Thermodynamic limit of Nekrasov partition function for 5-brane web with O5-plane

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Mainly based on the work with Xiaobin Li (SWJTU)
[arXiv:2102.09482]

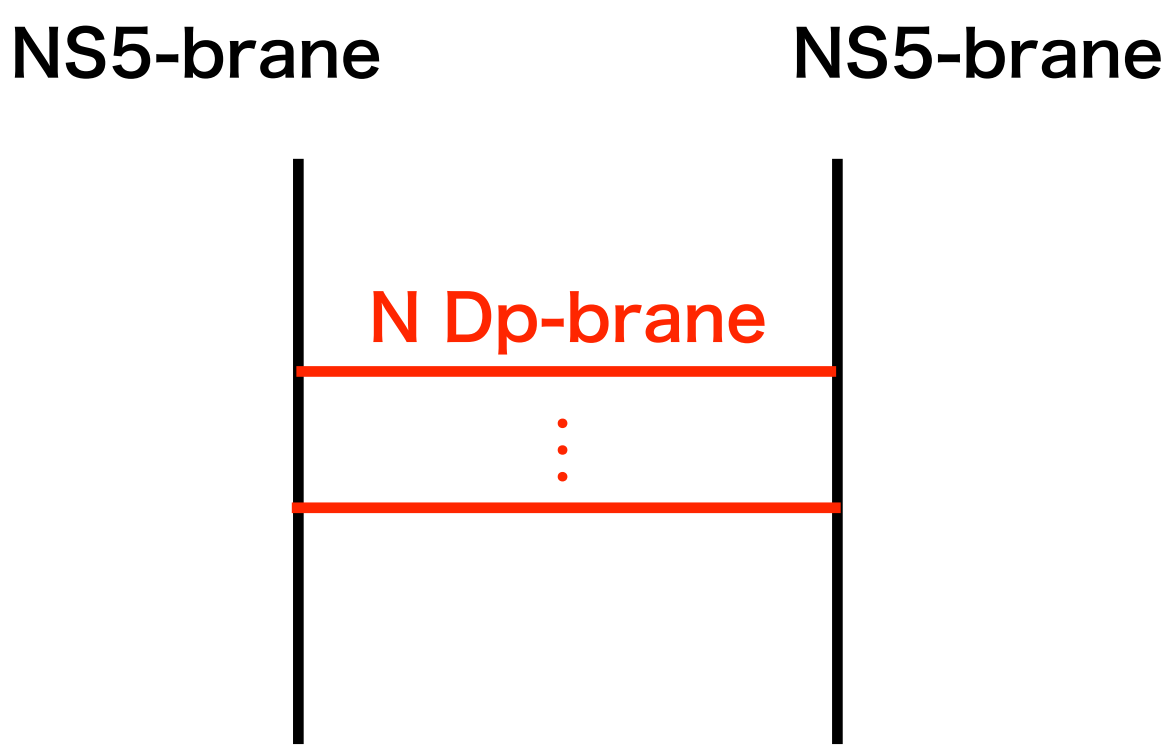
§1. Introduction

Study 5d N=1 SUSY gauge theory based on 5-brane web diagram

(Some of the) 5-brane web diagram is a variation of

Hanany-Witten type brane set-up

p dimensional SU(N) gauge theory with 8 SUSY
is realized by the following
Hanany-Witten type brane set-up



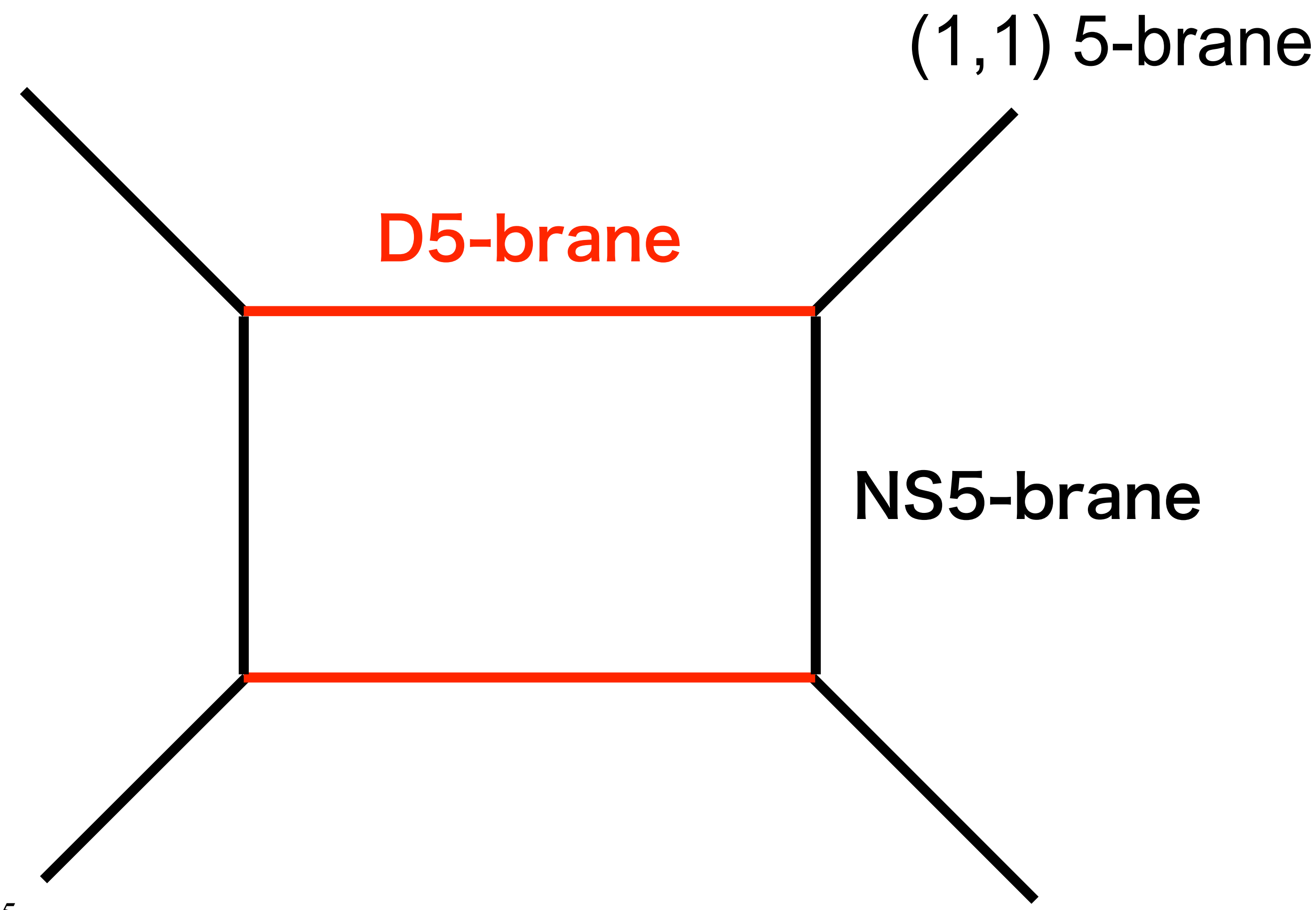
	0	1	…	p-1	p	…	6	7	8	9
NS5-brane	-	-		-	-	·	·	·	·	·
Dp-brane	-	-		-	·	-	-	·	·	·

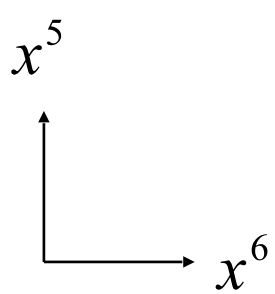
p=3 [Hanany, Witten ’96]

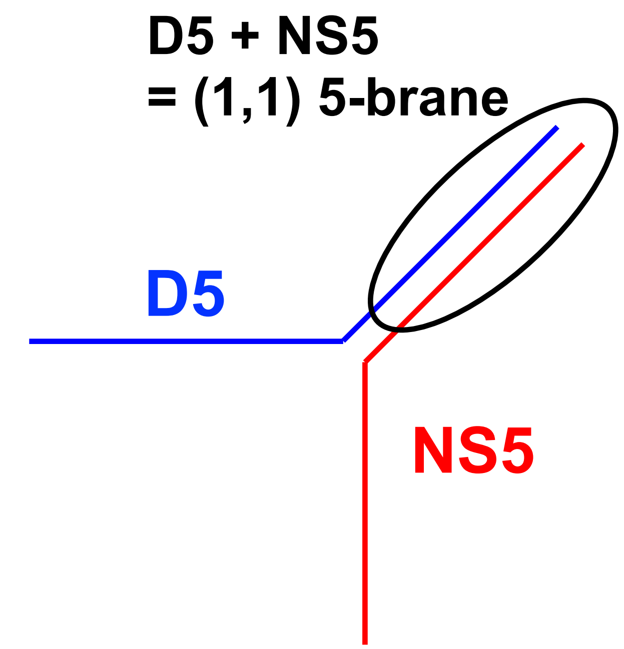
p=4 [Witten ’97]

“(p,q) 5-brane” is necessary to realize 5d theory

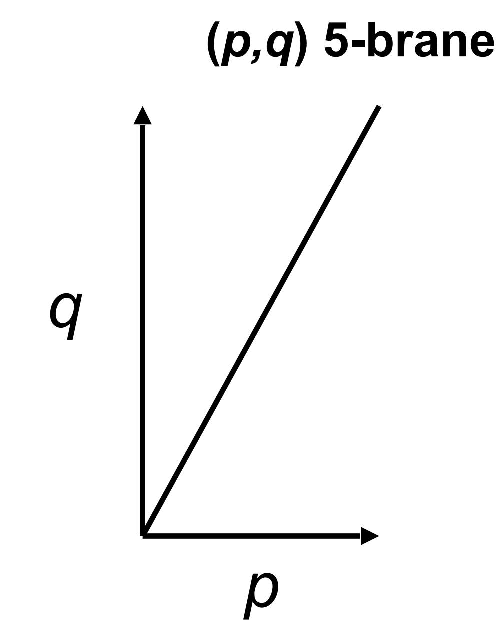
ex. 5D N=1 SU(2) SYM







[Aharony, Hanany ’97]
[Aharony, Hanany, Kol ’97]



	0	1	2	3	4	5	6	7	8	9
5-brane	-	-	-	-	-	web		·	·	·

Power of 5-brane web diagram

- **Existence of UV fixed point**

It is expected that the number of hypermultiplets have upper bound in order for the given 5d gauge theory to have UV fixed point.

Approach from field theory
[Seiberg '97], [Intriligator, Morrisson, Seiberg '97], [Yonekura '15], [Jefferson, Kim, Vafa, Zafrir '17], …

Approach from string theory (5-brane web)
[Aharony, Hanany '97], [Aharony, Hanany, Kol '97], [DeWolfe, Hanany, Iqbal, Katz '99], [Benini, Benvenuti, Tachikawa 09'], [Bergman, Zafrir '14'15], [Hayashi, S.S.Kim, K.Lee, Taki, FY '15], ⋯

Existence of 5-brane web → Existence of fixed point

Power of 5-brane web diagram

- **UV duality**

  Two different 5d N=1 gauge theories have identical UV fixed point.

  - $SU(N)^{M-1} \Leftrightarrow SU(M)^{N-1}$ (c.f. Fiber-base duality)

    [Katz, Mayer, Vafa '97], [Aharony, Hanany, Kol '97] [Bao, Pomoni, Taki, FY '11], ...

  - $Sp(N) \Leftrightarrow SU(N+1)_{N+3}$

    [Gaiotto, H.C.Kim '15], [Hayashi, S.S.Kim, K.Lee, Taki, FY '15], ...

  - $G_2 \Leftrightarrow SU(3)_7$

    [Jefferson, Katz, H.C. Kim, Vafa '17], [Hayashi, S.S.Kim, K.Lee, Taki, FY '18], ...,

Power of 5-brane web diagram

- **Cubic prepotential**

5d N=1 gauge theories have exact cubic prepotential.
[Seiberg '97], [Intriligator, Morrisson, Seiberg '97], [Morrisson, Seiberg '97],...

It can be reproduced by computing the area of the faces in the 5-brane web diagram.
[Aharony, Hanany, Kol '97], ...

It is checked in various examples.
[Hayashi, S.S.Kim, K.Lee, Taki, FY '17, '18, '19], [Closset, del Zotto, Saxena '18] , ...

Power of 5-brane web diagram

- **Higgs branch at infinite coupling**

Unlike Coulomb branch, Higgs branch does not get quantum effect in general. However, for 5d gauge theories extra direction opens up at the UV fixed point, where gauge coupling is infinite.
[Cremonesi, Ferlito, Hanany, Mekareeya '15],

This can be captured by the decomposition of 5-brane web to its sub-web.
[Cabrera, Hanany, Yagi '18], [Akhond, Carta, Dwivedi, Hayashi, S.S. Kim, FY '20, '21], [Bourget, Grimminger, Hanany, Sperling, Zafrir, Zhong '20], [Bourget, Eckhard, Schafer-Nameki, van Beest '20], …

Power of 5-brane web diagram

- **Seiberg-Witten curve**

By taking T-duality and by uplifting to M-theory, we obtain single M5-brane, which is identified as Seiberg-Witten curve.

[Witten ’97], [Brandhuber, Itzhaki, Sonnenschein, Theisen, Yankielowicz ’97], [Aharony, Hanany, Kol ’97], …

Generalizing to the 5-brane web diagram with O5-plane is discussed with careful treatment of the boundary conditions.

[Landsteiner, Lopez, Lowe '97], [Hayashi, S.S.Kim, K.Lee, Taki, FY ’17], [X.Li, FY ’21]

⇒ Section 2 of this talk

Power of 5-brane web diagram

- **Nekrasov partition function**

5-brane web is mapped to a (toric) Calabi-Yau 3-fold.
[Leung, Vafa ’02]

Topological string partition function of Calabi-Yau 3-fold is computed by using topological vertex.
[Aganagic, Klemm, Marino, Vafa ’03]

Topological string partition function agrees with Nekrasov partition function for 5d gauge theory on S^1
[Nekrasov ’03], [Iqbal, Kashani-Poor '02, '03], [Eguchi, Kanno '03, '04] [Hollowood, Iqbal, Vafa '03], [J. Zhou, ’03], ...

Generalization to 5-brane web diagram with O5-plane is proposed and checked.
[S.S. Kim, FY ’17], [Hayashi, S.S.Kim, K.Lee, Taki, FY ’18, ’19], [Hayashi, Zhu, 20], [H.C. Kim, M. Kim, S.S.Kim ’21], [X. Li, FY ’21]…

⇒ **Section 3 of this talk**

5-brane web diagram

Seiberg-Witten curve
(Seiberg-Witten prepotential)

$$F$$

Nekrasov partition function
(Topological string partition function)

$$Z$$

[Nekrasov ’03], [Nekrasov, Okounkov ’03], [Nakajima, Yoshioka ’03,],[Etingof ’04], ...

Thermodynamic limit

$$F = -\lim_{\hbar \to 0} \hbar^2 \log Z \qquad (\varepsilon_1 = -\varepsilon_2 = \hbar)$$

5-brane web diagram with O5-plane

Boundary conditions
[Hayashi,S.S.Kim, K.Lee, Taki, FY '17]

§2

§3

Topological vertex formalism
with O5-plane
[S.S. Kim, FY'17]

Seiberg-Witten curve
(Seiberg-Witten prepotential)

F

Nekrasov partition function
(Topological string partition function)

Z

§4

[X. Li, FY '21]

Thermodynamic limit

$$F = -\lim_{\hbar \to 0} \hbar^2 \log Z \qquad (\varepsilon_1 = -\varepsilon_2 = \hbar)$$

Plan of this talk

§1 Introduction (Done)

§2 Seiberg-Witten curve from 5-brane web diagram with O5-plane

§3 Nekrasov partition function from 5-brane web diagram with O5-plane

§4 Deriving Seiberg-Witten prepotential from Nekrasov partition function (Thermodynamics limit)

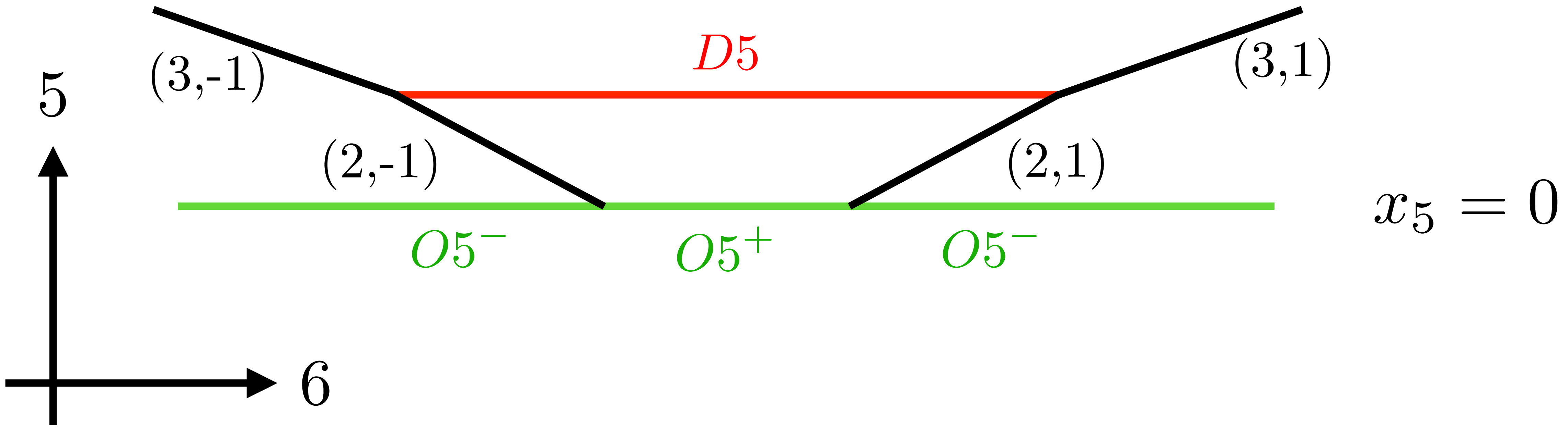
§5 Conclusion

§2 Seiberg-Witten curve from 5-brane web diagram with O5-plane

IIB → IIA → M

	0	1	2	3	4	5	6	7	8	9
5-brane	—	—	—	—	—	web		·	·	·
O5-plane	—	—	—	—	—	·	—	·	·	·

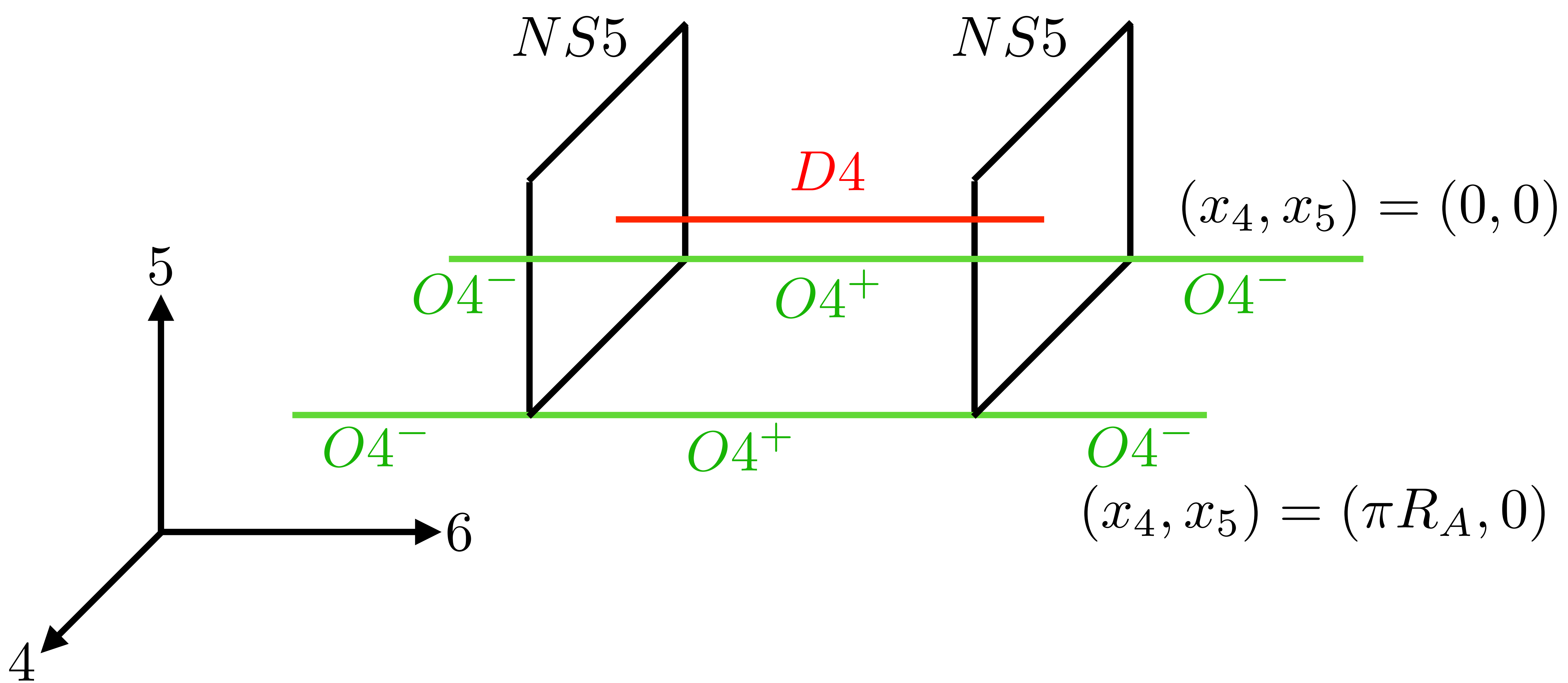
ex. 5D N=1 Sp(1) SYM



We take T-dual along direction “4”

IIB → IIA → M

	0	1	2	3	4	5	6	7	8	9
NS5-brane	—	—	—	—	—	—	·	·	·	·
D4-brane	—	—	—	—	·	·	—	·	·	·
O4-plane	—	—	—	—	·	·	—	·	·	·



IIB → IIA → M

	0	1	2	3	4	5	6	7	8	9	11
M5-brane	—	—	—	—	x_4	x_5	x_6	·	·	·	x_{11}
OM5-plane	—	—	—	—	·	·	—	·	·	·	—

M5-brane: Riemann surface in (x_4, x_5, x_6, x_{11})-space

$$t^2 + P(w)t + Q(w) = 0.$$

$$P(w) = P(w^{-1}), \quad Q(w) = Q(w^{-1})$$

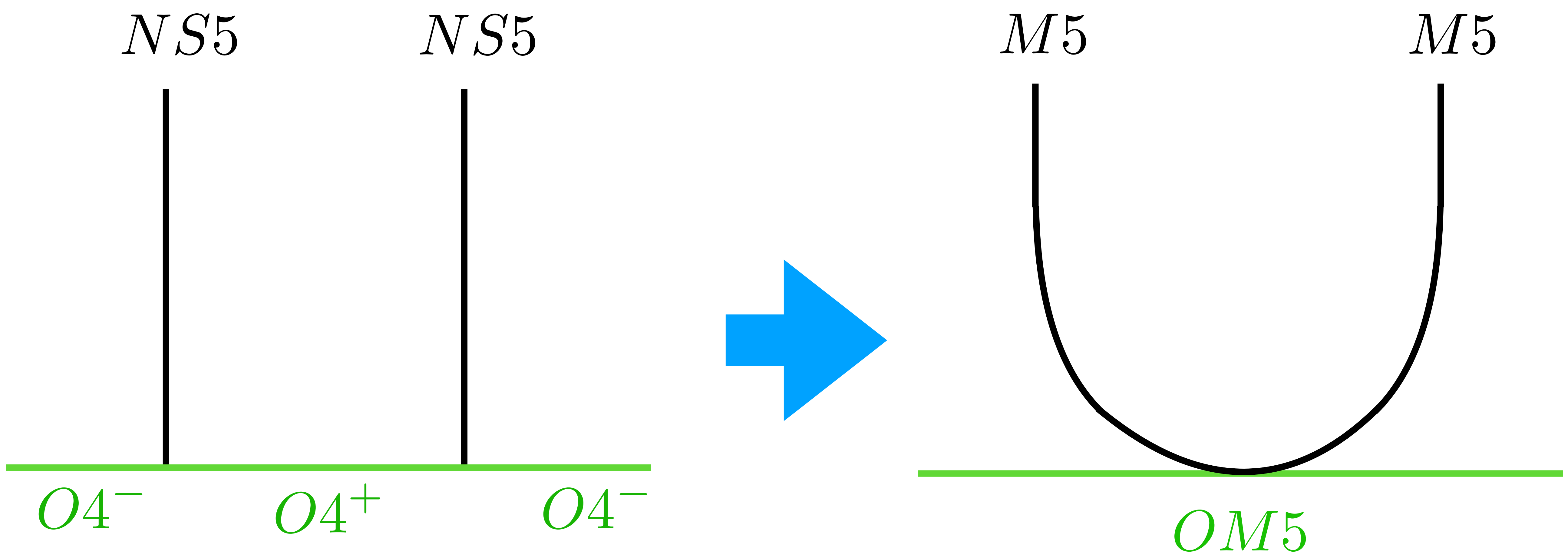
$$w = e^{-\frac{x^5 + ix_4}{R_A}} \qquad t = e^{-\frac{x^6 + ix_{11}}{R_M}}$$

→ **Seiberg-Witten curve**

Boundary condition at OM5-plane

The curve has a double root at OM5-plane

cf [Hori '96]
[Landsteiner, Lopez, Lowe 97']



Seiberg-Witten curve for 5d Sp(N) gauge theory with flavors

$$t^2 + P(w)t + Q(w) = 0.$$

$$P(w) = P(w^{-1}), \quad Q(w) = Q(w^{-1})$$

Boundary conditions the two OM5-planes at w=1, -1

$$P(1)^2 - 4Q(1) = 0, \quad P(-1)^2 - 4Q(-1) = 0$$

[Hayashi, S.S.Kim, K.Lee, Taki, FY '17], [X. Li, FY '21]

In the past literatures the following condition was imposed

$$P(1) = 0, \quad P(-1) = 0$$

[Brandhuber, Itzhaki, Sonnenschein, Theisen, Yankielowicz '97]

Seiberg-Witten prepotential is obtained from Seiberg-Witten curve

Seiberg-Witten curve: $t^2 + P(w)t + Q(w) = 0.$

$$\lambda_{SW} := -\frac{1}{2\pi i\beta}\log w\, d(\log t),$$

$$\oint_{A_I}\lambda_{SW} = a_I, \quad (I = 1, 2, \cdots, N, N+1), \qquad \sum_{I=1}^{N+1} a_I = -m_0 + \frac{1}{2}\sum_{j=1}^{N_f} m_j,$$

$$\oint_{B_I}\lambda_{SW} = \frac{1}{2\pi i\beta}\left(\frac{\partial F}{\partial a_I} - \frac{\partial F}{\partial a_{I+1}}\right) \quad (I = 1, 2, \cdots, N-1), \qquad \oint_{B_N}\lambda_{SW} = \frac{1}{2\pi i\beta}\frac{\partial F}{\partial a_N}.$$

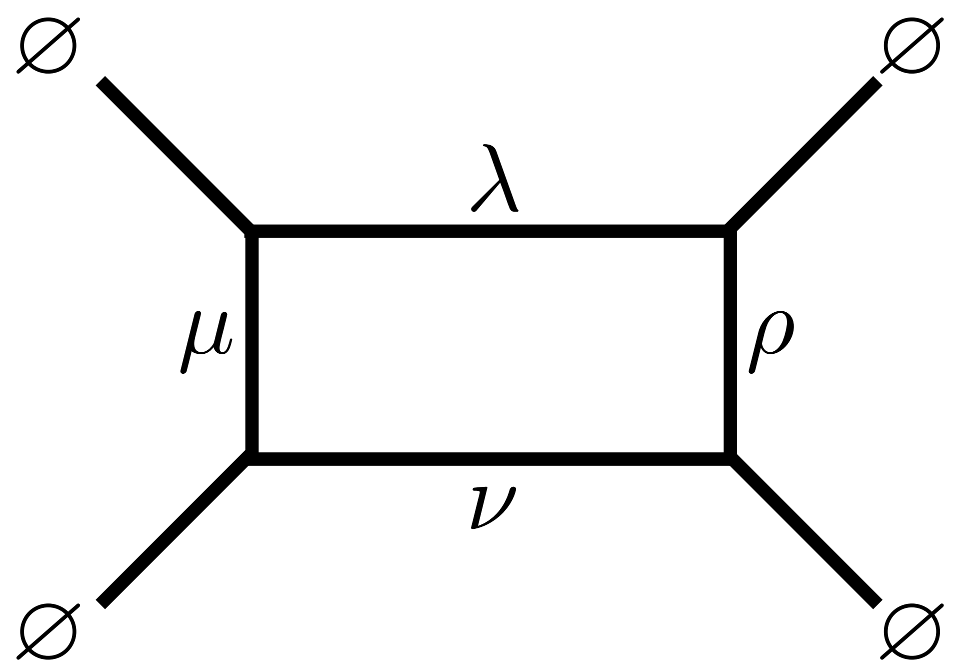
↓

Seiberg-Witten prepotential: F

§3 Nekrasov partition function from 5-brane web diagram with O5-plane

Topological vertex formalism

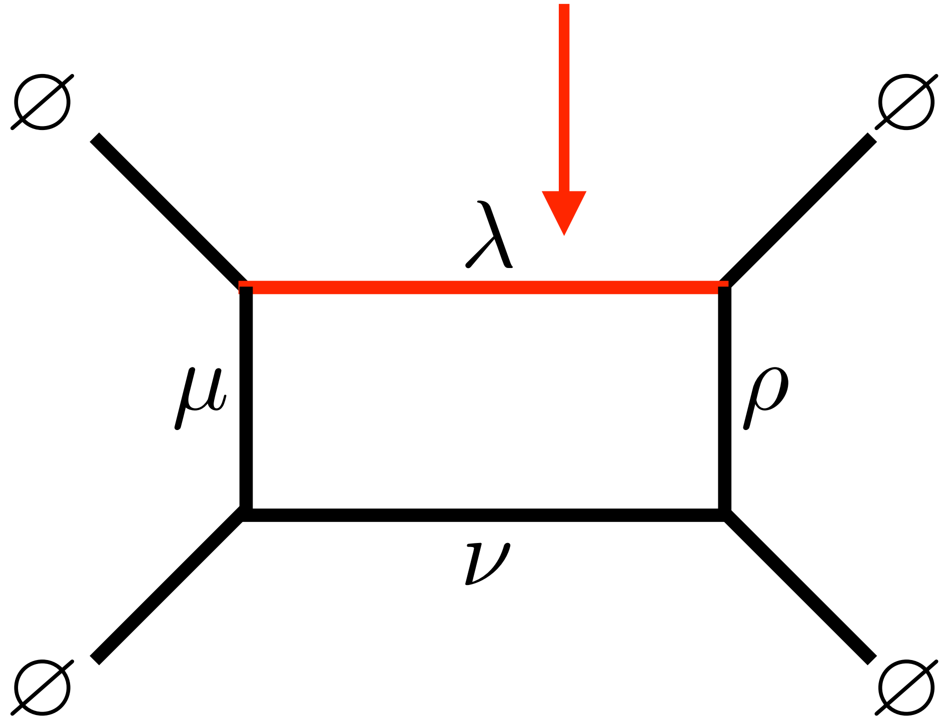
Assign different Young diagrams to each edge



Topological vertex formalism

Assign different Young diagrams to each edge

Edge Factor: $E_\lambda(g, Q)$



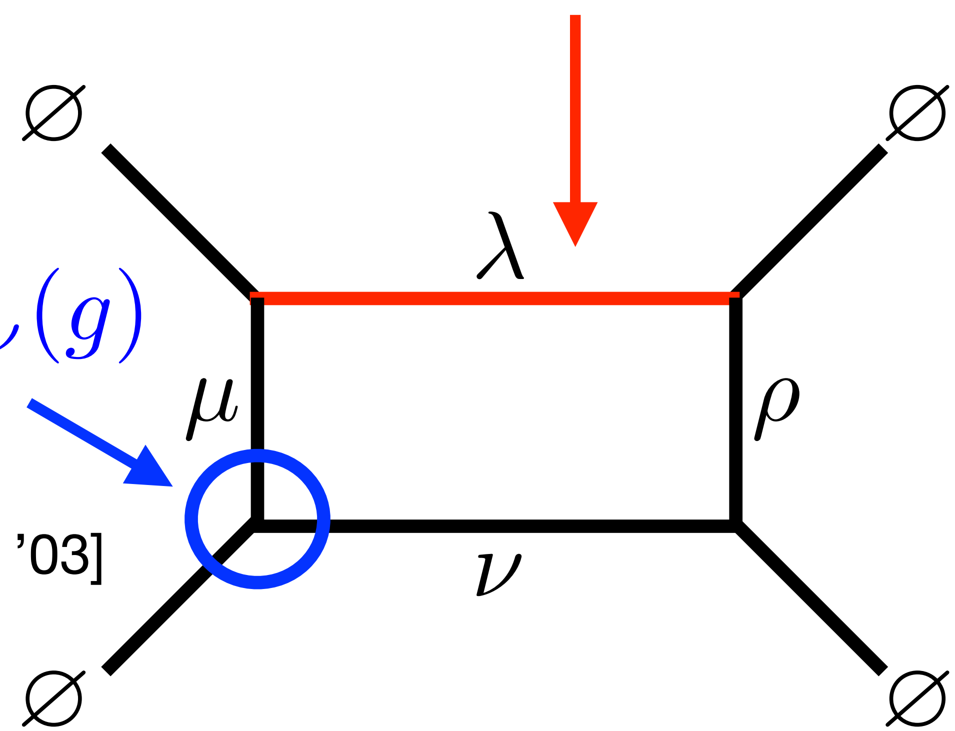
$Q = \exp\left(-\text{“Length”}\right)$

Coulomb moduli, gauge coupling, masses

Topological vertex formalism

Assign different Young diagrams to each edge

Edge Factor: $E_\lambda(g, Q)$



Vertex Factor: $C_{\varnothing\mu\nu}(g)$

(topological vertex)

[Aganagic, Klemm, Marino, Vafa '03]

$Q = \exp(-\text{"Length"})$

Coulomb moduli, gauge coupling, masses

Topological vertex formalism

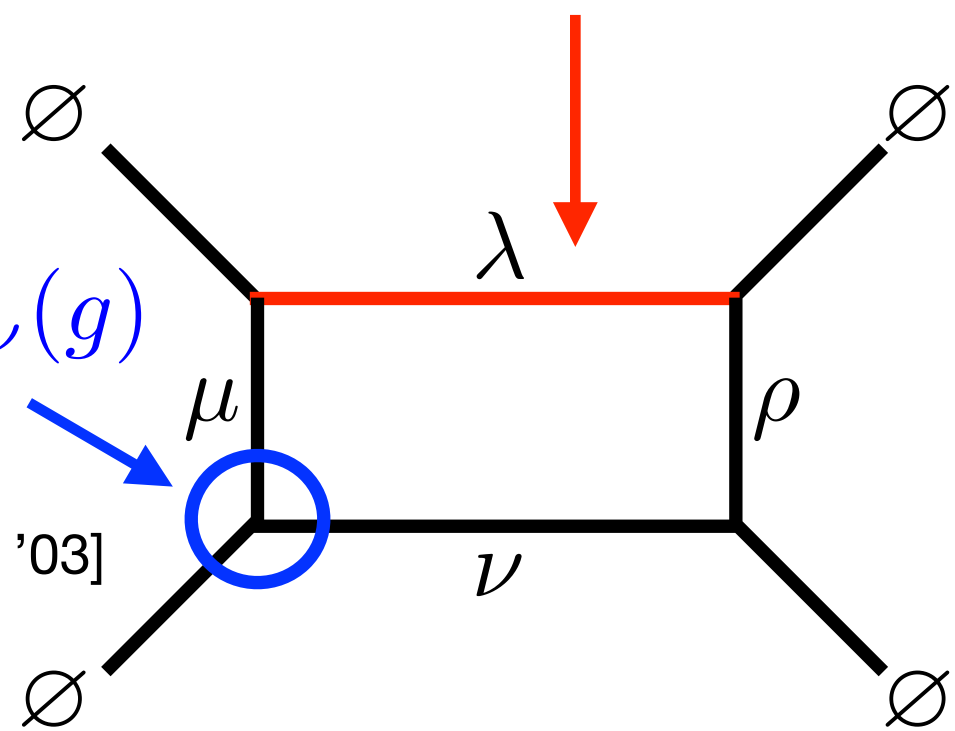
Assign different Young diagrams to each edge

Edge Factor: $E_\lambda(g, Q)$

Vertex Factor: $C_{\varnothing\mu\nu}(g)$

(topological vertex)

[Aganagic, Klemm, Marino, Vafa ’03]



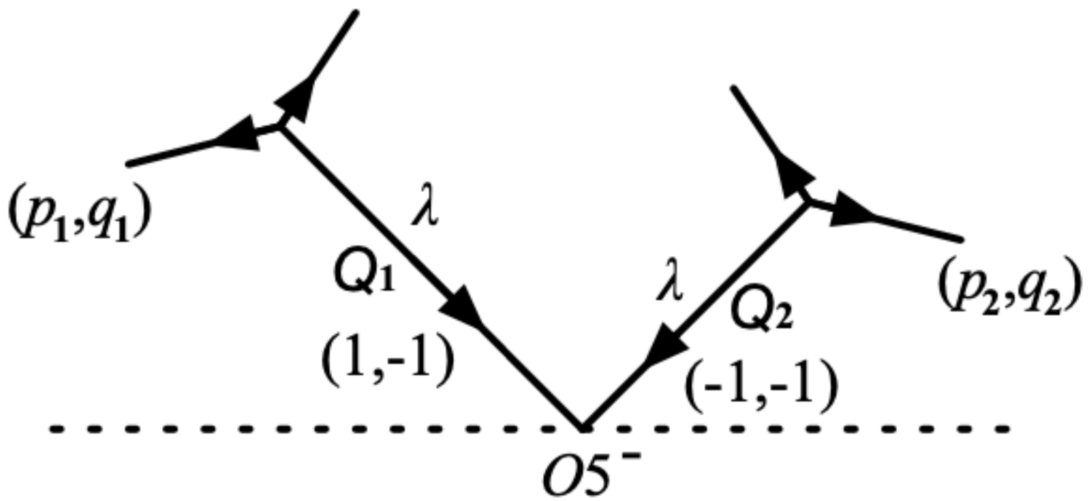
$Q = \exp\left(-\text{“Length”}\right)$

Coulomb moduli, gauge coupling, masses

$$Z = \sum_{\lambda,\mu,\nu,\rho\cdots} \prod (\text{Edge factor}) \cdot \prod (\text{Vertex factor}).$$

Topological vertex formalism with O5-plane

1. As for the two edges intersecting at the O5-planes, assign the the O5-plane factor to them



$$I_{\boldsymbol{\lambda}}(Q_1, Q_2; n') = (Q_1 Q_2)^{|\boldsymbol{\lambda}|}(-1)^{n'|\boldsymbol{\lambda}|} g^{\frac{n'}{2}(||\boldsymbol{\lambda}^T||^2 - ||\boldsymbol{\lambda}||^2)},$$

2. As for the part away from the O5-plane, the rule is identical.

Partition function for 5d Sp(N) gauge theory with N_f flavors

$$Z = C' \prod_{1 \le I < J \le 2N+2} \left(\frac{R_{\varnothing\varnothing}(e^{-\beta(a_I - a_J)})}{R_{\varnothing\varnothing}(e^{-\beta(-a_I + a_J)})} \right)^{-\frac{1}{2} c_I c_J}$$
$$\sum_{\{\boldsymbol{\mu}\}} \left(\prod_{I=1}^{2N+2} e^{-\beta(N+2) c_I \left(a_I |\boldsymbol{\mu}_I| - \hbar \|\boldsymbol{\mu}_I\|^2\right)} \right) \left(\prod_{I=1}^{2N+2} \prod_{J=1}^{2N+2} \left(R_{\boldsymbol{\mu}_I \boldsymbol{\mu}_J^T}(e^{-\beta(a_I - a_J)}) \right)^{-\frac{1}{2} c_I c_J} \right)$$
$$\left(\prod_{i=1}^{N_f} \prod_{I=1}^{2N+2} \left(R_{\varnothing \boldsymbol{\mu}_I^T}(e^{-\beta(m_i - a_I)}) \right)^{c_I} \right),$$

$$R_{\boldsymbol{\lambda}\boldsymbol{\mu}}(Q) := \prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \left(1 - Q g^{i+j-\lambda_i - \mu_j - 1} \right)$$

$$c_I = \begin{cases} 1 & (1 \le I \le N+1) \\ -1 & (N+2 \le I \le 2N+2). \end{cases} \qquad \boldsymbol{\mu}_I := \boldsymbol{\mu}_{2N+3-I}^T. \qquad a_I := -a_{2N+3-I}.$$

$$\sum_{I=1}^{N+1} a_I + m_0 - \frac{1}{2} \sum_{i=1}^{N_f} m_j = 0$$

§4 Deriving Seiberg-Witten prepotential from Nekrasov partition function

5-brane web diagram with O5-plane

Boundary conditions
[Hayashi,S.S.Kim, K.Lee, Taki, FY '17]

§2
(Done)

§3
(Done)

Topological vertex formalism
with O5-plane
[S.S. Kim, FY'17]

Seiberg-Witten curve
(Seiberg-Witten prepotential)

F

Nekrasov partition function
(Topological string partition function)

Z

§4

[X. Li, FY '21]

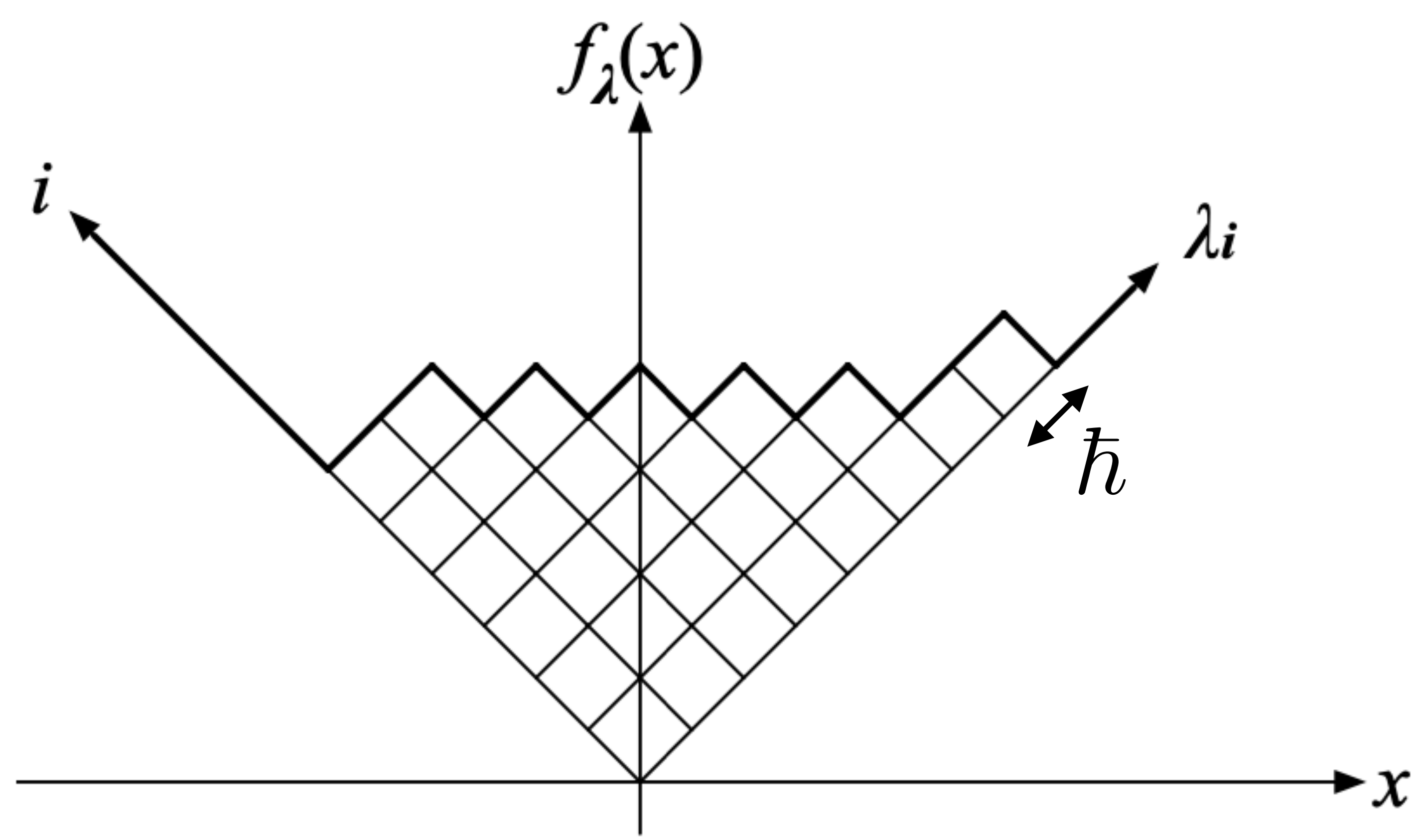
Thermodynamic limit

$$F = -\lim_{\hbar \to 0} \hbar^2 \log Z \qquad (\varepsilon_1 = -\varepsilon_2 = \hbar)$$

Step1. Rewrite in terms of profile function

Sum over Young diagram → Sum over profile function

$$\sum_{\lambda} \qquad\qquad \sum_{f_\lambda(x)}$$



Step2. Take the thermodynamic limit

$$\hbar \to 0$$

Sum over profile function → "Path integral" over profile function

$$\sum_{f_\lambda(x)} \qquad\qquad \int \mathcal{D}f$$

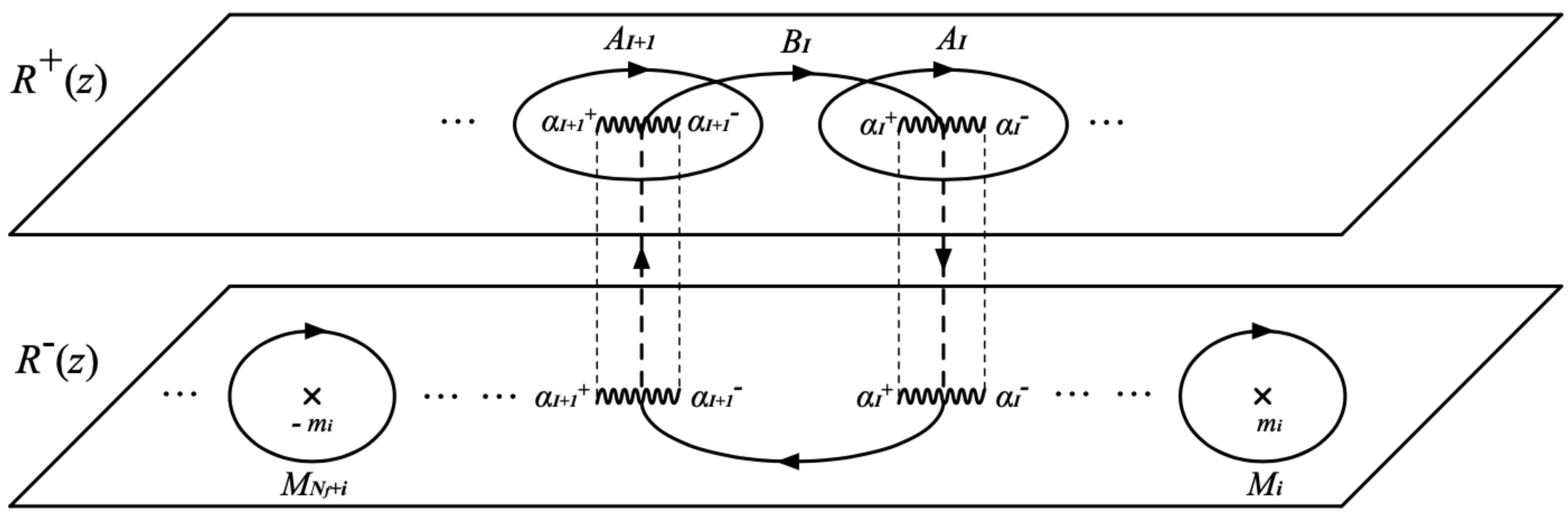
$$Z = \int \mathcal{D}f \; e^{-\frac{1}{\hbar^2}(\mathcal{E}[f]+\xi(\mathrm{Constraint}))} = e^{-\frac{1}{\hbar^2}\underbrace{\mathcal{E}[f_*]}_{F}+\mathcal{O}(\hbar^{-1})}$$

f_* **is the function which extremize the exponent.**

Step3. Introduce resolvent

$$R(z) := \int_{-\infty}^{\infty} \frac{f''_*(x)}{1-e^{z-x}} dx$$

Analytic continuation shows that the resolvent is defined on a <u>Riemann surface</u>



→ Seiberg-Witten curve !

Step4. Reproduce Seiberg-Witten solution

By finding a proper function of the resolvent,

$$t(w = e^{-z}) = t(R(z))$$

we find the constraints identified as the Seiberg-Witten curve

$$t^2 + P(w)t + Q(w) = 0.$$

Especially, we reproduced the “boundary condition”

$$P(1)^2 - 4Q(1) = 0, \quad P(-1)^2 - 4Q(-1) = 0$$

Step5. Reproduce Seiberg-Witten solution by evaluating integral over cycles

$$\lambda_{SW}^{\pm} = \frac{\beta z}{4\pi i}\left(R^{\pm}(z) - \sum_{i=1}^{N_f} \frac{2e^{-\beta(m_i+z)}}{1-e^{-\beta(m_i+z)}} - 4(N+2)\right) dz,$$

$$\oint_{A_I} \lambda_{SW} = a_I, \quad (I = 1, 2, \cdots, N, N+1), \qquad \sum_{I=1}^{N+1} a_I = -m_0 + \frac{1}{2}\sum_{j=1}^{N_f} m_j,$$

$$\oint_{B_I} \lambda_{SW} = \frac{1}{2\pi i\beta}\left(\frac{\partial F}{\partial a_I} - \frac{\partial F}{\partial a_{I+1}}\right) \quad (I = 1, 2, \cdots, N-1), \qquad \oint_{B_N} \lambda_{SW} = \frac{1}{2\pi i\beta}\frac{\partial F}{\partial a_N}.$$

§5 Summary

We studied 5d N=1 Sp(N) gauge theory with N_f flavor by using 5-brane web diagram with O5-plane.

We find Seiberg-Witten curve from 5-brane web. Especially, we discuss the boundary condition originated from the O5-plane.

We compute Nekrasov partition function by using topological vertex formalism with O5-plane.

We take thermodynamic limit of the prepotential and reproduced the Seiberg-Witten curve (prepotential).