Gedanken Experiments and the Weak Cosmic Censorship Conjecture

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Strings and Related Topics @ PCFT, USTC

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Cosmic Censorship Conjecture

Cosmic Censorship Conjecture (CCC):
was proposed to save the predictability of GR Penrose

- Weak CCC:
  Singularity should be hidden behind the future event horizon, so as not to influent the spacetime outside the black hole.
- Strong CCC:
  Cauchy horizon should be destroyed by perturbations, so that wormhole travelers would not be influenced by the timelike singularity.

Weak Cosmic Censorship Conjecture

- A rigorous proof of WCCC is still absent.
- Various counterexamples have been proposed, while not quite "physical". Choptuik, Christodoulou, ...
- Gedanken experiment: an operational demonstration

  Trying to destroy the horizon by throwing matter into extremal or near-extremal black holes. Wald *et al*

Gedanken experiment

Version I Wald 1974

4D extremal Kerr-Newman black hole

$$M^2 = (J/M)^2 + Q^2$$

Trying to overcharge or overspin it by dropping test particles:

- Particles with large q or j/m couldn't fall in due to electromagnetic or centrifugal repulsion.
- Spinning particles with large s/m couldn't fall in due to gravitational spin-spin repulsive force.

WCCC preserved.

Gedanken experiment

Version I

Reinterpretation: for RN black hole, test particle energy is bounded from below if it could enter the black hole

$$E \;=\; -(mu_a + qA_a)\xi^a \;\geq\; q\,\Phi_H \;=\; q$$

where $\Phi_H = 1$ is horizon electromagnetic potential for extremal RN black hole, hence

$$M + E \;\geq\; Q + q$$

Gedanken experiment

Challenge Hubeny 1999

For near-extremal RN black hole characterized by

$$\epsilon = \frac{\sqrt{M^2 - Q^2}}{M}$$

since $\Phi_H = Q/r_+ \approx 1 - \epsilon$ with $r_+ = M + \sqrt{M^2 - Q^2}$, one have $E \geq q(1 - \epsilon)$, hence

$$(M + E) - (Q + q) \approx -\epsilon q + \frac{M\epsilon^2}{2}$$

Seems WCCC is violated if $q \geq \epsilon M/2$?

- "Violation" occurs at $\mathcal{O}(q^2)$, test particle assumption is invalid, other effects such as electromagnetic and gravitational self-force might enter.

Gedanken experiment

Version II Sorce-Wald 2017

- For arbitrary matter that enters black hole, a complete analysis of contribution to BH mass was performed via Iyer-Wald formulation.
- Taking care of 2^{nd} order effects, the Hubeny-type "violation" would not occur, WCCC is preserved for Einstein-Maxwell theory given the *Null Energy Condition* for matter.

Gedanken experiment

Version II

Iyer-Wald formulation: establishes that BH entropy is the diffeomorphism Noether charge for the horizon-generating Killing field. Iyer-Wald 1994

$$\delta \mathbf{L} \;=\; \mathbf{E}(\phi)\delta\phi \,+\, \mathrm{d}\mathbf{\Theta}(\phi, \delta\phi)$$

from the surface term, one defines the conserved symplectic current

$$\mathbf{\Omega}(\phi, \delta_1\phi, \delta_2\phi) \;=\; \delta_1\mathbf{\Theta}(\phi, \delta_2\phi) \,-\, \delta_2\mathbf{\Theta}(\phi, \delta_1\phi)$$

For any vector ξ^a one can construct the associated Noether current

$$\mathbf{J}_\xi \;=\; \mathbf{\Theta}(\phi, \mathcal{L}_\xi\phi) \,-\, \mathrm{i}_\xi\mathbf{L}$$

which is closed and can be written as

$$\mathbf{J}_\xi \;=\; \mathrm{d}\mathbf{Q}_\xi \,+\, \xi^a\mathbf{C}_a$$

Gedanken experiment

Version II

Variation of Noether current gives rise to *linear variational identity* :

$$\int_{\partial\Sigma} \delta\mathbf{Q}_\xi - \mathrm{i}_\xi \mathbf{\Theta}(\phi, \delta\phi) = \int_\Sigma \mathbf{\Omega}(\phi, \delta\phi, \mathcal{L}_\xi \phi) - \int_\Sigma \delta\mathbf{C}_\xi - \int_\Sigma \mathrm{i}_\xi (\mathbf{E}\,\delta\phi)$$

Assuming on-shell and ξ is Killing vector $\partial_t + \Omega_{\mathrm{H}}\partial_\varphi$, the ADM mass, angular momentum and entropy for black hole are identified as

$$\begin{aligned} \mathcal{M} &\equiv \int_\infty \delta\mathbf{Q}_t - \mathrm{i}_t \mathbf{\Theta}(\phi, \delta\phi) \\ \mathcal{J} &\equiv \int_\infty \delta\mathbf{Q}_\varphi - \mathrm{i}_\varphi \mathbf{\Theta}(\phi, \delta\phi) \\ T_{\mathrm{H}}\delta S &\equiv \int_B \delta\mathbf{Q}_\xi - \mathrm{i}_\xi \mathbf{\Theta}(\phi, \delta\phi) \end{aligned}$$

Linear variational identity reduce to

$$\delta\mathcal{M} - \Omega_{\mathrm{H}}\delta\mathcal{J} - T_{\mathrm{H}}\,\delta S = -\int_\Sigma \delta\mathbf{C}_\xi \tag{1}$$

Gedanken experiment

Version II

- Constraints turn out to be always proportional to EOMs.
- Even (1) was derived without matter, since linearized EOMs are not enforced, one could simply replace $\delta \mathbf{C}_\xi$ with matter perturbation so that (1) is also valid in presence of matter.
- From now on only consider $\delta\phi$ vanishing near internal boundary of Σ, then (1) with $T_{\mathrm{H}}\delta S = 0$ holds for both extremal and non-extremal BHs.

Gedanken experiment

Version II

A second variation of linear variational identity gives

$$\mathcal{E}_\Sigma(\phi;\delta\phi) = \int_{\partial\Sigma} \left[\delta^2 \mathbf{Q}_\xi - \mathrm{i}_\xi \delta\mathbf{\Theta}(\phi,\delta\phi)\right] + \int_\Sigma \delta^2 \mathbf{C}_\xi + \int_\Sigma \mathrm{i}_\xi \left(\delta\mathbf{E} \wedge \delta\phi\right) \tag{2}$$

in which Wald's *canonical energy*

$$\mathcal{E}_\Sigma(\phi;\delta\phi) \equiv \int_\Sigma \mathbf{\Omega}(\phi,\delta\phi,\mathcal{L}_\xi\delta\phi)\,.$$

- Using the energy constraint given by identity (1)(2), Sorce-Wald showed both *extremal* and *near-extremal* black holes could not be overcharged or overspun, up to 2^{nd} order perturbation of matter.

Beyond Einstein-Maxwell theory ?

4D higher derivative gravity

- Quantum corrections could leave low-energy relics in the form of higher-order derivative terms beyond Einstein-Maxwell.

- Regard WCCC as a physical principle to constrain the higher-order EFTs.

$$I = \int d^4x \sqrt{-g} \left(\frac{1}{2\kappa} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \Delta L \right) ,$$

where

$$\begin{aligned} \Delta L \;\; = \;\; & c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} + c_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \\ & + c_4 R F_{\mu\nu} F^{\mu\nu} + c_5 R_{\mu\nu} F^{\mu\rho} F^{\nu}{}_{\rho} + c_6 R_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \\ & + c_7 F_{\mu\nu} F^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} + c_8 F_{\mu\nu} F^{\nu\rho} F_{\rho\sigma} F^{\sigma\mu} \, . \end{aligned}$$

Charged black holes

Charged non-spinning solutions (linear correction to RN black hole):

Motl et al, 2007

$$
\begin{aligned}
-g_{tt} \;=\; & 1-\frac{\kappa m}{r}+\frac{\kappa q^2}{2r^2}+c_2\left(\frac{\kappa^3 m q^2}{r^5}-\frac{\kappa^3 q^4}{5r^6}-\frac{2\kappa^2 q^2}{r^4}\right) \\
& +c_3\left(\frac{4\kappa^3 m q^2}{r^5}-\frac{4\kappa^3 q^4}{5r^6}-\frac{8\kappa^2 q^2}{r^4}\right)+c_4\left(-\frac{6\kappa^2 m q^2}{r^5}+\frac{4\kappa^2 q^4}{r^6}+\frac{4\kappa q^2}{r^4}\right) \\
& +c_5\left(\frac{4\kappa^2 q^4}{5r^6}-\frac{\kappa^2 m q^2}{r^5}\right)+c_6\left(\frac{\kappa^2 m q^2}{r^5}-\frac{\kappa^2 q^4}{5r^6}-\frac{2\kappa q^2}{r^4}\right) \\
& +c_7\left(-\frac{4\kappa q^4}{5r^6}\right)+c_8\left(-\frac{2\kappa q^4}{5r^6}\right)+\mathcal{O}(c_i^2)
\end{aligned}
$$

with gauge potential [Credit: Jiang et al, 2101.10172]

$$
A_t \;=\; -\frac{q}{r}-\frac{q^3}{5r^5}\times\left[c_2\kappa^2+4c_3\kappa^2+10c_4\kappa+c_5\kappa-c_6\kappa\left(9-\frac{10mr}{q^2}\right)-16c_7-8c_8\right]
$$

Charged black holes

- Singularity will be hidden by a horizon if (absorbing the correction to metric function as mass shift)

$$m \geq \sqrt{\frac{2}{\kappa}}|q|\left(1 - \frac{4}{5q^2}c_0\right)\,, \qquad c_0 \,\equiv\, c_2 + 4c_3 + \frac{c_5}{\kappa} + \frac{c_6}{\kappa} + \frac{4c_7}{\kappa^2} + \frac{2c_8}{\kappa^2}$$

- For extremal solution, location of degenerate horizon ($g_{tt} = 0$ and $\mathrm{d}g_{tt}/\mathrm{d}r = 0$)

$$r_{\mathrm{H}}^{\mathrm{ext}} \;=\; \frac{m\,\kappa}{2} + \frac{4}{5m}\left(c_2 + 4c_3 + \frac{10c_4 + c_5 + c_6}{\kappa} - \frac{16c_7 + 8c_8}{\kappa^2}\right)$$

Electrostatic potential on extremal horizon

$$\Phi_{\mathrm{H}}^{\mathrm{ext}} = -\,(\xi^a A_a)\,|_{\mathcal{H}} = \sqrt{\frac{2}{\kappa}}\left(1 + \frac{4c_0'}{5q^2}\right)\,, \quad c_0' = c_2 + 4c_3 + \frac{c_5}{\kappa} + \frac{c_6}{\kappa} + \frac{4c_7}{\kappa^2} + \frac{2c_8}{\kappa^2} = c_0$$

Gedanken experiments and WCCC

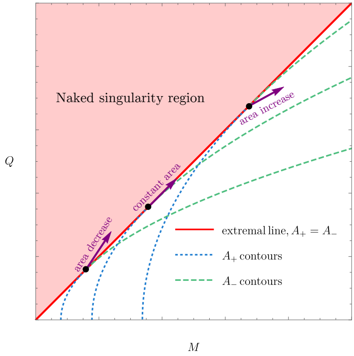
Consider throwing matter into an extremal black hole.

- Assuming *stability* of our family of solutions: in-falling matter finally turns original extremal BH into a one-parameter family solutions $(m(w), q(w))$.
- At first order in w , WCCC holds only if

$$\delta m - \sqrt{\frac{2}{\kappa}}\left(1 + \frac{4c_0}{5q^2}\right)\delta q \geq 0 \tag{3}$$

Gedanken experiments and WCCC

It turns out (3) coincides with the the requirement that horizon area must increase as matter fall into extremal black holes.



Extremality contour & constant area contours

Gedanken experiments: Test particle

For regular solution (m, q), consider a test particle with mass δm_0 and charge δq_0 falling in from infinity with action

$$S_p = 4\pi \int d\tau \, (\delta m_0 - \delta q_0 u_a A^a)$$

canonical momentum $p^a = \delta m_0 \, u^a - \delta q_0 \, A^a$ is conserved along trajectory, applying $\xi_a p^a =$ const to infinity and horizon,

$$\delta m_0 \left(u_a^{\mathrm{H}} \xi^a \right) - \Phi_{\mathrm{H}}^c \delta q_0 = \delta m_0 \, (u_a^{\infty} \xi^a) = -\delta E_\infty$$

Final space-time is parameterized by $(m + \delta m, q + \delta q)$. Conservation of charge and ADM mass (gravitational radiation neglectable) gives $\delta q = \delta q_0$, $\delta m = \delta E_\infty$, hence

$$\delta m - \Phi_{\mathrm{H}}^c \delta q = -\delta m_0 \left(u_a^{\mathrm{H}} \xi^a \right) \geq 0$$

or

$$\delta m \geq \sqrt{\frac{2}{\kappa}} \left(1 + \frac{4c_0'}{5q^2} \right) \delta q \qquad (4)$$

Gedanken experiments: Sorce-Wald method

Using Iyer-Wald formulation to derive the "law of energy conservation" for in-falling process, identify the variation of constraint with matter perturbation

$$(\delta \mathbf{C}_a)_{bcd} \;=\; \boldsymbol{\epsilon}_{ebcd}\,(\delta T^e{}_a + A_a\,\delta j^e)$$

Linear variational identity (1) gives

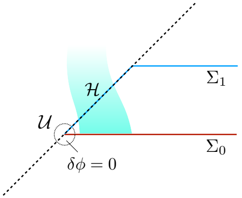
$$\delta \mathcal{M} \;=\; -\int_\Sigma \boldsymbol{\epsilon}_{abcd}\,\xi^a\,(\delta T^e{}_a + A_a\,\delta j^e)$$

- δj^e, $\delta T^e{}_a$: associated current and stress tensor of in-falling matter passing through hypersurface Σ (chosen to be $\mathcal{H} \cup \Sigma_1$)

Gedanken experiments: Sorce-Wald method

Assuming all the matter fall into black hole far earlier than joint moment of $\mathcal{H}$ and Σ_1, replace integral on Σ_1 with $\mathcal{H}$ and obtain

$$\delta\mathcal{M} - \Phi_H \int_{\mathcal{H}} \epsilon_{abcd}\, \delta j^a = -\int_{\mathcal{H}} \epsilon_{ebcd}\, \xi^a \delta T^e{}_a$$



- on horizon $\xi^a \propto n^a$, RHS is non-negative due to *null energy condition*
- charge crossing the horizon: $\delta\mathcal{Q} = \int_{\mathcal{H}} \epsilon_{abcd}\, \delta j^a$

hence

$$\delta\mathcal{M} - \Phi^c_H \delta\mathcal{Q} \geq 0$$

Gedanken experiments: Sorce-Wald method

Explicit form of $\mathbf{Q}_\xi$:

$$(\mathbf{Q}_\xi)_{c_3 c_4} = \boldsymbol{\epsilon}_{abc_3 c_4} \left(M^{abc}\, \xi_c - E^{abcd}\, \nabla_{[c}\, \xi_{d]} \right)$$

with

$$\begin{aligned} M^{abc} &\equiv -2\nabla_d E^{abcd} + E_F^{ab} A^c \\ E^{abcd} &\equiv \frac{\delta L}{\delta R_{abcd}} \\ E_F^{ab} &\equiv \frac{\delta L}{\delta F_{ab}} \end{aligned}$$

Gedanken experiments: Sorce-Wald method

$\delta\mathcal{M}$ and $\delta\mathcal{Q}$ for black hole in higher theory might not be the same as the ones for RN black hole, i.e., δM and δQ .

- Corrections to $\mathbf{Q}_\xi$ due to the higher-dimension Lagrangian ΔL fall off too quickly to contribute to the ADM mass, hence $\delta\mathcal{M} = \delta M$.
- Straightforward calculation using $\mathbf{C}_a$ gives $\delta\mathcal{Q} = \delta Q + \mathcal{O}(c_i^2)$.

Hence linear variational identity gives

$$\delta m \geq \sqrt{\frac{2}{\kappa}}\left(1 + \frac{4c_0'}{5q^2}\right)\delta q \tag{4}$$

Conclusion

- Since $c_0 = c_0'$, constraint obtained from "law of energy conservation" (4) coincides with WCCC holding condition (3), i.e., WCCC always be valid for in-falling matter satisfying NEC at linear order.

- For near-extremal black holes, a check of WCCC at 2^{nd} order is required. (Work in progress...)

Discussion

In fact, one can prove that WCCC is preserved for nonrotating extremal BHs in all n-dim. diffeomorphism-covariant theories of gravity and U(1) gauge field.

Condition for extremal solution not becoming singular is (generalization of (3), if assume $(dM/dQ)_{\mathrm{ext}} > 0$ and non-extremal BHs have $M > M_{\mathrm{ext}}(Q)$)

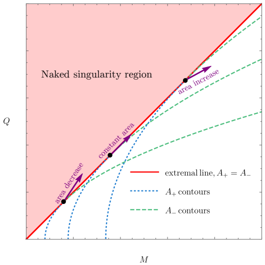
$$\delta M - \left(\frac{dM}{dQ}\right)_{\mathrm{ext}} \delta Q \;\geq\; 0$$

According to 1st law $\delta M = T\delta S + \Phi_{\mathrm{H}}\delta Q$,
for extremal BH with $T = 0$,

$$\left(\frac{dM}{dQ}\right)_{\mathrm{ext}} \;=\; \Phi_{\mathrm{H}}$$

coincides with constraint from variational identity

$$\delta M - \Phi_{\mathrm{H}}\delta Q \;\geq\; 0$$



Discussion

Weak gravity conjecture (WGC): Vafa et al, 2005-2006

Finiteness of the number of stable particles not protected by symmetry

- $M/|Q| < 1$ due to corrections from higher dimension operators

  Motl et al, 2006

$$c_2 + 4c_3 + \frac{c_5}{\kappa} + \frac{c_6}{\kappa} + \frac{4c_7}{\kappa^2} + \frac{2c_8}{\kappa^2} \geq 0$$

- Shift of thermodynamic entropy of extremal black hole due to higher-dimension operators must be positive gives the same constraint. Cheung et al,2018

It would be interesting to compare the 2nd order result from WCCC with this WGC bound...

THANK YOU !