General Properties of Light Rings and quasi-black holes

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1. PRD 103, 104031 (2021)（arXiv:2011.02211）
2. arXiv:2108.08967

2021年9月30日 USTC

General properties of light-rings

1.Motivation

- A light ring (LR) is a circular photon orbit outside a black hole or an ultracompact object (UCO）
- Light ring is an important feature of curved spacetimes.
- Light rings play an important role in gravitational wave observations and black hole photographs.
- Light rings could be observational evidence for event horizons [V. Cardoso, et.al. PRL(2014)]

2.Example: Schwarzschild black hole

The metric for the Schwarzschild black hole is:

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

Due to the spherical symmetry, we can always assume that the light moves on the equatorial plane $\theta = \pi/2$. The four-momentum is

$$p^a = \dot{t}\left(\frac{\partial}{\partial t}\right)^a + \dot{r}\left(\frac{\partial}{\partial r}\right)^a + \dot{\phi}\left(\frac{\partial}{\partial \phi}\right)^a$$

where $\dot{t} = \frac{\partial t}{\partial \lambda}$ and λ is an affine parameter of the null geodesic.

There are two conserved quantities, energy and angular momentum associated with the two Killing vector fields $\left(\frac{\partial}{\partial t}\right)^a$ and $\left(\frac{\partial}{\partial \phi}\right)^a$:

$$E = -g_{ab}p^a \left(\frac{\partial}{\partial t}\right)^b = (1 - 2M/r)\dot{t}$$

$$L = g_{ab}p^a \left(\frac{\partial}{\partial \phi}\right)^b = r^2\dot{\phi}$$

For null curves, we have

$$0 = g_{ab}p^a p^b = -(1 - 2M/r)\dot{t}^2 + (1 - 2M/r)^{-1}\dot{r}^2 + r^2\dot{\phi}^2$$

Thus, the radial geodesic equation can be solved as

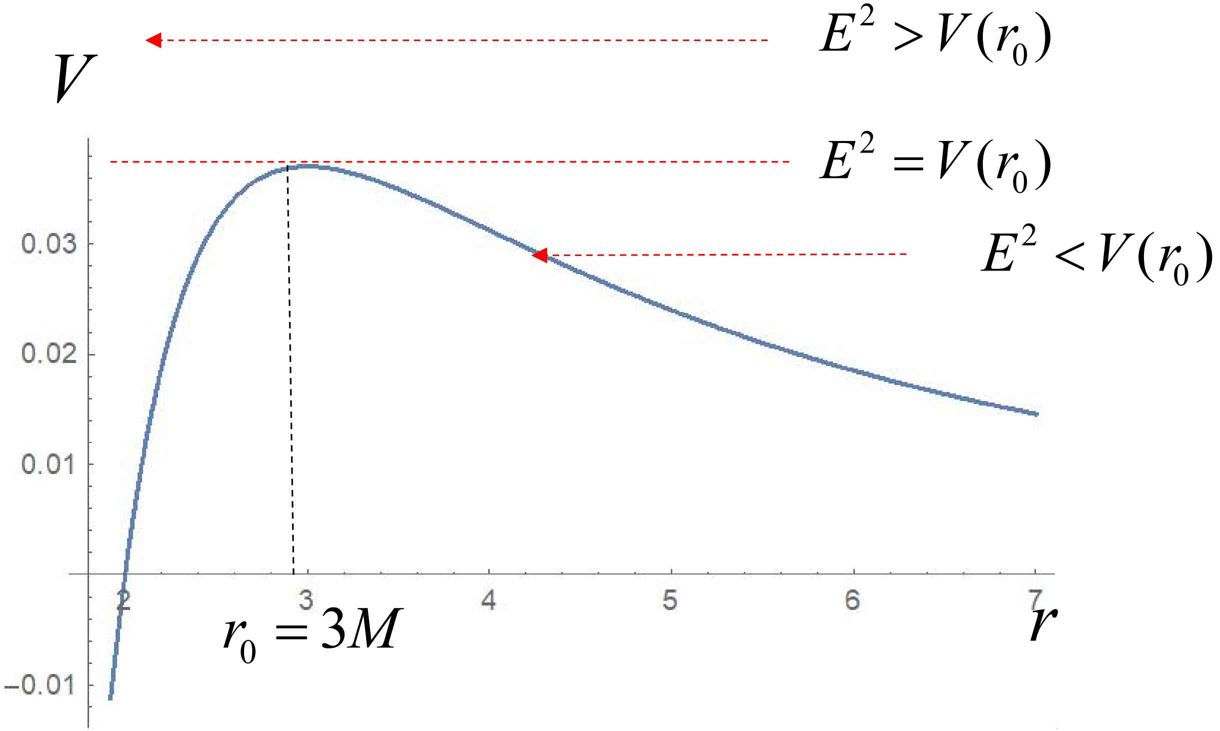
$$\dot{r}^2 = E^2 - V(r)$$

with $$V(r) = \frac{L^2}{r^2} - \frac{2ML^2}{r^3}$$

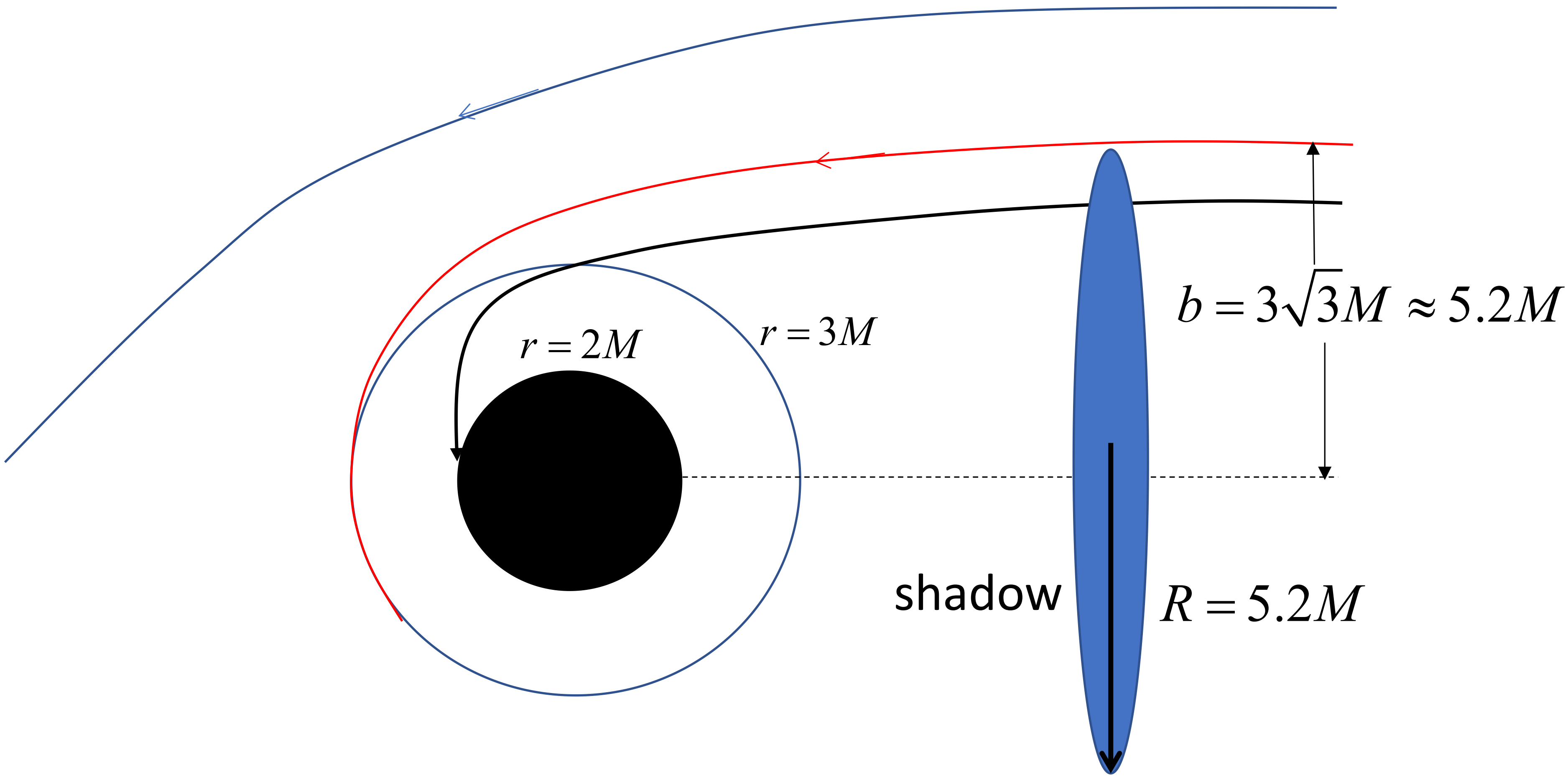
The light ring $r = r_0$ satisfies

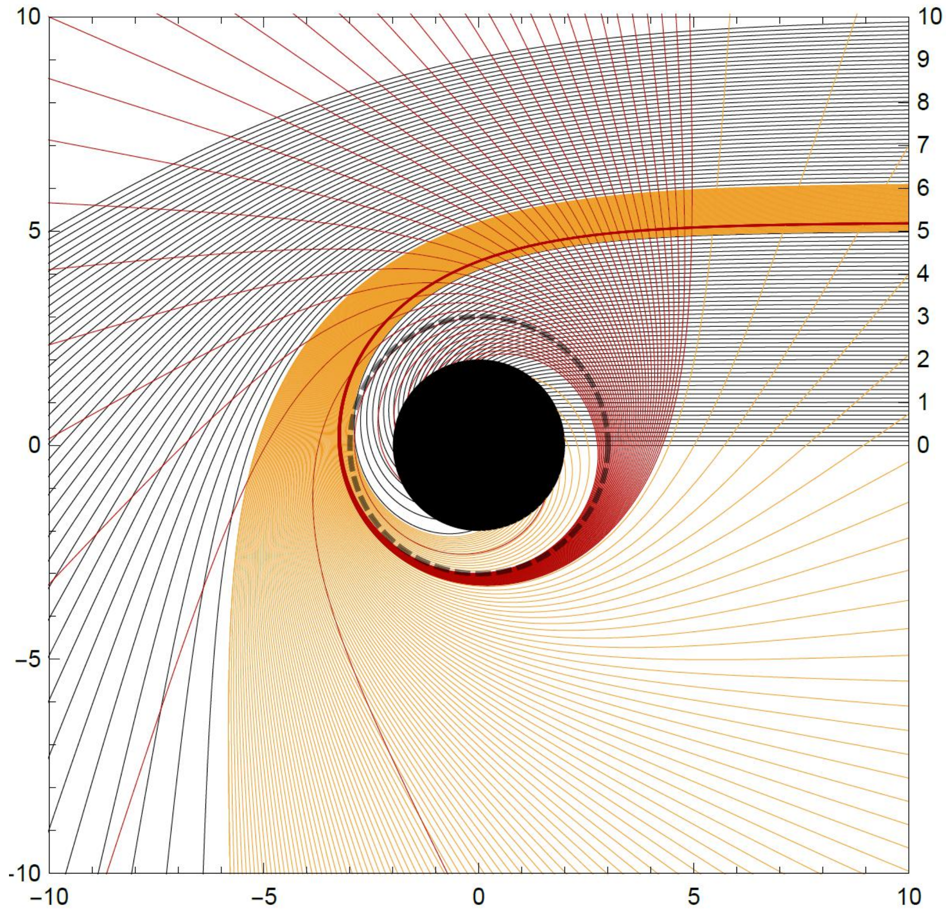
$$\dot{r}^2 \;=\; E^2 - V(r_0) = 0$$

$$\left.\frac{\partial V}{\partial r}\right|_{r=r_0} \;=\; 0$$



The unstable light ring is located at $r = 3M$.


$b = 3\sqrt{3}M \approx 5.2M$
$r = 3M$
$r = 2M$
shadow
$R = 5.2M$

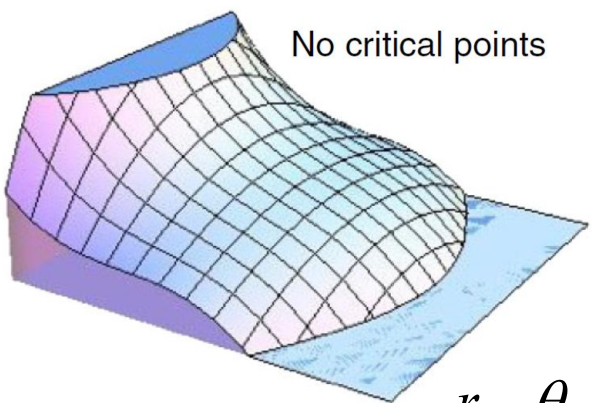


From Wald et al. 2019

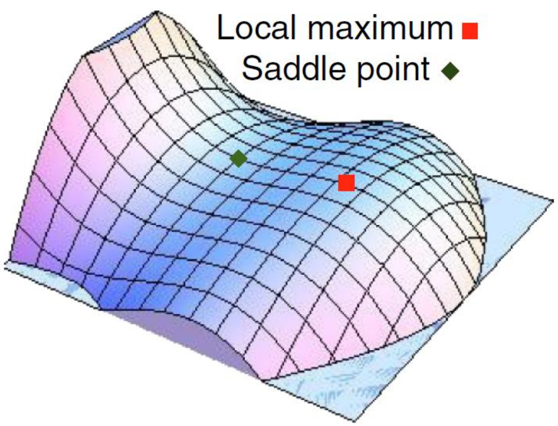
Do the light rings exist in general in curved spacetimes?

3. Previous works

- P. Cunha et.al shows that light rings always come in pairs for UCO (PRL 2017)， one being a saddle point and the other a local extremum. A topological argument was used in the proof.



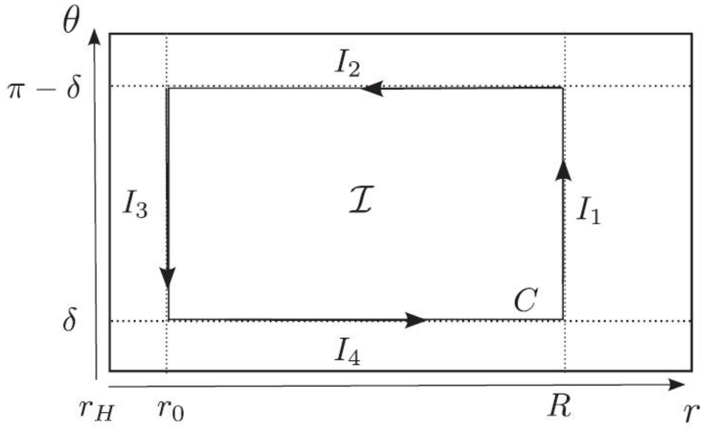
$r-\theta$



- Very recently, the authors employed this topological argument to a stationary black hole and found that at least one standard LR exists out-side the nonextremal horizon for each rotation sense (PRL 2020).

Unresolved issues

- The topological argument relies on the formation history of UCO
- The previous works cannot answer that whether a light ring must exist on the equatorial plane.



- It cannot answer whether the radial or the angular direction is stable.
- The topological argument does not apply to extremal black holes.

4. General setup

- A stationary spacetime with two Killing vector fields:

$$ds^2 = g_{tt}(r,\theta)dt^2 + g_{rr}(r,\theta)dr^2 + 2g_{t\phi}(r,\theta)dtd\phi + g_{\theta\theta}(r,\theta)d\theta^2 + g_{\phi\phi}(r,\theta)d\phi^2$$

- In general, the 4-momentum of a photon is

$$p^a = \dot{t}\left(\frac{\partial}{\partial t}\right)^a + \dot{r}\left(\frac{\partial}{\partial r}\right)^a + \dot{\theta}\left(\frac{\partial}{\partial \theta}\right)^a + \dot{\phi}\left(\frac{\partial}{\partial \phi}\right)^a$$

With $p^a p_a = 0$

- The conservation of energy and angular momentum:

$$E = -g_{ab}p^a\left(\frac{\partial}{\partial t}\right)^b = -g_{tt}\dot{t} - g_{t\phi}\dot{\phi}\,,$$

$$L = g_{ab}p^a\left(\frac{\partial}{\partial \phi}\right)^b = g_{\phi\phi}\dot{\phi} + g_{t\phi}\dot{t}\,,$$

- Therefore $g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2 + V(r,\theta) = 0$

Where $V(r,\theta) = -\frac{1}{D}\left(E^2 g_{\phi\phi} + 2ELg_{t\phi} + L^2 g_{tt}\right)$

$$D \equiv g_{t\phi}^2 - g_{tt}g_{\phi\phi}$$

Define $\sigma = \dfrac{E}{L}$. Then

$$V = -\frac{L^2 g_{\phi\phi}}{D}\left(\sigma - H_+\right)\left(\sigma - H_-\right)$$

$$H_\pm = \frac{-g_{t\phi} \pm \sqrt{D}}{g_{\phi\phi}} .$$

For the H_+ branch, assume that a LR is located at $\{r = r_+, \theta = \theta_+\}$, which satisfies

$$\partial_r H_+\Big|_{(r_+,\theta_+)} = 0, \qquad \partial_\theta H_+\Big|_{(r_+,\theta_+)} = 0, \qquad \sigma = H_+(r_+, \theta_+)$$

Similarly for the H_- branch.

- The H_+ branch corresponds to LR counter-rotating with the black hole and the H_- branch corresponds to LR co-rotating with the black hole.
- To find LRs, we shall analyze the angular direction and the radial direction, respectively.

Stability

$$\partial_m^2 V(r_{\mathrm{LR}}, \theta_{\mathrm{LR}}) = \partial_m^2 H_+(r_{\mathrm{LR}}, \theta_{\mathrm{LR}})(H_+ - H_-)\,.$$

$$\partial_m^2 V(r_{\mathrm{LR}}, \theta_{\mathrm{LR}}) = \partial_m^2 H_-(r_{\mathrm{LR}}, \theta_{\mathrm{LR}})(H_- - H_+)\,.$$

where, $m \in \{r, \theta\}$

Since $H_+ > H_-$, we see that $\partial_m^2 H_+$ has the same sign as $\partial_m^2 V$ and $\partial_m^2 H_-$ has the opposite sign.

So the maximum of H_+ and the minimum of H_- correspond to unstable orbits.

5. Existence of LR for Black holes

- Axis-symmetric black holes--angular direction

  (fixed r)

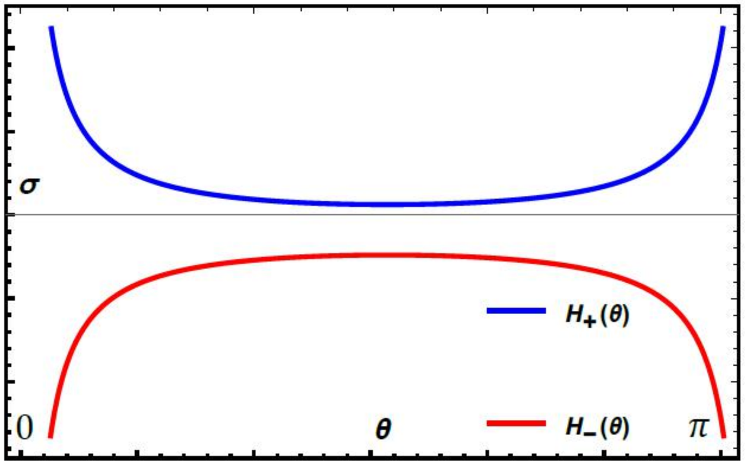
$\rho \equiv \sqrt{g_{\phi\phi}}$ goes to zero when $\theta \to 0$ and $\theta \to \pi$.

$$\begin{cases} \partial_\theta \rho > 0 & \theta \to 0\,, \\ \partial_\theta \rho < 0 & \theta \to \pi\,. \end{cases}$$

Thus we find $$H_\pm \simeq \pm \frac{1}{\rho} \to \pm\infty\,,$$

and $$\partial_\theta H_\pm \sim \mp \frac{\partial_\theta \rho}{\rho^2} \sim \begin{cases} \mp\infty & \theta \to 0 \\ \pm\infty & \theta \to \pi, \end{cases} .$$

So, for any fixed r, $H_{\pm}$ can be viewed as a function of θ ranging from 0 to π,



For each given r, there always exists a $\theta = \theta_{+}$ such that $H_{+}(r, \theta_{+})$ is a minimum in the θ direction. In this way, we obtain a function $\theta = \theta_{+}(r)$. Similarly, we have $\theta_{-}(r)$ for H_{-}.

In asymptotically flat spacetimes, $H_{\pm} \to \pm\frac{1}{r\sin\theta}$ as $r \to \infty$. Thus, we have

$$\theta_{\pm}(r \to \infty) = \pi/2\,.$$

When the spacetime possesses a parity reflection symmetry $\theta \to \pi - \theta$, H_+ satisfies $H_+(\theta) = H_+(\pi - \theta)$. Thus,

$$\left.\frac{\partial H_+}{\partial \theta}\right|_{\theta} = -\left.\frac{\partial H_+}{\partial \theta}\right|_{\pi-\theta}$$

One the equatorial plane,

$$\left.\frac{\partial H_+}{\partial \theta}\right|_{\pi/2} = -\left.\frac{\partial H_+}{\partial \theta}\right|_{\pi/2} = 0$$

Therefore, $\theta_\pm(r) = \pi/2$.

An orbit confined on the equatorial plane always has an extremum in the angular direction.

Radial direction (fixed θ)

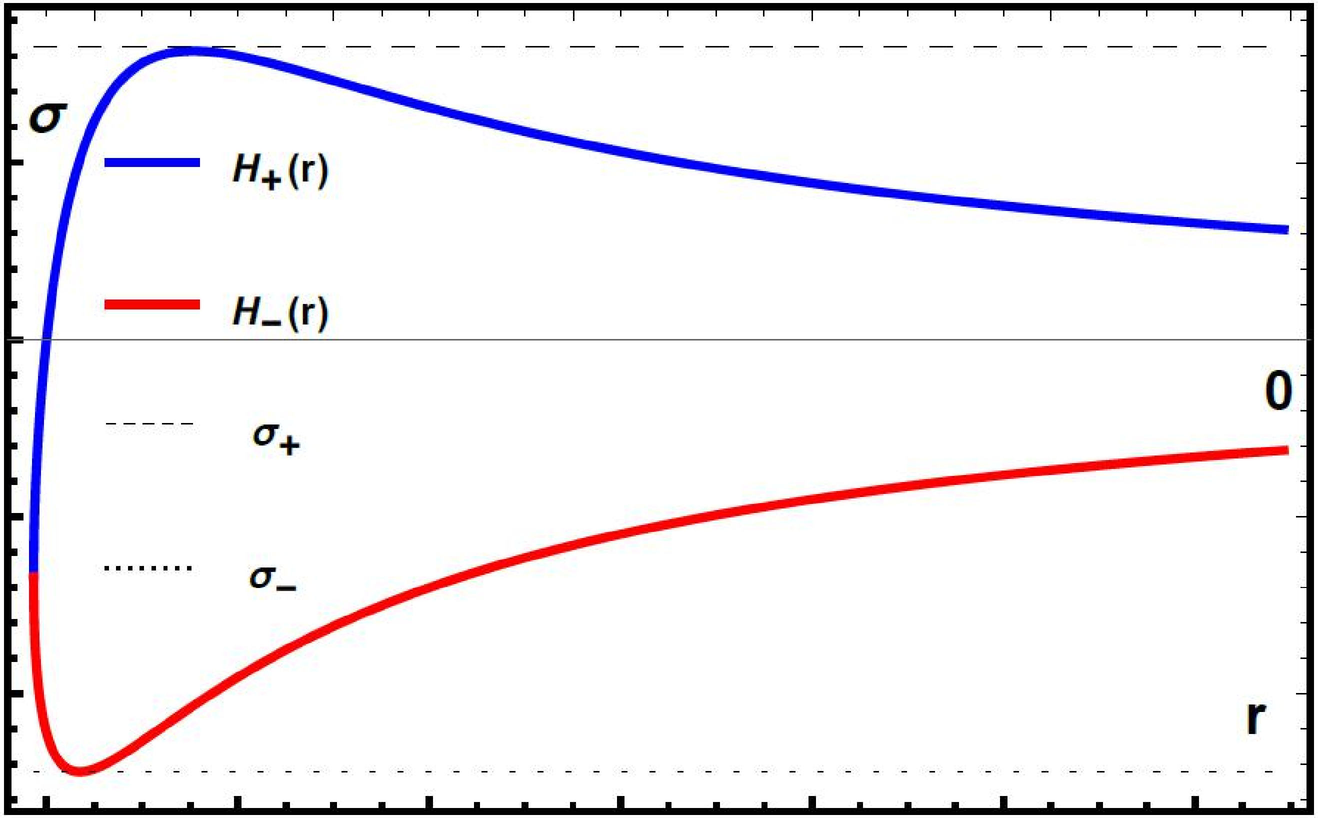
- At infinity $r \to \infty$, we have

$$g_{t\phi} \to 0,\ g_{tt} \to -1 \text{ and } g_{\phi\phi} \to r^2$$

$$H_\pm \to \pm\frac{1}{r\sin\theta} \to 0^\pm$$

- On the horizon $D|_{r_h} = 0$, and D is always positive outside the horizon

$$H_\pm|_{r_h} = -\frac{g_{t\phi}}{g_{\phi\phi}}\bigg|_{r_h} < 0\,.$$



So there exists at least one maximum for H_+

H_- is always negative outside the horizon.

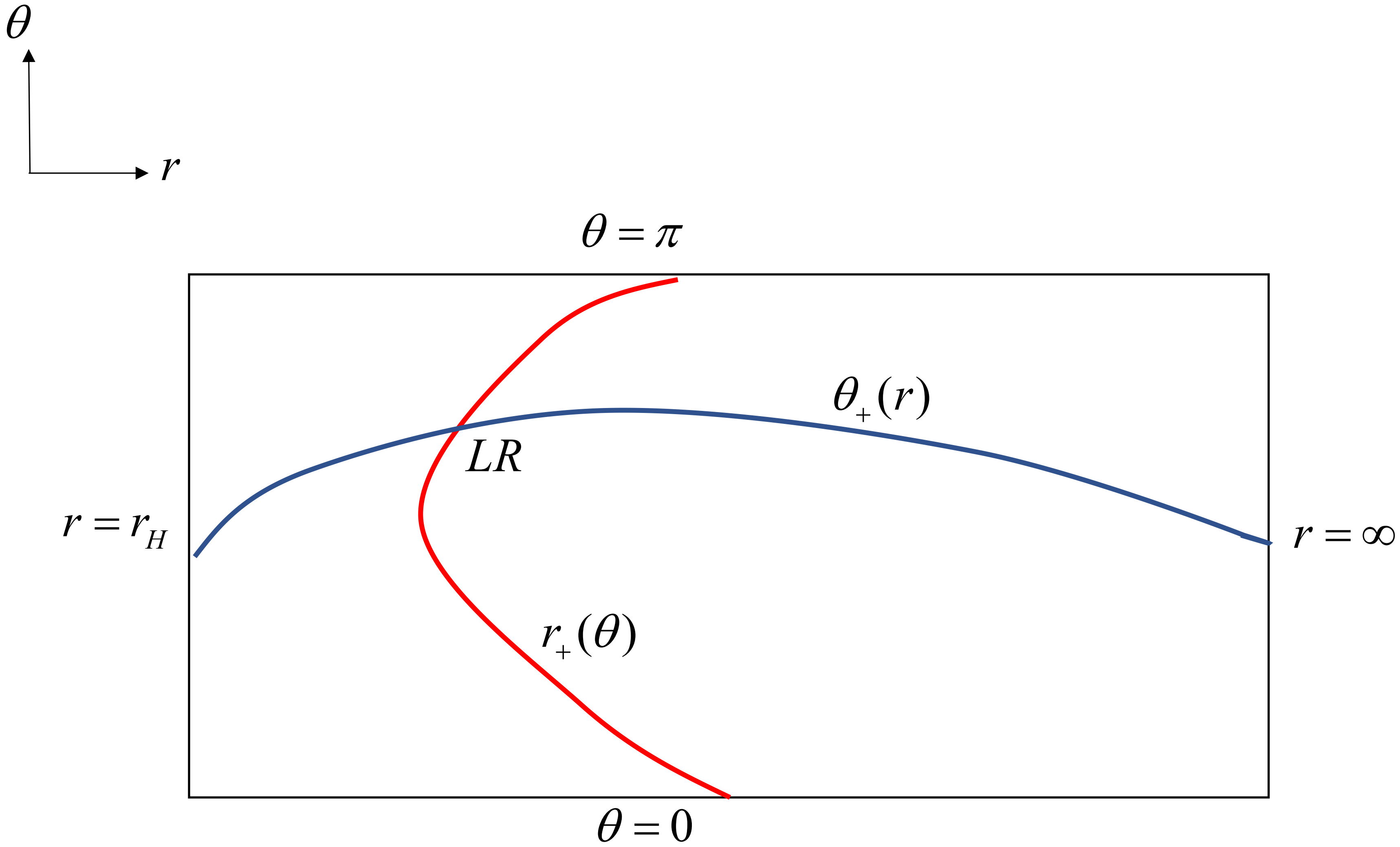
$$H'_-(r_h) \sim -\frac{D'(r_h)}{2\sqrt{D(r_h)}g_{\phi\phi}(r_h)} + \frac{g_{t\phi}(r_h)g'_{\phi\phi}(r_h) - g'_{t\phi}(r_h)g_{\phi\phi}(r_h)}{g^2_{\phi\phi}(r_h)}$$

One can show that $D'(r_h) > 0$ for nonextremal horizons

Thus, $H'_-(r_h) \to -\infty$

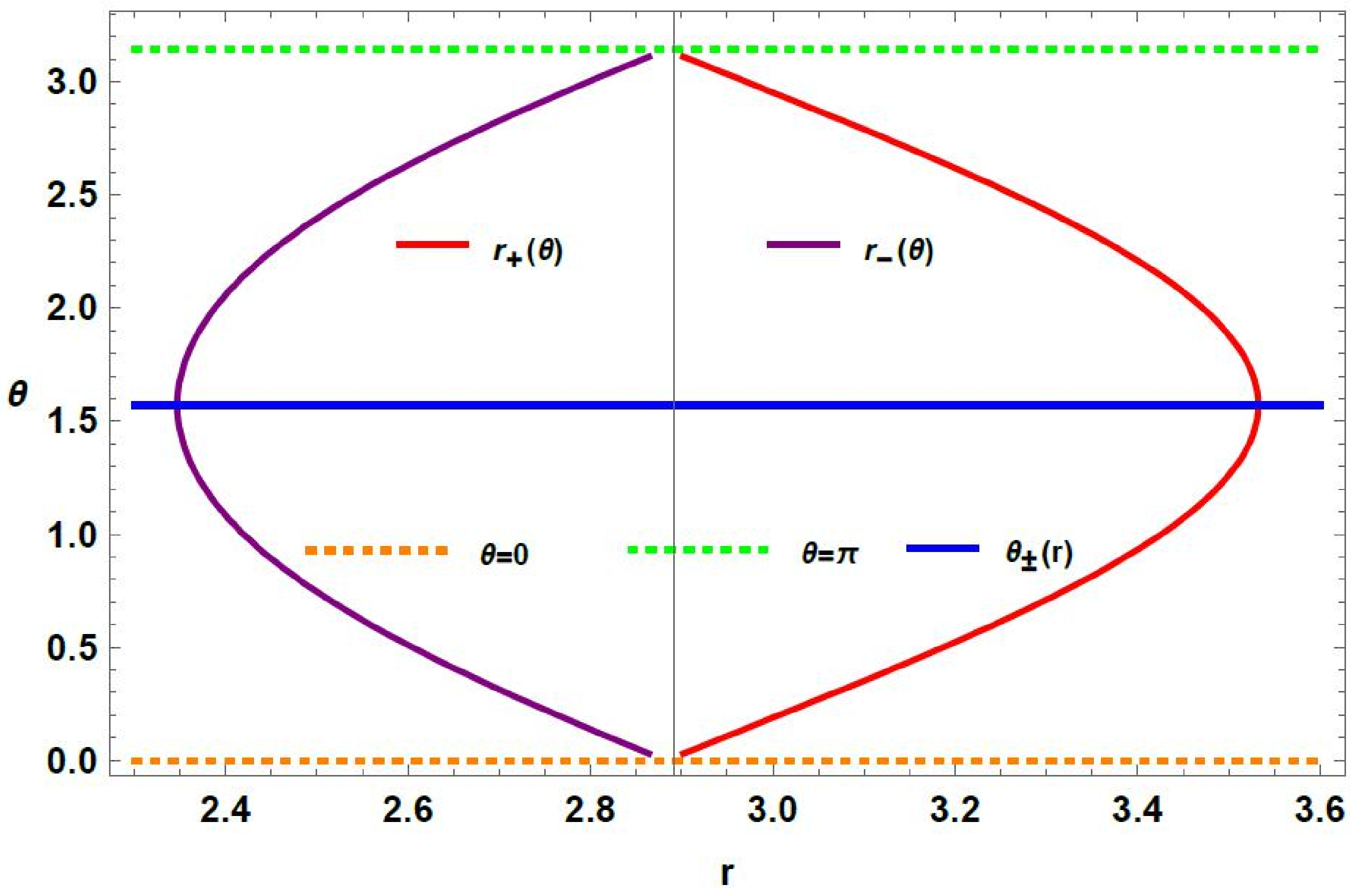
There exists at least one minimum for H_- (maximum for V)

The above argument holds for any constant θ, i.e., given any θ, there exists $r = r_+ > r_h$ such that H_+ takes an maximum value in the r direction. Hence, we have a function $r_+(\theta)$ defined in the range $0 \leq \theta \leq \pi$.

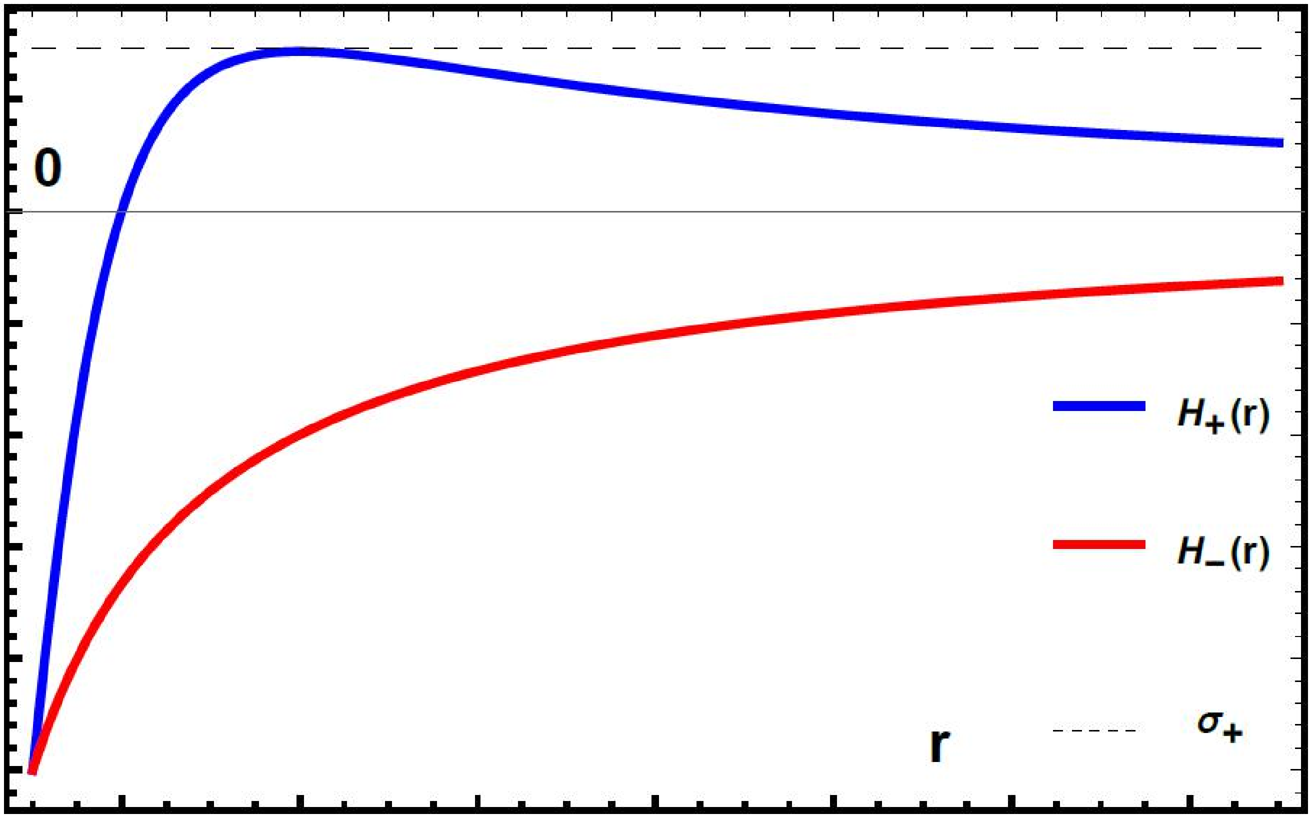


θ
r
$\theta = \pi$
$\theta_+(r)$
LR
$r = r_H$
$r = \infty$
$r_+(\theta)$
$\theta = 0$

The functions $r_{\pm}(\theta)$ and $\theta_{\pm}(r)$ for the Kerr black hole



- For extremal Kerr black holes $D'(r_h) = 0$



6. LRs in horizonless spacetime

- Horizonless spacetimes are interesting because they can represent ultracompact objects (UCO)
- The behavior of $H_{\pm}$ at infinity is the same as BH,

$$H_{\pm} \to 0^{\pm}$$

- Assume that the spacetime is regular,

$g_{tt} < 0 \text{ and } g_{t\phi} > 0$ everywhere.

- Near the center r=0, assume

$$g_{\phi\phi} \to r^2 \qquad g_{tt} \to -k^2 \qquad g_{t\phi} \to pr^s$$

$$H_\pm \sim \frac{-pr^s \pm \sqrt{p^2 r^{2s} + k^2 r^2}}{r^2}$$

For $s > 1$, $\quad H_\pm \sim \frac{-pr^s \pm \sqrt{k^2 r^2}}{r^2} \sim \pm\frac{1}{r} \to \pm\infty\,.$

For $s = 1$, $\quad H_\pm \sim \frac{\pm\sqrt{p^2 + k^2} - p}{r} \to \pm\infty\,.$

For $0 < s < 1$,

$$H_+ \sim \frac{-pr^s + pr^s\left(1+\alpha^2 r^{2-2s}\right)^{1/2}}{r^2} \sim \frac{\alpha^2 p}{2} r^{-s} \to \infty$$

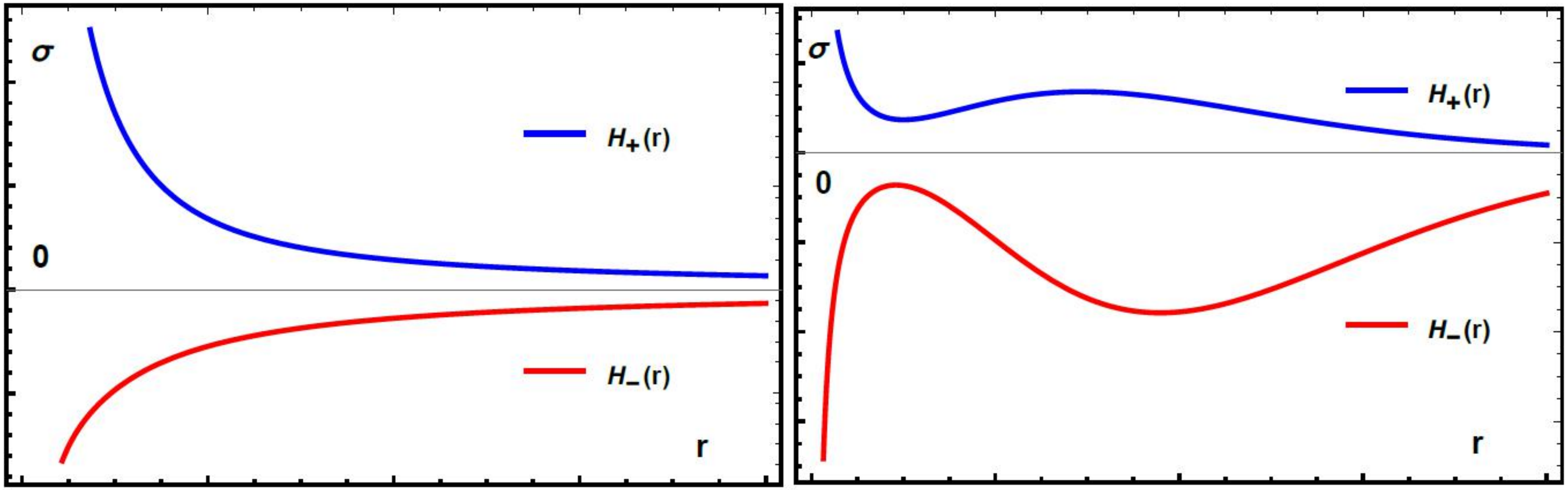
$$H_- \sim -\left(\frac{2}{r^{2-s}} + \frac{\alpha^2}{2r^s}\right)p \to -\infty . \qquad \alpha \equiv \frac{k}{p}.$$

for $s = 0$, $$H_\pm \sim \frac{-p \pm \sqrt{p^2 + k^2 r^2}}{r^2} = \frac{-1 \pm \sqrt{1+\alpha^2 r^2}}{r^2} p ,$$

$H_- \to -\infty$ $\qquad$ $H_+ \to \frac{\alpha^2}{2} p > 0$

$H'_+(0^+) = 0$, and $H''_+(0^+) = -\dfrac{\alpha^4}{4} p < 0$,

Hence, H'_+ become negative just away from $r = 0$



- There is either no LR or there are even number of LRs for each branch.

7. Conclusions

- We have shown that there are at least two LRs outside a non-extreme stationary black hole. The outermost LRs are unstable along the radial direction and stable along the angular direction.
- For an extremal stationary black hole, we find there is at least one LR.
- When the spacetime possesses a reflection symmetry, there must exist a LR on the equatorial plane.

- For horizonless spacetimes, we have proved that if LR exists, there are at least two LRs, with the outer one being unstable in the radial direction and the inner one being stable.
- In our arguments, only some generic conditions have been used, for instance, the asymptotically flat condition and the behaviors of the metric near the horizon or the center of star. The results could apply to most gravity theories.

Quasi-black holes and stability of spacetimes

1. Motivations

- Cardoso et al. (2014) conjectured that ultracompact objects possessing LRs may not be stable. They calculated the quasi-normal modes of ultracompact stars and found that lone-lived modes near the stable light ring.
- If the conjecture is true, then the LRs can be the signature of black holes.
- Can we have a one-parameter family of horizonless solutions where the LRs appear at certain parameter?
- For this purpose, we revisit the quasi-black hole solutions, constructed by Lemos, et. al. (2000).
- The one-parameter family of static solutions are everywhere nonsingular but can come arbitrarily close to a black hole at certain critical parameters.

2. Quasi-black holes

The metric

$$ds^2 = -\frac{dt^2}{U(R)^2} + U(R)^2[dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2)]\,.$$

describes a spacetime consisting of extremal dust, i.e., $\rho = \rho_e$.

By using Einstein's equation, one can show that

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial U}{\partial R}\right) = -4\pi U^3\rho\,.$$

Choose $$U(R) = 1 + \frac{q}{\sqrt{R^2 + c^2}},$$

where q is the total charge of the spacetime.

When $c \to 0$, the solution reduces to the extremal RN black hole

and $R = 0$ is the black hole horizon.

Then the areal radius r is related to R by

$$r = RU = R + \frac{qR}{\sqrt{R^2 + c^2}}$$

For $c=0$, The horizon $R = 0$ corresponds to $r = q$.

when $c \neq 0$ quasi-black hole spacetime describes a spherically symmetric compact star with $R = 0$ $(r = 0)$ being its center.

3. Light rings of QBH

On the equatorial plane $\theta = \pi/2$, the four-momentum of a photon takes the forr

$$p^a = \dot{t}\left(\frac{\partial}{\partial t}\right)^a + \dot{R}\left(\frac{\partial}{\partial r}\right)^a + \dot{\phi}\left(\frac{\partial}{\partial \phi}\right)^a .$$

The conserved energy and angular momentum are given by

$$\begin{aligned} E &= -g_{ab}p^a\left(\frac{\partial}{\partial t}\right)^b = -g_{tt}\dot{t} = \frac{1}{U^2}\dot{t}\,, \\ L &= g_{ab}p^a\left(\frac{\partial}{\partial \phi}\right)^b = g_{\phi\phi}\dot{\phi} = U^2R^2\sin^2\theta\dot{\phi}\,. \end{aligned}$$

In addition, we have the null condition

$$0 = g_{ab}p^a p^b\,.$$

We get the radial equation

$$\dot{R}^2 + \frac{L^2}{R^2 U^4} = E^2 \,.$$

Define the potential

$$V(R) = \frac{L^2}{R^2 U^4} \,.$$

Let $z^2 = R^2 + c^2$ and then

$$V(R) = \frac{L^2 z^4}{(z^2 - c^2)(z + q)^4} \,.$$

The light rings occur at $V'(R) = 0$ and $E^2 = V(R)$, which gives

$$z^3 - qz^2 + 2qc^2 = 0 \,. \tag{3.8}$$

For $c=0$,there are two solutions $z=0$ and $z=q$, or $R=0$ and $R=q$.

$R=q$ is just the light ring located outside the black hole horizon.

$R=0$ (or $r=q$) is the null geodesic normal to the horizon, which is not the light ring.

For $c\neq 0$, we let

$$y = z - q/3 > -q/3\,.$$

Then Eq. (3.8) becomes

$$y^3 - \frac{1}{3}y + 2c^2 - \frac{2}{27} = 0\,.$$

where we have set $q=1$.

To solve the cubic equation, we define

$$\Delta = \left(\frac{v}{2}\right)^2 + \left(\frac{u}{3}\right)^3 = c^2\left(c^2 - \frac{2}{27}\right).$$

$\Delta = 0$ yields a critical constant $K \equiv \frac{2}{27}$.

When $c^2 > K$, Eq. (3.10) has only one real root taking in this form

$$y_d = \frac{-1 - \sqrt{3}i + i(i + \sqrt{3})\left(1 - 27c^2 + 3\sqrt{-6c^2 + 81c^4}\right)^{2/3}}{6\left(1 - 27c^2 + 3\sqrt{-6c^2 + 81c^4}\right)^{1/3}}.$$

But this root does not give a light ring because $r < 0$.

For $\Delta = 0$, that is, $c^2 = K$. We find the roots of the cubic equation read

$$y_{m1} = y_{m2} = 1/3 \quad \text{and} \quad y_{m3} = -2/3\,.$$

So there are two degenerate light rings.

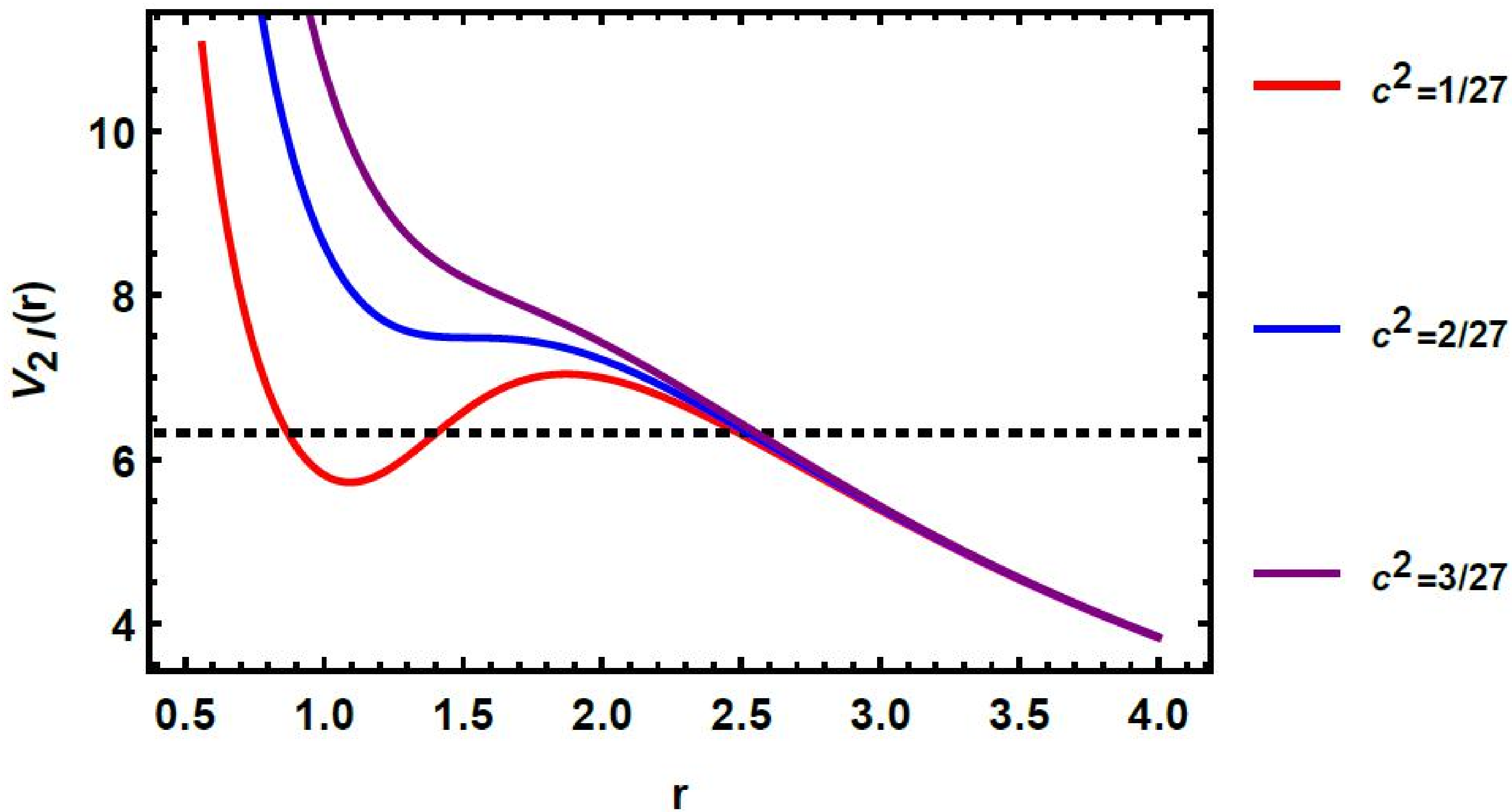
For $\Delta < 0$, that is $c^2 < K$. We have three different real roots,

$$y_w = \frac{2}{3}\cos\frac{\Theta + 2w\pi}{3}$$

where $w = -1, 0, 1$ and

$$\Theta = \arccos(1 - 27c^2).$$

r_0 and r_1 correspond to two LRs.



$V_{2l}(r)$
10
8
6
4
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
r
c^2=1/27
c^2=2/27
c^2=3/27

Now we pay special attention to the regime $c \to 0$, i.e., the black hole limit.

We can expand the roots to the order of c^2 and find

$$\begin{aligned} R_0 &= q - \frac{5}{2}c^2/q, \\ R_{-1} &= c\,, \end{aligned}$$

Since

$$r = RU = R + \frac{qR}{\sqrt{R^2 + c^2}}\,,$$

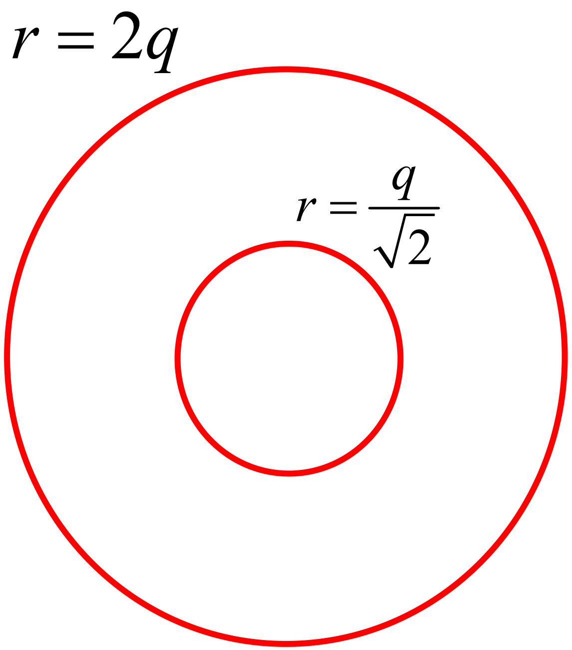
We see that as $c \to 0$,

$$\begin{aligned} r_0 &\to 2q \\ r_{-1} &\to q/\sqrt{2} \end{aligned}$$

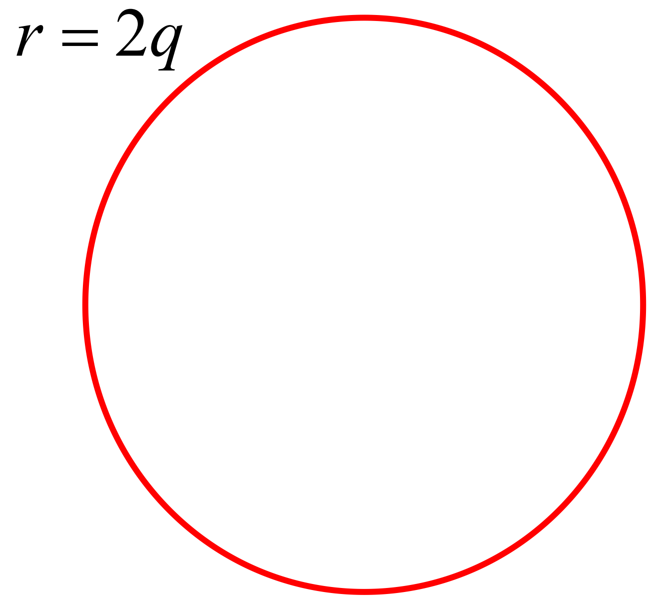
Note that r_0 just reduces to the light ring of the extremal RN black hole,

while r_{-1} does not have the black hole correspondence in the $c \to 0$ limit.

Quasi-black hole $c \to 0$



Extremal RN black hole $c=0$



4. Long-lived quasi-normal modes (QNM) of the QBH spacetime

The master equation

$$\left[\frac{\partial^2}{\partial_t^2} - \frac{\partial^2}{\partial_{r_*}^2} + V_{sl}(r)\right]\Psi(r,t) = 0\,,$$

where $s = 2$ corresponds to gravitational perturbations.

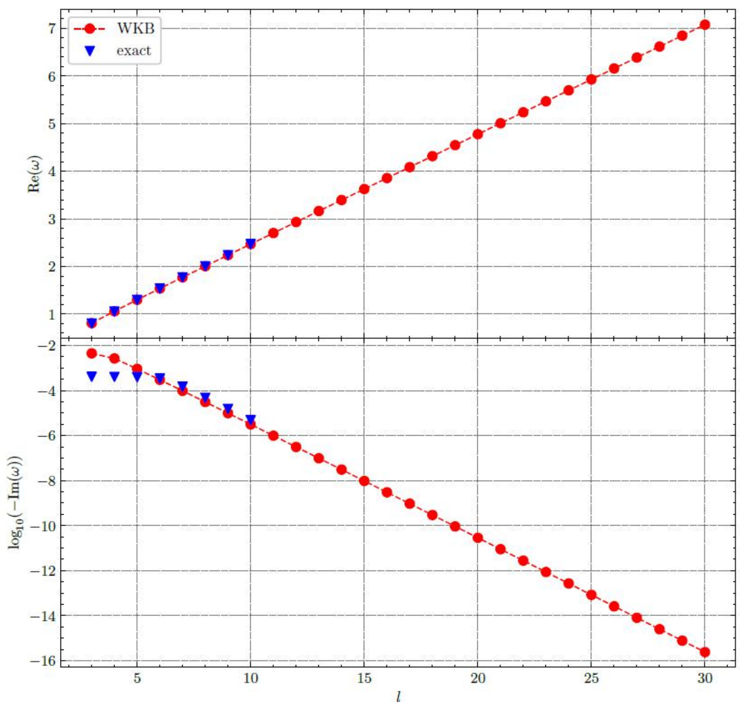
Assuming a time dependence $\Psi(r,t) = \psi(r)e^{-i\omega t}$,

the radial function $\psi(r)$ satisfies a Schrödinger-like equation

$$\frac{d^2\psi}{dr_*^2} + [\omega^2 - V_{sl}(r)]\psi = 0,$$

Write $\omega = \omega_R + i\omega_I$. As we assume $\Psi(r,t) = \psi(r)e^{-i\omega t}$,

the amplitude of perturbation will grow exponentially when $\omega_I > 0$



5. Conclusions

- We found a critical parameter for the quasi-black hole at which the light rings just appear.
- In the black hole limit, we found one unstable LR which coincides with the RN black hole, while the stable LR does not have the black hole correspondence .
- We calculated the quasinormal modes of the quasi-black holes. Both the WKB result and the numerical result show that long-live modes survive for the range where light rings exist, indicating that horizonless spacetimes with light rings are unstable.
- Our work provides a strong and explicit example that light rings could be direct observational evidence for black holes.

Thank You！