The Torus partition function and Wormhole in TTbar deformed theory

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Outline

- **Introduction to TTbar deformation**
- **TTbar flow effects on partition function[2^{nd} order] & KdV [1^{st} order]**
- **Wormhole induced by TTbar**
- **Conclusions & Future Problems**

Basic introduction of the $T\bar{T}$ deformation

Definition:

We focus on two dimensional spacetime with complex coordinates

$$
\begin{aligned}
z =& x + it, \\
\bar{z} =& x - it,
\end{aligned}
$$

then in $\{ z, \ \bar{z} \}$ coordinates,

$$
\det[T_{\mu\nu}] = -4\left(T\bar{T} - \Theta^2\right),
$$

Zamolodchikov '04

Our notations are following

$$
T := T_{zz}, \quad \bar{T} := T_{\bar{z}\bar{z}}, \quad \Theta := T_{z\bar{z}}.
$$

Deformation of TTbar

$$\mathcal{L}^{(\lambda+\delta\lambda)} = \mathcal{L}^{(\lambda)} + \delta\lambda\, T\bar{T}$$

$$\frac{dS(\lambda)}{d\lambda} = \int d^2x\, T\bar{T}(x).$$

Smirnov, Zamolodchikov '16

$$T^{\lambda}_{\mu\nu} = \frac{2}{\sqrt{g}}\frac{\delta S^{\lambda}}{\delta g^{\mu\nu}} = 2\frac{\partial \mathcal{L}^{\lambda}}{\partial g^{\mu\nu}} - g_{\mu\nu}\mathcal{L}^{\lambda}.$$

Example:

From free bosons to Nambu-Goto

N-free Boson

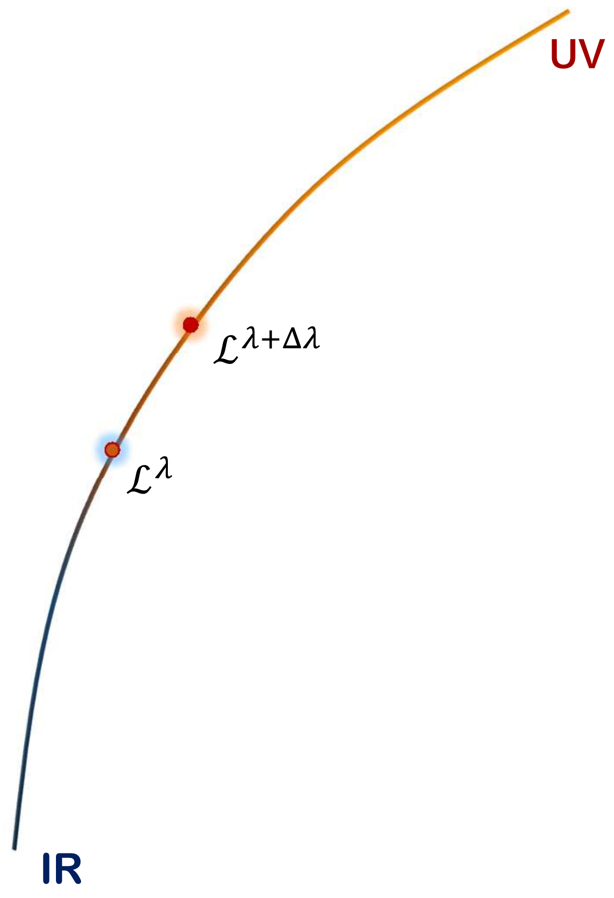
$$\mathcal{L}^0 = \partial\phi\bar{\partial}\phi$$

Before the deformation: (Free boson)

N+2 dimensional String

$$\mathcal{L}^\lambda = \frac{1}{2\lambda}\sqrt{4\lambda\partial\phi\bar{\partial}\phi + 1} - \frac{1}{2\lambda}$$

(Nambu-Goto action
- Noncritical String theory)



Cavaglia,Negro,Szecsenyi,Tateo ‘16

$TTbar$-deformed

torus partition functions

The deformation of correlation function

$$\langle \mathcal{O}_1(z_1,\bar{z}_1)\mathcal{O}_2(z_2,\bar{z}_2)\cdots\mathcal{O}_n(z_n,\bar{z}_n)\rangle_\lambda = \lambda\int d^2z\langle T\bar{T}(z,\bar{z})\mathcal{O}_1(z_1,\bar{z}_1)\mathcal{O}_2(z_2,\bar{z}_2)\cdots\mathcal{O}_n(z_n,\bar{z}_n)\rangle$$

Energy-Momentum conservation

$$\langle \mathcal{O}_1(z_1,\bar{z}_1)\mathcal{O}_2(z_2,\bar{z}_2)\cdots\mathcal{O}_n(z_n,\bar{z}_n)\rangle_\lambda = \lambda\int d^2z\Big(\sum_{i=1}^{n}\big(\frac{h_i}{(z-z_i)^2}+\frac{\partial_{z_i}}{z-z_i}\big)\Big)\Big(\sum_{i=1}^{n}\big(\frac{\bar{h}_i}{(\bar{z}-\bar{z}_i)^2}+\frac{\partial_{\bar{z}_i}}{\bar{z}-\bar{z}_i}\big)\Big) \times \langle \mathcal{O}_1(z_1,\bar{z}_1)\mathcal{O}_2(z_2,\bar{z}_2)\cdots\mathcal{O}_n(z_n,\bar{z}_n)\rangle,$$

Conformal Ward Identity

Go beyond the 1st order?

- **Conformal symmetry does not hold.**

- **Wald Identity breaks down due to higher order deformation of TTbar.**

- **Solve the effective action order by order and go further.**

TTbar deformation on torus:

Field theory defined on torus

has (anti-) periodic boundary condition

$$\text{Bosons}: \phi(z+1) = \phi(z), \qquad \phi(z+\tau) = \phi(z)$$

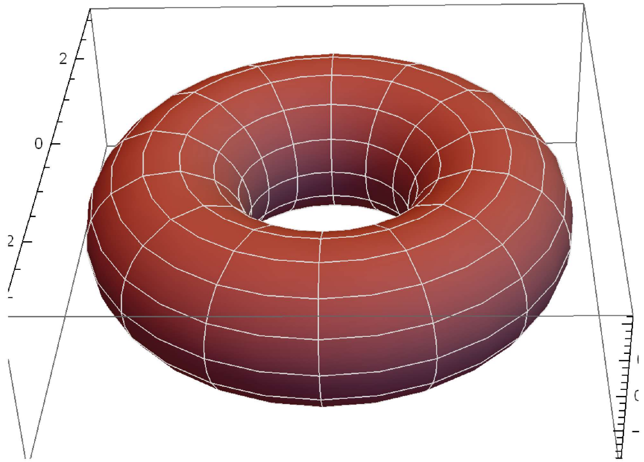
$$\text{Fermions}: \psi(z+1) = e^{2\pi i\nu}\psi(z), \quad \psi(z+\tau) = e^{2\pi i u}\psi(z)$$

$$(\nu, u) = \{(0,0),(0,1/2),(1/2,0),(1/2,1/2)\}$$

Green functions of free theories are comprised

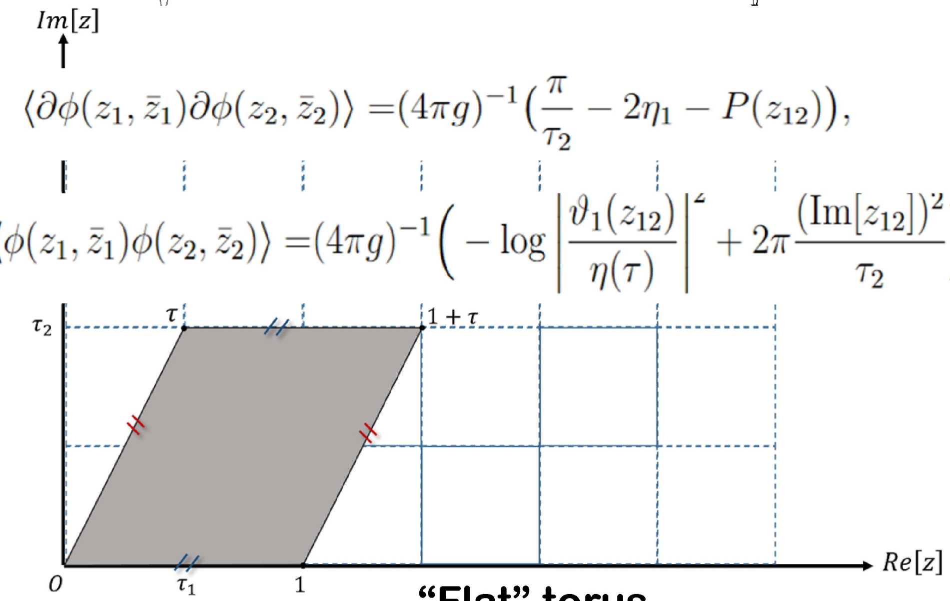
of (anti) double periodic functions

(*Weierstrass* elliptic functions, *Jacobi* theta functions)



$$\langle \partial\phi(z_1,\bar z_1)\partial\phi(z_2,\bar z_2)\rangle = (4\pi g)^{-1}\left(\frac{\pi}{\tau_2} - 2\eta_1 - P(z_{12})\right),$$

$$\langle \phi(z_1,\bar z_1)\phi(z_2,\bar z_2)\rangle = (4\pi g)^{-1}\left(-\log\left|\frac{\vartheta_1(z_{12})}{\eta(\tau)}\right|^2 + 2\pi\frac{(\mathrm{Im}[z_{12}])^2}{\tau_2}\right)$$



"Flat" torus

Path integral formulation:

perturbation approach

The deformed partition function in the Lagrangian path integral formalism

$$Z^{\lambda} = \int [\mathrm{d}\phi] \; \exp\Big\{ - \int_{\mathcal{T}^2} \mathcal{L}^{\lambda}[\phi] \Big\}.$$

$$\frac{\mathrm{d}\mathcal{L}^{\lambda}}{\mathrm{d}\lambda} = \frac{1}{2}\epsilon^{\mu\nu}\epsilon^{\rho\sigma}T^{\lambda}_{\mu\rho}T^{\lambda}_{\nu\sigma};$$

TTbar deformed Partition function

$$
\begin{aligned}
Z^\lambda &= \int \mathcal{D}\phi \, \mathrm{e}^{-\int_{\mathcal{M}} \mathcal{L}^\lambda[\phi]} \\
&= Z^{(0)} - \lambda Z^{(0)} \int_{\mathcal{M}} \langle \mathcal{L}^{(1)} \rangle + \frac{\lambda^2}{2} Z^{(0)} \Big(\int_{\mathcal{M}} \int_{\mathcal{M}'} \langle \mathcal{L}^{(1)}(x) \mathcal{L}^{(1)}(x') \rangle - \int_{\mathcal{M}} \langle \mathcal{L}^{(2)} \rangle \Big) + \mathcal{O}(\lambda^3) \\
&\equiv Z^{(0)} + \lambda Z^{(1)} + \frac{\lambda^2}{2} Z^{(2)} + \mathcal{O}(\lambda^3),
\end{aligned}
$$

Deformed Partition function up to second order

$$
\begin{aligned}
Z^{(0)} &= \int \mathcal{D}\phi \, \mathrm{e}^{-\int_{\mathcal{M}} \mathcal{L}^{(0)}[\phi]}, \\
Z^{(1)} &= - Z^{(0)} \int_{\mathcal{M}} \langle \mathcal{L}^{(1)} \rangle, \\
Z^{(2)} &= Z^{(0)} \Big(\int_{\mathcal{M}} \int_{\mathcal{M}'} \langle \mathcal{L}^{(1)}(x) \mathcal{L}^{(1)}(x') \rangle - \int_{\mathcal{M}} \langle \mathcal{L}^{(2)} \rangle \Big).
\end{aligned}
$$

1st order deformation

$$\mathcal{L}^{(1)} = -\frac{1}{\pi^2} T^{(0)} \bar{T}^{(0)} = -4g^2 (\partial\phi\bar{\partial}\phi)^2,$$

2nd order deformation

$$\mathcal{L}^{(2)} = -\frac{1}{\pi^2} (T^{(0)} \bar{T}^{(1)} + \bar{T}^{(0)} T^{(1)}) = 32g^3 (\partial\phi\bar{\partial}\phi)^3.$$

Grassmanian nature leads to vanishing for fermionic theory

Regulized Z

$$\mathcal{Z}^{(2)} = \mathcal{Z}^{(0)} \left(\int_{\mathrm{T}_1^2} \int_{\mathrm{T}_2^2} \langle \mathcal{L}^{(1)}(x_1) \mathcal{L}^{(1)}(x_2) \rangle - \int_{\mathrm{T}^2} \langle \mathcal{L}^{(2)} \rangle - \frac{2}{\lambda^2} \int_{\mathrm{T}^2} \langle \mathcal{L}_{ct} \rangle \right)$$

SH, Yuan Sun, Yu-Xuan Zhang, 2011.02901

$$\mathcal{L}_{\mathrm{FB,ct}} = \lambda^2 \cdot \left\{ \frac{8g^2}{\pi\epsilon^2} (\partial\phi\bar{\partial}\phi)^2 + \frac{1}{24\pi^3\epsilon^6} \right\}$$

Free bosons

$$\left(Z^{(1)} = -Z^{(0)} \int_{\mathcal{T}^2} \langle \mathcal{L}^{(1)} \rangle, \right)$$

$$\left(Z^{(2)} = Z^{(0)} \Big(\int_{\mathcal{T}^2} \int_{\mathcal{T}'^2} \langle \mathcal{L}^{(1)}(x) \mathcal{L}^{(1)}(x') \rangle - \int_{\mathcal{T}^2} \langle \mathcal{L}^{(2)} \rangle \Big). \right)$$

$$S_{FB} = \frac{g}{2} \int_{\mathcal{T}^2} \mathrm{d}^2 x \partial_\mu \phi \partial^\mu \phi,$$

$$Z^{(0)} = \frac{1}{\sqrt{\tau_2} |\eta(\tau)|^2}$$

Free Boson

The additional term contribution comes from $\int_{\mathcal{T}^2} \langle \mathcal{L}^{(2)} \rangle$

$$Z^{(1)} = \frac{1}{Z^{(0)}} \tau_2 \partial_\tau \partial_{\bar{\tau}} Z^{(0)},$$

O. Aharony, S. Datta, A. Giveon, Y. Jiang and D. Kutasov, 18 1806.07426

$$Z^{(2)} = 16(\tau_2 \partial_\tau^2 \partial_{\bar{\tau}}^2 + i\tau_2 (\partial_\tau^2 \partial_{\bar{\tau}} - \partial_{\bar{\tau}}^2 \partial_\tau)) Z^{(0)} + (72 \partial_\tau \partial_{\bar{\tau}} - 6\tau_2^{-2}) Z^{(0)}.$$

Free bosons

$$\left(Z^{(1)} = -Z^{(0)} \int_{\mathcal{T}^2} \langle \mathcal{L}^{(1)} \rangle,\right)$$

$$\left(Z^{(2)} = Z^{(0)} \left(\int_{\mathcal{T}^2} \int_{\mathcal{T}'^2} \langle \mathcal{L}^{(1)}(x) \mathcal{L}^{(1)}(x') \rangle - \int_{\mathcal{T}^2} \langle \mathcal{L}^{(2)} \rangle \right).\right)$$

$$S_{FB} = \frac{g}{2} \int_{\mathcal{T}^2} d^2x \partial_\mu \phi \partial^\mu \phi,$$

$$\langle \partial\phi(z_1, \bar{z}_1) \bar{\partial}\phi(z_2, \bar{z}_2) \rangle = (4\pi g)^{-1} \left(\pi \delta^{(2)}(z_{12}) - \frac{\pi}{\tau_2} + \sum_{\{m,n\} \neq \{0,0\}} \pi \delta^{(2)}(z_{12} - (m + n\tau)) \right),$$

$$Z^{(0)} = \frac{1}{\sqrt{\tau_2} |\eta(\tau)|^2}$$

Free Boson

$$Z^{(1)} = \frac{1}{Z^{(0)}} \tau_2 \partial_\tau \partial_{\bar{\tau}} Z^{(0)},$$ O. Aharony, S. Datta, A. Giveon, Y. Jiang and D. Kutasov, 18

$$Z^{(2)} = 16(\tau_2 \partial_\tau^2 \partial_{\bar{\tau}}^2 + i\tau_2(\partial_\tau^2 \partial_{\bar{\tau}} - \partial_{\bar{\tau}}^2 \partial_\tau)) Z^{(0)} + (72\partial_\tau \partial \ldots \tau_2^{-2}$$

Generic operator: KdV

$$P_s^\lambda = \frac{1}{2\pi}\int_0^L \left(\mathrm{d}z T_{s+1}^\lambda + \mathrm{d}\bar{z}\Theta_{s-1}^\lambda\right)$$

$$\begin{aligned}
&\langle \mathcal{O}^\lambda \rangle_{\text{tor.}}^\lambda \\
=&\frac{1}{\mathcal{Z}^\lambda}\int \mathcal{D}\phi \mathcal{O}^\lambda \exp\left\{-\int_{\mathrm{T}^2}\mathcal{L}^\lambda\right\} \\
=&\langle \mathcal{O}^{(0)}\rangle_{\text{tor.}} + \lambda\cdot\left\{\langle \mathcal{O}^{(1)}\rangle_{\text{tor.}} + \langle \mathcal{O}^{(0)}\rangle_{\text{tor.}}\int_{\mathrm{T}^2}\langle \mathcal{L}^{(1)}\rangle_{\text{tor.}} - \int_{\mathrm{T}_1^2}\langle \mathcal{O}\mathcal{L}^{(1)}(z_1,\bar{z}_1)\rangle_{\text{tor.}}\right\} + O(\lambda^2)
\end{aligned}$$

Operator with flow effect:

$$\mathcal{O}^\lambda = \sum_{n=0}^\infty \frac{\lambda^n}{n!}\mathcal{O}^{(n)}, \ \langle \mathcal{O}\rangle_{\text{tor.}} \equiv \mathrm{Tr}\left[e^{-\beta H}\right]^{-1}\cdot \mathrm{Tr}\left[e^{-\beta H}\mathcal{O}\right].$$

$$\langle T^\lambda\rangle_{\text{tor.FB}}^\lambda = \left(\eta_1 - \frac{\pi}{2\tau_2}\right) + \lambda\cdot\left(\frac{2|\eta_1|^2}{\pi} - \frac{1}{2\tau_2}\left(\eta_1+\bar{\eta}_1\right) + \left(\frac{2}{\pi}\tau_2\bar{\eta}_1 - 1\right)i\partial_\tau\eta_1\right) + O(\lambda^2)$$

$$\langle \Theta^\lambda\rangle_{\text{tor.FB}}^\lambda = \lambda\cdot\left(-\frac{|\eta_1|^2}{\pi} + \frac{2}{\tau_2}\left(\eta_1+\bar{\eta}_1\right) - \frac{3\pi}{4\tau_2^2}\right) + O(\lambda^2).$$

SH, Yuan Sun, Yu-Xuan Zhang, 2011.02901

Resulting VEV of KdV

$$P_s^\lambda = \frac{1}{2\pi}\int_0^L \left(\mathrm{d}z T_{s+1}^\lambda + \mathrm{d}\bar{z}\Theta_{s-1}^\lambda\right)$$

$$\langle P_1^\lambda\rangle_{\mathrm{o,FB}}^\lambda = \frac{\pi}{12} + \lambda\cdot\frac{\pi^2}{72} + O(\lambda^2), \quad \text{(periodic B.C.)}$$

$$\langle P_1^\lambda\rangle_{\mathrm{o,DF}}^\lambda = \frac{\pi}{12} + \lambda\cdot\frac{\pi^2}{72} + O(\lambda^2), \quad \text{(antiperiodic B.C.)}$$

$$\langle P_1^\lambda\rangle_{\mathrm{o,MF}}^\lambda = \frac{\pi}{24} + \lambda\cdot\frac{\pi^2}{288} + O(\lambda^2), \quad \text{(antiperiodic B.C.)}$$

SH, Yuan Sun, Yu-Xuan Zhang, 2011.02901

$$\langle P_1^\lambda\rangle_{\mathrm{o,DF}}^\lambda = -\frac{\pi}{6} + \lambda\cdot\frac{\pi^2}{18} + O(\lambda^2), \quad \text{(periodic B.C.)}$$

$$\langle P_1^\lambda\rangle_{\mathrm{o,MF}}^\lambda = -\frac{\pi}{12} + \lambda\cdot\frac{\pi^2}{72} + O(\lambda^2). \quad \text{(periodic B.C.)}$$

V.S. M. Asrat, 2002.04824

Wormhole induced by 1D TTbar deformation

2D TTbar

Massive Gravity

A. J. Tolley, 1911.06142

$$S_\lambda = S_{\rm grav}[g_{\mu\nu}, \gamma_{\mu\nu}] + S_0[g_{\mu\nu}, \psi].$$

$$g_{\mu\nu} = \delta_{ab} e^a_\mu e^b_\nu \,, \quad \gamma_{\mu\nu} = \delta_{ab} f^a_\mu f^b_\nu \,,$$

$$S_{\rm grav}[e^a_\mu, f^a_\mu] = \frac{1}{2\pi^2\lambda} \int d^2x \, \epsilon^{\mu\nu} \epsilon_{ab} (e^a_\mu - f^a_\mu)(e^b_\nu - f^b_\nu)$$

EOM of f^a_μ

$$T^\mu_a \equiv \frac{2\pi}{\det(f^a_\mu)} \frac{\delta S_\lambda[e, f, \psi]}{\delta f^a_\mu} = -\frac{2}{\pi\lambda \det(f^a_\mu)} \epsilon^{\mu\nu} \epsilon_{ab} (e^b_\nu - f^b_\nu)$$

V.S.

EOM of e^a_μ

$$\frac{1}{\pi^2\lambda} \epsilon^{\mu\nu} \epsilon_{ab} (e^{*b}_\nu - f^b_\nu) + \frac{\delta S_0[e, \psi]}{\delta e^{*a}_\mu} = 0,$$

$T^a_{0\mu}$

on-shell value of $S_{\rm grav}$

$$\frac{dS_\lambda}{d\lambda} = -\frac{1}{4} \int d^2x \, \det(T^a_\mu)$$

Same as TTbar flow eq.

Massive Gravity

1D TTbar

$$S[e_\mu, v^\mu, \phi] = S_{\text{grav}}[e_\mu, v^\mu] + S_0[e_\mu, \phi]$$

$$S_0 = \int dt e_t \left(\frac{1}{e_t} p \partial_t \phi - H_0(\phi, p) \right)$$

$$S_{\text{grav}}[e_\mu, v^\mu] = \frac{1}{\lambda} \int dt e_t B(e_t v^t)$$

$$v^T = \frac{dT}{dt}, \quad e_T = 1, \quad e_t = \frac{dT}{dt}$$

$$\frac{dT}{dt} B'\left(\frac{dT}{dt}\right) + B\left(\frac{dT}{dt}\right) - \lambda H_0 = 0.$$

One H deformation

SH, Zhuoyu Xian2104.03852

TTbar deformed Hamiltonian

$$f(H) = \frac{1 - \sqrt{1 - 8H\lambda}}{4\lambda},$$

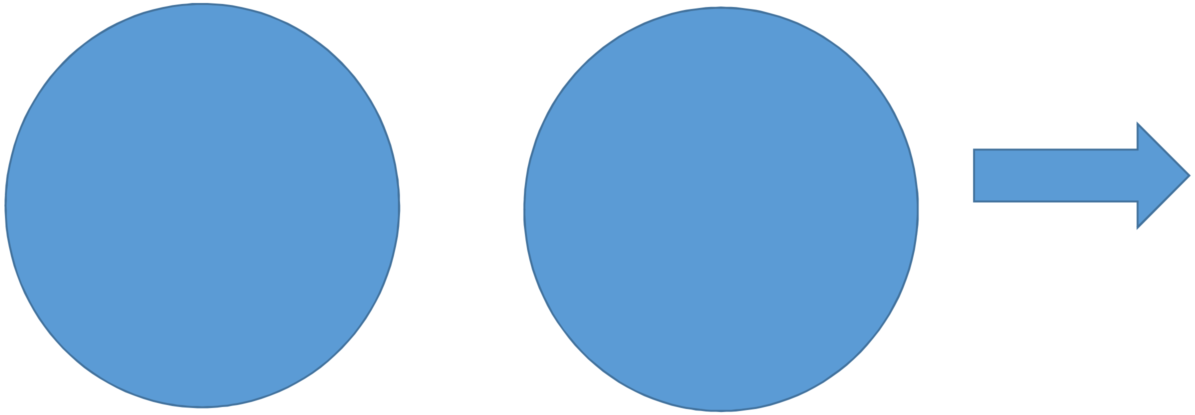
D. J. Gross, J. Krutho, A. Rolph and E. Shaghoulian,1912.06132

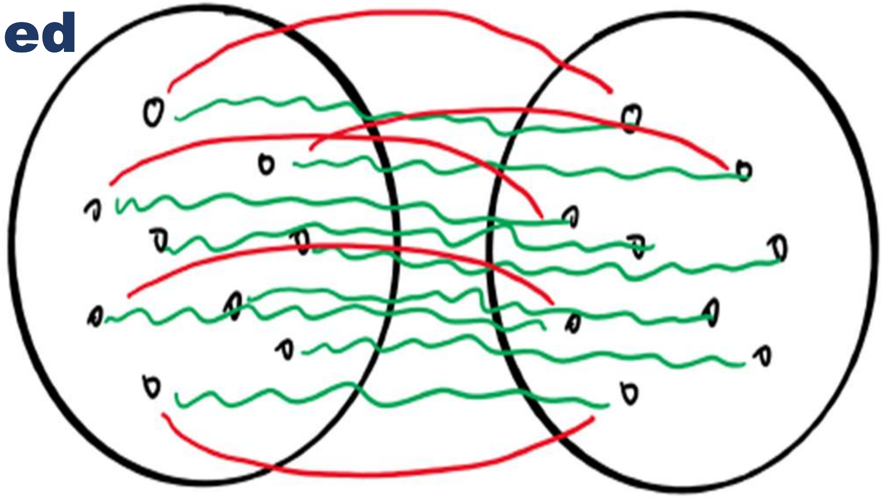
$$B(x) = \frac{(x-1)^2}{8x^2}.$$

H=H_L+H_R

$$f(H) = \frac{1-\sqrt{1-8H\lambda}}{4\lambda},$$

TTbar deformed





$$\mathcal{L}_\lambda = \frac{\sqrt{(1+4\lambda\sum_s \phi'_s\phi'_s)(1-8\lambda V(\vec{\phi}))} - 1}{4\lambda}$$

TFD under the $T\bar{T}$ deformation

Consider a deformation

$$H_\lambda = f(H_L + H_R), \quad e.g.\ f(H) = H + 2\lambda H^2.$$

$T\bar{T}$ deformed TFD: Deform and evolve the TFD $|\Psi\rangle$ with H_λ

$$|\Psi_\lambda\rangle = \sum_E e^{-\beta f(E)/2} |E\rangle_L |E\rangle_R, \quad \rho_\lambda = \sum_E e^{-\beta f(E)} |E\rangle \langle E|$$

$$|\Psi_\lambda(t)\rangle = e^{-itf(H_0)} |\Psi_\lambda\rangle, \quad \rho_\lambda(t) = \rho_\lambda.$$

Normalization $\left|\tilde{\Psi}\right\rangle = |\Psi\rangle / \sqrt{Z(\beta)}$, $\left|\tilde{\Psi}_\lambda\right\rangle = |\Psi_\lambda\rangle / \sqrt{Z_\lambda(\beta)}$.

Causal correlation caused by the $T\bar{T}$ deformation

Up to 1st order $$f(H) = H + 2\lambda H^2$$

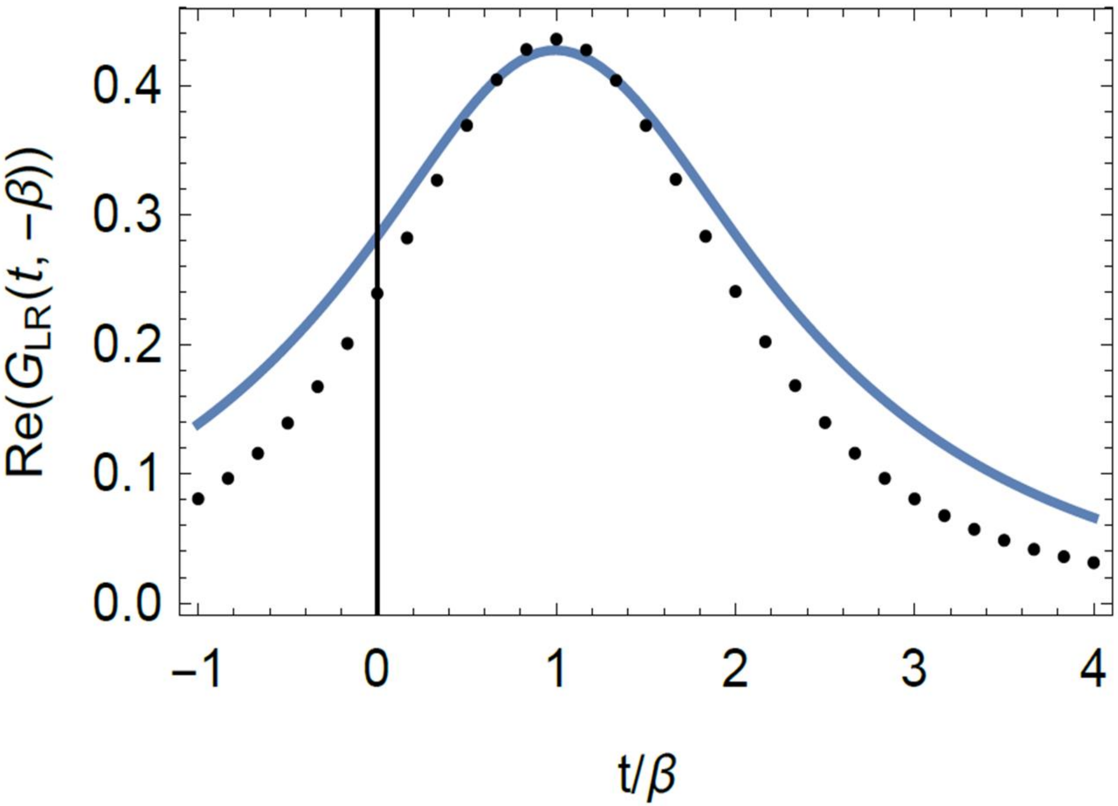
$$O_L = O \otimes 1, \quad O_R = 1 \otimes O^T.$$

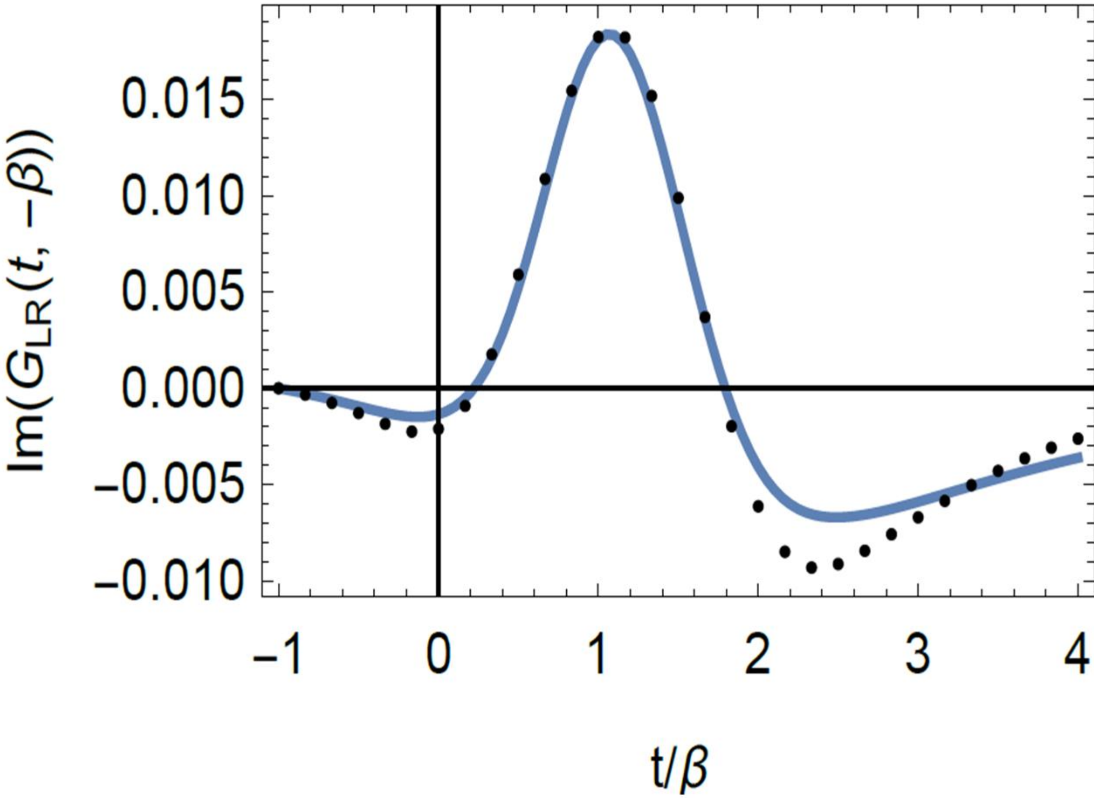
the deformed retarded correlator

$$\begin{aligned} G^R_{LR}(t_1, t_2) &= -i\Theta(t_-)\left\langle \tilde{\Psi} \middle| [O_L(t_1), O_R(t_2)] \middle| \tilde{\Psi} \right\rangle = 2\Theta(t_-)\mathrm{Im}\left\langle \tilde{\Psi} \middle| O_L(t_1) O_R(t_2) \middle| \tilde{\Psi} \right\rangle \\ &= -i\Theta(t_-)\left\langle \tilde{\Psi} \middle| \left(-4i\lambda t_- \dot{O}^{(0)}_L(t_1)\dot{O}^{(0)}_R(t_2) + \mathcal{O}[\lambda^2]\right) \middle| \tilde{\Psi} \right\rangle \\ &= -4\lambda t_- \Theta(t_-) G''_W(-t_+;\beta) + \mathcal{O}[\lambda^2], \end{aligned}$$

Non-vanishing means the causal correlation.

$G''_W(-t_+;\beta)$ is maximized at $t_+ = 0$. So the signal comes out from QM_L around the time $t_1 = -t_2$.





$G_{LR}(t, -\beta)$ for the SYK model.

Summary and future works

Summary

- **The second order correction contains the TTbar flow effect in path integral formalism.**

Partition function & KdV operator

- **Reformalize the 1D TTbar deformation to realize the quantum transversible wormhole.**

Future works

Generic higher order correlation functions on 2D Rieman surface in deformed theory.
Formulate the JT+CFTs in terms of TTbar deformation
2D Bosonization of TTbar…
Correlation functions in Lorentz-breaking theory [M. Guica, Cardy…], …
How can we do TTbar deformation of the gravity theory

Thank you for your attention!