Tetrahedron Instantons

Strings and Related Topics

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Background

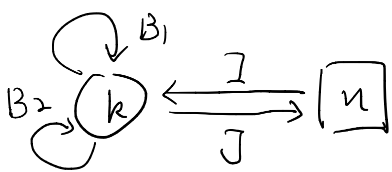
Instantons on $\mathbb{R}^4$: $\quad * F \pm F = 0 \quad$ (Self-dual YM)

$n \sim$ rank of gauge group G

$k \sim \int_{\mathbb{R}^4} \mathrm{Tr}\, F \wedge F \quad$ (instanton number)

Moduli space $\mathcal{M}^G_{n,k}$

ADHM construction: $\mathcal{M}^{SU(n)}_{n,k}$, algebraic. HK quotient



$$\mathcal{M}_{n,k} = \left\{ (B_i, I, J) \,\middle|\, \begin{array}{l} \sum [B_i, B_i^\dagger] + II^\dagger - J^\dagger J = 0 \\ [B_1, B_2] + IJ = 0 \end{array} \right\} \Big/ U(k)$$

String theory: $\quad k\,D1 - n\,D5$

$\mathcal{M}_{n,k} \simeq$ Higgs branch of theory on $D1$

Moduli space $\mathcal{M}_{n,k}$ non-compact

Nakajima: $\tilde{\mathcal{M}}_{n,k}$ resolve the singularity of $\mathcal{M}_{n,k}$

- $\tilde{\mathcal{M}}_{n,k} = \left\{ (B_i, I, J) \,\middle|\, \begin{array}{l} \sum [B_i, B_i^\dagger] + II^\dagger - J^\dagger J = \zeta \mathbb{1} \\ [B_1, B_2] + IJ = 0 \end{array} \right\} / U(k) \Rightarrow \{ I = 0 / J = 0 \}$
- $\tilde{\mathcal{M}}_{n,k} \simeq$ moduli space of torsion free sheaves on $\mathbb{C}^2$

Nekrasov - Schwarz: $\tilde{\mathcal{M}}_{n,k}$ IIB + non-zero background B-fields

$\tilde{\mathcal{M}}_{n,k}$: symmetries $U(1) \times U(1) \subset SO(4)$ rotation of $\mathbb{R}^4$

$SU(n)$: rotation of gauge orientation at ∞

Instanton partition function (Nekrasov)

$$Z_{\tilde{\mathcal{M}}_{n,k}} = \sum_{k \in \mathbb{Z}_+} q^k Z_k \qquad Z_k \sim \int_{\tilde{\mathcal{M}}_{n,k}} \mathbb{1}$$

Instanton partition function

$$Z_{\hat{\mathfrak{M}}_{n,k}}$$

- $\longrightarrow$ SW prepotential
- $\longrightarrow$ A-model topological string
- $\xrightarrow{\text{AGT}}$ Virasoro / W-alg conformal blocks
- $\xrightarrow{\text{N.S.}}$ Quantum integrable system

Nekrasov: Ω-background, equivariant localization ("IR")

Recent: $Z_{\hat{\mathfrak{M}}_{n,k}}$ as elliptic genus of 2d susy theory (HKKP 1406.6793 ...)
("UV")

advantage: only require the 2d susy effective theory
not detail of $\hat{\mathfrak{M}}_{n,k}$

Generalization of YM instantons in higher dim

D1 − D7 : ADHM of instantons in 6 dim (Solutions of DUY)

D1 − D9 − $\overline{\text{D9}}$: magnificient four (Nekrasov)

$Z_{\mathcal{X}}$: generating function of Donaldson − Thomas invariants on CY_3 / CY_4

Example: $k D1 - n D7$ on $R^{1,1} \times \mathbb{C}^3 \times \mathbb{R}^2 \Rightarrow$ DT on $\mathbb{C}^3$

(ℨ B P T , ...)

$k D1 - n D9 - n \overline{D9}$ on flat $R^{1,1} \times \mathbb{C}^4 \Rightarrow$ DT on $\mathbb{C}^4$

(Nekrasov ...)

Tetrahedron instantons

Instantons:

$[D1 - D5]$

$[D1 - D7]$ $\xleftarrow{(n,0,0,0)}$

$[D1 - D9 - \overline{D9}]$ $\xrightarrow[\text{condensation}]{\text{tachyon}}$

all D_{2d+1} branes are parallel (supported on the same direction)

Our setup

		z_1		z_2		z_3		z_4		
	0	1	2	3	4	5	6	7	8	9
$k\,D1$	—	×	×	×	×	×	×	×	×	—
$n_{123}\,D7$	—	—	—	—	—	—	—	×	×	—
$n_{124}\,D7$	—	—	—	—	—	×	×	—	—	—
$n_{134}\,D7$	—	—	—	×	×	—	—	—	—	—
$n_{234}\,D7$	—	×	×	—	—	—	—	—	—	—

flat spacetime: $\mathbb{R}^{1,1} \times \mathbb{C}^4$

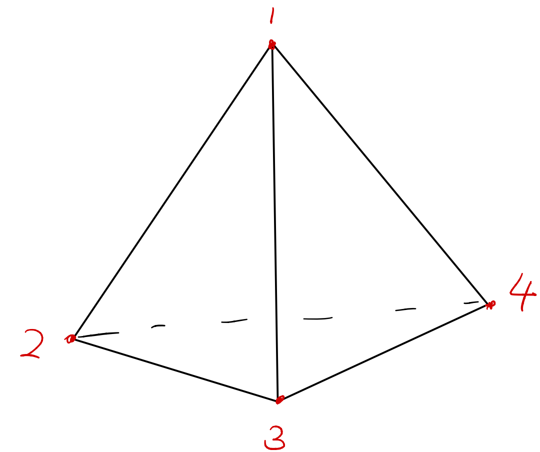
B-field: $B = \sum_{i=1}^{4} b_i \, dx^{2i-1} \wedge dx^{2i}$

effective theory on D1: (0,2) SUSY

Tetrahedron instantons

Our setup

	0	1	2	3	4	5	6	7	8	9
		z_1		z_2		z_3		z_4		
k D1	—	×	×	×	×	×	×	×	×	—
n_{123} D7	—	—	—	—	—	—	—	×	×	—
n_{124} D7	—	—	—	—	—	×	×	—	—	—
n_{134} D7	—	—	—	×	×	—	—	—	—	—
n_{234} D7	—	×	×	—	—	—	—	—	—	—



flat background : $R^{1,1} \times \mathbb{C}^4$

B-field: $B = \sum_{i=1}^{4} b_i \, dx^{2i-1} \wedge dx^{2i}$

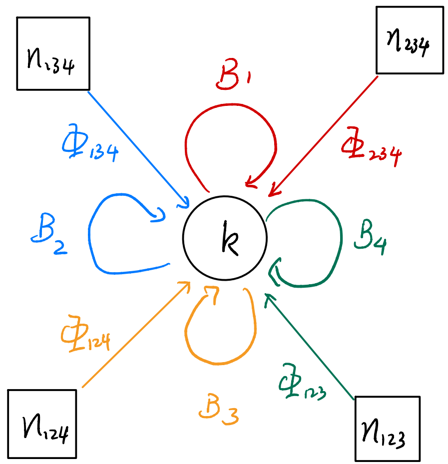
effective theory on D1 : (0,2) Susy

• : $\mathbb{C}_i$ $i = 1, 2, 3, 4$

— : $\mathbb{C}_i \times \mathbb{C}_j$

F : $\mathbb{C}_i \times \mathbb{C}_j \times \mathbb{C}_k \sim$ D7's

Effective theory on D1's : (0,2)



$A = \{1, 2, 3, 4\}$

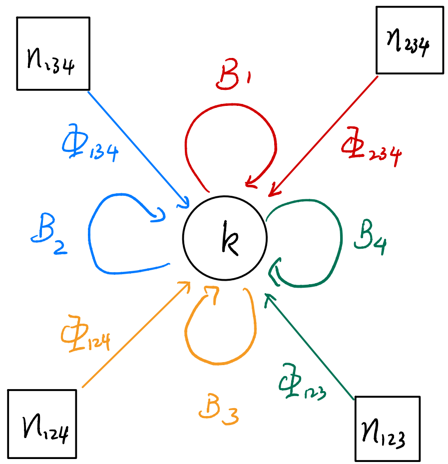
$\check{A} = \{234, 134, 124, 123\}$

$\vec{n} = \{n_{234}, n_{134}, n_{124}, n_{123}\}$

field contents

	$\mathcal{N} = (0,2)$	gauge $U(k)$	flavor $U(n_{\check{A}})$
D1-D1	vector γ	adj	1
	chiral B_A	adj	1
D1-D7$_{\check{A}}$	chiral $\Phi_{\check{A}}$	k	$\overline{n}_{\check{A}}$
	Fermi $\Psi_{\check{A}}$	k	$\overline{n}_A$

Effective theory on D1's : (0, 2)



Scalar potential:

$$V = \mathrm{Tr}\Big(\sum_{i\in A} [B_i, B_i^\dagger] + \sum_{i\in \check{A}} \Phi_i \Phi_i^\dagger - r\,\mathbb{1} \Big)^2$$

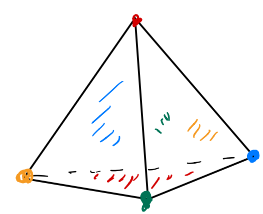
$$+ \sum_{i\in A} \mathrm{Tr}\,|B_i \Phi_{\check{i}}|^2 + \sum_{i,j\in A} \mathrm{Tr}\,|[B_i, B_j]|^2$$

$$\mathrm{Tr}\,|B\Phi|^2 + \mathrm{Tr}\,|B\Phi|^2 + \mathrm{Tr}\,|B\Phi|^2 + \mathrm{Tr}\,|B\Phi|^2$$

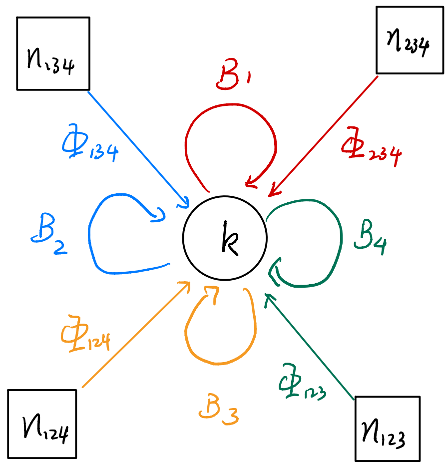
$A = \{1, 2, 3, 4\}$

$\check{A} = \{234, 134, 124, 123\}$

$\vec{n} = \{n_{234}, n_{134}, n_{124}, n_{123}\}$



Effective theory on D1's : (0,2)



$$A = \{1, 2, 3, 4\}$$

$$\check{A} = \{234, 134, 124, 123\}$$

$$\vec{n} = \{n_{234}, n_{134}, n_{124}, n_{123}\}$$

Moduli space of vacua $\mathcal{M}_{\vec{n},k}$

$$\mathcal{M}_{\vec{n},k} = \{(\vec{B}, \vec{\Phi}) \mid V = 0\} / U(k)$$

$$\vec{B} = \{B_1, B_2, B_3, B_4\}$$

$$\vec{\Phi} = \{\Phi_{234}, \Phi_{134}, \Phi_{124}, \Phi_{123}\}$$

$$V = 0 \iff \begin{cases} \sum [B_i, B_i^\dagger] + \sum [\Phi_i, \Phi_i^\dagger] - r\,1 = 0 \\ [B_i, B_j] = 0 \\ B_i \Phi_{\check{i}} = 0 \end{cases}$$

$\mathcal{M}_{\vec{n},k}$: ADHM of tetrahedron instantons

Example: 1 - instanton

B_i : complex numbers $\qquad [B_i, B_j] = 0$

Φ_j : complex vectors.

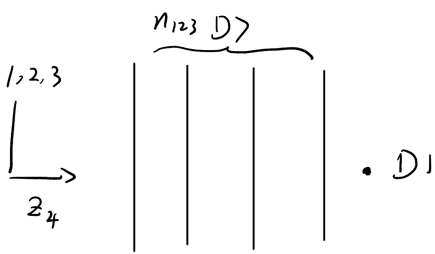
$\vec{n} = (n_{123}, 0, 0, 0)$: 1 - instanton on $\mathbb{C}^3$ with rank n_{123}

$$\Phi_{123} \neq 0 \xrightarrow{B\Phi = 0} B_4 = 0, \quad B_1 \; B_2 \; B_3 \text{ arbitrary}$$

$$\mathcal{M}_{(n_{123}, 0, 0, 0)} \simeq \mathbb{C}^3 \times \mathbb{CP}^{n_{123}-1}$$

B_i's / position of D1 in D7

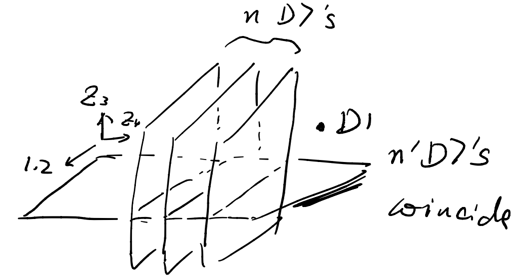
relative distance between D7's



Example: 1-instanton

$\vec{n} = (n_{123} , n_{124} , 0 , 0)$ instanton with defects

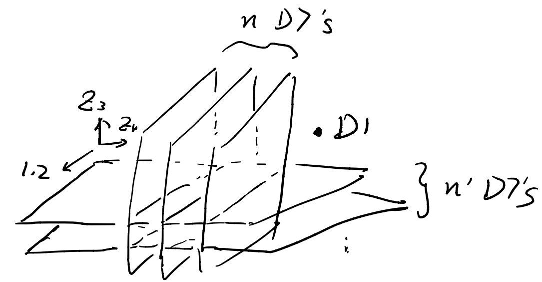
$n_{123} = n$, $n_{124} = n'$



$$\mathcal{M}_{(n,n',0,0),1} \simeq \mathbb{C}^2 \times \mathbb{C}^* \times \mathbb{CP}^{n-1}$$

$$\cup\ \mathbb{C}^2 \times \mathbb{C}^* \times \mathbb{CP}^{n'-1} \quad n \leftrightarrow n'$$

$$\cup\ \mathbb{C}^2 \times \mathbb{CP}^{n+n'-1}$$



I : n $D7_{(123)}$ "physical"

n' $D7_{(124)}$ "defects" in $D7_{(123)}$

II : n' $D7_{(124)}$ "physical"

n $D7_{(123)}$ "defects" in $D7_{(124)}$

$A \simeq B$

Instanton partition function

$$\mathcal{Z}_{\vec{n},k} = \text{elliptic genus of } (0,2) \text{ theory}$$

$$\mathcal{Z}_k = \int [d\phi]_{U(k)} \, \mathcal{Z}^{1-1} \prod_{i=1}^{4} \mathcal{Z}^{1-7_i}$$

theory:	2d $\longrightarrow$	1d $\longrightarrow$	0d
	$\downarrow$	$\downarrow$	$\downarrow$
inv.	elliptic $\longrightarrow$	K-theory $\longrightarrow$	$\int_{\bar{\mathcal{M}}_{\vec{n},k}} \mathbb{1}$
$\mathcal{Z}$:	θ-functions	trigonometry	rational

$$\mathcal{Z}_{\vec{n}} = \sum_{k \geq 0} q^k \, \mathcal{Z}_{\vec{n},k}$$

Now we have $\mathcal{M}_{\vec{n},k}$ from String theory

Q: What is the INSTANTON?

- What is the geometry / equations which gives our moduli space?

Our tetrahedron instanton

A: Instantons on $\mathbb{C}^3$ with different defects

B: Instantons on $\mathbb{C}^4$ with "tachyon condensation"

By construction, $\mathcal{M}_{\vec{n},k}$ is symmetric under permutations of n's. So is $Z_{\vec{n},k}$

$+ A \Rightarrow$ moduli space of $U(n_{123})$ instantons with $n_{124}, n_{134}, n_{234}$ defects stay the same when you permute n's

Geometric interpretation I

$(n, 0, 0, 0)$ k-instantons on $\mathbb{C}^3$

$\mathcal{M}_{(n,0,0,0),k} \simeq$ moduli spece of torsion free sheaf I on $\mathbb{C}^3$ with Chern character $(n, 0, 0, -k)$

Exact sequence: $0 \longrightarrow I \longrightarrow [\mathcal{O}_{\mathbb{C}^3}^{\oplus n} \longrightarrow \mathcal{O}_{k\,pt}] \longrightarrow 0$

$$\begin{array}{cc} D7 & D1 \end{array}$$

$$\mathcal{M}_I \simeq \mathcal{M}_{(n,0,0,0),k}$$

when $n=1$ $\quad I$: ideal sheaf

$$\mathcal{M}_{(1,0,0,0)k} \simeq \mathrm{Hilb}_{\mathbb{C}^3}(k \text{ pts})$$

Geometric interpretation I

$(n_{123}, n_{124}, n_{134}, n_{234})$ D7 − defect D7 − D1

Exact sequence: $0 \longrightarrow I \longrightarrow [\mathcal{O}_{\mathbb{C}^3}^{\oplus n_{123}} \longrightarrow F] \longrightarrow 0$

F contains sheaves support on defect D7 / D1

$\mathcal{M}_{\vec{n},k} \simeq$ moduli space of complex of sheaves

$$[\mathcal{O}_{\mathbb{C}^3}^{\oplus n_{123}} \longrightarrow F]$$

- Don't know def of F yet ???
- permutation symmetry in $\vec{n}$ $\Rightarrow$ Isom of moduli spaces of different "sheaves"
- Instanton partition function $\Rightarrow$ gen. DT inv. of $\mathcal{M}_I$

Geometric interpretation II

$k=1 \quad (n, n', 0, 0)$

$$\mathcal{M}_{(n,n',0,0)1} \simeq \mathbb{C}^2 \times \mathbb{C}^* \times \mathbb{CP}^{n-1} \cup \mathbb{C}^2 \times \mathbb{C}^* \times \mathbb{CP}^{n'-1} \cup \mathbb{C}^2 \times \mathbb{CP}^{n+n'-1}$$

recall moduli space of vortices with charge k of $U(n+n')$ gauge theory

$$\mathcal{V}_{n+n',k} \simeq \{ (B, \Phi) \mid [B, B^\dagger] + \Phi\Phi^\dagger = r\mathbb{1} \} / U(k)$$

we can show

T_i fixed points

$$\mathcal{M}_{(n,n',0,0)1} \simeq \mathbb{C}^2 \times \left(\mathcal{V}^{T_1}_{n+n',1} \cup \mathcal{V}^{T_2}_{n+n',1} \right)$$

where T_1, T_2 are two particular $U(1)$ action on $\mathcal{V}_{n+n',1}$

- $\mathcal{M}_{\vec{n},k}$ as fixed points of some $\mathcal{M}$ under certain action?

Geometric interpretation II

tetrahedron instanton

$$\mathcal{M}_{\vec{n},k} \simeq \left\{ (B_i, \Phi_i) \;\middle|\; \begin{array}{l} \sum_i [B_i, B_i^\dagger] + \sum_i [\Phi_i, \Phi_i^\dagger] = r\mathbb{1}, \\ [B_i, B_j] = 0 \qquad B_i \Phi_i^\vee = 0 \end{array} \right\}$$

magnificent four

$$\mathcal{M}_{m4} \simeq \left\{ (B_i, \Phi_i) \;\middle|\; \begin{array}{l} \sum_i [B_i, B_i^\dagger] + \sum_i [\Phi_i, \Phi_i^\dagger] = r\mathbb{1}, \\ [B_i, B_j] = 0 \end{array} \right\}$$

Q: $\mathcal{M}_{\vec{n},k} \simeq \bigcup_i \mathcal{M}_{m4}^{T_i}$?

Summary

Tetrahedron instanton

1. systems of normal crossing D7's and D1's
2. Instantons on $\mathbb{C}^3$ with codim$_{\mathbb{C}} = 1$ defects
3. moduli space $\simeq$ Higgs branch of a 2d (0,2) theory
4. Instanton partition function = elliptic genus
5. understanding moduli spaces and invariants of many geometric objects.

Future

1. $\mathbb{C}^3 \longrightarrow CY_3$. $\mathbb{C}^3/\Gamma$, toric , etc ...

   especially X is a toric CY_3 with compact 4-cycles

2. $\underset{\text{IIB}}{DT} = \underset{\text{M-theory}}{GV}$ for tetrahedron instantons

3. Precise mathematical formulation

Thank you!