Some results of one-loop reduction

Bo Feng

based on work with Binhong Wang, Tingfei Li, Xiaodi Li

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- The perturbative calculation of scattering amplitude is crucial for higher energy physics. using Feynman diagrams.
- The tradition way to do the calculation is to use the Feynman diagrams, but it is well known now, this method is not efficient in many situations.
- In last thirty years, various techniques have been developed to speed the computation. Now one-loop computation is considered as solved problem and the frontier is the two loop and higher, as we will hear a lot in this workshop.
- However, in this talk, I will discuss some problems left in the one-loop calculation.

Some efficient one-loop computation algorithms:

- OPP method: [Ossola, Papadopoulos, Pittau, 2006]
- Unitarity cut method: [Bern, Dixon, Dunbar , Kosower, 1994][Britto, Buchbinder, Cachazo, B.F, 2005] [C. Anastasiou, R. Britto, B.F, Z. Kunszt, P. Mastrolia, 2006]
- Forde's method: [D. Forde, 2007]
- Generalized OPP method: [R.K. Ellis, W.T. Giele, Z. Kunszt, 2007]
- ACK method: [N. Arkani-Hammed, F. Cachazo, J. Kaplan, 2008]

- For one-loop computation, the well established method is the reduction method.
- Now we are all known that the reduction can be divided into two categories: the reduction at the integrand level and the reduction at the integral level.
- The reduction at the integrand level is nothing, but division and separation of polynomial, for which the powerful mathematical tool is the "computational algebraic geometry".

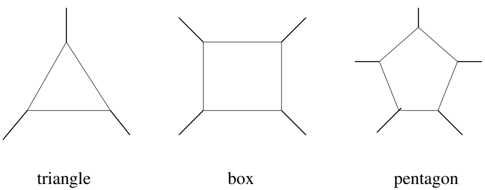
- One well known algorithm for reduction at the integrand level is the OPP method.
- OPP method has the advantage that it is easy to be implemented into program, both numerically and analytically.
- The disadvantage of OPP method is that we need to compute coefficients of spurious terms, although they do not contribute at the integral level. For practical applications, it is not a big problem since for the renormalizable theories, the spurious terms are not so much.

- However, from theoretical point of view, it is not satisfied, since the number of spurious terms increasing with the increasing of power of ℓ in numerator. Thus for arbitrary higher and higher power in numerator, there are more and more terms to be calculated, and the efficiency will be lost.

- For the reduction at the integral level, the typical algorithm is the celebrating PV-reduction method.
- For this method, we need to calculate the coefficients of masters only and the spurious terms will never show up.
- Although the algorithm of the original PV-reduction method is clear, its implement is not so easy.
- A better realization of reduction at the integral level is the **Unitarity cut method.**

PV reduction

- The mast basis are given by



- For massless inner line, there is no tadpole and massless bubble.

Unitary cut

Some facts regarding the one-loop amplitudes:

- The singular behavior of one-loop amplitudes is much more complicated than the tree-level: we have branch cuts as well as higher dimension singular surface.
- Under the expansion into basis, all branch cuts are given by scalar basis while coefficients are rational functions.
- Applying above observation we have unitarity cut method: taking imaginary part at both sides $\mathrm{Im}(I) = \sum_i c_i \mathrm{Im}(I_i)$ and comparing both sides we can get c_i if each $\mathrm{Im}(I_i)$ is unique.

Unitary cut

- The good point for this method is that the input is the multiplication of on-shell tree-level amplitudes of both sides. Especially when we combine the BCFW recursion relation.
- The difficulty is how to evaluate $\mathrm{Im}(I)$? This is solved by holomorphic anomaly: reducing integration into reading out residues of poles

  [Cachazo, Svrcek, Witten, 2004] [Britto, Buchbinder, Cachazo, Feng, 2005]
- Current status: Now we have well defined algebraic steps to extract coefficients from tree-level input.

- Example: Triangle

$$
\begin{aligned}
&\mathrm{Tri}[K_s, K] \\
= \; & \frac{1}{2}\frac{(K^2)^{N+1}}{(-\beta\sqrt{1-u})^{N+1}(\sqrt{-4q_s^2K^2})^{N+1}}\frac{1}{(N+1)!\left\langle P_{s,1}\ P_{s,2}\right\rangle^{N+1}} \\
& \frac{d^{N+1}}{d\tau^{N+1}}\left(\frac{\langle\ell|K|\ell]^{N+1}}{(K^2)^{N+1}}\mathcal{T}^{(N)}(\widetilde{\ell})\cdot D_s(\widetilde{\ell})\Big|_{\left\{\begin{array}{lcl}|\ell] & \to & |Q_s(u)|\ell\rangle \\ |\ell\rangle & \to & |P_{s,1}-\tau P_{s,2}\rangle\end{array}\right.}\right. \\
& \left. +\{P_{s,1}\leftrightarrow P_{s,2}\}\right)\Big|_{\tau\to 0}
\end{aligned}
$$

- Advantage: (1) we can get the wanted coefficients without calculating the spurious terms; (2) we can deal with arbitrary higher power in numerator.

However, there are some unsatisfied parts of unitarity cut method. In this talk we will discuss following two aspects:

- (A) The unitarity cut for higher poles
- (B) The tadpole coefficients

We consider the reduction of

$$\mathcal{M}[\ell] \equiv \int \frac{d^D\ell}{(2\pi)^{D/2}} \frac{\mathcal{N}[\ell]}{\prod_{j=1}^{n}((\ell - K_j)^2 - m_j^2 + i\epsilon)^{a_j}}, \quad a_i \geq 1$$

- By general theory, we know that

$$\mathrm{Im}(\mathcal{M}[\ell]) = \sum_t c_t \mathrm{Im}(\mathcal{I}_t[\ell])$$

- The $\mathrm{Im}(\mathcal{I}_t[\ell])$ is known, so we need to find $\mathrm{Im}(\mathcal{M}[\ell])$

To use the unitarity cut method, we use a trick by noticing that

$$\int \frac{d^D\ell}{(2\pi)^{D/2}} \frac{\mathcal{N}[\ell]}{\prod_{j=1}^n((\ell-K_j)^2-m_j^2+i\epsilon)^{a_i}}$$

$$= \left\{ \prod_{j=1}^n \frac{1}{(a_j-1)!} \frac{d^{a_j-1}}{d\eta_j^{a_j-1}} \int \frac{d^D\ell}{(2\pi)^{D/2}} \frac{\mathcal{N}[\ell]}{\prod_{j=1}^n((\ell-K_j)^2-m_j^2-\eta_j+i\epsilon)} \right\} |_{\eta_j\to 0}$$

thus

$$Re[L]+iIm[L]=\left\{ \prod_{j=1}^n \frac{1}{(a_j-1)!} \frac{d^{a_j-1}}{d\eta_j^{a_j-1}} (Re[R]+iIm[R]) \right\} |_{\eta_j\to 0}$$

Since the η_i's are real numbers, we have

$$Re[L] + iIm[L] = \left\{ \prod_{j=1}^{n} \frac{1}{(a_j - 1)!} \frac{d^{a_j - 1}}{d\eta_j^{a_j - 1}} Re[R] \right\} |_{\eta_j \to 0}$$

$$+ i \left\{ \prod_{j=1}^{n} \frac{1}{(a_j - 1)!} \frac{d^{a_j - 1}}{d\eta_j^{a_j - 1}} Im[R] \right\} |_{\eta_j \to 0}$$

so finally

$$Im[L] = \left\{ \prod_{j=1}^{n} \frac{1}{(a_j - 1)!} \frac{d^{a_j - 1}}{d\eta_j^{a_j - 1}} Im[R] \right\} |_{\eta_j \to 0}$$

- For general $\mathcal{N}[\ell]$, we know the expansion

$$Im[R] = \sum_t c_t Im(\mathcal{I}_t[\ell])$$

- The action of $\frac{d}{d\eta}$ will act on both c_t and $Im(\mathcal{I}_t[\ell])$.
- Since the analytic function c_t's are known, the unknown piece is the action of $\frac{d}{d\eta}$ on $Im(\mathcal{I}_t[\ell])$ and its expansion. In another words, we just need to consider the reduction of general power with $\mathcal{N}[\ell] = 1$ for $n \leq 5$.

Example I: bubble

$$\int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} \frac{1}{(p^2 - M_1^2)^a((p-K)^2 - M_2^2)^b}$$

- The imaginary part is given by

$$\mathcal{C}[\mathcal{I}_2] = (K^2)^{-1+\epsilon}\Delta^{\frac{1}{2}-\epsilon}\int_0^1 \mathrm{d}u u^{-1-\epsilon}\sqrt{1-u}$$

where

$$\Delta[K; M_1, M_2] = (K^2)^2 + ({M_1}^2)^2 + ({M_2}^2)^2$$
$$-2{M_1}^2{M_2}^2 - 2K^2{M_1}^2 - 2K^2M_2^2$$

- By our trick

$$\mathcal{C}[I_2(n+1,m+1)] = \frac{1}{m!n!}\left(\frac{\partial}{\partial M_2^2}\right)^m \left(\frac{\partial}{\partial M_1^2}\right)^n \mathcal{C}[I_2(1,1)]$$

thus

$$c_{2\to 2}(n+1,m+1) = \frac{1}{m!n!\Delta^{\frac{1}{2}-\epsilon}}\left(\frac{\partial}{\partial M_2^2}\right)^m \left(\frac{\partial}{\partial M_1^2}\right)^n \Delta^{\frac{1}{2}-\epsilon}$$

Recurrence relation:

$$
\begin{aligned}
I_3(1,1,n_3) &= \frac{1}{(n_3-1)!}\frac{d^{n_3-1}}{d(m_1^2)^{n_3-1}} I_3(1,1,1) \\
= & \frac{1}{(n_3-1)}\frac{d}{d(m_1^2)}\frac{1}{(n_3-2)!}\frac{d^{n_3-2}}{d(m_1^2)^{n_3-2}} I_3(1,1,1) \\
= & \frac{1}{(n_3-1)}\frac{d}{d(m_1^2)} I_3(1,1,n_3-1) \\
= & \frac{1}{(n_3-1)}\frac{d}{d(m_1^2)} \Big\{ c_{3\to 3}(1,1,n_3-1)\mathcal{I}_3 \\
& + \sum_{i=1}^{3} c_{3\to 2;\bar{i}}(1,1,n_3-1)\mathcal{I}_{2;\bar{i}} + \ldots \Big\}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{1}{(n_3-1)} \frac{dc_{3\to 3}(1,1,n_3-1)}{d(m_1^2)} \mathcal{I}_3 + \frac{c_{3\to 3}(1,1,n_3-1)}{(n_3-1)} I_3(1,1,2) \\
& + \sum_{i=1}^{3} \frac{dc_{3\to 2;\bar{i}}(1,1,n_3-1)}{(n_3-1)d(m_1^2)} \mathcal{I}_{2;\bar{i}} \\
& + \frac{c_{3\to 2;\bar{1}}(1,1,n_3-1)}{(n_3-1)} I_{2;\bar{1}}(1,2) + \frac{c_{3\to 2;\bar{2}}(1,1,n_3-1)}{(n_3-1)} I_{2;\bar{2}}(2,1) + \ldots
\end{aligned}
$$

Thus we derive

$$
\begin{aligned}
c_3(1,1,n_3) &= \frac{1}{(n_3-1)}\frac{dc_{3\to 3}(1,1,n_3-1)}{d(m_1^2)} + \frac{c_{3\to 3}(1,1,n_3-1)}{(n_3-1)} c_{3\to 3}(1,1,2) \\
c_{3\to 2;\bar{1}}(1,1,n_3) &= \frac{c_{3\to 3}(1,1,n_3-1)}{(n_3-1)} c_{3\to 2;\bar{1}}(1,1,2) + \frac{1}{(n_3-1)}\frac{dc_{3\to 2;\bar{1}}(1,1,n_3-1)}{d(m_1^2)} \\
&\quad + \frac{c_{3\to 2;\bar{1}}(1,1,n_3-1)}{(n_3-1)} c_{2\to 2;\bar{1}}(1,2) \\
c_{3\to 2;\bar{2}}(1,1,n_3) &= \frac{c_{3\to 3}(1,1,n_3-1)}{(n_3-1)} c_{3\to 2;\bar{2}}(1,1,2) + \frac{1}{(n_3-1)}\frac{dc_{3\to 2;\bar{2}}(1,1,n_3-1)}{d(m_1^2)} \\
&\quad + \frac{c_{3\to 2;\bar{2}}(1,1,n_3-1)}{(n-1)} c_{2\to 2;\bar{2}}(2,1) \\
c_{3\to 2;\bar{3}}(1,1,n_3) &= \frac{c_{3\to 3}(1,1,n_3-1)}{(n_3-1)} c_{3\to 2;\bar{3}}(1,1,2) + \frac{1}{(n_3-1)}\frac{dc_{3\to 2;\bar{3}}(1,1,n_3-1)}{d(m_1^2)}
\end{aligned}
$$

Thus the key calculation is for scalar integral with one and only one propagator having power 2.

Further simplification—- The dihedral symmetry D_n:

- By momentum shifting $p \to p + K_1$ we get

$$\begin{aligned} & I_3(n_1, n_2, n_3)[K_1, K_2, K_3; M_1, M_2, m_1] \\ = & \int \frac{d^{4-2\epsilon}p^4}{(2\pi)^{4-2\epsilon}} \frac{1}{((p+K_1)^2 - M_1^2)^{n_1}(p^2 - M_2^2)^{n_2}((p-K_2)^2 - m_1^2)^{n_3}} \\ = & I_3(n_2, n_3, n_1)[K_2, K_3, K_1; M_2, m_1, M_1] \end{aligned}$$

- We can also consider the variable changing $p \to -p$ to get

$$\begin{aligned} & I_3(n_1, n_2, n_3)[K_1, K_2, K_3; M_1, M_2, m_1] \\ = & \int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}} \frac{1}{(p^2 - M_1^2)^{n_1}((p+K_1)^2 - M_2^2)^{n_2}((p-K_3)^2 - m_1^2)^{n_3}} \\ = & I_3(n_1, n_3, n_2)[K_3, K_2, K_1; M_1, m_1, M_2] \end{aligned}$$

- Thus only $I_n(1, ..., 1, 2)$ needed to be calculated.

For triangle, we need to compute only $I_3(1,1,2)$. Let us show the calculation for the cut K_1:

- $$\mathcal{C}_{K_1}(I_3(1,1,2)) = -\left(\frac{4K_1^2}{\Delta[K_1,M_1,M_2]}\right)^{\epsilon}\frac{1}{\sqrt{\Delta_{3;m=0}}}\frac{\partial}{\partial m_1^2}Tri^{(0)}(Z)$$

- With a little algebra we have

$$\frac{\partial}{\partial m_1^2}Tri^{(0)}(Z) = \frac{2K_1^2}{\sqrt{\Delta_{3;m=0}\Delta[K_1,M_1,M_2]}}\left(\frac{2(1-2\epsilon)}{1-Z^2}Bub^{(0)} + \frac{2Z\epsilon}{1-Z^2}Tri^{(0)}(Z)\right)$$

Thus

$$c_{3\to 3;K_1}(1,1,2) = \frac{4K_1^2}{\sqrt{\Delta_{3;m=0}\Delta[K_1,M_1,M_2]}}\frac{Z\epsilon}{1-Z^2}$$

and

$$c_{3\to 2;\bar{3};K_1}(1,1,2) = -\frac{4K_1^2}{\Delta[K_1,M_1,M_2]\Delta_{3;m=0}}\frac{1-2\epsilon}{1-Z^2}$$

- One of the big problem of unitarity cut method is that tadpole coefficients can not be found by this way.
- There are proposal using the single cut, but the calculation is still complicated.
- In this talk, I will present a method to give the analytic expression of tadpole coefficients

We want to find the tadpole coefficient of integral

$$I_{n+1}^{(m)}[R;\{K_i\};M_0,\{M_i\}] \equiv \int \frac{d^D\ell}{(2\pi)^D} \frac{(2\ell\cdot R)^m}{(\ell^2-M_0^2)\prod_{j=1}^{n}((\ell-K_j)^2-M_j^2)}$$

This expression is general. By setting $R=\sum_{i=1}^{m}\alpha_i R_i$ into (**??**) and expanding the result to find the coefficients of $\alpha_1...\alpha_m$, it is easy to see that we will get the reduction of

$$I_{n+1}^{\mu_1\cdots\mu_m} = \int \frac{d^D\ell}{(2\pi)^D} \frac{\ell^{\mu_1}\ell^{\mu_2}\ldots\ell^{\mu_m}}{P_0\cdots P_n},$$

We will focus on

$$I_{n+1}^{(m)} = C_0(m, n+1) \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{(\ell^2 - M_0^2)} + \ldots$$

and others can be obtained by momentum shifting.

- To find the C_0, we will use a trick, i.e., to establish some differential equations by using following differential operators:

$$\widehat{D}_i \equiv K_i \cdot \frac{\partial}{\partial R}, \; i = 1, \ldots, n; \qquad \widehat{T} \equiv \eta^{\mu\nu} \frac{\partial}{\partial R^\mu} \frac{\partial}{\partial R^\nu}$$

- $$K_1^{\mu}\frac{\partial}{\partial R^{\mu}}I_{n+1}^{(m)} = \int\frac{d^D\ell}{(2\pi)^D}\frac{m(2\ell\cdot R)^{m-1}(2K_1\cdot\ell)}{(\ell^2-M_0^2)\prod_{j=1}^n((\ell-K_j)^2-M_j^2)}$$
$$= \int\frac{d^D\ell}{(2\pi)^D}\frac{m(2\ell\cdot R)^{m-1}}{\prod_{j=1}^n((\ell-K_j)^2-M_j^2)}$$
$$-\int\frac{d^D\ell}{(2\pi)^D}\frac{m(2\ell\cdot R)^{m-1}}{(\ell^2-M_0^2)\prod_{j=2}^n((\ell-K_j)^2-M_j^2)}$$
$$+(M_0^2+K_1^2-M_1^2)\int\frac{d^D\ell}{(2\pi)^D}\frac{m(2\ell\cdot R)^{m-1}}{(\ell^2-M_0^2)\prod_{j=1}^n((\ell-K_j)^2-M_j^2)}$$
$$= mI_{n+1;\bar{0}}^{(m-1)} - mI_{n+1;\bar{1}}^{(m-1)} + mf_1 I_{n+1}^{(m-1)}$$

- Using

$$\widehat{D}_j I_{n+1}^{(m)} = \left\{ \widehat{D}_j C_0(m, n+1) \right\} \int \frac{d^D \ell}{(2\pi)^D} \frac{1}{(\ell^2 - M_0^2)} + \ldots$$

and comparing the tadpole coefficients, we have the equation

$$\widehat{D}_j C_0(m, n+1) = -m C_0(m-1, n+1; \bar{j}) \\ + m f_j C_0(m-1, n+1)$$

- Similarly

$$\eta^{\mu\nu}\frac{\partial}{\partial R^{\mu}}\frac{\partial}{\partial R^{\nu}}I_{n+1}^{(m)} = \int\frac{d^D\ell}{(2\pi)^D}\frac{m(m-1)(2\ell\cdot R)^{m-2}(4\ell^2)}{(\ell^2-M_0^2)^2\prod_{j=1}^{n}((\ell-K_j)^2-M_j^2)}$$

$$= 4m(m-1)M_0^2\int\frac{d^D\ell}{(2\pi)^D}\frac{(2\ell\cdot R)^{m-2}}{(\ell^2-M_0^2)^2\prod_{j=1}^{n}((\ell-K_j)^2-M_j^2)}$$

$$+\int\frac{d^D\ell}{(2\pi)^D}\frac{4m(m-1)(2\ell\cdot R)^{m-2}}{\prod_{j=1}^{n}((\ell-K_j)^2-M_j^2)}$$

$$= 4m(m-1)M_0^2 I_{n+1}^{(m-2)} + 4m(m-1)I_{n+1;\bar{0}}^{(m-2)}$$

thus

$$\widehat{T}C_0(m,n+1) = 4m(m-1)M_0^2 C_0(m-2,n+1)$$

- To continue the study, we are not solve the differential equations directly, but noticing that it can be expand as following

$$C_0(m,n+1)=(M_0^2)^{-n}\sum_{\{i_k\}}c^{(m)}_{i_1,i_2,i_3,\ldots i_n}(M_0^2 s_{00})^{\frac{m-\sum i_k}{2}}\prod_{k=1}^{n}s_{0k}^{i_k}$$

we extend the definition domain of $i_k, k=0,1,...,n$ to $\mathbb{Z}$ but keep in mind that $c^{(m)}_{i_1,i_2,\ldots,i_n}$ vanishes if one index i_k meets $|i_k-\frac{m}{2}|>\frac{m}{2}$ or $m-\sum_{k=1}^{n}i_k$ is odd. Using this expansion, we transfer the differential equation to the algebraic recurrence relation

Example I: Tadpole coefficients of tensor tadpole

- $$\begin{aligned} \widehat{T} C_0(m,1)[R;M_0] = \widehat{T}\left(c^{(m)}(M_0^2)^{\frac{m}{2}} s_{00}^{\frac{m}{2}}\right) \\ = \quad & c^{(m)}(M_0^2)^{\frac{m}{2}}(Dm + m(m-2))s_{00}^{\frac{m-2}{2}} \\ = \quad & 4m(m-1)M_0^2 C_0(m-2,1) = 4m(m-1)M_0^2 c^{(m-2)}(M_0^2)^{\frac{m-2}{2}} s_{00}^{\frac{m-2}{2}} \end{aligned}$$

  which leads to the recurrence relation

  $$c^{(m)} = \frac{4(m-1)}{(D+m-2)} c^{(m-2)}$$

  Using the initial condition $c^{(0)} = 1$, we get immediately for

  $$c^{(m=even)} = 2^m \frac{(m-1)!!}{\prod_{i=1}^{\frac{m}{2}}(D+2(i-1))}, \qquad c^{(m=odd)} = 0$$

Example II: Tadpole coefficients of tensor bubble
With the expansion

$$C_0(m,2)=\sum_i c_i^{(m)}(M_0^2)^{-1}(M_0^2 s_{00})^{\frac{m-i}{2}} s_{01}^i$$

we have

- By D_1, we get immediately

$$(i+1)\beta_{11}c_{i+1}^{(m)}+(m-i+1)c_{i-1}^{(m)}=m\alpha_1 c_i^{(m-1)}-m\delta_{0,i}c^{(m-1)}$$

Replacing i with $i+1$, then we solve out $c_{i+2}^{(m)}$

$$c_{i+2}^{(m)}=\frac{1}{(i+2)\beta_{11}}\left(m\alpha_1 c_{i+1}^{(m-1)}-m\delta_{0,i+1}c^{(m-1)}-(m-i)c_i^{(m)}\right)$$

where $c^{(m)}$ is the tadpole expansion coefficients, and $\alpha_i=\frac{f_i}{M_0^2}, \beta_{ij}=\frac{K_i\cdot K_j}{M_0^2}$. We just need to calculate $c_0^{(m=2r)}$.

- By T, we have

$$r(D+2r-2)c_0^{(2r)}+\beta_{11}c_2^{(2r)}=4r(2r-1)c_0^{(2r-2)}$$

for it contains another unknown terms $c_2^{(2r)}$, we need to cancel $c_2^{(2r)}$. Here we can use iteratively to write $c_2^{(2r)}$ as following

$$\begin{aligned} c_2^{(2r)} &= \frac{r}{\beta_{11}}\left(\alpha_1 c_1^{(2r-1)}-c_0^{(2r)}\right)\\ &= \frac{r}{\beta_{11}}\left(\alpha_1\frac{2r-1}{\beta_{11}}\left(\alpha_1 c_0^{(2r-2)}-c^{(2r-2)}\right)-c_0^{(2r)}\right)\end{aligned}$$

then we have

$$c_0^{(2r)}=\frac{2r-1}{2r+D-3}\left(\left(4-\frac{\alpha_1^2}{\beta_{11}}\right)c_0^{(2r-2)}+\frac{\alpha_1}{\beta_{11}}c^{(2r-2)}\right)$$

With the recurrence relations of $c_{i+2}^{(m)}$ and $c_0^{(2r)}$, and the boundary condition $c_0^{(0)} = 0$, one can obtain all coefficients for arbitrary rank. Here are some examples:

- $m = 1$

$$c_1^{(1)} = \frac{1}{\beta_{11}} \left(\alpha_1 c_0^{(0)} - c^{(0)} \right) = \frac{-1}{\beta_{11}}$$

- $m = 2$

$$c_0^{(2)} = \frac{1}{D-1} \left(\left(4 - \frac{\alpha_1^2}{\beta_{11}} \right) c_0^{(0)} + \frac{\alpha_1}{\beta_{11}} c^{(0)} \right) = \frac{\alpha_1}{(D-1)\beta_{11}}$$

$$c_2^{(2)} = \frac{1}{2\beta_{11}} \left(2\alpha_1 c_1^{(1)} - 2c_0^{(2)} \right) = -\frac{\alpha_1 D}{(D-1)\beta_{11}^2}$$

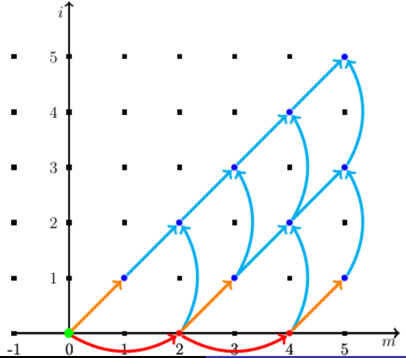
- $m = 3$

$$c_1^{(3)} = \frac{3}{\beta_{11}} \left(\alpha_1 c_0^{(2)} - c^{(2)} \right) = \frac{3\left(4\beta_{11} - 4D\beta_{11} + \alpha_1^2 D\right)}{(D-1)D\beta_{11}^2}$$

- $m = 3$

$$c_3^{(3)} = \frac{1}{3\beta_{11}}\left(3\alpha_1 c_2^{(2)} - 3c_1^{(3)}\right) = -\frac{8\beta_{11} - 8D\beta_{11} + \alpha_1^2 D^2 + 2\alpha_1^2 D}{(D-1)D\beta_{11}^3}$$

The process of calculation is shown as the figure below



Example III: Tadpole coefficients of tensor triangle
With the expansion

$$C_0(m,3)=\sum_{i,j}(M_0^2)^{-2}(M_0^2 s_{00})^{\frac{m-i-j}{2}} s_{01}^i s_{02}^j$$

we have

- by D_1, D_2, we get

$$\begin{aligned} & m\alpha_1 c_{i,j}^{(m-1)} - m\delta_{i,0} c_j^{(m-1)}[0,2] \\ = \; & (m+1-i-j)c_{i-1,j}^{(m)} + (i+1)\beta_{11} c_{i+1,j}^{(m)} + (j+1)\beta_{12} c_{i,j+1}^{(m)} \\ & m\alpha_2 c_{i,j}^{(m-1)} - m\delta_{j,0} c_i^{(m-1)}[0,1] \\ = \; & (m+1-i-j)c_{i,j-1}^{(m)} + (i+1)\beta_{12} c_{i+1,j}^{(m)} + (j+1)\beta_{22} c_{i,j+1}^{(m)} \end{aligned}$$

then we solve out $c^{(m)}_{i+1,j}$ and $c^{(m)}_{i,j+1}$

$$c^{(m)}_{i+1,j} = \frac{1}{(i+1)\Delta(1,2)}\Big((m+1-i-j)(\beta_{12}c^{(m)}_{i,j-1} - \beta_{22}c^{(m)}_{i-1,j})$$
$$+ m(\beta_{12}\delta_{0,j}c_i^{(m-1)}[0,1] - \beta_{22}\delta_{0,i}c_j^{(m-1)}[0,2]) + m(\alpha_1\beta_{22} - \alpha_2\beta_{12})c^{(m-1)}_{i,j}\Big)$$
$$c^{(m)}_{i,j+1} = \frac{1}{(j+1)\Delta(1,2)}\Big((m+1-i-j)(\beta_{12}c^{(m)}_{i-1,j} - \beta_{11}c^{(m)}_{i,j-1})$$
$$+ m(\beta_{12}\delta_{0,i}c_j^{(m-1)}[0,2] - \beta_{11}\delta_{0,j}c_i^{(m-1)}[0,1]) + m(\alpha_2\beta_{11} - \alpha_1\beta_{12})c^{(m-1)}_{i,j}\Big)$$

- by T, we get

$$r(2r+D-2)c^{(2r)}_{0,0} + \beta_{11}c^{(2r)}_{2,0} + \beta_{22}c^{(2r)}_{0,2} + \beta_{12}c^{(2r)}_{1,1} = 4r(2r-1)c^{(2r-2)}_{0,0}$$

Finally we get another recurrence relation

$$c_{0,0}^{(2r)} = \frac{2r-1}{(2r+D-4)\Delta(1,2)}\Big[\left(2\alpha_1\alpha_2\beta_{12}-\alpha_2^2\beta_{11}-\alpha_1^2\beta_{22}+4\Delta(1,2)\right)c_{0,0}^{(2r-2)} + (\alpha_2\beta_{11}-\alpha_1\beta_{12})\, c_0^{(2r-2)}[0,1] + (\alpha_1\beta_{22}-\alpha_2\beta_{12})\, c_0^{(2r-2)}[0,2]\Big]$$

where for simplicity we denote $\Delta(i_1, i_2, \cdots, i_n; j_1, j_2, \cdots, j_n)$ as the determinant of a $n \times n$ massless matrix A with entry $A_{ab} = \beta_{i_a,j_b}$ and $\Delta(i_1, i_2, \cdots, i_n) \equiv \Delta(i_1, i_2, \cdots, i_n; i_1, i_2, \cdots, i_n)$. With the three recurrence relations of $c_{i+1,j}^{(m)}, c_{i,j+1}^{(m)}, c_{0,0}^{(2r)}$ and the boundary condition $c_{0,0}^{(0)} = 0$ we can get all coefficients for any rank. Here are some examples:

- $m = 1$

$$c_{1,0}^{(1)} = 0$$

- $m = 2$

$$c_{0,0}^{(2)} = 0$$

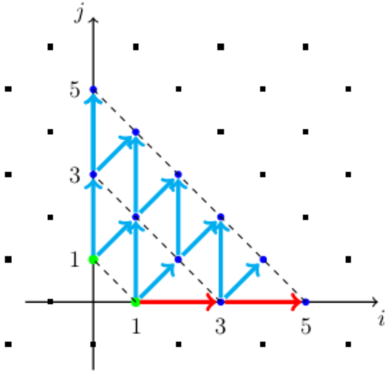
$$c_{0,2}^{(2)} = \frac{1}{2\Delta(1,2)}\Big(-2\beta_{11}c_{0,0}^{(2)} + 2\beta_{12}c_1^{(1)}[0,2] + 2(\alpha_2\beta_{11} - \alpha_1\beta_{12})c_{0,1}^{(1)}\Big) = -\frac{\beta_{12}}{\beta_{22}\Delta(1,2)}$$

$$c_{1,1}^{(2)} = \frac{1}{\Delta(1,2)}\Big(2\beta_{12}c_{0,0}^{(2)} - 2\beta_{22}c_1^{(1)}[0,2] + 2(\alpha_1\beta_{22} - \alpha_2\beta_{12})c_{0,1}^{(1)}\Big) = \frac{2}{\Delta(1,2)}$$

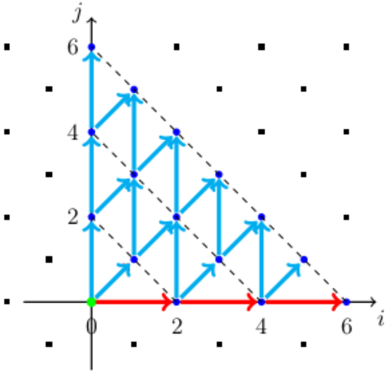
- $m = 3$

$$c_{1,0}^{(3)} = \frac{3}{\Delta(1,2)}\Big((\beta_{12}c_0^{(2)}[0,1] - \beta_{22}c_0^{(2)}[0,2]) + (\alpha_1\beta_{22} - \alpha_2\beta_{12})c_{0,0}^{(2)}\Big)$$
$$= \frac{3\alpha_1\beta_{12} - 3\alpha_2\beta_{11}}{(D-1)\beta_{11}\Delta(1,2)}$$
$$c_{1,2}^{(3)} = \frac{1}{2\Delta(1,2)}\Big(2(\beta_{12}c_{0,1}^{(3)} - \beta_{11}c_{1,0}^{(3)}) + 3(\alpha_2\beta_{11} - \alpha_1\beta_{12})c_{1,1}^{(2)}\Big)$$
$$= \frac{3\alpha_2 D}{(D-1)\beta_{22}\Delta(1,2)} + \frac{3(D+1)\beta_{12}\,(\alpha_2\beta_{12} - \alpha_1\beta_{22})}{(D-1)\beta_{22}\Delta(1,2)^2}$$
$$c_{3,0}^{(3)} = \frac{1}{3\Delta(1,2)}\Big(-2\beta_{22}c_{1,0}^{(3)} + 3\beta_{12}c_2^{(2)}[0,1] + 3(\alpha_1\beta_{22} - \alpha_2\beta_{12})c_{2,0}^{(2)}\Big)$$
$$= \frac{\alpha_2\,(2\beta_{11}\beta_{22} + (D-1)\beta_{12}^2) - \alpha_1(D+1)\beta_{12}\beta_{22}}{(D-1)\beta_{11}\Delta(1,2)^2} - \frac{\alpha_1 D\beta_{12}}{(D-1)\beta_{11}^2\Delta(1,2)}$$

The process of calculation is shown as the figure below



(a) $m = 5$



(b) $m = 6$

Example IV: Tadpole coefficients of tensor box
With the expansion

$$C_0(m,4) = \sum_{i,j,k} (M_0^2)^{-3} (M_0^2 s_{00})^{\frac{m-i-j-k}{2}} c_{ijk}^{(m)} s_{01}^i s_{02}^j s_{03}^k$$

we have

- by D_1, D_2, D_3, we finally get

$$c_{i+1,j,k}^{(m)} = \frac{1}{(i+1)\Delta(1,2,3)} \Big(\Delta_{1,1}^{(3)} O_1^{(m)}(i,j,k) + \Delta_{1,2}^{(3)} O_2^{(m)}(i,j,k) + \Delta_{1,3}^{(3)} O_3^{(m)}(i,j,k) \Big) \quad (1)$$

Other recurrence relations can be got by permutation of $\{1,2,3\}, \{i,j,k\}$.

where we have defined

$$O_1^{(m)}(i,j,k) = m\left(\alpha_1 c_{i,j,k}^{(m-1)} - \delta_{0,i} c_{jk}^{(m-1)}[0,2,3]\right) - (m+1-i-j-k)c_{i-1,j,k}^{(m)}$$
$$O_2^{(m)}(i,j,k) = m\left(\alpha_2 c_{i,j,k}^{(m-1)} - \delta_{0,j} c_{ik}^{(m-1)}[0,1,3]\right) - (m+1-i-j-k)c_{i,j-1,k}^{(m)}$$
$$O_3^{(m)}(i,j,k) = m\left(\alpha_3 c_{i,j,k}^{(m-1)} - \delta_{0,k} c_{ij}^{(m-1)}[0,1,2]\right) - (m+1-i-j-k)c_{i,j,k-1}^{(m)}$$

- by T, and using the three recurrences of D_1, D_2, D_3, we finally get

$$c_{0,0,0}^{(2r)} = \frac{2r-1}{D+2r-5}\left[(4-\boldsymbol{\alpha}^T G^{-1}\boldsymbol{\alpha})c_{0,0,0}^{(2r-2)} + \boldsymbol{\alpha}^T G^{-1}\boldsymbol{c}^{(2r-2)}[0,1,2,3]\right]$$

  where we have defined G as the massless Gram matrix with $G_{ij} = \beta_{ij}$ and $\boldsymbol{\alpha}$ as the column vector $\{\alpha_i\}, i = 1,2,...,n$, and the column vector $\boldsymbol{c}^{(m)}[0,1,2,...,n] = \{c^{(m)}[0,1,2,...,n;\bar{i}], i = 1,2,...,n\}$, With the boundary condition $c^{(0)}_{0,0,0} = 0$ The four recurrence relations are sufficient to determine the tadpole coefficient of any rank, the first nontrivial case is $m = 3$.

$$
\begin{aligned}
c^{(3)}_{1,1,1} &= \frac{1}{\Delta(1,2,3)} \left[\Delta^{(3)}_{1,1} O^{(3)}_{1}(0,1,1) + \Delta^{(3)}_{1,2} O^{(3)}_{2}(0,1,1) + \Delta^{(3)}_{1,3} O^{(3)}_{3}(0,1,1) \right] \\
&= -\frac{6}{\Delta(1,2,3)} \\
c^{(3)}_{1,2,0} &= \frac{1}{\Delta(1,2,3)} \left(\Delta^{(3)}_{1,1} O^{(3)}_{1}(0,2,0) + \Delta^{(3)}_{1,2} O^{(3)}_{2}(0,2,0) + \Delta^{(3)}_{1,3} O^{(3)}_{3}(0,2,0) \right) \\
&= \frac{3\beta_{23}}{\beta_{22}\Delta(1,2,3)} + \frac{3\beta_{12}\Delta^{(3)}_{1,3}}{\beta_{22}\Delta^{(3)}_{3,3}\Delta(1,2,3)} \\
c^{(3)}_{3,0,0} &= \frac{1}{3\Delta(1,2,3)} \left(\Delta^{(3)}_{1,1} O^{(3)}_{1}(2,0,0) + \Delta^{(3)}_{1,2} O^{(3)}_{2}(2,0,0) + \Delta^{(3)}_{1,3} O^{(3)}_{3}(2,0,0) \right) \\
&= \frac{\beta_{13}\Delta^{(3)}_{1,2}}{\beta_{11}\Delta^{(3)}_{2,2}\Delta(1,2,3)} + \frac{\beta_{12}\Delta^{(3)}_{1,3}}{\beta_{11}\Delta^{(3)}_{3,3}\Delta(1,2,3)}
\end{aligned}
$$

where we have used:

$$O_1^{(3)}(0,1,1) = 3\left(\alpha_1 c_{0,1,1}^{(2)} - c_{1,1}^{(2)}[0,2,3]\right) = -\frac{6}{\Delta(2,3)}$$

$$O_2^{(3)}(0,1,1) = 3\alpha_2 c_{0,1,1}^{(2)} - 4c_{0,0,1}^{(3)} = 0$$

$$O_3^{(3)}(0,1,1) = 3\alpha_3 c_{0,1,1}^{(2)} - 4c_{0,1,0}^{(3)} = 0$$

$$O_1^{(3)}(0,2,0) = 3\left(\alpha_1 c_{0,2,0}^{(2)} - c_{2,0}^{(2)}[0,2,3]\right) = \frac{3\beta_{23}}{\beta_{22}\Delta_{1,1}^{(3)}}$$

$$O_2^{(3)}(0,2,0) = 3\alpha_2 c_{0,2,0}^{(2)} - 2c_{0,1,0}^{(3)} = 0$$

$$O_3^{(3)}(0,2,0) = 3\left(\alpha_3 c_{0,2,0}^{(2)} - c_{0,2}^{(2)}[0,1,2]\right) = \frac{3\beta_{12}}{\beta_{22}\Delta_{3,3}^{(3)}}$$

$$O_1^{(3)}(2,0,0) = 3\alpha_1 c_{2,0,0}^{(2)} - 2c_{1,0,0}^{(3)} = 0$$

$$O_2^{(3)}(2,0,0) = 3\left(\alpha_2 c_{2,0,0}^{(2)} - c_{2,0}^{(2)}[0,1,3]\right) = \frac{3\beta_{13}}{\beta_{11}\Delta_{2,2}^{(3)}}$$

$$O_3^{(3)}(2,0,0) = 3\left(\alpha_3 c_{2,0,0}^{(2)} - c_{2,0}^{(2)}[0,1,2]\right) = \frac{3\beta_{12}}{\beta_{11}\Delta_{3,3}^{(3)}}$$

Example V: Tadpole coefficients of tensor pentagon
With the expansion

$$C_0(m,5) = \sum_{i,j,k,l} (M_0^2)^{-4} (M_0^2 s_{00})^{\frac{m-i-j-k-l}{2}} c_{ijkl}^{(m)} s_{01}^i s_{02}^j s_{03}^k s_{04}^l$$

we have

- by D_1, D_2, D_3, D_4, we finally get

$$c_{i+1,j,k,l}^{(m)} = \frac{1}{(i+1)\Delta(1,2,3,4)} \Big[\Delta_{11}^{(4)} O_1^{(m)}(i,j,k,l) + \Delta_{12}^{(4)} O_2^{(m)}(i,j,k,l) + \Delta_{13}^{(4)} O_3^{(m)}(i,j,k,l) + \Delta_{14}^{(4)} O_4^{(m)}(i,j,k,l) \Big]$$

Other recurrence relations can be got by permutation of $\{1,2,3,4\}, \{i,j,k,l\}$.

where we have defined

$$O_1^{(m)}(i,j,k,l) = m\left(\alpha_1 c_{i,j,k,l}^{(m-1)} - \delta_{0,i} c_{j,k,l}^{(m-1)}[0,2,3,4]\right) - (m+1-i-j-k-l)c_{i-1,j,k,l}^{(m)}$$
$$O_2^{(m)}(i,j,k,l) = m\left(\alpha_2 c_{i,j,k,l}^{(m-1)} - \delta_{0,j} c_{i,k,l}^{(m-1)}[0,1,3,4]\right) - (m+1-i-j-k-l)c_{i,j-1,k,l}^{(m)}$$
$$O_3^{(m)}(i,j,k,l) = m\left(\alpha_3 c_{i,j,k,l}^{(m-1)} - \delta_{0,k} c_{i,j,l}^{(m-1)}[0,1,2,4]\right) - (m+1-i-j-k-l)c_{i,j,k-1,l}^{(m)}$$
$$O_4^{(m)}(i,j,k,l) = m\left(\alpha_4 c_{i,j,k,l}^{(m-1)} - \delta_{0,l} c_{i,j,k}^{(m-1)}[0,1,2,3]\right) - (m+1-i-j-k-l)c_{i,j,k,l-1}^{(m)}$$

- by T, and using the three recurrences of D_1, D_2, D_3, D_4, we finally get

$$c_{0,0,0,0}^{(2r)} = \frac{(2r-1)}{D+2r-6}\left[(4-\boldsymbol{\alpha}^T G^{-1}\boldsymbol{\alpha})c_{0,0,0,0}^{(2r-2)} + \boldsymbol{\alpha}^T G^{-1}\boldsymbol{c}^{(2r-2)}[0,1,2,3,4]\right]$$

With the boundary condition $c^{(0)}_{0,0,0,0} = 0$ The four recurrence relations are sufficient to determine the tadpole coefficient of any rank, the first nontrivial case is $m = 4$. The nonzero expansion coefficients are

$$
\begin{aligned}
c^{(4)}_{4,0,0,0} &= \frac{1}{4\Delta(1,2,3,4)}\Big[\Delta^{(4)}_{1,1}O^{(4)}_1(3,0,0,0) + \Delta^{(4)}_{1,2}O^{(4)}_2(3,0,0,0) \\
&\quad + \Delta^{(4)}_{1,3}O^{(4)}_3(3,0,0,0) + \Delta^{(4)}_{1,4}O^{(4)}_4(3,0,0,0)\Big] \\
&= \frac{-1}{\beta_{11}\Delta(1,2,3,4)}\Bigg\{\frac{\Delta^{(4)}_{1,2}\left(\beta_{13}\Delta(1,4)\Delta(3,4;1,3) + \beta_{14}\Delta(1,3)\Delta(3,4;4,1)\right)}{\Delta(1,3)\Delta(1,4)\Delta(1,3,4)} \\
&\quad + \frac{\Delta^{(4)}_{1,3}\left(\beta_{12}\Delta(1,4)\Delta(2,4;1,2) + \beta_{14}\Delta(1,2)\Delta(2,4;4,1)\right)}{\Delta(1,2)\Delta(1,4)\Delta(1,2,4)} \\
&\quad + \frac{\Delta^{(4)}_{1,4}\left(\beta_{12}\Delta(1,3)\Delta(2,3;1,2) + \beta_{13}\Delta(1,2)\Delta(2,3;3,1)\right)}{\Delta(1,2)\Delta(1,3)\Delta(1,2,3)}\Bigg\}
\end{aligned}
$$

$$c^{(4)}_{1,3,0,0} = \frac{1}{\Delta(1,2,3,4)}\Big[\Delta^{(4)}_{1,1}O^{(4)}_{1}(0,3,0,0)+\Delta^{(4)}_{1,2}O^{(4)}_{2}(0,3,0,0)$$
$$+\Delta^{(4)}_{1,3}O^{(4)}_{3}(0,3,0,0)+\Delta^{(4)}_{1,4}O^{(4)}_{4}(0,3,0,0)\Big]$$
$$= \frac{-4}{\beta_{22}\Delta(1,2,3,4)}\times$$
$$\Bigg\{\frac{\Delta^{(4)}_{1,1}\left[\beta_{23}\Delta(2,4)\Delta(3,4;2,3)+\beta_{24}\Delta(2,3)\Delta(3,4;4,2)\right]}{\Delta(2,3)\Delta(2,4)\Delta(2,3,4)}$$
$$+\frac{\Delta^{(4)}_{1,3}\left[\beta_{12}\Delta(2,4)\Delta(4,1;1,2)+\beta_{24}\Delta(1,2)\Delta(4,1;2,4)\right]}{\Delta(1,2)\Delta(2,4)\Delta(1,2,4)}$$
$$+\frac{\Delta^{(4)}_{1,4}\left[\beta_{12}\Delta(2,3)\Delta(3,1;1,2)+\beta_{23}\Delta(1,2)\Delta(3,1;2,3)\right]}{\Delta(1,2)\Delta(2,3)\Delta(1,2,3)}\Bigg\}$$

$$
\begin{aligned}
c^{(4)}_{2,2,0,0} &= \frac{1}{2\Delta(1,2,3,4)}\Big[\Delta^{(4)}_{1,1}O^{(4)}_1(1,2,0,0)+\Delta^{(4)}_{1,2}O^{(4)}_2(1,2,0,0) \\
&\quad +\Delta^{(4)}_{1,3}O^{(4)}_3(1,2,0,0)+\Delta^{(4)}_{1,4}O^{(4)}_4(1,2,0,0)\Big] \\
&= \frac{-6}{\beta_{22}\Delta(1,2)\Delta(1,2,3,4)}\left\{\Delta^{(4)}_{1,3}\frac{(\beta_{24}\Delta(1,2)+\beta_{12}\Delta(2,4;1,2))}{\Delta(1,2,4)}\right. \\
&\quad \left. +\Delta^{(4)}_{1,4}\frac{(\beta_{23}\Delta(1,2)+\beta_{12}\Delta(2,3;1,2))}{\Delta(1,2,3)}\right\} \\
c^{(4)}_{2,1,1,0} &= \frac{12\Delta^{(4)}_{1,4}}{\Delta(1,2,3)\Delta(1,2,3,4)}, \quad c^{(4)}_{1,1,1,1}=\frac{24}{\Delta(1,2,3,4)} \qquad (2)
\end{aligned}
$$

Other expansion coefficients can be got by using the permutation symmetry.

Final remarks:

- Our method for tadpole is nothing, but the traditional PV-reduction method with a little deformation
- It can also be applied to find coefficients of other basis, such as bubble, triangle, box and pentagon.
- The generalization to higher loops is possible, but there are some technical difficulties.

Our method can be applied to other master integrals by changing the boundary conditions.

- The reduction coefficient of bubble $I_2[0,1]$.

$$c_0^{(0)}[0,1]=1,c^{(m)}=c_{0,0}^{(0)}=c_{0,0,0}^{(0)}=c_{0,0,0,0}^{(0)}=0$$

- The reduction coefficient of triangle $I_3[0,1,2]$.

$$c_{0,0}^{(0)}[0,1,2]=1,c^{(m)}=c_i^{(m)}=c_{0,0,0}^{(0)}=c_{0,0,0,0}^{(0)}=0$$

- The reduction coefficient of box $I_4[0,1,2,3]$.

$$c_{0,0,0}^{(0)}[0,1,2,3]=1,c^{(m)}=c_i^{(m)}=c_{i,j}^{(m)}=c_{0,0,0,0}^{(0)}=0$$

- The reduction coefficient of triangle $I_5[0,1,2,3,4]$.

$$c_{0,0,0,0}^{(0)}[0,1,2,3,4]=1,c^{(m)}=c_i^{(m)}=c_{i,j}^{(m)}=c_{i,j,k}^{(m)}=0$$

Example : Bubble coefficients of tensor triangle for $I_2[0,1]$.

- $m = 1$

$$c_{1,0}^{(1)} = \frac{1}{\Delta(1,2)}\left(\beta_{12}c_0^{(0)}[0,1] - \beta_{22}c_0^{(0)}[0,2]\right) = \frac{\beta_{12}}{\Delta(1,2)}$$

- $m = 2$

$$c_{0,0}^{(2)} = \frac{1}{(D-2)\Delta(1,2)}\left(\left(\alpha_2\beta_{11} - \alpha_1\beta_{12}\right)c_0^{(2r-2)}[0,1]\right)$$
$$= \frac{\alpha_2\beta_{11} - \alpha_1\beta_{12}}{(D-2)\Delta(1,2)}$$

$$c_{1,1}^{(2)} = \frac{1}{\Delta(1,2)}\left(2\beta_{12}c_{0,0}^{(2)} + 2(\alpha_1\beta_{22} - \alpha_2\beta_{12})c_{0,1}^{(1)}\right)$$
$$= \frac{2\alpha_2(D-1)\beta_{11}\beta_{12} - 2\alpha_1\beta_{12}^2}{(D-2)\Delta(1,2)^2} - \frac{2\alpha_1\beta_{11}\beta_{22}}{\Delta(1,2)^2}$$

$$c_{2,0}^{(2)} = \frac{1}{2\Delta(1,2)}\left(2\beta_{12}c_1^{(1)}[0,1] - 2\beta_{22}c_{0,0}^{(2)} + 2(\alpha_1\beta_{22} - \alpha_2\beta_{12})c_{1,0}^{(1)}\right)$$
$$= \frac{\beta_{12}\left(\alpha_1(D-1)\beta_{11}\beta_{22} - (D-2)\Delta(1,2)\right) - \alpha_2(D-1)\beta_{11}^2\beta_{22}}{(D-2)\beta_{22}\Delta(1,2)^2}$$

Thanks a lot of your attention !