Holographic Complexity from ~~Second~~ Group Cohomology.

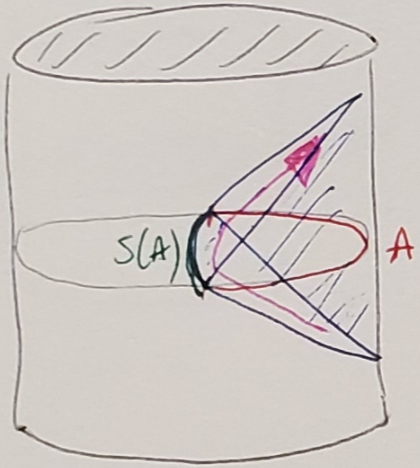
w/ J. de Boer
L. Lamprou
Z. Wang
Bowen Chen

w/ Gabriel Wong
Robert Raussendorf

THANKS TO:
Janet Hung

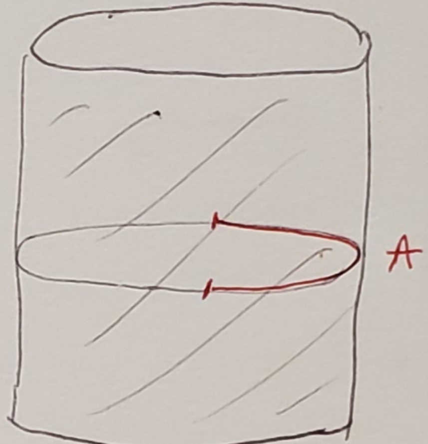
AdS$_3$/CFT$_2$

AdS



$\rightleftarrows$

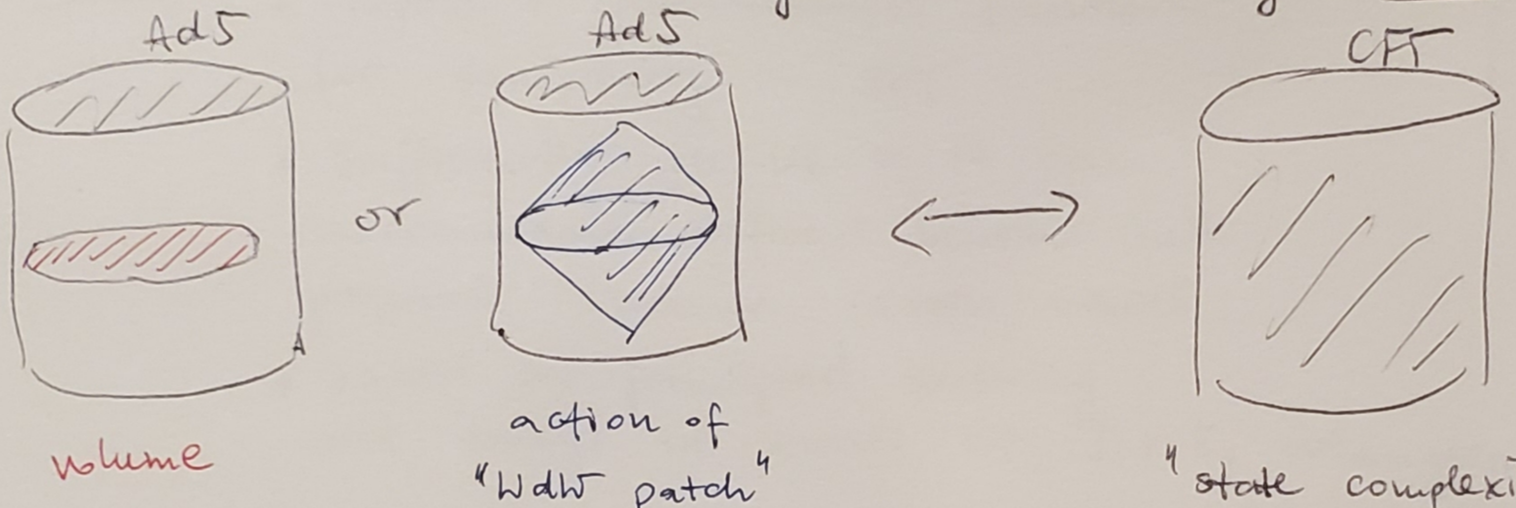
CFT



geometry of spacetime	$\longleftrightarrow$	geometry of information	
pure AdS	$\longleftrightarrow$	$\lvert 0\rangle_{CFT}$	
"entanglement wedge of A"	$\longleftrightarrow$	$\rho_A = \mathrm{tr}_{\bar{A}} \lvert 0\rangle\langle 0\rvert$	• BC et al, 2012 • Almheiri, Dong, Harlow 2016
minimal / extremal area	$\longleftrightarrow$	$S(A) = -\mathrm{tr}\,\rho_A \log \rho_A$	• Ryu-Takayanagi, 2006
$\perp$ boost generator	$\longleftrightarrow$	$\rho_A = e^{-H_{mod}}$	• JLMS, 2015

etc...

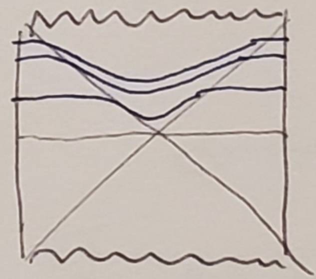
This talk is about another entry in this dictionary: <u>complexity</u>.



Susskind, 2014 + 2015

<u>Motivation</u>:

- mostly from Black Holes
- you can make interesting combinations of these "shockwave geometries" and the relation persists



longer and longer
$\sim$
"complexity" of e^{-iHt} in CFT
proportional to t

<u>BUT</u>: (1) What is complexity in the CFT? ← many proposals...
(2) Even in the bulk:

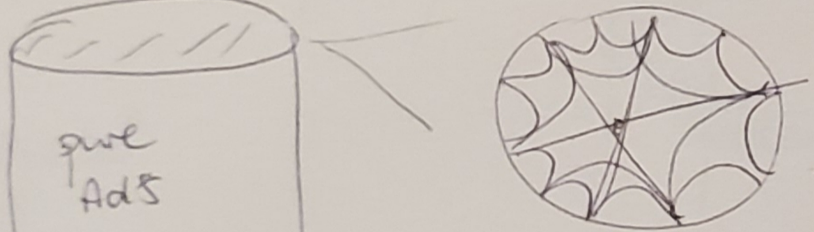
$$\text{complexity} = \#\,\text{volume} + \#\,\text{action} + \#\,\text{volume}^2$$
$$+ \text{ exotic combinations?}$$

Goal:

- Propose a fundamental framework for complexity in CFT
- Sufficiently flexible so it can accommodate different sensible bulk proposals: volume, action, others
- based on principled reasoning and things we know: RT, JLMS, entanglement wedges...

$\Big\}\ \mathcal{H}^2(G, \mathbb{R})$

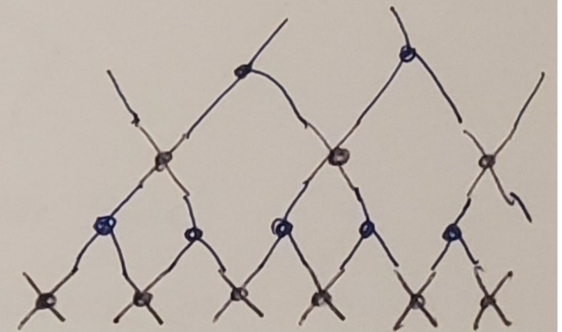
Extra heuristic: TENSOR NETWORKS



hyperbolic disk

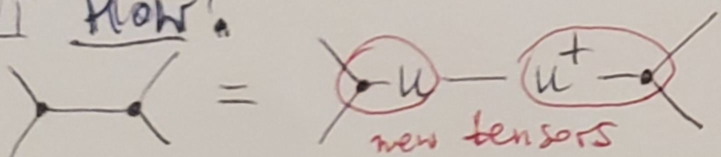
$\longleftrightarrow \quad |0\rangle \in \mathcal{H}_{CFT}$

Swingle, 2009: SIMILAR!



"MERA" tensor network
Vidal, 2005

- if space ~ ~~tensore~~ network then:
- complexity ~ volume ~ counting tensors.
- BUT HOW?



"gauge freedom" in tensor network ~ diffeomorphism

 $\longleftrightarrow$ tensor with $n=4$ indices

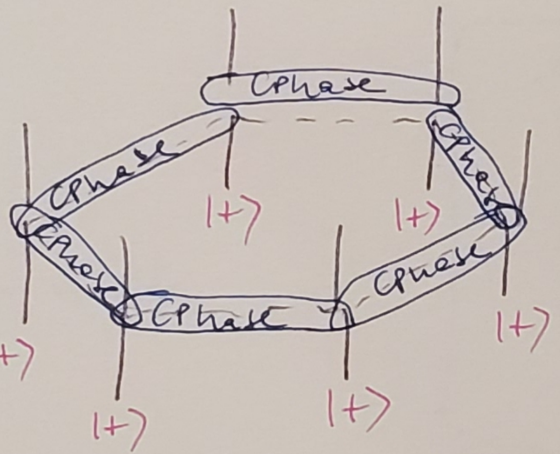
 $\longleftrightarrow$ index contraction

<u>Plan:</u> • I will tell you about a different system, which is understood in terms of $H^2(G, U(1))$ ▷ "cluster state" $|\phi\rangle$ in SYMMETRY-PROTECTED TOPOLOGICALLY (SPT) ORDERED PHASE

• We'll see that the relevance of group cohomology to the cluster state has the same heuristics as complexity in holography

$$\boxed{|\phi\rangle \in (\text{qubit})^{\otimes 6}}$$

$$X|+\rangle = |*\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$



$= |\phi\rangle$

"cluster state"

"CPhase"

$\updownarrow$

2-qubit gate, which is in Z-basis $\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&-1\end{pmatrix}$

- c.f. resource for MEASUREMENT-BASED QUANTUM COMPUTATION (MBQC), Raussendorf 2001+
- SPT order
- preparation of $|\phi\rangle$ is like a layer of MERA....

Need to understand the cluster state in a better way:

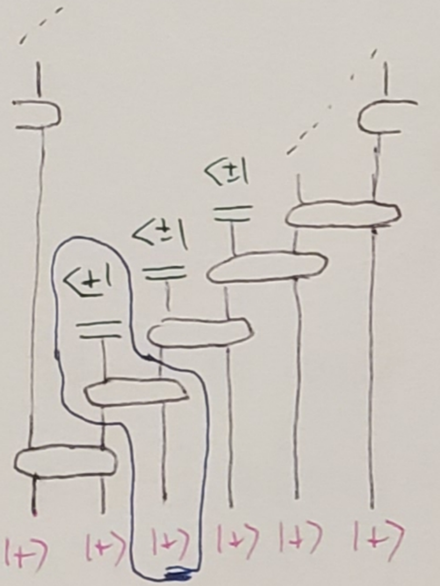
CHEAT SHEET!

$$H = \frac{X+Z}{\sqrt{2}}$$

$$H^2 = 1$$

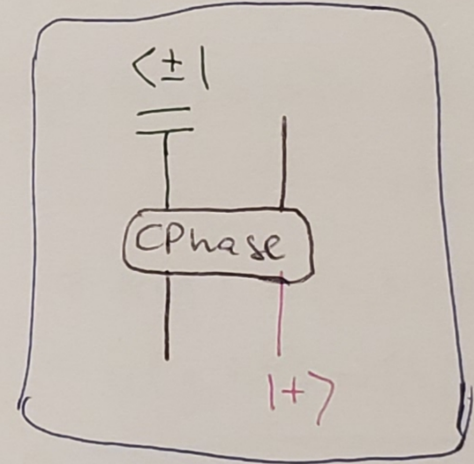
$$HZH = X$$

$$HXH = Z$$



$$= \langle \pm\pm \cdots \pm | \phi \rangle$$

This basic unit repeats itself.



$$= A[\pm]$$

$$\boxed{\begin{aligned} A[+] &= H \\ A[-] &= HZ \end{aligned}}$$

"HALF-TELEPORTATION"

$$\boxed{|\phi\rangle = \sum_{\pm, \ldots} \mathrm{tr}\Big(A[\pm]\, A[\pm]\, A[\pm]\, A[\pm]\, A[\pm]\, A[\pm]\Big) \, |\pm\pm\pm\pm\pm\pm\rangle}$$

$|\pm \cdots \pm\rangle$ has non-vanishing amplitude if $A[\pm] \cdots A[\pm]$ has non-vanishing trace.

$$A[+]A[+] \equiv A[++] = \mathbb{1}$$
$$A[+]A[-] = A[+-] = Z$$
$$A[-]A[+] = A[-+] = X$$
$$A[-]A[-] = A[--] = XZ$$

elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$

What is their algebra?

$$A[--]\,A[--] = -A[++]$$
$$A[+-]\,A[-+] = -A[--]$$
$$A[+-]\,A[--] = -A[-+]$$

all others like $\mathbb{Z}_2 \times \mathbb{Z}_2$ group multiplication

$\therefore$ this is a <u>projective</u> representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$:

$$A(g)\,A(h) = \omega(g,h)\,A(g,h)$$

- the $\omega(g,h)$ cannot be chosen at will

$$A(g)\,A(h)\,A(k) = \omega(g,h)\,A(gh)\,A(k)$$
$$= \omega(g,h)\,\omega(gh,k)\,A(ghk)$$
$$= A(g)\,\omega(h,k)\,A(hk)$$
$$= \omega(g,hk)\,\omega(h,k)\,A(ghk)$$

$$\Rightarrow \boxed{\omega(g,h)\,\omega(gh,k) = \omega(g,hk)\,\omega(h,k)}$$

2-COCYCLE CONDITION

$\omega \in H^2(G, U(1))$ where $G = \mathbb{Z}_2 \times \mathbb{Z}_2$

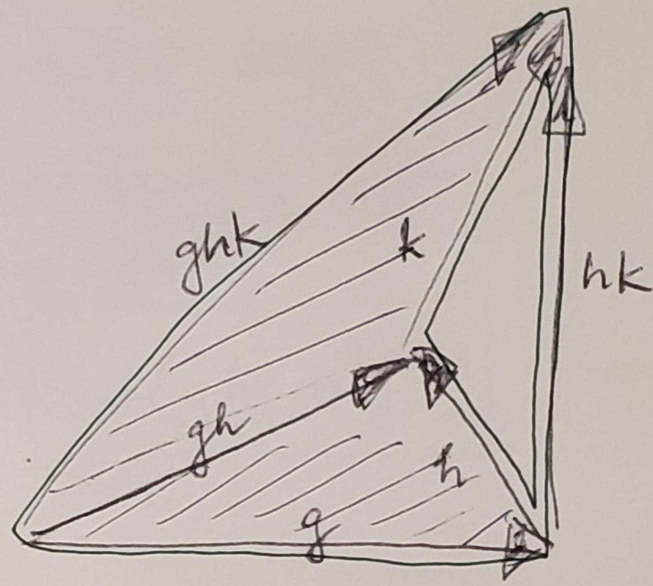
Xie Chen, Xiaogang Wen et al. 2011 $\rightarrow$

- projective representations of G
- SPT phases

$\Big\}$ classified by $H^2(G, U(1))$

WHY "CO-CYCLE"?

- think of $\omega(g,h)$ as "flux" through a side of this tetrahedron



TETRAHEDRON

$$\omega(g,h)\,\omega(gh,k) = \omega(g,hk)\,\omega(h,k)$$

FLUX IN = FLUX OUT

- no "source" inside tetrahedron
- ω (flux) is a "harmonic function" over simplices of G-elements

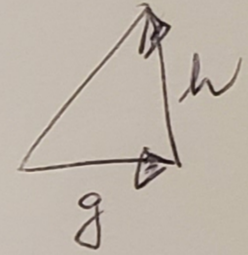
$$\Rightarrow \boxed{\omega \in H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1))}$$

We need one last thing about the cluster state:

- It can be prepared by a Hartle-Hawking type path integral of a topological field theory whose action is:

$$\boxed{e^{iS} = \omega(g,h)\ \omega(gh, h^{-1}g^{-1})}$$

over



- Let's calculate it:

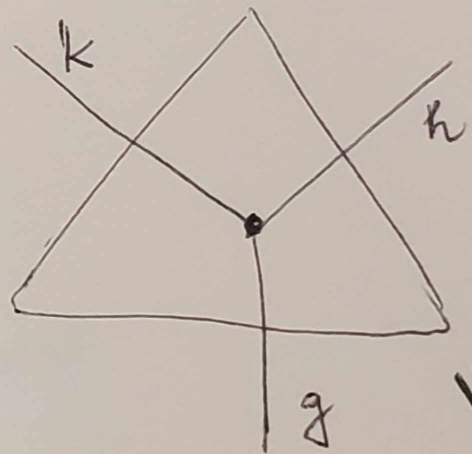
$$|\Phi\rangle = \sum_{g,h,k \in \mathbb{Z}_2\times\mathbb{Z}_2} \mathrm{tr}\left(A[g]A[h]A[k]\right)|g,h,k\rangle$$

$$= \sum_{g,h,k} \mathrm{tr}\left(A[gh]\,A[k]\right) \omega(g,h)\,|g,h,k\rangle$$

$$= \sum_{g,h,k} \omega(g,h)\,\omega(gh,k)\ \underbrace{\mathrm{tr}\,A[ghk]}_{\text{sets } ghk=1}\ |g,h,k\rangle$$

$$= \sum_{g,h} \omega(g,h)\,\omega(gh,h^{-1}g^{-1})\ |g,h,(gh)^{-1}\rangle$$

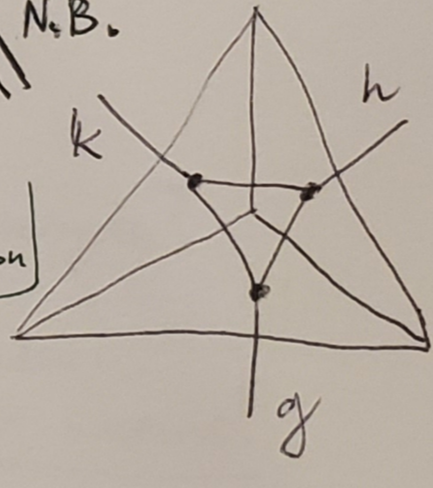
$$\text{"="} \int \mathcal{D}[g(\text{edge})]\ e^{iS}\ |g(\text{edge})\rangle = \text{"HH state"}$$



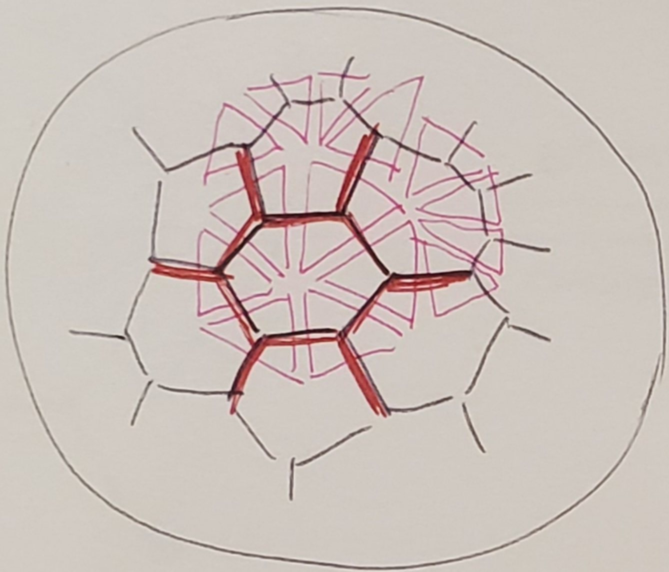
$g,h,k = \begin{matrix}++\\+-\\-+\\--\end{matrix}$

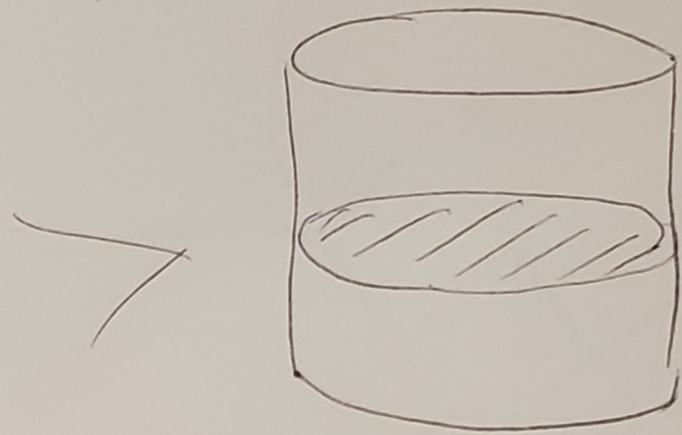
(independent of triangulation (TQFT) because of cocycle condition)

|| N.B.



FINALLY BACK TO HOLOGRAPHY:





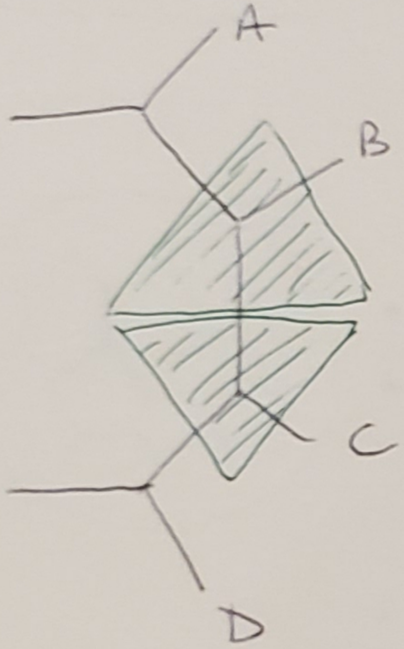
- a network with trivalent vertices
- dual to a triangulation

- a state on the red lines ⟷ at a cutoff is like our cluster state

- complexity ~ action (Susskind)
  action ~ cocycle (cluster state) ⟹ complexity ~ cocycle?

- "cocycle" ⟷ independent of triangulation ⟹ independence of RG scheme!

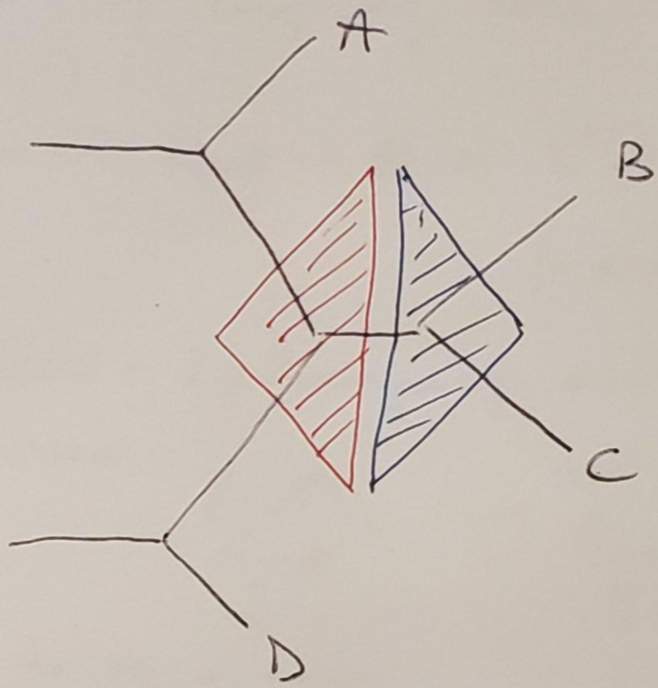
LOCAL RG:



BY COCYCLE

=

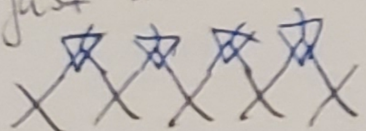
CONDITION

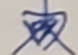


FINER CUTOFF: A, B, C, D

 $\in$ state complexity

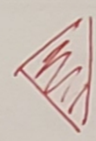
Just like in MERA:



 also perform $\mathcal{H}$-coarse graining

$\Rightarrow$ cohomology complexity $\sim$ counting tensors

COARSER CUTOFF: A, BC, D

 $\in$ state complexity

 $\leftrightarrow$ $\mathcal{H}$ coarse-graining

$B \otimes C \to BC$

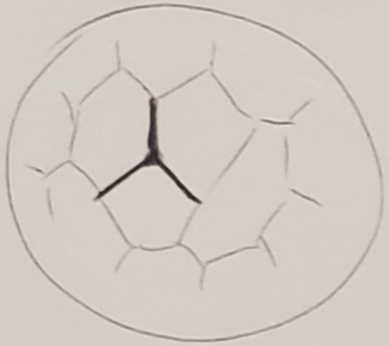
 represents the "complexity of RG"

represents difference of complexities at distinct cutoffs

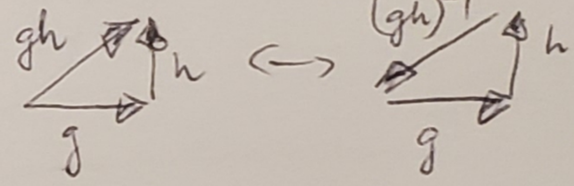
I promised: COHOMOLOGY gives a notion of complexity with

- the right heuristics ✓
- with rules! ? ← what is G in $H^2(G, \mathbb{R})$

for $\mathbb{R}$-valued complexity



- junction of bulk line segments - Ryu-Takayanagi surfaces
- we want a flat g-connection where g is associated to Ryu-Takayanagi surfaces



- it's natural to take G to be a group that can map any entanglement wedge to any other in CFT: $\boxed{\rho_A \xrightarrow{G} \rho_B \quad \forall A, B}$
- I presented a construction of such a g from modular parallel transport in ~~previous~~ talks elsewhere

← relies on $H_{mod} \xleftrightarrow{JLMS}$ boosts.

This is where the "rules" for complexity come from:

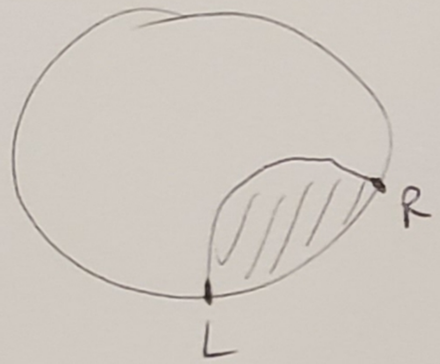
$$\boxed{\text{complexity} \in H^2(G, \mathbb{R}) \quad \text{where} \quad \rho_A \xleftrightarrow{G} \rho_B}$$

N.B.: Any cocycle defines A MEASURE of complexity!

Example:

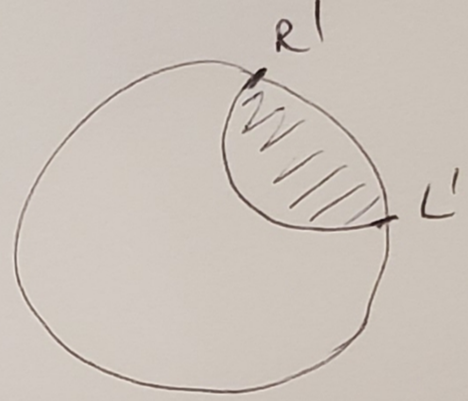
$\boxed{\text{group } G \text{ must reshuffle all admissible } \rho_A\text{'s}}$

- take global state $|0\rangle_{CFT}$
- dual to pure AdS_3



$\xrightarrow{g \in G}$

L_{-1}, L_0, L_{+1}
can do it!
and they preserve
$|0\rangle_{CFT}$



<u>Claim:</u>

complexity of $|0\rangle_{CFT} \in H^2(SL(2,\mathbb{R}), \mathbb{R})$

cf. 2004.11377 where we constructed a cocycle for complexity without knowing it...

$G = SL(2,\mathbb{R})$
and we need
a flat $sl(2,\mathbb{R})$ connection
for this to work. $\Longleftrightarrow$ Chern-Simons description of AdS_3 gravity!

THANK YOU!