-M Greuel via Wikimedia Commons
Peter W. Higgs

Unifying Gravity?

Quantum mechanics

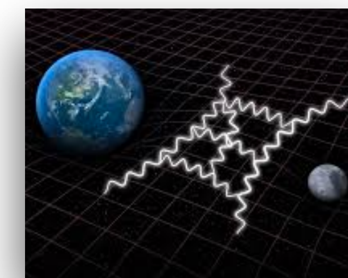
Special relativity

Newton Gravity

→ Quantum field theory

→ General relativity

? → Quantum gravity puzzle



Gravity is not renormalizable.

Renormalizability

QED is renormalizable -> we know the full theory

$$\mathscr{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^{\mu}D_{\mu} - m)\psi$$

Gravity： needs infinitely many “counter terms”:

$$\mathscr{L} = \sqrt{g}(R+c_1R^2 + c_2R_{\mu\nu}R^{\mu\nu} + c_3R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \cdots)$$

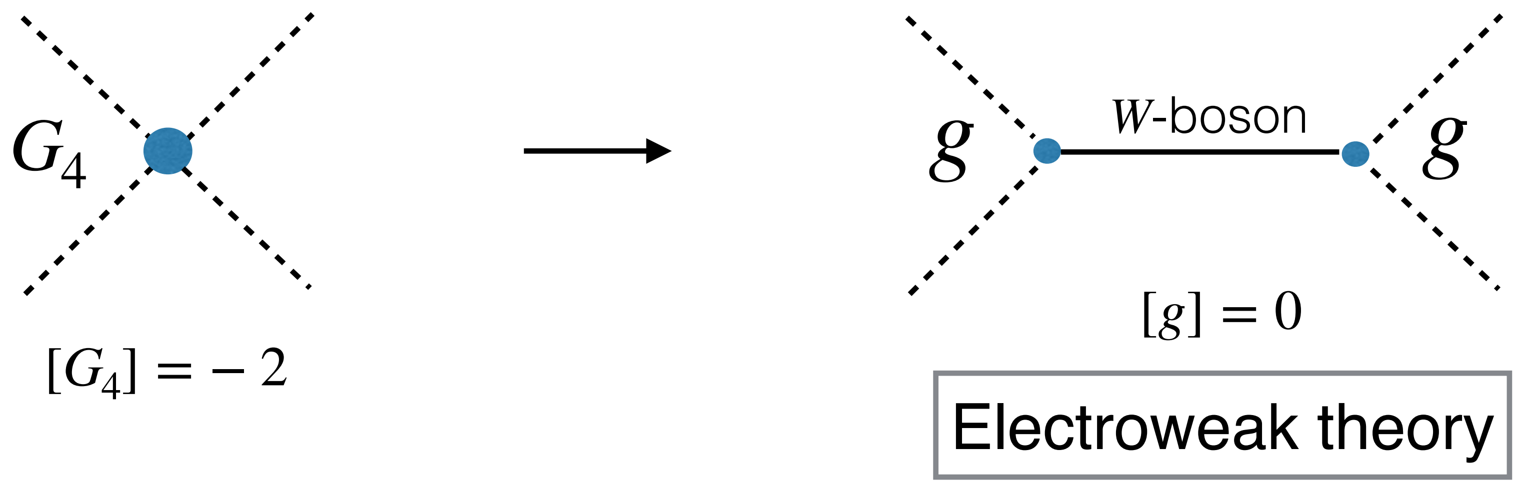
This may be understood from the dimension of the coupling：

Gravity coupling has mass dimension -2.

High-dimensional local operators (as counter terms) appear at high orders.

Renormalizability

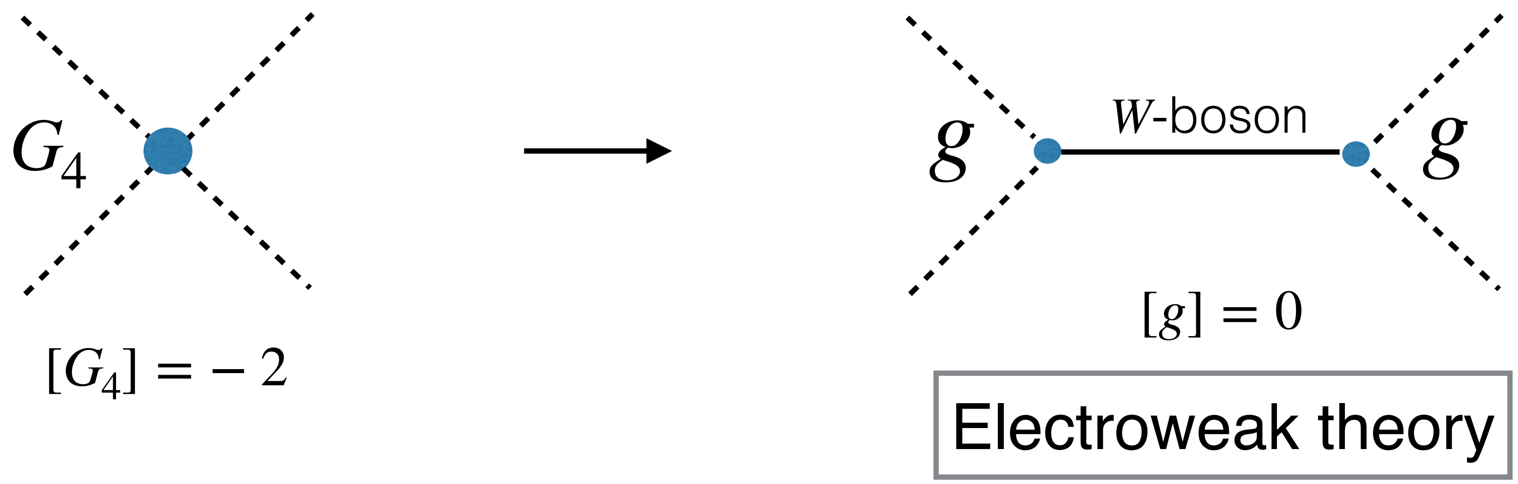
An analogy："four-fermion effective theory"，coupling dimension is -2



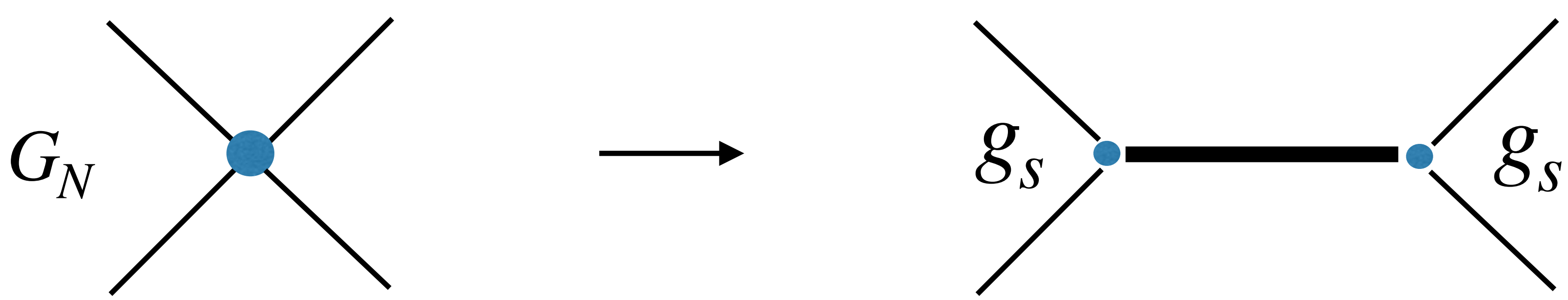
Four-fermion theory is an "effective theory" of the more fundamental electroweak theory.

Renormalizability

An analogy：“four-fermion effective theory”，coupling dimension is -2



Gravity is also an effective theory of some fundamental theory



Renormalizability

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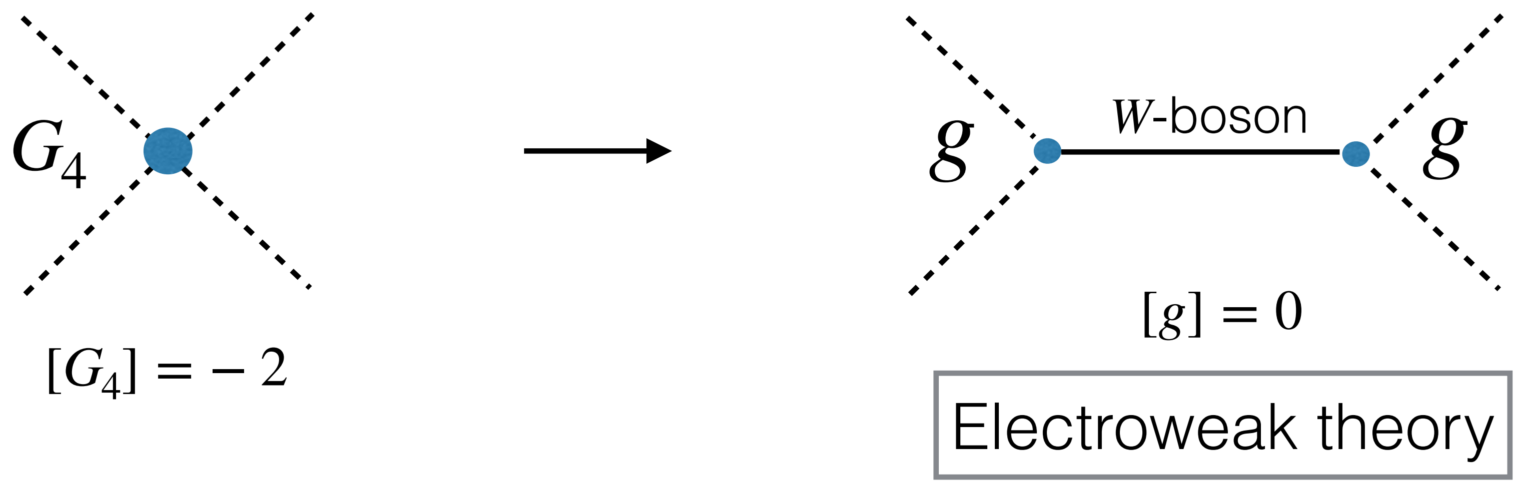
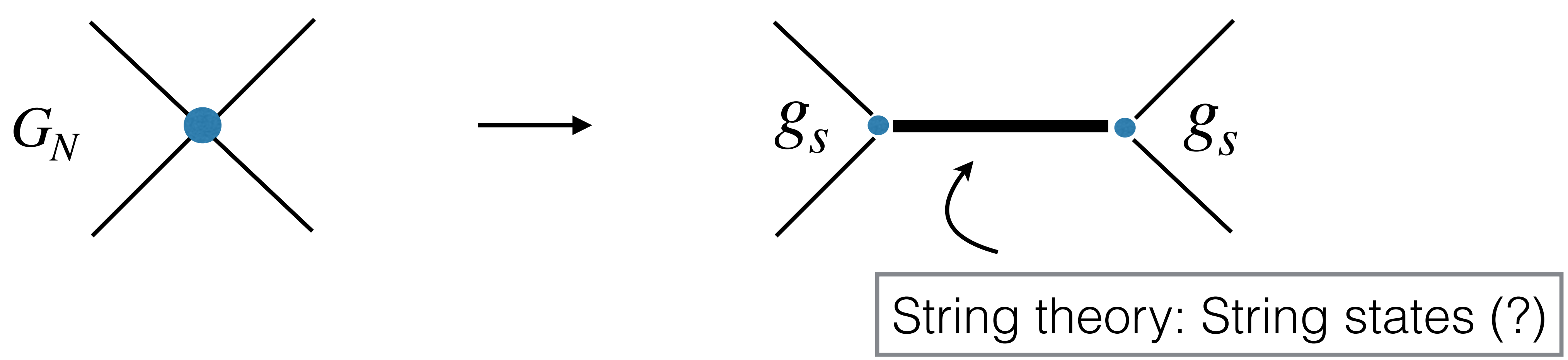


Gravity is also an effective theory of some fundamental theory



Effective theory

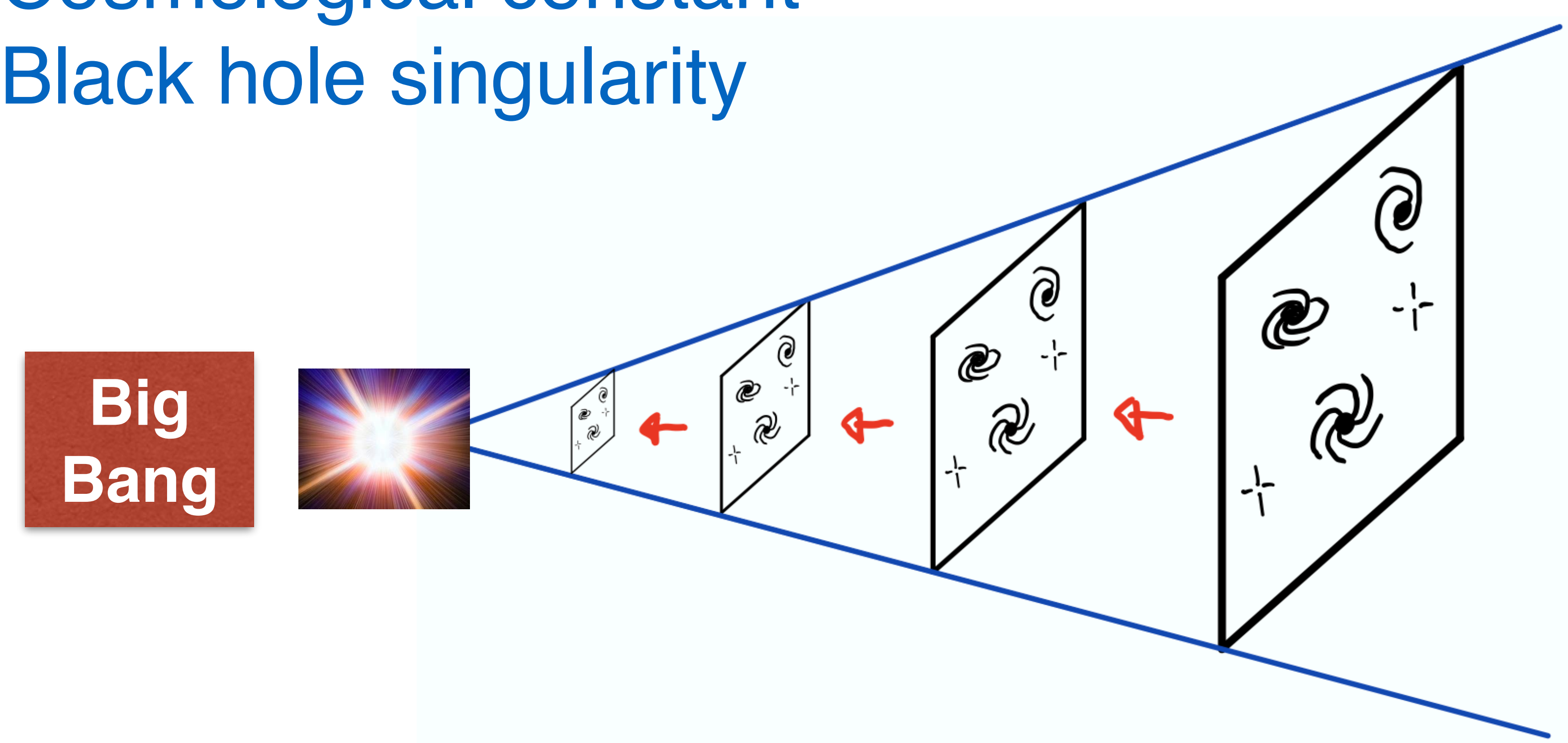
Effective theory only works up to certain energy scale.
Above such energy scale, the theory is meaningless.

Quantum gravity

Effective theory only works up to certain energy scale.
Above such energy scale, the theory is meaningless.

A fundamental quantum gravity theory is necessary:

- Origin of the universe
- Cosmological constant
- Black hole singularity

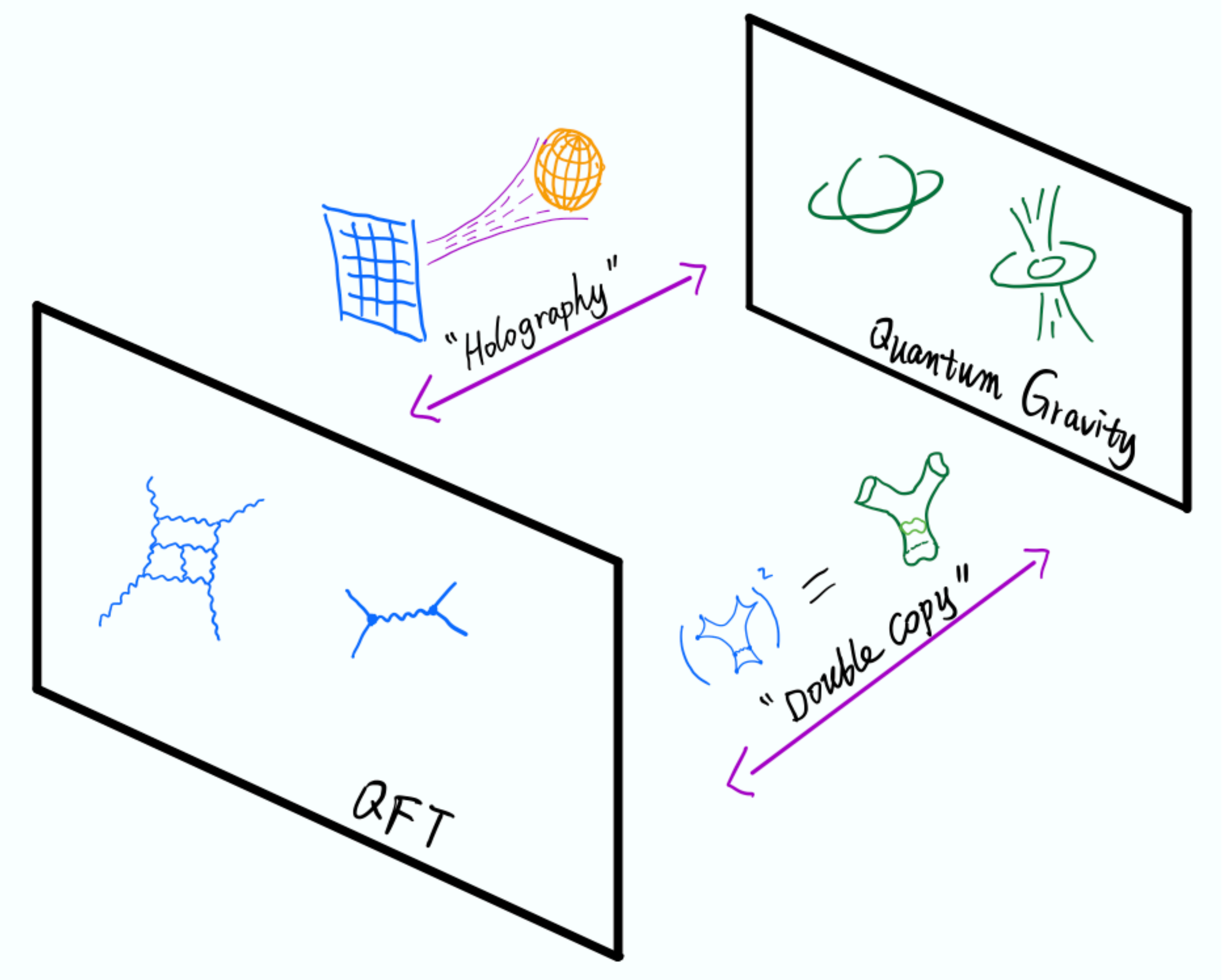


Toward a quantum theory of gravity

String theory,
Loop quantum gravity,
Asymptotic safety,
…

Toward a quantum theory of gravity

Gauge-gravity duality



Outline

Motivation

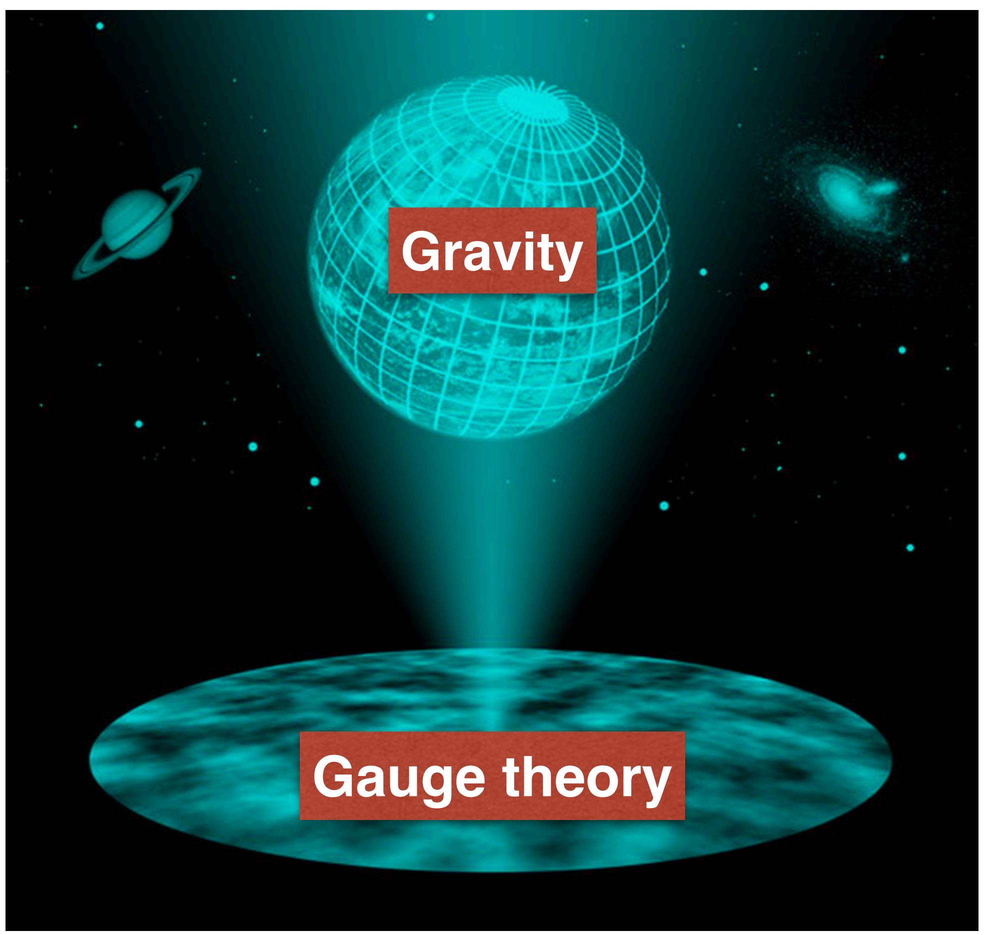
Color-kinematics duality

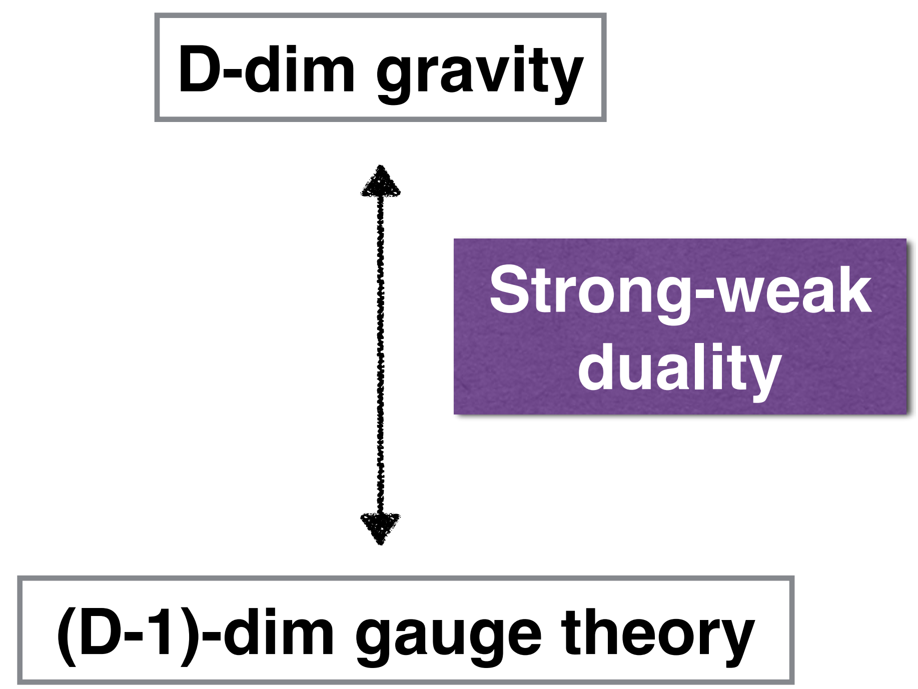
New form factor results

Summary and discussion

Gauge-gravity duality: holography

AdS/CFT correspondence

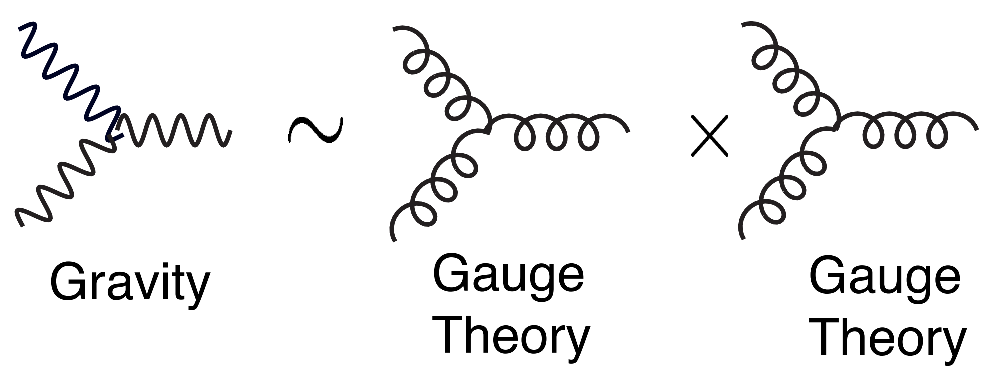


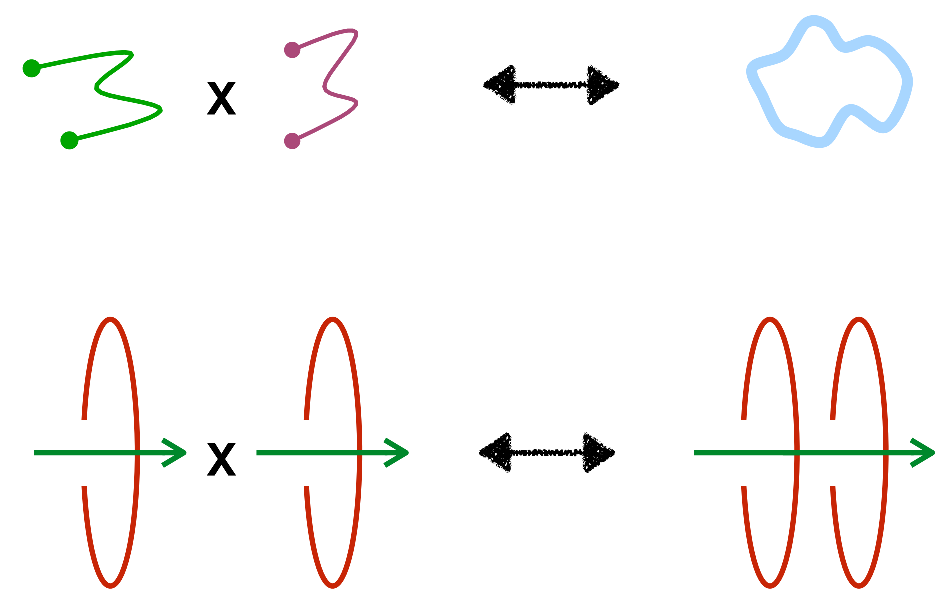


Gauge-gravity duality: double copy

Gravity ⟷ **(Yang-Mills)^2**

Weak-weak duality





KLT relation

CLNS - 85/667
September 1985

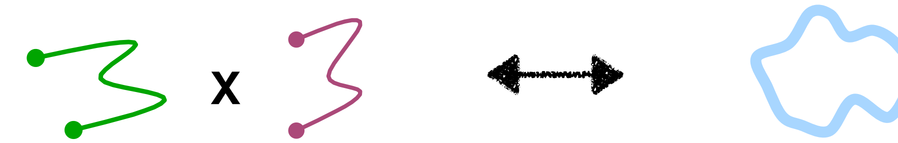
A Relation Between Tree Amplitudes of Closed and Open Strings

H. Kawai, D.C. Lewellen, and S.-H.H. Tye

Newman Laboratory of Nuclear Studies
Cornell University
Ithaca, New York 14853

ABSTRACT

We derive a formula which expresses any closed string tree amplitude in terms of a sum of the products of appropriate open string tree amplitudes. This formula is applicable to the heterotic string as well as to the closed bosonic string and type II superstrings. In particular, we demonstrate its use by showing how to write down, without any direct calculation, all four-point heterotic string tree amplitudes with massless external particles.



$$A^{(4)}_{closed} = -\pi\kappa^2 \sin(\pi\kappa_1\cdot\kappa_2) A^{(4)}_{open}(s,t)\ \bar{A}^{(4)}_{open}(s,u)$$

$$A^{(5)}_{closed} = \pi\kappa^3 A^{(5)}_{open}(12345)\bar{A}^{(5)}_{open}(21435)\sin(\pi\kappa_1\cdot\kappa_2)\sin(\pi\kappa_3\cdot\kappa_4)$$
$$+\pi\kappa^3\ A^{(5)}_{open}(13245)\bar{A}^{(5)}_{open}(31425)\sin(\pi k_1\cdot k_3)\sin(\pi k_2\cdot k_4).$$

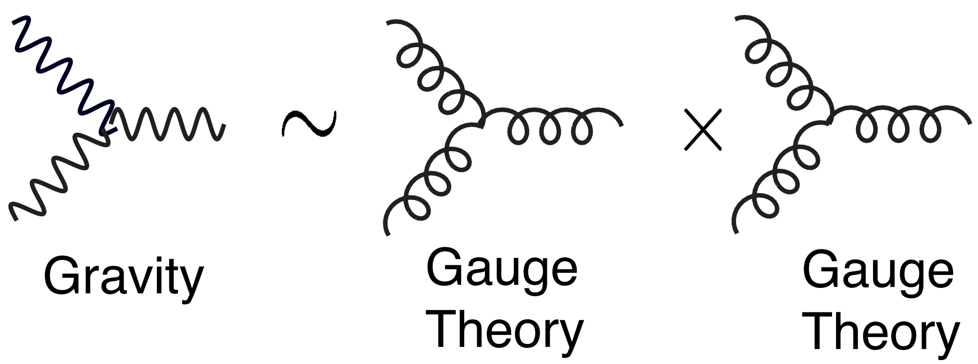
↓ Field theory limit

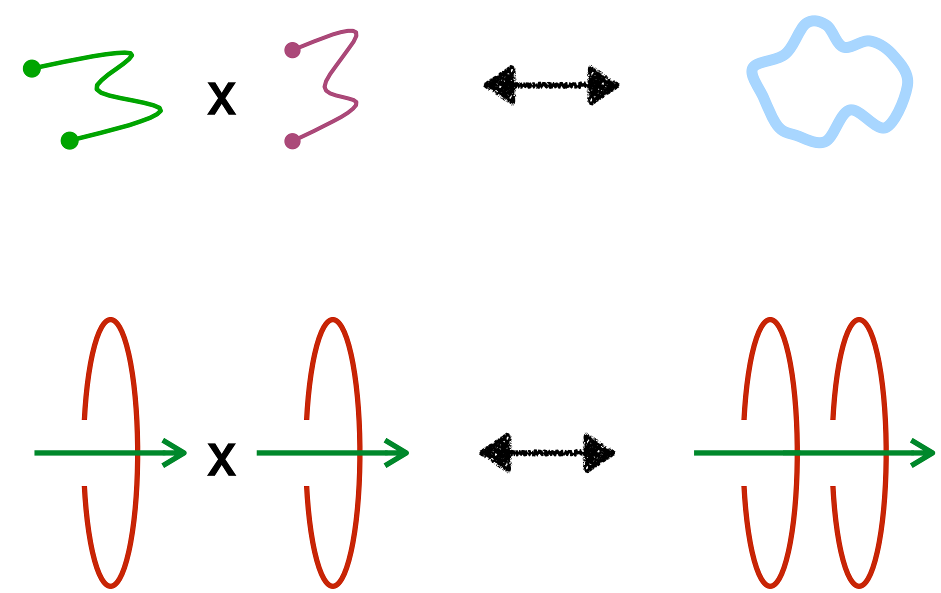
$$M_4^{\rm tree}(1,2,3,4) = -is_{12}A_4^{\rm tree}(1,2,3,4)\,A_4^{\rm tree}(1,2,4,3)\,,$$

$$M_5^{\rm tree}(1,2,3,4,5) = is_{12}s_{34}A_5^{\rm tree}(1,2,3,4,5)A_5^{\rm tree}(2,1,4,3,5)$$
$$+\,is_{13}s_{24}A_5^{\rm tree}(1,3,2,4,5)\,A_5^{\rm tree}(3,1,4,2,5)$$

Gauge-gravity duality: double copy

Gravity ⟷ (Yang-Mills)^2





[Kawai, Lewellen, Tye 1986]

[Bern, Carrasco, Johansson 2008]

[Cachazo, He, Yuan 2013]

Color-kinematics duality

Color-kinematics duality

In 2008 Bern, Carrasco and Johansson proposed a duality between color and kinematics factors:

[Bern, Carrasco, Johansson 2008]

Duality

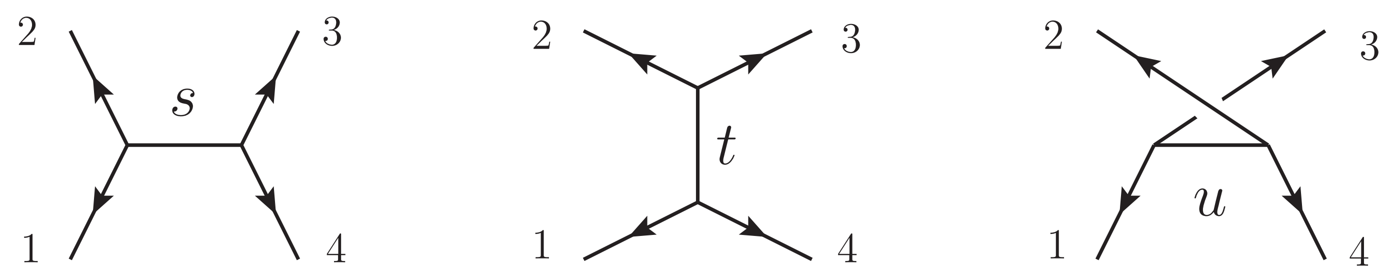
Color factor ⟷ Kinematic factor

(conjecture)

$$\tilde{f}^{abc} = \mathrm{Tr}([T^a, T^b]T^c) \qquad\qquad s_{ij} = (p_i + p_j)^2$$

Gauge symmetry **Spacetime symmetry**

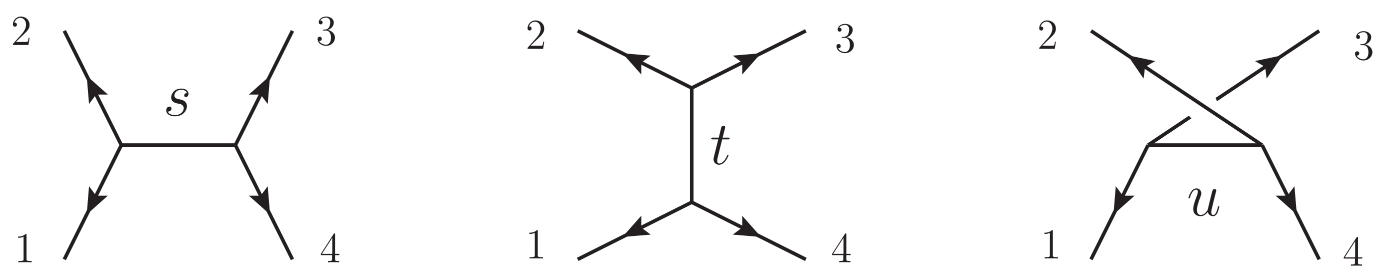
Example: 4-pt amplitude



$$A_4(1,2,3,4) = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u}$$

$$c_s = \tilde{f}^{a_1 a_2 b}\tilde{f}^{b a_3 a_4}, \quad c_t = \tilde{f}^{a_2 a_3 b}\tilde{f}^{b a_4 a_1}, \quad c_u = \tilde{f}^{a_1 a_3 b}\tilde{f}^{b a_2 a_4}$$

Example: 4-pt amplitude



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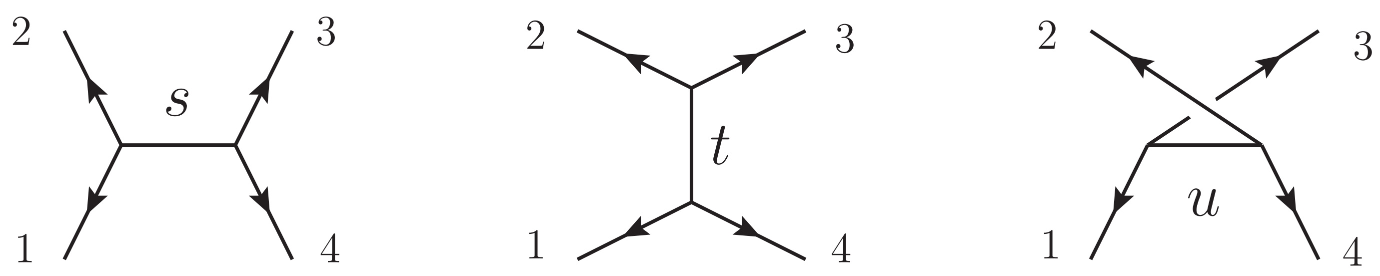
$$c_s = c_t + c_u$$

Jacobi identity

Example: 4-pt amplitude



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$$c_s = c_t + c_u \quad \Rightarrow \quad n_s = n_t + n_u$$

Jacobi identity

dual Jacobi relation

Not trivial !

Color-kinematics duality

If the gauge amplitude satisfies CK duality, one can directly construct gravity amplitude：

$$A_4(1,2,3,4) = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u} \longrightarrow M_4(1,2,3,4) = \frac{n_s n_s}{s} + \frac{n_t n_t}{t} + \frac{n_u n_u}{u}$$

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Gauge invariance, via double copy, implies the diffeomorphism invariance in gravity:

$$n_i \to n_i + \delta_i \,, \qquad \delta_i = n_i|_{\varepsilon_j \to p_j}$$

$$\sum_i \frac{c_i \delta_i}{D_i} = 0 \quad \xrightarrow[n_i = n_j + n_k]{c_i = c_j + c_k} \quad \sum_i \frac{n_i \delta_i}{D_i} = 0$$

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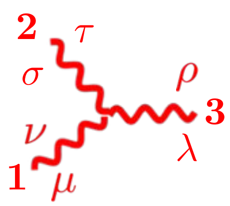
“double-copy” can be used also at **high loops**:

$$A^{(\ell)} \sim \sum_i \int \frac{C_i \times N_i}{\prod D} \longrightarrow M^{(\ell)} \sim \sum_i \int \frac{N_i \times N_i}{\prod D}$$

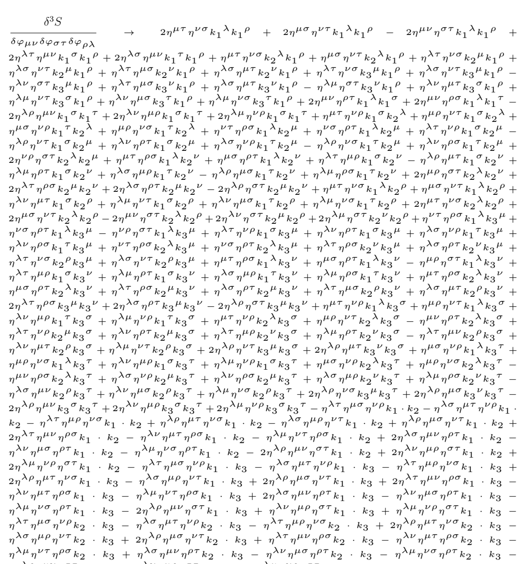
Gauge x Gauge ⟵ CK-duality ⟶ Gravity

High-loop graviton amplitudes

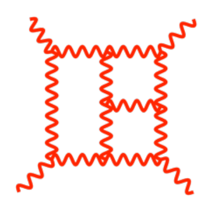
Very difficult using Feynman diagram:



3-vertex
more than
100 terms

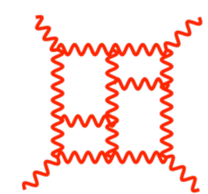


3-loop



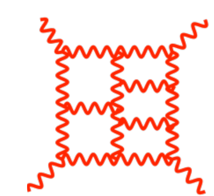
$\sim 10^{20}$ terms

4-loop



$\sim 10^{26}$ terms

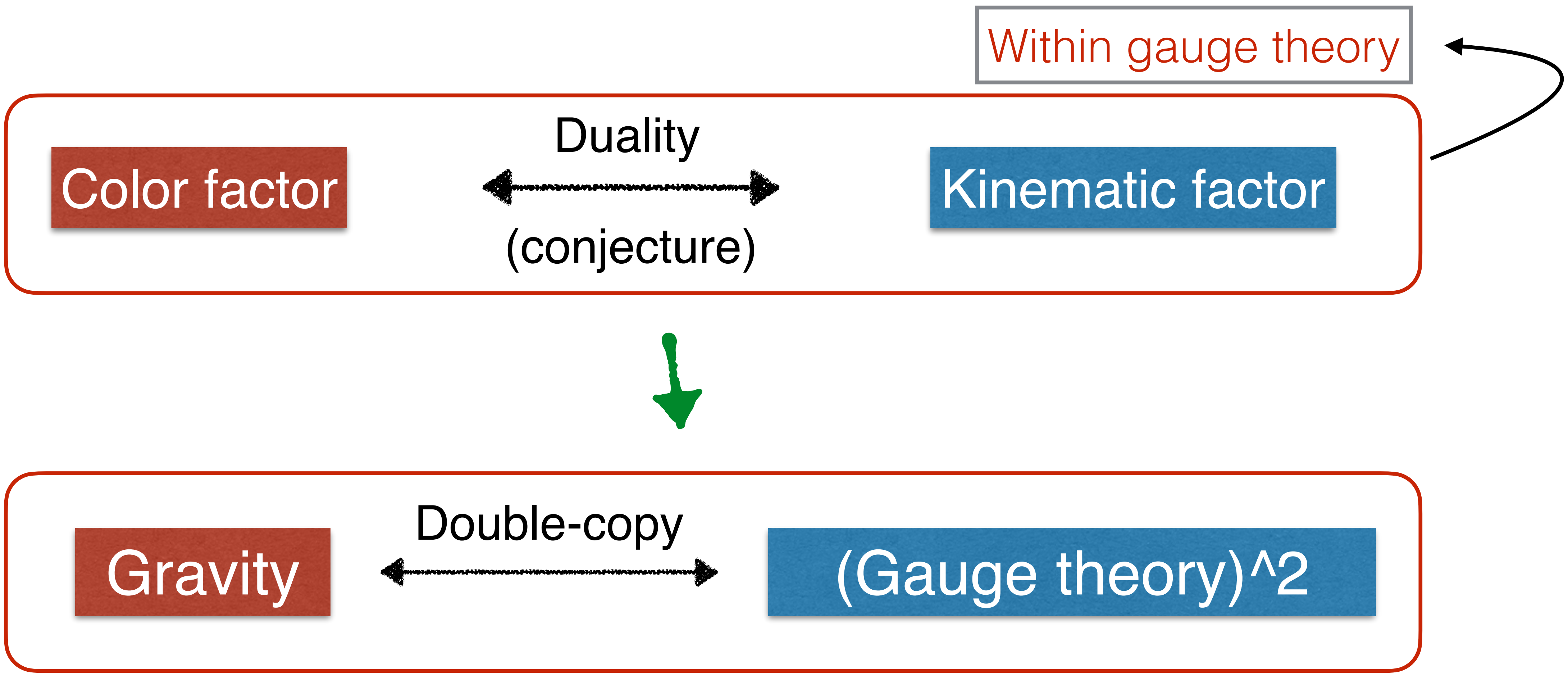
5-loop



$\sim 10^{31}$ terms

More than the number of atoms in human's brain!

CK-duality v.s. Double-copy



By studying the simpler gauge theory, one may understand the far more complicated gravity theory.

Color-kinematics duality

Proved at tree-level:

- String Monodromy relation Bjerrum-Bohr et.al 2009; Stieberger 2009
- BCFW recursion Feng, Huang, Jia 2010

Still a conjecture at loop level, relying on explicit constructions:

- 4-loop 4-point amplitudes in N=4 Bern, et.al, 2012
- 5-loop Sudakov form factor in N=4 G. Yang, 2016
- 2-loop 5-point amplitudes in pure YM O'Connell and Mogull 2015

It is usually non-trivial to find CK dual solution at high loops.

New 3-loop solutions for form factors

with 24 free parameters!

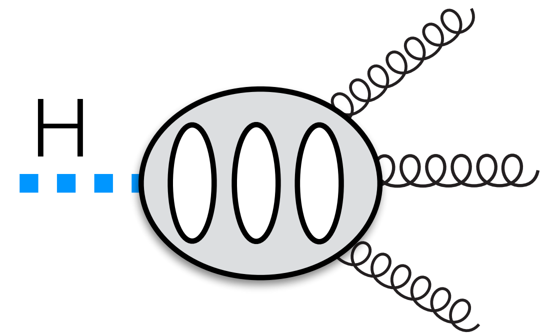
arXiv: 2106.05280 with Guanda Lin, Siyuan Zhang

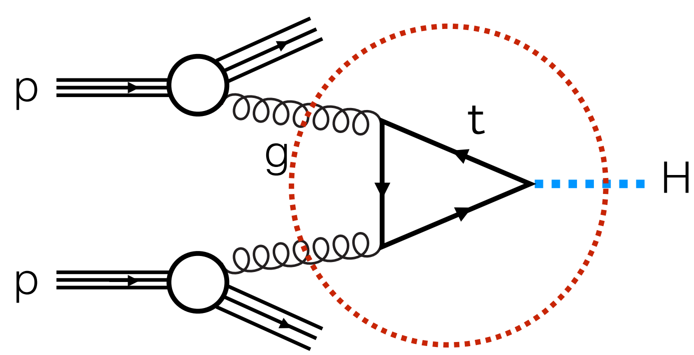
Three-loop form factors

We consider three-loop three-point form factor in N=4 SYM:

$$\mathcal{F}_{\mathcal{O}_i,n} = \int d^4x\, e^{-iq\cdot x}\langle p_1, p_2, p_3 \,|\, \mathrm{tr}(F^2)(x) \,|\, 0\rangle$$

It is a N=4 version of Higgs+3-gluon amplitudes in QCD:





$$m_t \to \infty$$

$$\mathcal{L}_{\text{eff}} = \hat{C}_0 H \,\mathrm{tr}(F^2) + \mathcal{O}\left(\frac{1}{m_{\mathrm{t}}^2}\right)$$

Three-loop form factors

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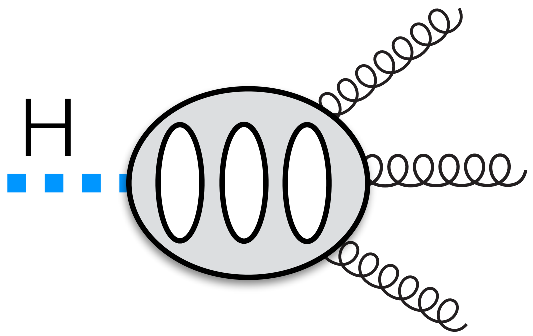


Maximal transcendentality principle

N=4 SYM ⟷ **QCD**

N=4 result provides the maximally transcendental part in QCD

Strategy of high-loop computation

CK-duality

Ansatz of the loop integrand

Strategy of high-loop computation

CK-duality

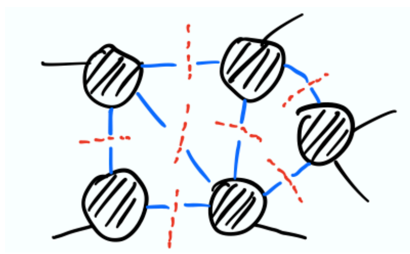
Ansatz of the loop integrand

Unitarity cuts

Strategy of high-loop computation

CK-duality

Ansatz of the loop integrand



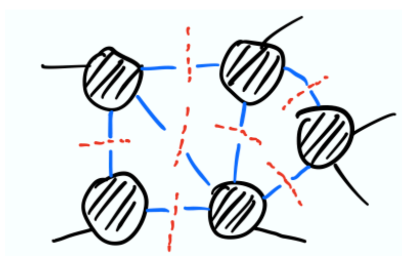
Unitarity cuts

Solving linear equations

Strategy of high-loop computation

CK-duality

Ansatz of the loop integrand



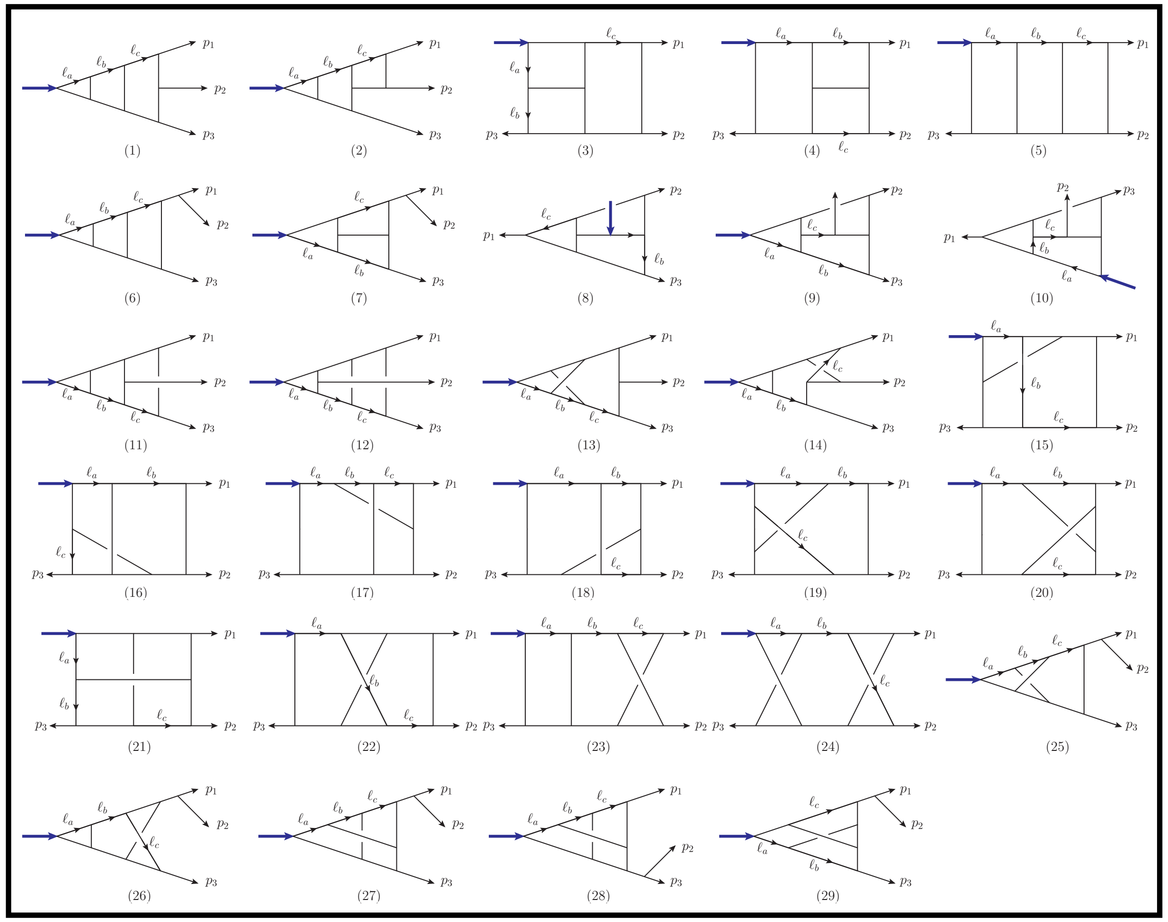
Unitarity cuts

Solving linear equations

Main challenge: it is not clear whether the solution exists

Ansatz

3-loop integral topologies:



Ansatz

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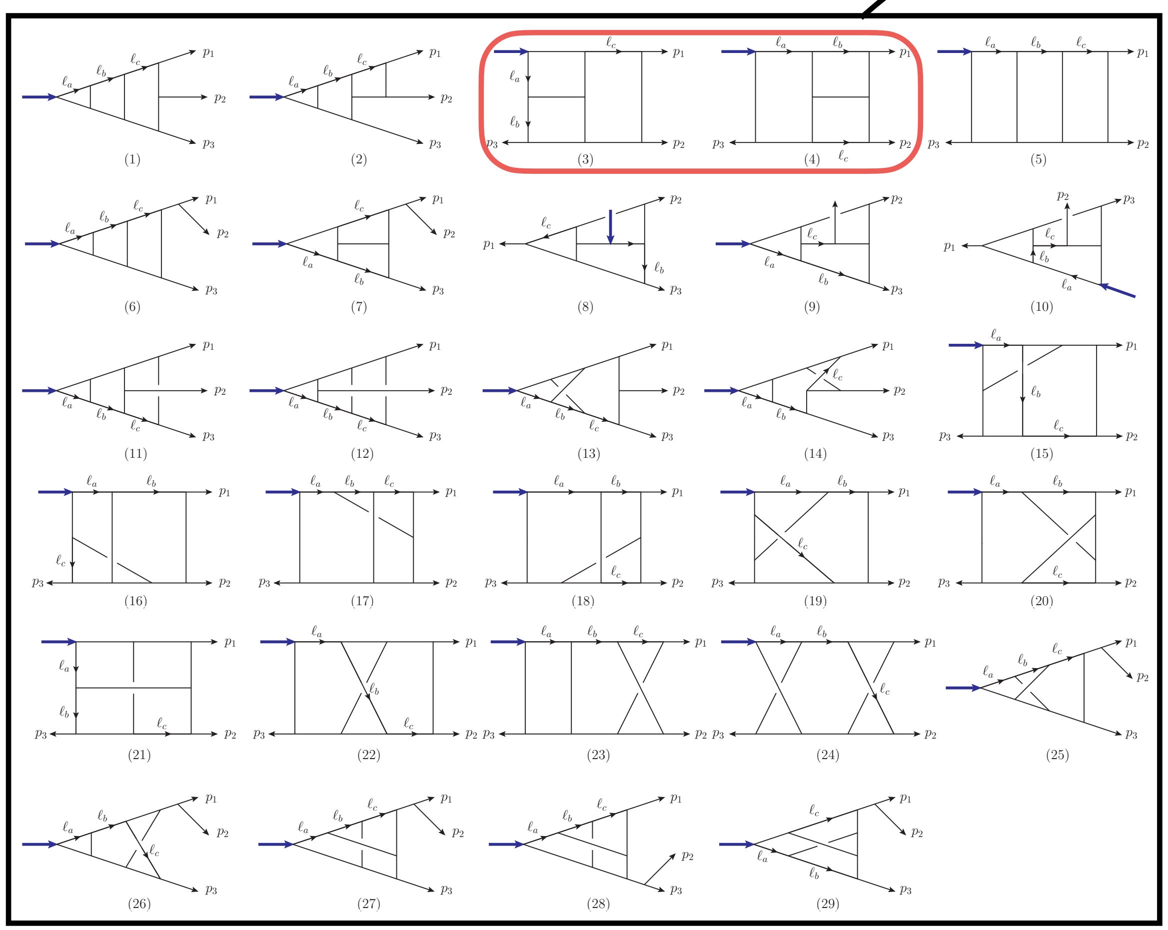
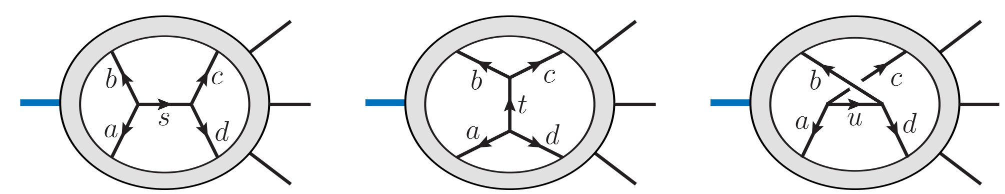


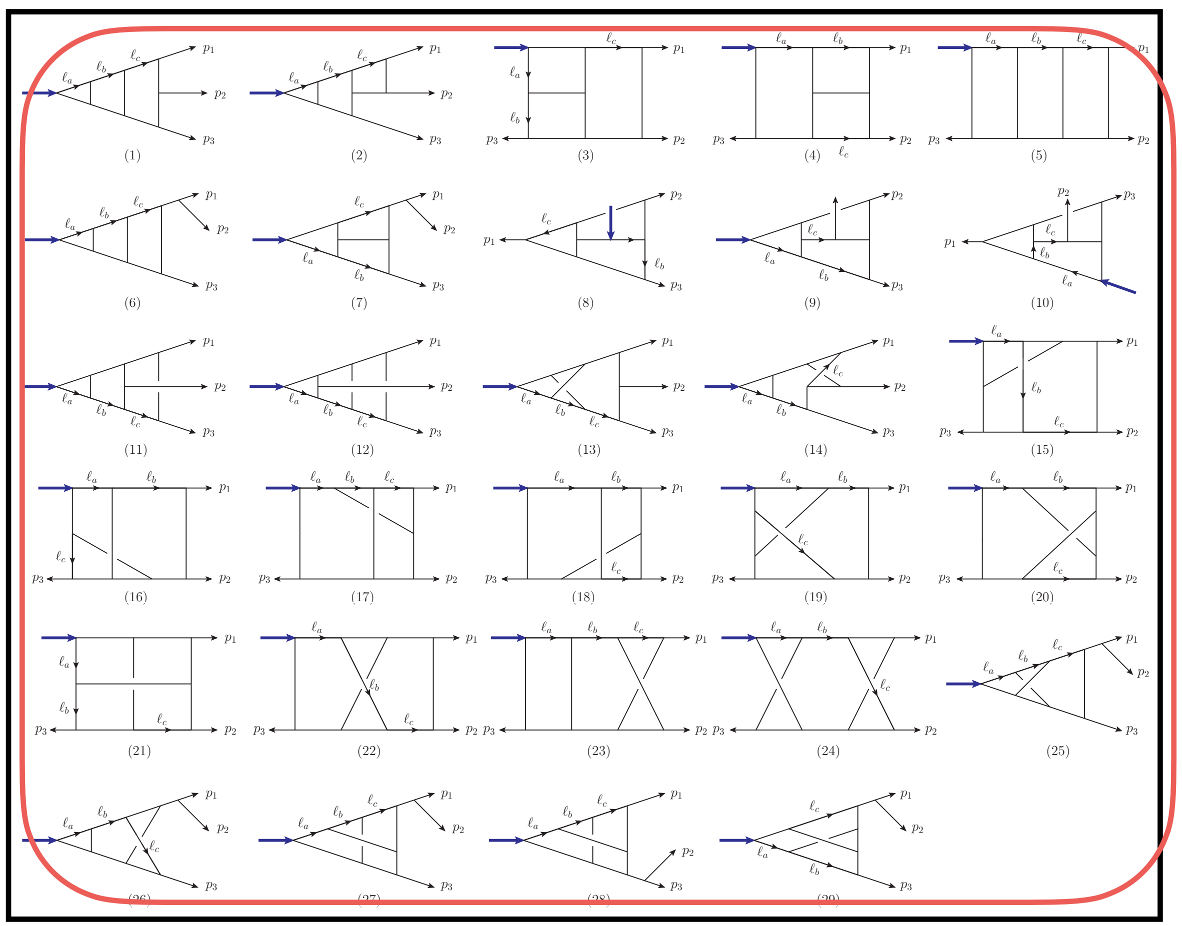
2 master integrals

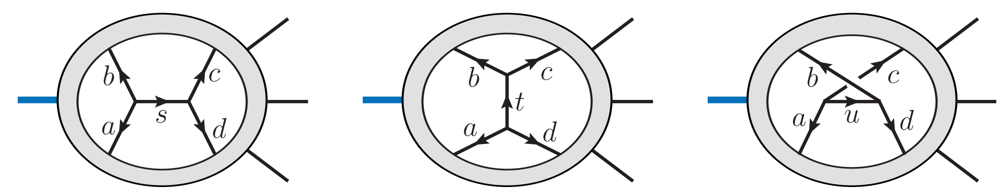


$$N_s + N_t + N_u = 0$$

Ansatz

3-loop integral topologies:





$$N_s + N_t + N_u = 0$$

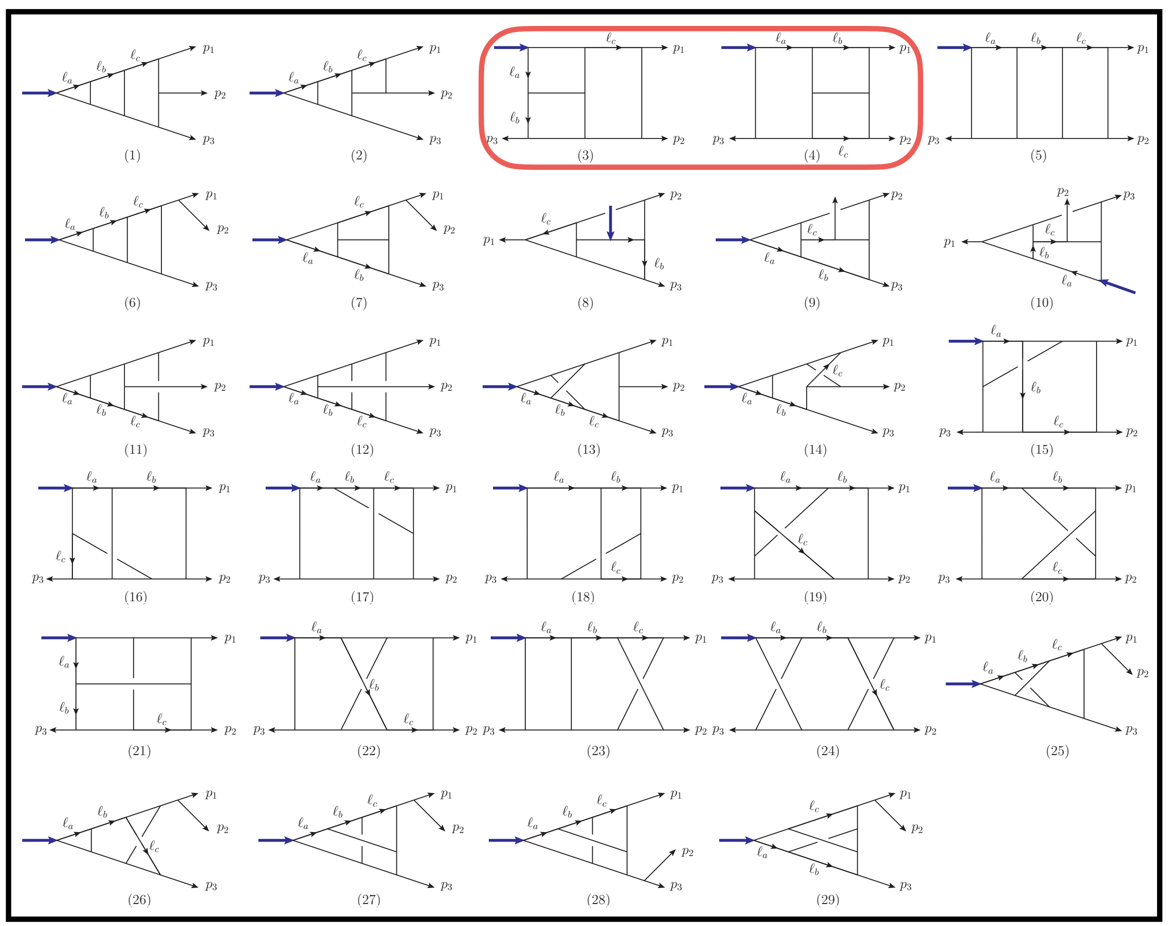
↓

$$\boldsymbol{F}^{(3)}_{\mathcal{O}_2,3} = \mathcal{F}^{(0)}_{\mathcal{O}_2,3} \sum_{\sigma_3} \sum_i \int \prod_{j=1}^{3} d^D \ell_j \frac{1}{S_i} \frac{C_i\, N_i}{\prod_{\alpha_i} P^2_{\alpha_i}}$$

And ansatz with 316 parameters

Solving ansatz

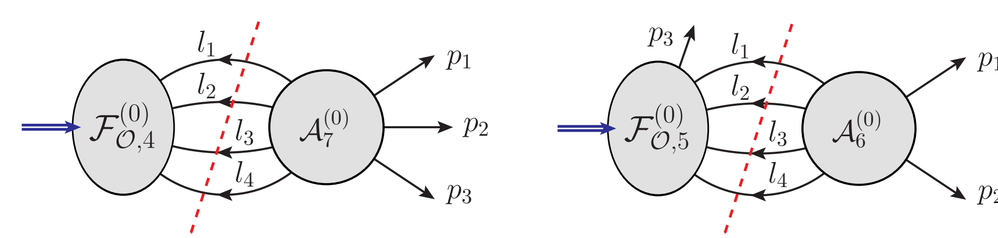
3-loop integral topologies:



Solve ansatz：

Symmetry constraints and Unitarity cuts

Most complicated cuts



(a) Cut$_1$: $\mathcal{F}_4$-$\mathcal{A}_7$ cut

(b) Cut$_2$: $\mathcal{F}_5$-$\mathcal{A}_6$ cut

$$\boldsymbol{F}^{(3)}_{\mathcal{O}_2,3} = \mathcal{F}^{(0)}_{\mathcal{O}_2,3} \sum_{\sigma_3} \sum_i \int \prod_{j=1}^{3} d^D \ell_j \frac{1}{S_i} \frac{C_i\, N_i}{\prod_{\alpha_i} P^2_{\alpha_i}}$$

Final solution with 24 free parameters

Integration and checks

All free parameters cancel at integrand level.

Color decomposition:

$$\boldsymbol{F}^{(3)}_{\mathcal{O}_2,3} = \mathcal{F}^{(0)}_{\mathcal{O}_2,3} \sum_{\sigma_3} \sum_i \int \prod_{j=1}^{3} d^D \ell_j \frac{1}{S_i} \frac{C_i\, N_i}{\prod_{\alpha_i} P^2_{\alpha_i}}$$

$$\downarrow$$

$$\boldsymbol{F}^{(3)}_{\mathcal{O}_2,3} = \mathcal{F}^{(0)}_{\mathcal{O}_2,3} f_{123}\big(N_c^3 \mathcal{I}^{(3)}_{\mathcal{O}_2} + 12 N_c \mathcal{I}^{(3)}_{\mathcal{O}_2,\mathrm{NP}}\big)$$

Integration and checks

$$\boldsymbol{F}^{(3)}_{\mathcal{O}_2,3} = \mathcal{F}^{(0)}_{\mathcal{O}_2,3} f_{123} \big(N_c^3 \mathcal{I}^{(3)}_{\mathcal{O}_2} + 12 N_c \mathcal{I}^{(3)}_{\mathcal{O}_2,\mathrm{NP}} \big)$$

Three-loop IR divergences provide important check,
(which are also a research frontier).

Planar part is given by the BDS ansatz: Bern, Dixon, Smirnov 2005

$$\mathcal{I}^{(3)}(\epsilon) = -\frac{1}{3}\Big(\mathcal{I}^{(1)}(\epsilon)\Big)^3 + \mathcal{I}^{(2)}(\epsilon)\mathcal{I}^{(1)}(\epsilon) + f^{(3)}(\epsilon)\mathcal{I}^{(1)}(3\epsilon) + \mathcal{R}^{(3)} + C^{(3)} + O(\epsilon)$$

Integration and checks

Numerical result for planar part:

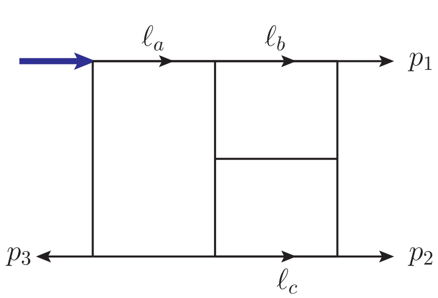
Form factor	$\mathcal{I}^{(3)}_{\mathrm{tr}(\phi^2)}$						
(s_{12}, s_{23}, s_{13})	ϵ^{-6}	ϵ^{-5}	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
(-2,-2,-2)	-4.49999	9.35749	-22.6136	55.8891	-77.252	92.943	-336.51
est. error	6×10^{-7}	2.3×10^{-5}	3.3×10^{-4}	0.0021	0.012	0.078	0.59

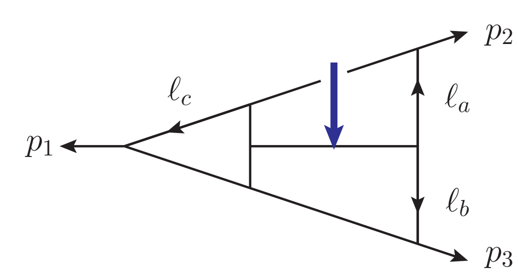
universal IR divergence
(determined by lower loops)

Consistent with OPE bootstrap results (-336.7)

Dixon, McLoed, Wilhelm 2020

Integral examples





~100,000 CPU core hours

FIESTA
Smirnov 2015
pySecDec
Borowka et.al 2017

Integration and checks

$$\boldsymbol{F}^{(3)}_{\mathcal{O}_2,3} = \mathcal{F}^{(0)}_{\mathcal{O}_2,3} f_{123} \left(N_c^3 \mathcal{I}^{(3)}_{\mathcal{O}_2} + 12 N_c \mathcal{I}^{(3)}_{\mathcal{O}_2,\mathrm{NP}} \right)$$

Full color IR structure:

$$\boldsymbol{F}(p_i, a_i, \epsilon) = \boldsymbol{Z}(p_i, \epsilon) \boldsymbol{F}^{\mathrm{fin}}(p_i, a_i, \epsilon) \qquad \boldsymbol{Z} = \mathcal{P} \exp \left[- \sum_{\ell=1}^{\infty} g^{2\ell} \left(\text{dipole terms} + \frac{1}{\ell \epsilon} \boldsymbol{\Delta}^{(\ell)} \right) \right]$$

Non-dipole term

$$\boldsymbol{\Delta}_3^{(3)} = \alpha \sum_i \sum_{j<k, j,k \neq i} \tilde{f}_{abe} \tilde{f}_{cde} (\mathbf{T}_i^a \mathbf{T}_i^d + \mathbf{T}_i^d \mathbf{T}_i^a) \mathbf{T}_j^b \mathbf{T}_k^c$$

Almelid, Duhr, Gardi 2015

Non-dipole term

- Starting at 3-loop
- Sub-leading in Nc (non-planar)
- IR starting at 1/eps

Integration and checks

Numerical result for non-planar part:

Non-dipole prediction

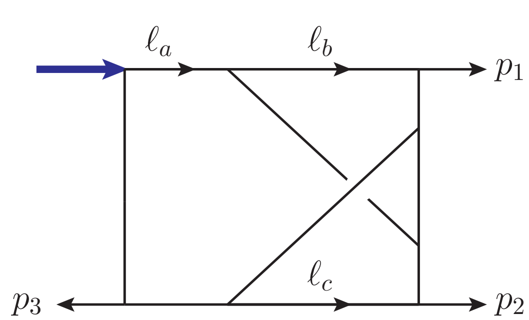
$-2(\zeta_5 + 2\zeta_2\zeta_3) = -9.983$

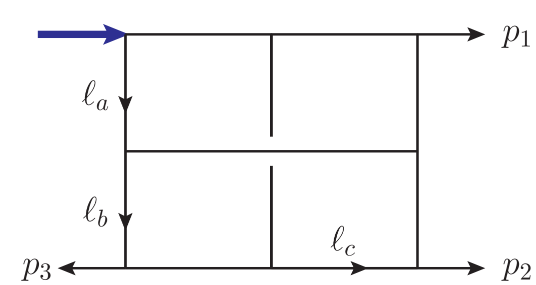
Form factor	$\mathcal{I}^{(3)}_{\mathrm{tr}(\phi^2),\mathrm{NP}}$						
(s_{12}, s_{23}, s_{13})	ϵ^{-6}	ϵ^{-5}	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
(-2,-2,-2)	-2.3×10^{-7}	5.8×10^{-6}	3.8×10^{-5}	5.6×10^{-4}	-0.001	-9.989	-265.314
est. error	1.2×10^{-6}	2.4×10^{-5}	3.0×10^{-4}	2.5×10^{-3}	0.02	0.1848	1.757

zero as expected

New non-planar finite remainder !

Integral examples

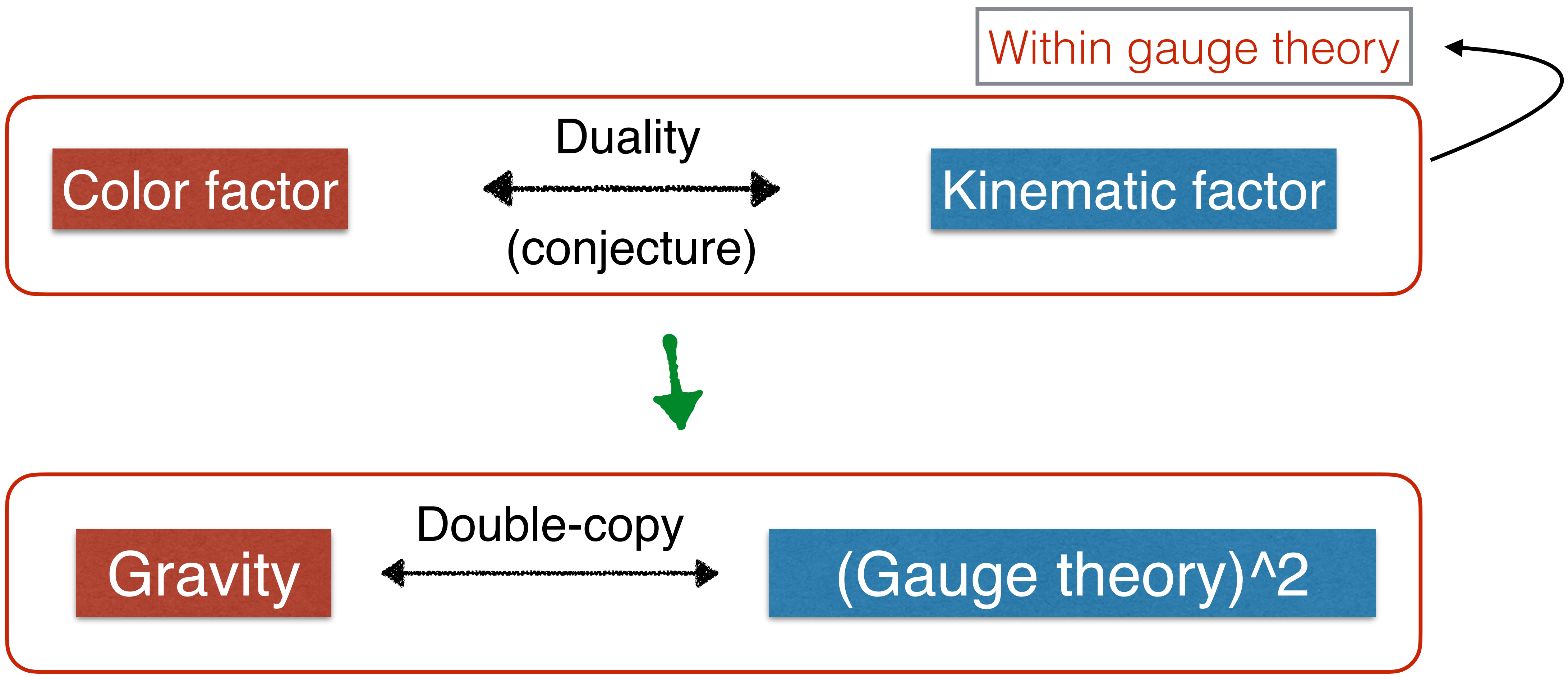




Much harder than planar cases:
- choosing proper UT numerators
- several 1000,000 CPU core hours (FIESTA)

Summary and discussion

CK-duality v.s. Double-copy



By studying the simpler gauge theory, one may understand the far more complicated gravity theory.

Summary

We obtain 3-loop form factors via CK duality and unitarity cut:

$$\boldsymbol{F}^{(3)}_{\mathcal{O}_2,3} = \mathcal{F}^{(0)}_{\mathcal{O}_2,3} \sum_{\sigma_3} \sum_i \int \prod_{j=1}^{3} d^D \ell_j \frac{1}{S_i} \frac{C_i N_i}{\prod_{\alpha_i} P^2_{\alpha_i}}$$

$$C_s + C_t + C_u = 0 \qquad N_s + N_t + N_u = 0$$

The CK-dual numerators contain a large number of free parameters:

$\mathrm{tr}(\phi^2)$: 24 parameters $\qquad$ $\mathrm{tr}(\phi^3)$: 10 parameters

We evaluate the integrals numerically and find consistent IR and planar finite results.

Free parameters

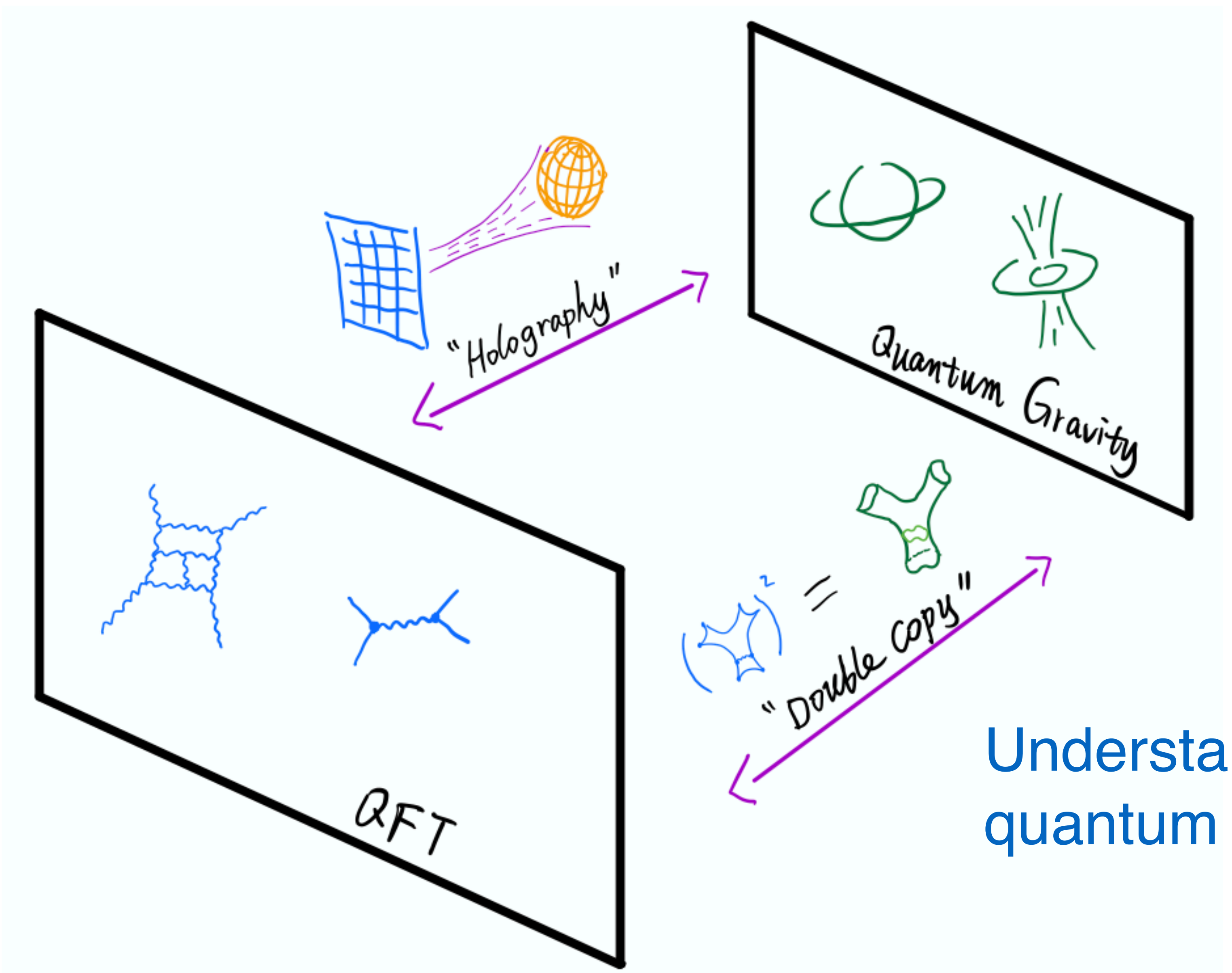
It is usually non-trivial to find CK dual solution at high loops.

Large number of free parameters ->
form factor is a nice arena for applying CK duality, and probably the duality can be realized at higher loops.

These deformation correspond to "CK-duality preserving generalized GT". Is there any deep interpretation?

Does the double-copy of form factor have gravity correspondence?

Gauge-gravity duality

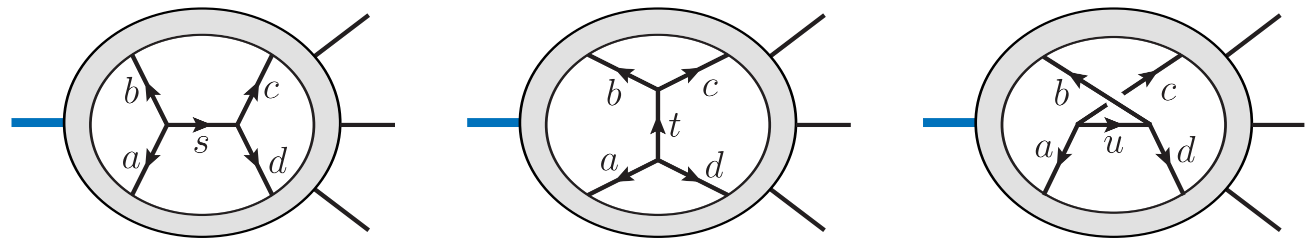


Understand better the duality and quantum theory of gravity

Thank you!

Extra slides

Generalized gauge transformation



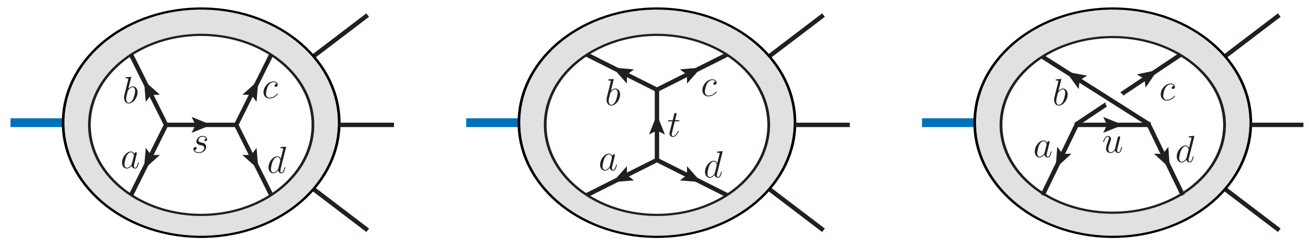
Generalized GT: $N_s \to N_s + s\Delta, \quad N_t \to N_t + t\Delta, \quad N_u \to N_u + u\Delta$

The result does not change since $C_s + C_t + C_u = 0$

They usually break CK duality: $s + t + u \neq 0$

New relations in form factors

Jacobi relation

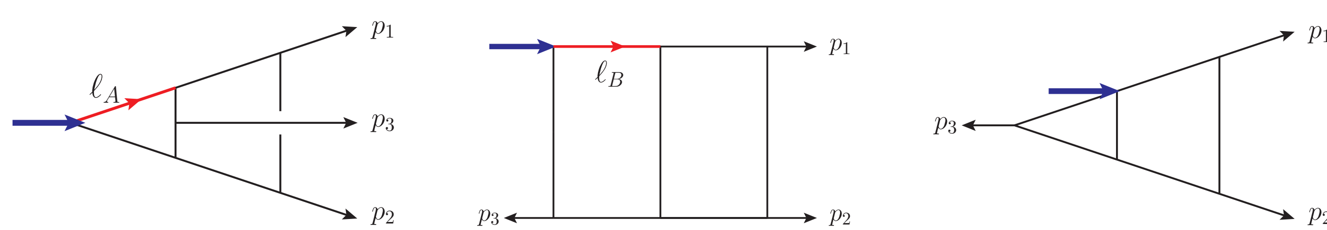


Generalized GT: $N_s \to N_s + s\Delta, \quad N_t \to N_t + t\Delta, \quad N_u \to N_u + u\Delta$

The result does not change since $C_s + C_t + C_u = 0$

Beyond Jacobi

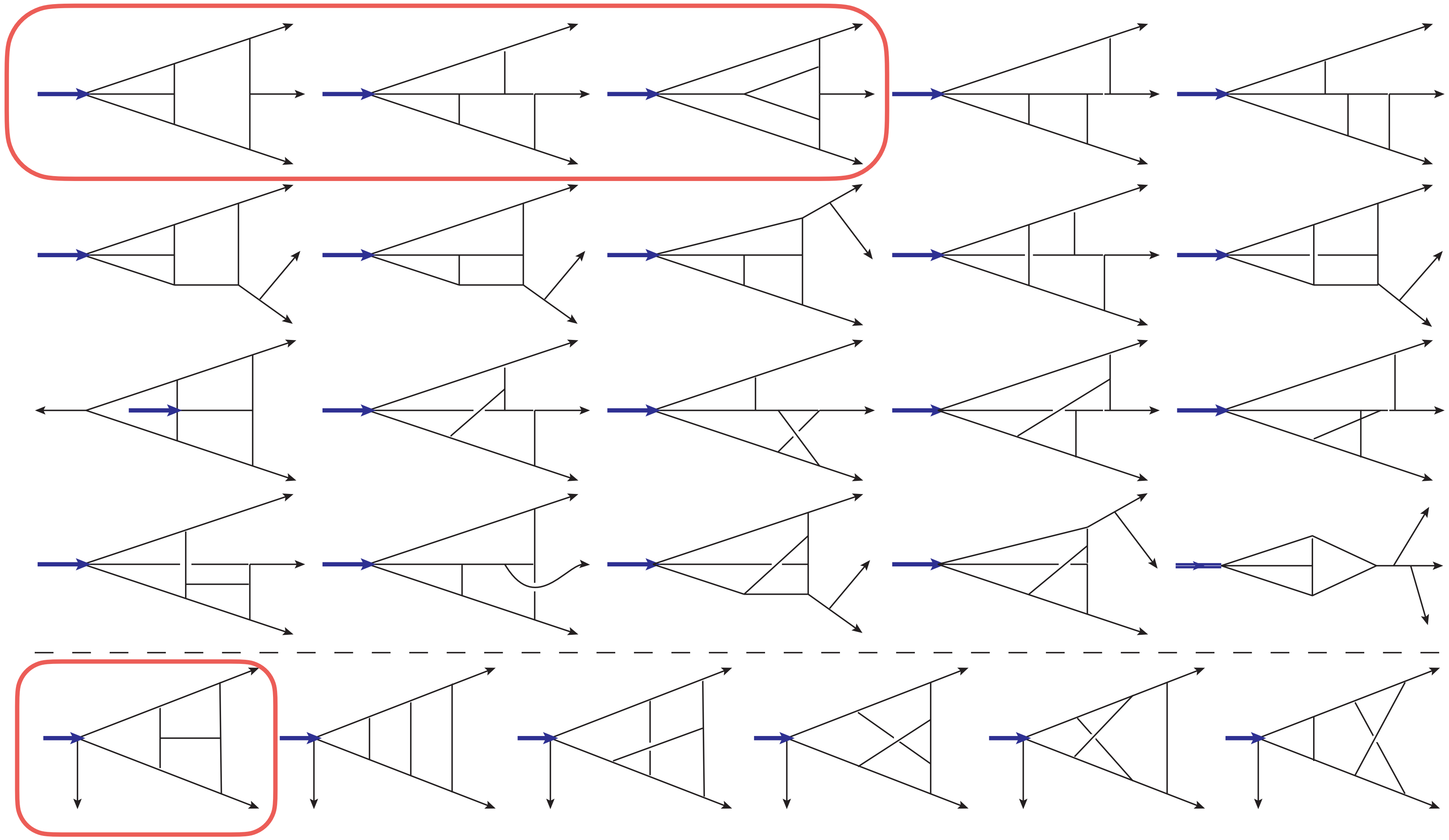
New in form factor



(a) (b) (c)

Generalized GT: $N_{\rm a} \to N_{\rm a} + \ell_A^2 \Delta, \quad N_{\rm b} \to N_{\rm b} - \ell_B^2 \Delta,$ with any Δ

The result does not change since $C_a + C_b = 0$



$\mathrm{tr}(\phi^3)$: 10 parameters

Unitarity

Unitarity $\longleftrightarrow$ **Conservation of probability**

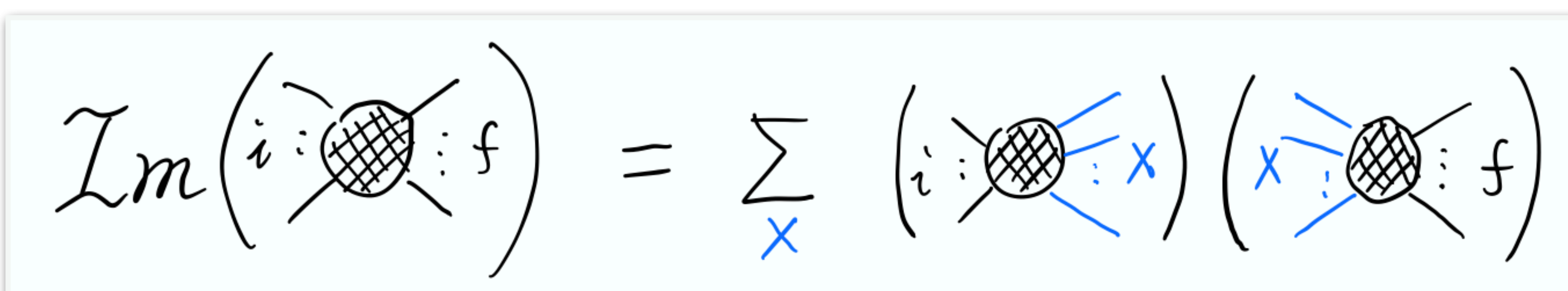
$$\sum_n |S_{m\to n}|^2 = 1 \qquad\qquad S^\dagger S = 1$$

Generalized optical theorem

$$S = 1 + iT \xrightarrow{S^\dagger S = 1} i(T^\dagger - T) = T^\dagger T$$

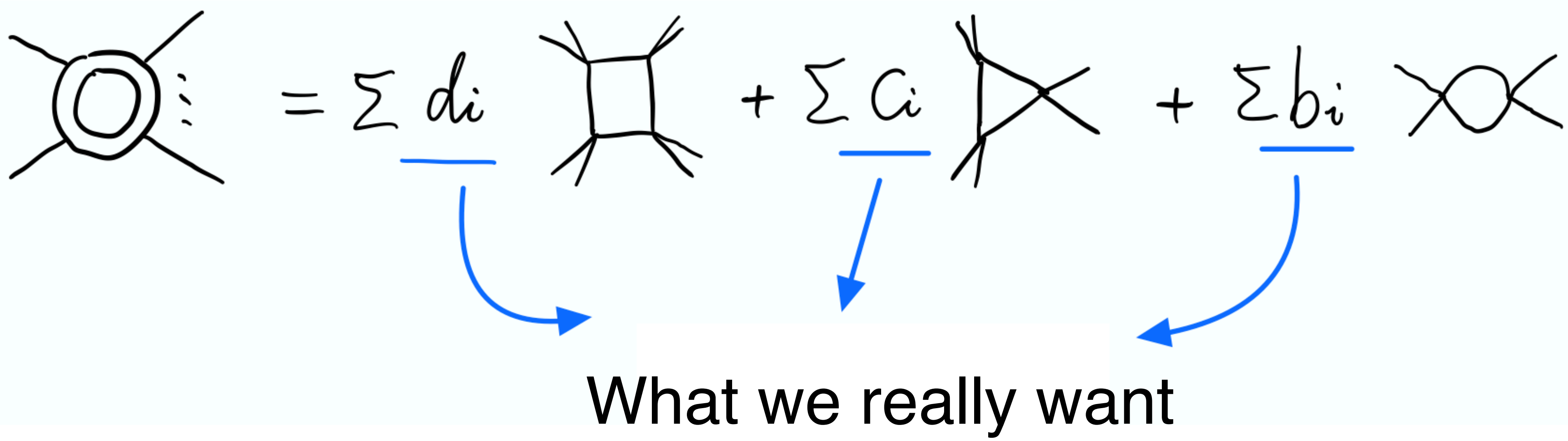
$$\langle f | T | i \rangle = A(i \to f) \qquad \sum_X |X\rangle\langle X| = 1$$

$$\mathrm{Im}\left[A(i \to f)\right] = \sum_X A^*(f \to X)\, A(i \to X)$$



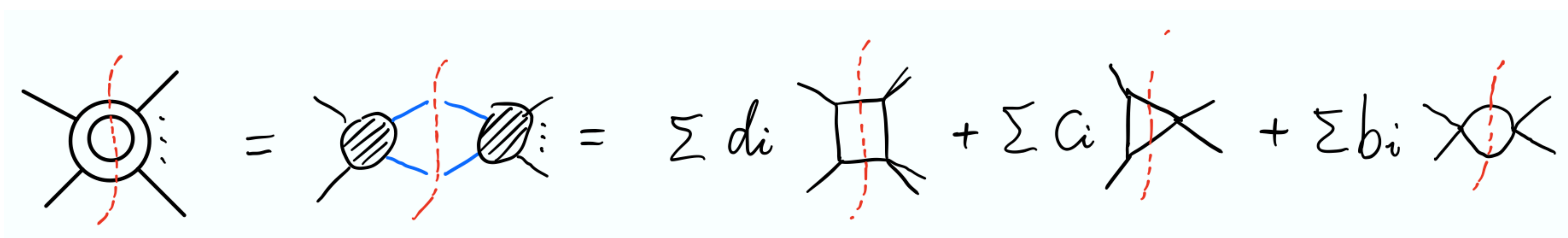
Unitarity cuts

Consider one-loop amplitudes:



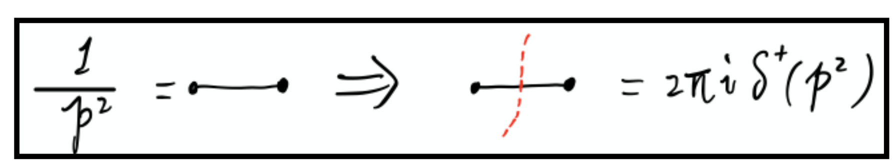
Unitarity cuts

We can perform unitarity cuts:



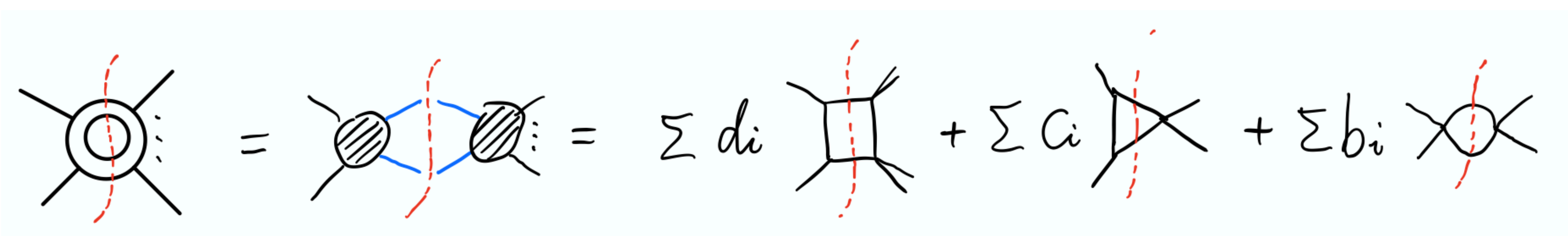
and from tree products, we derive the coefficients more directly.

Cutkosky cutting rule:

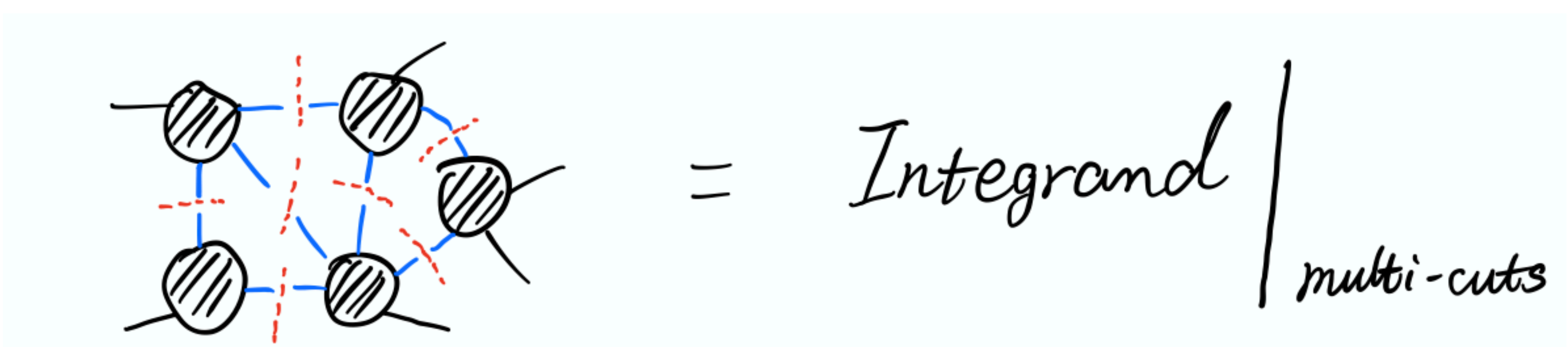


Unitarity cuts

We can perform unitarity cuts:



High-loop generalization:



Interpretation

Does the Jacobi relation for momentum have any physical meaning?

Self-dual YM/gravity

[Monteiro, O'Connell 2011]

$$F_{\mu\nu} = \frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$$

$$R_{\mu\nu\lambda\delta} = \frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}R^{\rho\sigma}{}_{\lambda\delta}$$

$$\Phi^a(k) = \frac{1}{2}\,g\int đp_1 đp_2\,\frac{F_{p_1p_2}{}^k\, f^{b_1b_2a}}{k^2}\,\Phi^{b_1}(p_1)\Phi^{b_2}(p_2)$$

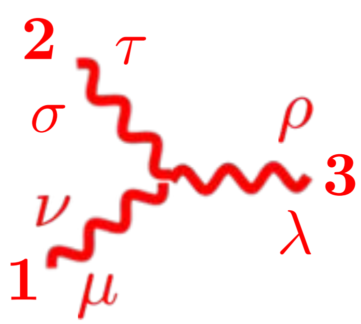
$$\phi(k) = \frac{1}{2}\,\kappa\int đp_1 đp_2\,\frac{X(p_1,p_2)F_{p_1p_2}{}^k}{k^2}\,\phi(p_1)\phi(p_2)$$

$$L_k = e^{-ik\cdot x}(-k_w\partial_u + k_u\partial_w)$$

$$[L_{p_1}, L_{p_2}] = iX(p_1,p_2)L_{p_1+p_2} = iF_{p_1p_2}{}^k L_k$$

Dual Jacobi relation can be understood from the algebra of area-preserving diffeomorphism.

Feynman diagram?



more than 100 terms

[DeWitt 1967]

$$
\begin{aligned}
&\frac{\delta^3 S}{\delta\varphi_{\mu\nu}\delta\varphi_{\sigma\tau}\delta\varphi_{\rho\lambda}} \to 2\eta^{\mu\tau}\eta^{\nu\sigma}{k_1}^{\lambda}{k_1}^{\rho} + 2\eta^{\mu\sigma}\eta^{\nu\tau}{k_1}^{\lambda}{k_1}^{\rho} - 2\eta^{\mu\nu}\eta^{\sigma\tau}{k_1}^{\lambda}{k_1}^{\rho} + \\
&2\eta^{\lambda\tau}\eta^{\mu\nu}{k_1}^{\sigma}{k_1}^{\rho} + 2\eta^{\lambda\sigma}\eta^{\mu\nu}{k_1}^{\tau}{k_1}^{\rho} + \eta^{\mu\tau}\eta^{\nu\sigma}{k_2}^{\lambda}{k_1}^{\rho} + \eta^{\mu\sigma}\eta^{\nu\tau}{k_2}^{\lambda}{k_1}^{\rho} + \eta^{\lambda\tau}\eta^{\nu\sigma}{k_2}^{\mu}{k_1}^{\rho} + \\
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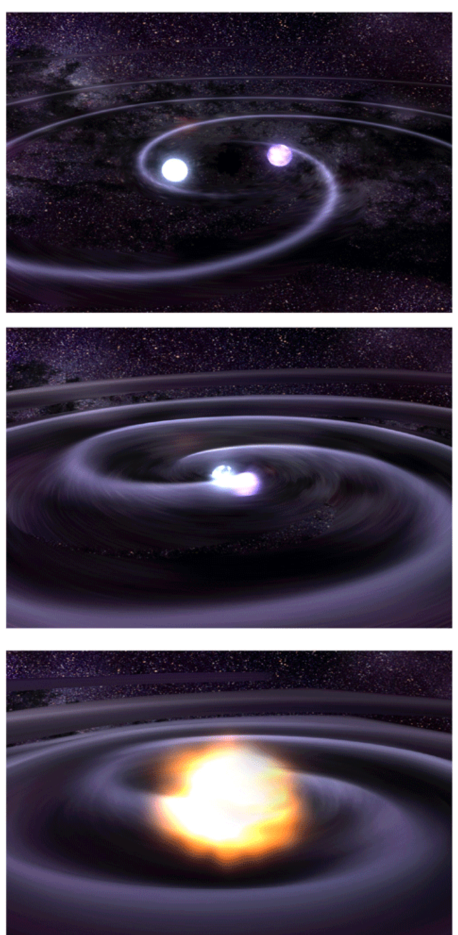
&2\eta^{\lambda\rho}\eta^{\mu\tau}\eta^{\nu\sigma}k_1\cdot k_3 - \eta^{\lambda\sigma}\eta^{\mu\rho}\eta^{\nu\tau}k_1\cdot k_3 + 2\eta^{\lambda\rho}\eta^{\mu\sigma}\eta^{\nu\tau}k_1\cdot k_3 + 2\eta^{\lambda\tau}\eta^{\mu\nu}\eta^{\rho\sigma}k_1\cdot k_3 - \\
&\eta^{\lambda\nu}\eta^{\mu\tau}\eta^{\rho\sigma}k_1\cdot k_3 - \eta^{\lambda\mu}\eta^{\nu\tau}\eta^{\rho\sigma}k_1\cdot k_3 + 2\eta^{\lambda\sigma}\eta^{\mu\nu}\eta^{\rho\tau}k_1\cdot k_3 - \eta^{\lambda\nu}\eta^{\mu\sigma}\eta^{\rho\tau}k_1\cdot k_3 - \\
&\eta^{\lambda\mu}\eta^{\nu\sigma}\eta^{\rho\tau}k_1\cdot k_3 - 2\eta^{\lambda\rho}\eta^{\mu\nu}\eta^{\sigma\tau}k_1\cdot k_3 + \eta^{\lambda\nu}\eta^{\mu\rho}\eta^{\sigma\tau}k_1\cdot k_3 + \eta^{\lambda\mu}\eta^{\nu\rho}\eta^{\sigma\tau}k_1\cdot k_3 - \\
&\eta^{\lambda\tau}\eta^{\mu\sigma}\eta^{\nu\rho}k_2\cdot k_3 - \eta^{\lambda\sigma}\eta^{\mu\tau}\eta^{\nu\rho}k_2\cdot k_3 - \eta^{\lambda\tau}\eta^{\mu\rho}\eta^{\nu\sigma}k_2\cdot k_3 + 2\eta^{\lambda\rho}\eta^{\mu\tau}\eta^{\nu\sigma}k_2\cdot k_3 - \\
&\eta^{\lambda\sigma}\eta^{\mu\rho}\eta^{\nu\tau}k_2\cdot k_3 + 2\eta^{\lambda\rho}\eta^{\mu\sigma}\eta^{\nu\tau}k_2\cdot k_3 + \eta^{\lambda\tau}\eta^{\mu\nu}\eta^{\rho\sigma}k_2\cdot k_3 - \eta^{\lambda\nu}\eta^{\mu\tau}\eta^{\rho\sigma}k_2\cdot k_3 - \\
&\eta^{\lambda\mu}\eta^{\nu\tau}\eta^{\rho\sigma}k_2\cdot k_3 + \eta^{\lambda\sigma}\eta^{\mu\nu}\eta^{\rho\tau}k_2\cdot k_3 - \eta^{\lambda\nu}\eta^{\mu\sigma}\eta^{\rho\tau}k_2\cdot k_3 - \eta^{\lambda\mu}\eta^{\nu\sigma}\eta^{\rho\tau}k_2\cdot k_3 - \\
&2\eta^{\lambda\rho}\eta^{\mu\nu}\eta^{\sigma\tau}k_2\cdot k_3 + 2\eta^{\lambda\nu}\eta^{\mu\rho}\eta^{\sigma\tau}k_2\cdot k_3 + 2\eta^{\lambda\mu}\eta^{\nu\rho}\eta^{\sigma\tau}k_2\cdot k_3
\end{aligned}
$$

Gravitational wave

Quantum amplitudes for classical gravity:

- PN and PM correction for classical potential
- Gravitational radiation



Point particle approximation

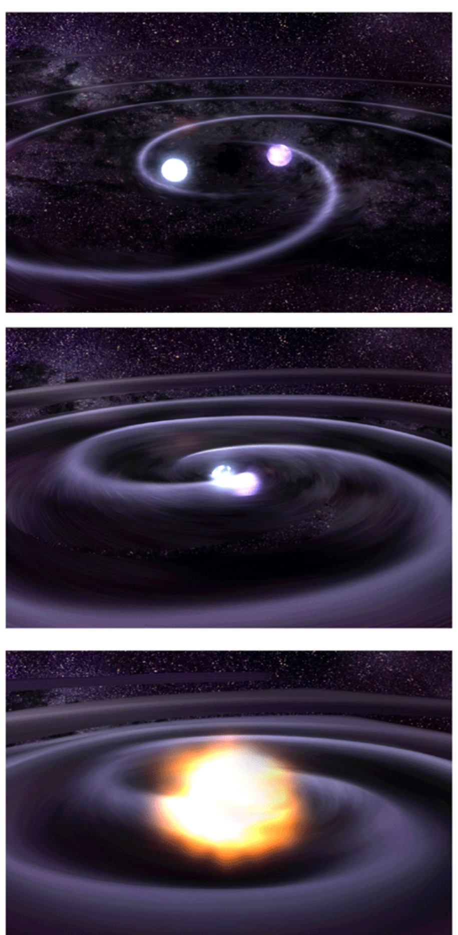
Classical limit

Classical potential or radiation

Gravitational wave

Quantum amplitudes for classical gravity:

- PN and PM correction for classical potential
- Gravitational radiation



First 3PM computation using amplitudes:

	0PN		1PN		2PN		3PN		4PN		5PN		6PN		7PN			
1PM	(1	+	v^2	+	v^4	+	v^6	+	v^8	+	v^{10}	+	v^{12}	+	v^{14}	+	$\dots)$	G^1
2PM			(1	+	v^2	+	v^4	+	v^6	+	v^8	+	v^{10}	+	v^{12}	+	$\dots)$	G^2
3PM					(1	+	v^2	+	v^4	+	v^6	+	v^8	+	v^{10}	+	$\dots)$	G^3
4PM							(1	+	v^2	+	v^4	+	v^6	+	v^8	+	$\dots)$	G^4
5PM									(1	+	v^2	+	v^4	+	v^6	+	$\dots)$	G^5
6PM											(1	+	v^2	+	v^4	+	$\dots)$	G^6
												⋮						

Z. Bern et.al, arXiv: 1908.01493

Interpretation

Does the Jacobi relation for momentum have any physical meaning?

Self-dual YM/gravity [Monteiro, O'Connell 2011]

$$F_{\mu\nu} = \frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$$

$$R_{\mu\nu\lambda\delta} = \frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}R^{\rho\sigma}{}_{\lambda\delta}$$

$$\Phi^a(k) = \frac{1}{2}\,g\int đp_1 đp_2\,\frac{F_{p_1p_2}{}^k\,f^{b_1b_2a}}{k^2}\,\Phi^{b_1}(p_1)\Phi^{b_2}(p_2)$$

$$\phi(k) = \frac{1}{2}\,\kappa\int đp_1 đp_2\,\frac{X(p_1,p_2)F_{p_1p_2}{}^k}{k^2}\,\phi(p_1)\phi(p_2)$$

$$L_k = e^{-ik\cdot x}(-k_w\partial_u + k_u\partial_w)$$

$$[L_{p_1}, L_{p_2}] = iX(p_1,p_2)L_{p_1+p_2} = iF_{p_1p_2}{}^k L_k$$

Dual Jacobi relation can be understood from the algebra of area-preserving diffeomorphism.

Classical solutions

Double copy for "Kerr-Schild" type solutions:

gravity

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$
$$\equiv \eta_{\mu\nu} + k_\mu k_\nu \phi$$

$$\eta^{\mu\nu} k_\mu k_\nu = 0 = g^{\mu\nu} k_\mu k_\nu$$

gauge

$$A^a_\mu = k_\mu \phi^a$$

Examples:

Schwarzschild BH ⟷ Coulomb particle

[Monteiro, O'Connell, White 2014]