Expansion of EYM Amplitudes in Gauge Invariant Vector Space

Xiaodi Li

Zhejiang University

with Bo Feng, Rijun Huang, Kang Zhou

1904.05997, 2005.06287

USTC, Oct. 29 2020

Content

Relations of Amplitudes

KK Relations

There are many relations for color-ordered YM amplitudes. The Kleiss-Kuijf relations

$$A_n^{\text{YM}}(1,\{\alpha\},n,\{\beta\}) = (-1)^{|\beta|} \sum_{\sigma\in\alpha\shuffle\beta^T} A_n^{\text{YM}}(1,\sigma,n),$$

reduce the number of independent color-ordered YM amplitudes into $(n-2)!$.

[Kleiss, Kuijf; 1989] [Del Duca, Dixon, Maltoni; 2000]

BCJ Relations

In 2008, Bern, Carrasco and Johansson find new relations,

$$A_n^{\text{YM}}(1,\beta_1,\cdots,\beta_r,2,\alpha_1,\cdots,\alpha_{n-r-3},n)=\sum_{\{\xi\}\in\{\beta\}\sqcup_{\mathcal{P}}\{\alpha\}}\mathcal{C}_{\{\alpha\},\{\beta\};\{\xi\}}A_n^{\text{YM}}(1,2,\{\xi\},n),$$

which reduces the independent number to $(n-3)!$.

The set of color-ordered YM amplitudes $\{A_n^{\text{YM}}(1,2,\{\xi\},n),\xi\in S_{n-3}\}$ is the minimal basis, called the *BCJ basis*.
[Bern, Carrasco, Johansson, 2008]

KLT Relations

KLT (Kawai, Lewellen and Tye) relations initially say that closed string amplitudes can be written as the sums of products of open string amplitudes,

$$A_{\rm c}^{(M)} = (\frac{1}{2}i)^{M-3}\pi\kappa^{M-2}\sum_{P,P'} A_{\rm o}^{(M)}\bar{A}_{\rm o}^{(M)}e^{i\pi F(P,P')}.$$

[Kawai, Lewellen and Tye; 1986]

KLT Relations

Taking filed theory limit $\alpha' \to 0$, we get the field theory version of the KLT relations as

$$A_n^{\mathrm{G}} = \sum_{\sigma,\tilde{\sigma}\in S_{n-3}} A_n^{\mathrm{YM}}(n-1,n,\sigma,1)\mathcal{S}[\sigma|\tilde{\sigma}]A_n^{\mathrm{YM}}(1,\tilde{\sigma},n-1,n),$$

where $\mathcal{S}[\sigma|\tilde{\sigma}]$ is the momentum kernel.
[Bjerrum-Bohr, Damgaard, Feng, Sondergaard; 2010]

CHY formalism

Through the scattering equations, Cachazo, He, and Yuan (CHY) give the tree-level amplitude of a specific theory as an integral over the n-punctured sphere

$$A_n(k,\epsilon,\tilde{\epsilon}) = \int d\mu_n \, \mathcal{I}_n(k,\epsilon,\tilde{\epsilon},\sigma).$$

- Here the measure part is universal, same for different theories

$$d\mu_n = \frac{d^n\sigma}{\mathrm{vol}SL(2,\mathbb{C})} \prod\nolimits'_a \delta(\sum_{b\neq a} \frac{k_a \cdot k_b}{\sigma_{ab}}),$$

  the integrations are localized by $n-3$ linearly independent delta functions completely.

[Cachazo, He, and Yuan; 2013, 2014]

The integrand depends on the specific theory, for example

$$\mathrm{PT}_n(\alpha)\mathrm{Pf}'\Psi_n, \quad \mathrm{Pf}'\Psi_n(\epsilon)\mathrm{Pf}'\Psi_n(\tilde{\epsilon}), \quad \mathrm{PT}_r(\alpha)\mathrm{Pf}\Psi_{n-r}(\epsilon)\mathrm{Pf}'\Psi_n(\tilde{\epsilon})$$

for YM, gravity, and single trace EYM.

- The Parke-Taylor factor $\mathrm{PT}_n(\alpha)$ is defined by

$$\mathrm{PT}_n(\alpha) = \frac{1}{\sigma_{\alpha_1\alpha_2}\sigma_{\alpha_2\alpha_3}\cdots\sigma_{\alpha_n\alpha_1}}.$$

- The reduced Pfaffian $\mathrm{Pf}'\Psi_n$ is

$$\mathrm{Pf}'\Psi_n = 2\frac{(-1)^{i+j}}{\sigma_{ij}}\mathrm{Pf}(\Psi^{ij}_{ij}).$$

The $2n \times 2n$ anti-symmetric matrix is

$$\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix},$$

where

$$A_{ab} = \begin{cases} \frac{k_a \cdot k_b}{\sigma_{ab}} & a \neq b, \\ 0 & a = b, \end{cases} \qquad B_{ab} = \begin{cases} \frac{\epsilon_a \cdot \epsilon_b}{\sigma_{ab}} & a \neq b, \\ 0 & a = b, \end{cases}$$

and

$$C_{ab} = \begin{cases} \frac{\epsilon_a \cdot k_b}{\sigma_{ab}} & a \neq b, \\ -\sum_{c=1, c\neq a} \frac{\epsilon_a \cdot k_c}{\sigma_{ac}} & a = b. \end{cases}$$

Generalized KLT Relations

Integrand $\mathcal{I}_n$ can factorize into two factors, $\mathcal{I} = \mathcal{I}_L \times \mathcal{I}_R$. Each factor multiplied by a PT factor can be viewed as an integrand of a new theory.

$$\mathcal{I}_n(k,\epsilon,\tilde{\epsilon},\sigma) = \mathcal{I}_n^{(L)}(k,\epsilon,\sigma) \times \mathcal{I}_n^{(R)}(k,\tilde{\epsilon},\sigma)$$

$\mathrm{PT}(1,\alpha,n-1,n)$ $\swarrow$ $\qquad$ $\searrow$ $\mathrm{PT}(1,\beta,n,n-1)$

$$\mathrm{PT}(1,\alpha,n-1,n) \times \mathcal{I}_n^{(L)}(\alpha) \qquad \mathrm{PT}(1,\beta,n,n-1) \times \mathcal{I}_n^{(R)}(\alpha)$$

After localize the integrations, we get the generalized KLT relations:

$$A_n = \sum_{\alpha,\beta} A_n^{(L)}(\alpha)\mathcal{S}[\alpha|\beta]A_n^{(R)}(\beta).$$

[Cachazo, He, and Yuan; 2013, 2014]

- For example, consider the integrand of single trace EYM amplitude

$$\mathcal{I}^{\rm EYM} = {\rm PT}_r(\alpha){\rm Pf}\Psi_{n-r}(\epsilon){\rm PT}_n(1,\tilde{\sigma},n-1,n) \times \frac{1}{{\rm PT}_n(\tilde{\sigma}){\rm PT}_n(\sigma)} \times {\rm Pf}'\Psi_n(\tilde{\epsilon}){\rm PT}_n(n-1,n,\sigma,1),$$

after localize the integrations,

$$A^{\rm EYM}_{r,n-r}(\alpha) = \sum_{\sigma,\tilde{\sigma}\in S_{n-3}} A^{\rm YM}_n(n-1,n,\sigma,1)\mathcal{S}[\sigma|\tilde{\sigma}]A^{\rm YMs}_{r,n-r}(\alpha|1,\tilde{\sigma},n-1,n).$$

- The KLT relation gives the expansion of EYM amplitude in BCJ basis of YM amplitudes, after summing $\tilde{\sigma}$,

$$A^{\rm EYM}_{r,n-r}(\alpha) = \sum_{\sigma} \mathcal{C}_\sigma(\alpha)A^{\rm YM}_n(n-1,n,\sigma,1).$$

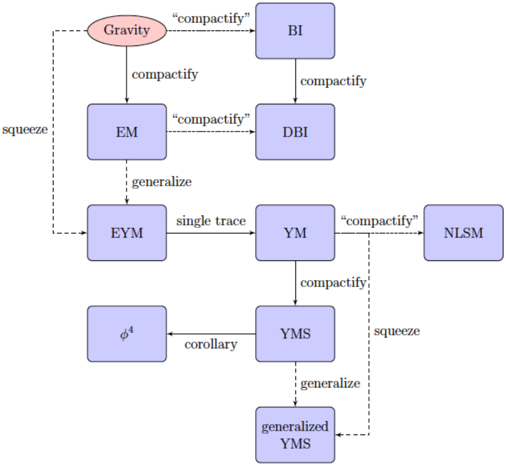
Web of different theories (1)

Cachazo, He, and Yuan introduced three operations on the integrands,

- dimensional reduction,
- generalized dimensional reduction,
- squeezing.

From the integrand of Einstein gravity, they got the integrands of many other different theories.
[Cachazo, He, and Yuan; 1412.3479]

Web of different theories (1)



[Cachazo, He, and Yuan; 1412.3479]

Differential Operators

- Cheung, Shen and Wen simply view the physical scattering amplitude A as a function of $k_i \cdot k_j, k_i \cdot \epsilon_j, \epsilon_i \cdot \epsilon_j$ for $i \neq j$ on the support of on-shell conditions.
- Physical differential operators should preserve some constraints, i.e., commuting with total momentum operator $\mathcal{P}_v = \sum_{i=1}^{n} k_i \cdot v$ and gauge invariance operator $\mathcal{G}_i = \sum_v (p_i \cdot v)\partial_{(\epsilon_i \cdot v)}$.
- They introduced three kinds of differential operators to transmute the amplitude of one theory into that of another theory.

[Cheung, Shen, Wen; 2017]

These differential operators are

- Trace operators, $\mathcal{T}_{ij} = \partial_{\epsilon_i \cdot \epsilon_j}$, reduce the spin of particles i, j by one, and put them in a new color order.
- Insertion operators, $\mathcal{T}_{ijl} = \partial_{\epsilon_j \cdot k_i} - \partial_{\epsilon_j \cdot k_l}$, reduce the spin of particle j by one, and insert it between particles i, l in a color order.
- Longitudinal operators, $\mathcal{L}_i = \sum_j (k_i \cdot k_j) \partial_{\epsilon_i \cdot k_j}$, reduce the spin of particle i by one and convert it to a longitudinal mode.

[Cheung, Shen, Wen; 2017]

- The trace operators are intrinsically gauge invariant, but insertion operators are effectively gauge invariant,

$$[\mathcal{T}_{ij}, \mathcal{G}_l] = 0, \quad [\mathcal{T}_{ijk}, \mathcal{G}_l] = \delta_{il}\mathcal{T}_{ij} - \delta_{kl}\mathcal{T}_{jk}.$$

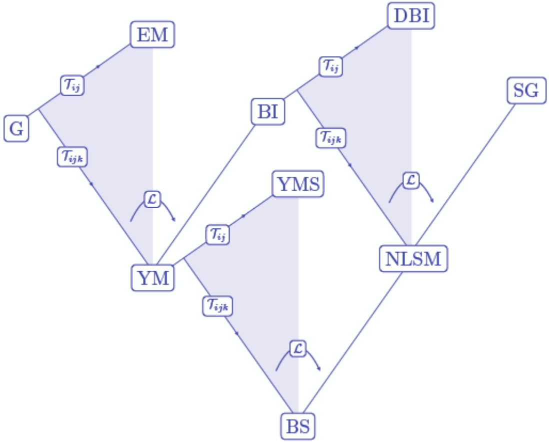
  It just means we can't apply insertion operators before trace operator.

- For example,

$$\mathcal{T}[i_1 \cdots i_n] A_{\mathrm{G}}(h_{i_1}, \cdots, h_{i_n}, \cdots) = A_{\mathrm{EYM}}(i_1, \cdots, i_n; \cdots),$$

  where $\mathcal{T}[i_1 \cdots i_n] = \left(\prod_{s=2}^{n-1} \mathcal{T}_{i_{s-1} i_s i_n}\right) \mathcal{T}_{i_1 i_n}$.

Web of different theories (2)



[Cheung, Shen, Wen; 1715.03025]

Expansion of EYM Amplitude

- In 2016, Stieberger and Taylor give the simple formula about the expansion of single trace EYM amplitude with one graviton

$$A^{\mathrm{EYM}}_{n,1}(1,\cdots,n;h)=\frac{\kappa}{g}\sum_{i=1}^{n-1}(\epsilon_h\cdot K_h)A^{\mathrm{YM}}_{n+1}(1,\cdots,i,h,i+1,\cdots,n),$$

with $K_h=k_1+\cdots+k_i$.
[Stieberger, Taylor; 2016]

- The result of Stieberger and Taylor is quickly generalized to the more general situations with gravitons, even double traces in CHY formalism or double-copy. [Nandan, Plefka, Schlotterer and Wen; 2016] [de la Cruz, Kniss, Weinzierl;2016] [Chiodaroli, Gunaydin, Johansson, Roiban; 2017]

- From a general ansatz, Fu, Du, Huang and Feng give a compact recursive formula for the expansion of EYM amplitudes with m gravitons

$$A^{\text{EYM}}_{n,m}(1,2,\cdots,n;\mathbf{H}) = \sum_{\shuffle} \sum_{\mathbf{h}|\tilde{h}=\mathbf{H}\setminus h_a} C_{h_a}(\mathbf{h}) \\ A^{\text{EYM}}_{n+m-|\tilde{h}|,|\tilde{h}|}(1,\{2,\ldots,n-1\}\shuffle\{\mathbf{h},h_a\},n;\tilde{h}),$$

  in KK basis with the help of gauge invariance. [Fu, Du, Huang, Feng; 2017]
- Quickly, Teng, Feng prove the formula in the CHY formalism. [Teng, Feng; 2017]
- Du, Feng, Teng generalize the expansion of single trace EYM amplitude to all multitirace tree level EYM amplitudes. [Du, Feng, Teng; 2017]

As we all known, the BCJ basis is the minimal basis, rather than KK basis. A nature question arises: **what is the expansion of single trace EYM amplitude in the BCJ basis?**

The Importance of Gauge Invariance

Gauge invariance is important!

The gauge invariance plays an important role in the expansion of EYM amplitude.

$$
\begin{aligned}
&A^{\text{EYM}}_{n,4}(1,\ldots,n;h_1,h_2,h_3,h_4) \qquad (3.41)\\
=&\sum_{\shuffle}(\epsilon_{h_1}\cdot Y_{h_1})A^{\text{EYM}}_{n+1,3}(1,\{2,\ldots,n-1\}\shuffle\{h_1\},n;h_2,h_3,h_4)\\
&+\sum_{i=2,3,4}\sum_{\shuffle}(\epsilon_{h_1}\cdot F_{h_i}\cdot Y_{h_i})A^{\text{EYM}}_{n+2,2}(1,\{2,\ldots,n-1\}\shuffle\{h_i,h_1\},n;\{h_2,h_3,h_4\}/\{h_i\})\\
&+\sum_{\substack{2\leq i,j\leq 4\\ i\neq j}}\sum_{\shuffle}(\epsilon_{h_1}\cdot F_{\sigma_{h_i}}\cdot F_{\sigma_{h_j}}\cdot Y_{\sigma_{h_j}})A^{\text{EYM}}_{n+3,1}(1,\{2,\ldots,n-1\}\shuffle\{\sigma_{h_j},\sigma_{h_i},h_1\},n;\{h_2,h_3,h_4\}/\{h_i,h_j\})\\
&+\sum_{\sigma\in S_3}\sum_{\shuffle}(\epsilon_{h_1}\cdot F_{\sigma_{h_2}}\cdot F_{\sigma_{h_3}}\cdot F_{\sigma_{h_4}}\cdot Y_{\sigma_{h_4}})A^{\text{YM}}_{n+4}(1,\{2,\ldots,n-1\}\shuffle\{\sigma_{h_2},\sigma_{h_3},\sigma_{h_4},h_1\},n)\,.
\end{aligned}
$$

Here $F_i^{\mu\nu}=\epsilon_i^{\mu}k_i^{\nu}-k_i^{\mu}\epsilon_i^{\nu}$ is the linearized field strength, which is manifestly gauge invariant.
So we need to consider the principle of gauge invariance more.
[Chih-Hao Fu, Yi-Jian Du, Rijun Huang and Bo Feng, 1702.08158]

First Principles

From first principles, sometimes we can determine the tree-level amplitudes uniquely.

Locality

The amplitude has only simple poles when the sum of a subset of momenta $K_{\mathcal{S}} = \sum_{i \in \mathcal{S}} k_i$ goes on shell.

Unitarity

The amplitude factorizes on the poles into the product of lower-point amplitudes, with an extra intermediate line.

Gauge Invariance

The amplitude satisfy the Ward identity $A(\epsilon_i \to k_i) = k_i^{\mu} A_{\mu} = 0$.

[Arkani-Hamed, Rodina, Trnka; 2016]

Gauge Invariance v.s. Locality and Unitarity

- Feynman diagrams make *locality* and *unitarity* manifest, but not gauge invariance. Only the sum of all Feynman diagrams is *gauge invariant*. For example,

$$A_4 \sim \frac{(\epsilon \cdot k)(\epsilon \cdot \epsilon)}{s} + \frac{(\epsilon \cdot k)(\epsilon \cdot \epsilon)}{t} + (\epsilon \cdot \epsilon)(\epsilon \cdot \epsilon).$$

- If we make the sum be gauge invariant manifestly, the locality and unitarity become obscure.

$$A_4 \sim \frac{F^4}{st}.$$

Determinacy of Gauge Invariance

Arkani-Hamed, Rodina and Trnka make a general ansatz compatible with locality, i.e., the singularity structure of cubic graphs:

$$\tilde{A}_n = \sum_{\Gamma} \frac{N_n^{\Gamma}(\epsilon_i, p_i; p^{n-2})}{P_{\sigma_1}^2 P_{\sigma_2}^2 \cdots P_{\sigma_{n-3}}^2}.$$

In the limit of one momentum being soft, they can prove the $\tilde{A}_n = A_n$ by requiring the *gauge invariance* inductively.

[Arkani-Hamed, Rodina, Trnka; 2016] [Rodina; 2016]

Arkani-Hamed, Rodina and Trnka even make a further conjecture:

Determinacy of Gauge Invariance

Simply specifying that singularities only occur when the sum of a subset of momenta goes on shell $P^2 \to 0$, gauge invariance uniquely fixes the amplitude, together with the usual mass dimension counting.

[Arkani-Hamed, Rodina, Trnka; 2016] [Rodina; 2016]

We want to know the consequences of gauge invariance only, not require the appearance of singularities, **then try to solve the constraints of gauge invariance solely.**

Physical constraints of amplitudes

Consider the color-ordered YM amplitude, a d-dimensional, parity even n-point gluon amplitude satisfies:

- on-shell conditions: $k_i^2 = 0$ for $i = 1, \cdots, n$,
- momentum conservation: $\sum_{i=1}^{n} k_i^\mu = 0$,
- multilinearity, $A = \epsilon_{1,\mu_1}\epsilon_{2,\mu_2}\cdots\epsilon_{n,\mu_n} I^{\mu_1\mu_2\cdots\mu_n}$,
- transversality: $k_i \cdot \epsilon_i = 0$, for $i = 1, \cdots, n$.

The most important

- gauge invariance: $A(\epsilon_i \to k_i) = 0$, for $i = 1, \cdots, n$.

Solving Gauge Invariance

1. The constraints of first four conditions are easily solved, only $n(n-3)/2$ independent $(k_i \cdot k_j)$, $n(n-2)$ $(k_i \cdot \epsilon_j)$ and $n(n-2)$ $(\epsilon_i \cdot \epsilon_j)$ ($i \neq j$) involved.
2. Then constructing all monomials of $(k_i \cdot \epsilon_j)$ and $(\epsilon_i \cdot \epsilon_j)$ satisfying the condition of mulitilinearity, which is a linearly independent basis, then a "possible" amplitude is a linear combination of these bases.
3. Imposing the conditions of gauge invariance results in n linear equations, transform them into the independent bases, then solve the system of linear equations.

[Barreiro, Medina; 2013] [Boels, Medina; 2016]

Solving Gauge Invariance (2)

TABLE I. Solving on-shell constraints: gluons.

Number of gluons	Minimal number of metrics	Number of terms in ansatz	Number of on-shell gauge invariant
4	1	27	1
	0	43	10
5	1	315	2
	0	558	142
6	1	4575	6
	0	8671	2364
7	1	79 275	24
	0	157 400	45 028
8	1	1 593 753	120

The two lines in Minimal number of metrics correspond to linear combinations with leading terms as $(\epsilon\cdot\epsilon)(\epsilon\cdot k)^{n-2}$ or $(\epsilon\cdot k)^n$.

[Boels, Medina; 1607.08246]

The idea of solving gauge invariance is simple, but very powerful.

Further, there is also a conjecture:

Determinacy of Gauge Invariance

A general function of n momenta and n polarizaiton vectors satisfying the gauge invariance, with leading term $(\epsilon \cdot \epsilon)(\epsilon \cdot k)^{n-2}$, must be the linear combination of the BCJ basis of YM amplitudes!

[Rodina, 1612.06342]

But the method of solving linear equations directly is not efficient and limited to the first several examples.

Content

Viewing EYM Amplitudes as Polynomials

For an EYM amplitude $A^{\text{EYM}}_{n,m}(1,2,\cdots,n;\{h_1,\cdots,h_m\})$, it contains $\{k_1^\mu,\cdots,k_n^\mu,k_{h_1}^\mu,\cdots,k_{h_m}^\mu\}$, and polarization vectors and tensors $\{\epsilon_1^\mu,\cdots,\epsilon_n^\mu,\epsilon_{h_1}^{\mu\nu},\cdots,\epsilon_{h_m}^{\mu\nu}\}$.

- Polarization tensors of gravitons factorize $\epsilon_{h_i}^{\mu\nu}=\epsilon_{h_i}^{\mu}\widetilde{\epsilon}_{h_i}^{\nu}$, and further $\epsilon_{h_i}\cdot\widetilde{\epsilon}_{h_j}$ doesn't exist.
- There are two sets of Lorentz contractions, $\{(k\cdot\epsilon_h),(\epsilon\cdot\epsilon_h)\}$ and $\{(k\cdot\widetilde{\epsilon}_h),(\epsilon\cdot\widetilde{\epsilon}_h)\}$, we view the amplitude $A^{\text{EYM}}_{n,m}$ as a polynomial of the first set contractions with coefficients of the latter.
- The polynomial $A^{\text{EYM}}_{n,m}$ is gauge invariant for ϵ_{h_i} and $\tilde{\epsilon}_{h_i}$ separately.

- Assuming the expansion $A^{\rm EYM}_{n,m} = \sum \mathcal{C}_\alpha(k,\epsilon_{h_i}) A^{\rm YM}_{n+m}(1,2,\{\alpha\},n)$. Since the amplitudes in the BCJ basis are linearly independent, if we require the gauge invariance of ϵ_{h_i} in $A_{n,m}$,

$$\sum \mathcal{C}_\alpha(\epsilon_{h_i}\to k_{h_i}) A^{\rm YM}_{n+m}(\{\alpha\}) = 0 \;\Rightarrow\; \mathcal{C}_\alpha(\epsilon_{h_i}\to k_{h_i}) = 0,$$

  then all expansion coefficients are gauge invariant for $(\epsilon_{h_1},\cdots,\epsilon_{h_m})$.

- Since the gauge invariance has strong constraints on the form of functions, then we want to solve the gauge invariance of $\mathcal{C}_\alpha$ as a function of $(k_1,\cdots,k_n,k_{h_1},\cdots,k_{h_m},\epsilon_{h_1},\cdots,\epsilon_{h_m})$.

Basic Mathematical Settings

A most general polynomial $\mathfrak{h}$ of n momenta and m polarization vectors ($m \leq n$) satisfying previous physical constraints is schematically described as

$$\mathfrak{h}_{n,m}(k_1,\ldots,k_n,\epsilon_1,\ldots,\epsilon_m) = \alpha_0 \boldsymbol{(\epsilon\cdot k)^m} + \\ \alpha_1(\epsilon\cdot\epsilon)(\epsilon\cdot k)^{m-2} + \cdots + \alpha_{\lfloor\frac{m}{2}\rfloor}(\epsilon\cdot\epsilon)^{\lfloor\frac{m}{2}\rfloor}(\epsilon\cdot k)^{m-2\lfloor\frac{m}{2}\rfloor}$$

with

- $\mathbb{B}[V] := \{(\epsilon\cdot\epsilon)^j(\epsilon\cdot k)^{m-2j},\ 0 \leq j \leq \lfloor\frac{m}{2}\rfloor\}$ as generating set,
- α_i's being rational functions of $k_i \cdot k_j$,
- $\alpha_0 \neq 0$.

Vector Space: $\mathcal{V}_{n,m}$

All such polynomials $\mathfrak{h}_{n,m}$ constitute the vector space $\mathcal{V}_{n,m}$.

And there are many such vector spaces for different n and m.

To avoid solving linear equations imposed by gauge invariance, we view the replacement $\epsilon_i \to k_i$ as a map among the vector spaces $\mathcal{V}_{n,m}$'s, called *gauge invariant map*.

A gauge invariant map is given by the gauge invariant operator $\mathcal{G}_i := \sum_v (v \cdot k_i)\partial_{v\cdot\epsilon_i}$, v representing all Lorentz vectors.

The kernel and image of the gauge invariant map $\mathcal{G}_i : \mathcal{V}_{n,s} \to \mathcal{V}^{(i)}_{n,s-1}$ are important,

- Ker $\mathcal{G}_i[\mathcal{V}_{n,s}] = \{f \in \mathcal{V}_{n,s} | f(\epsilon_i \to k_i) = 0\}$,
- the map is *surjective*, so Im $\mathcal{G}_i[\mathcal{V}_{n,s}] = \mathcal{V}^{(i)}_{n,s-1}$.

A physical polynomial we are interested in is gauge invariant for all its polynomial vectors.

Gauge Invariant Vector Space

All polynomials gauge invariant for all its polynomial vectors constitute a vector space, called *gauge invariant vector space*, given by

$$\mathcal{W}_{n,m} := \bigcap_{i=1}^{m} \text{Ker } \mathcal{G}_i[\mathcal{V}_{n,m}].$$

To characterize the gauge invariant space $\mathcal{W}_{n,m}$, we should known

- its dimension,
- the manifestly gauge invariant form of its vectors,
- the basis.

Dimension

Case: $m = 1$

For the case with $m=1$, $\mathcal{G}_1 : \mathcal{V}_{n,1} \to \mathcal{V}_{n,0}$, the dimension of $\mathcal{W}_{n,1}$

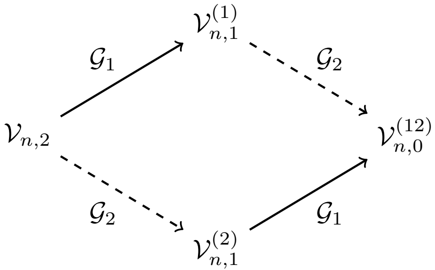
$$\begin{aligned}\dim \mathcal{W}_{n,1} = \dim \text{Ker } \mathcal{G}_1 &= \dim \mathcal{V}_{n,1} - \dim \text{Im } \mathcal{G}_1 \\ &= \dim \mathcal{V}_{n,1} - \dim \mathcal{V}_{n,0},\end{aligned}$$

the dimensions of $\dim \mathcal{V}_{n,1} = n-2, \dim \mathcal{V}_{n,0} = 1$.

The fundamental theorem of linear map is

$$\dim \text{Ker } \mathcal{G}_i = \dim \mathcal{V}_{n,s} - \dim \text{Im } \mathcal{G}_i = \dim \mathcal{V}_{n,s} - \dim \mathcal{V}^{(i)}_{n,s-1}.$$

Case: $m = 2$



The dimension

$$\begin{aligned}\dim \mathcal{W}_{n,2} &= \dim(\mathrm{Ker}\mathcal{G}_1 \cap \mathrm{Ker}\mathcal{G}_2) \\ &= \dim \mathrm{Ker}\mathcal{G}_1 + \dim \mathrm{Ker}\mathcal{G}_2 - \dim(\mathrm{Ker}\mathcal{G}_1 + \mathrm{Ker}\mathcal{G}_2)\end{aligned}$$

Case: $m = 2$

Proposition I: the splitting formula

$\text{Ker } \mathcal{G}_1 + \text{Ker } \mathcal{G}_2 = \text{Ker } \mathcal{G}_{12}$, with $\mathcal{G}_{12} = \mathcal{G}_1 \mathcal{G}_2$.

The physical meaning is: a polynomial which is gauge invariant for ϵ_1 and ϵ_2 *simultaneously* always can be divided into two parts, each of which is gauge invariant for one polarization vector.

Case: $m = 2$

Then the dimension can be calculated

$$\begin{aligned}\dim \mathcal{W}_{n,2} =& \dim(\mathrm{Ker}\mathcal{G}_1 \cap \mathrm{Ker}\mathcal{G}_2)\\ =& \dim \mathrm{Ker}\mathcal{G}_1 + \dim \mathrm{Ker}\mathcal{G}_2 - \dim \mathrm{Ker}\mathcal{G}_{12}\\ =& 2(\dim \mathcal{V}_{n,2} - \dim \mathcal{V}_{n,1}) - (\dim \mathcal{V}_{n,2} - \dim \mathcal{V}_{n,0})\\ =& \dim \mathcal{V}_{n,2} - 2\dim \mathcal{V}_{n,1} + \dim \mathcal{V}_{n,0}\end{aligned}$$

Case: $m = 3$

When generalize the method to $m = 3$, we meet problem for $m = 3$. In linear algebra, we only have

$$\begin{aligned} &\dim(U_1 + U_2 + U_3) \\ =&\dim U_1 + \dim U_2 + \dim U_3 - \dim(U_1 \cap U_2) - \dim((U_1 + U_2) \cap U_3), \end{aligned}$$

since in general $(U_1 + U_2) \cap U_3 \neq U_1 \cap U_3 + U_2 \cap U_3$.

Case: $m = 3$

Proposition II: the distribution formula

$$(\text{Ker } \mathcal{G}_1 + \text{Ker } \mathcal{G}_2) \cap \text{Ker } \mathcal{G}_3 = \text{Ker } \mathcal{G}_1 \cap \text{Ker } \mathcal{G}_3 + \text{Ker } \mathcal{G}_2 \cap \text{Ker } \mathcal{G}_3.$$

The physical meaning is: if a polynomial is gauge invariant for ϵ_3 and for ϵ_1, ϵ_2 *simultaneously*, then can be divided into two parts, one is gauge invariant for ϵ_1, ϵ_3, another is gauge invariant for ϵ_2, ϵ_3.

Then the dimension is

$$\begin{aligned}\dim \mathcal{W}_{n,3} =& \dim \mathrm{Ker}\ \mathcal{G}_1 + \dim \mathrm{Ker}\ \mathcal{G}_2 + \dim \mathrm{Ker}\ \mathcal{G}_3 - \dim \mathrm{Ker} \mathcal{G}_1\mathcal{G}_2 \\ & - \dim \mathrm{Ker}\ \mathcal{G}_1\mathcal{G}_3 - \dim \mathrm{Ker}\ \mathcal{G}_2\mathcal{G}_3 + \dim \mathrm{Ker}\ \mathcal{G}_1\mathcal{G}_2\mathcal{G}_3 \\ =& \dim \mathcal{V}_{n,3} - 3\dim \mathcal{V}_{n,2} + 3\dim \mathcal{V}_{n,1} - \dim \mathcal{V}_{n,0}.\end{aligned}$$

General Case

Then the dimension of $\mathcal{W}_{n,m}$

$$\dim \mathcal{W}_{n,m} = \sum_{s=0}^{m} (-1)^s \binom{m}{s} \dim \mathcal{V}_{n,m-s}$$

with

$$\dim \mathcal{V}_{n,m} = \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2i} \frac{(2i)!}{2^i\ (i!)} (n-2)^{m-2i}.$$

n	4	5	6	7	8	9
$\dim \mathcal{W}_{n,n}$	10	142	2364	45028	969980	23372550

TABLE I. Solving on-shell constraints: gluons.

Number of gluons	Minimal number of metrics	Number of terms in ansatz	Number of on-shell gauge invariant
4	1	27	1
	0	43	10
5	1	315	2
	0	558	142
6	1	4575	6
	0	8671	2364
7	1	79 275	24
	0	157 400	45 028
8	1	1 593 753	120

The two lines in Minimal number of metrics correspond to linear combinations with leading terms as $(\epsilon \cdot \epsilon)(\epsilon \cdot k)^{n-2}$ or $(\epsilon \cdot k)^n$.
[Boels, Medina; 1607.08246]

Manifestly Gauge Invariant Form

- Applying the gauge invariance in the form of operator equation $[\mathcal{T}_{ijk}, \mathcal{G}_l] = \delta_{il}\mathcal{T}_{ij} - \delta_{kl}\mathcal{T}_{jk}$, we can prove that **every vector of $\mathcal{W}_{n,m}$ $(m < n)$ can be written in the form of linear combinations of multiplications of** $(k_i \cdot f_{h_l} \cdots f_{h_s} \cdot k_j)$ with $f_a^{\mu\nu} = k_a^\mu \epsilon_a^\nu - \epsilon_a^\mu k_a^\nu$.

- Because

$$(Bf_iA)(Ck_i) = (Bf_iC)(Ak_i) + (Cf_iA)(Bk_i),$$

  all f-terms with more than two f can be split into fundamental f-terms: $(k_i \cdot f_a \cdot k_j)$, $(k_i \cdot f_a \cdot f_b \cdot k_j)$,

- Note that there is another kind of gauge invariant factors $\mathrm{Tr}(f \cdots f)$, but in fact we can also split them into the combinations of $(k_i \cdot f_{h_l} \cdots f_{h_s} \cdot k_j)$.

So we get the conclusion.

Proposition III: Gauge invariant vector

Every vector in $\mathcal{W}_{n,m}$ $(m < n)$ can be recast in a manifestly gauge invariant form, which is a linear combination of the multiplication of fundamental f-terms with the total number of field strength f in every monomial being m.

Gauge Invariant Basis

Proposition IV: Gauge invariant vector basis

The set of vectors

$$\left\{\prod_{i=1}^{s}(k_{n-1}\cdot f_{\alpha_{2i-1}}\cdot f_{\alpha_{2i}}\cdot k_{n-1})\prod_{i=2s+1}^{m}(k_{n-1}\cdot f_{\beta_i}\cdot k_j)\right\}$$

with $s=0,1,\cdots,\lfloor\frac{m}{2}\rfloor$ is the basis of $\mathcal{W}_{n,m}$ $(m\leq n-2)$.

- Momentum conservation eliminates k_n,
- $(k_{n-1}f_af_bk_{n-1})=(k_{n-1}f_bf_ak_{n-1}),(k_{n-1}f_ak_{n-1})=0,(k_{n-1}f_ak_a)=0.$

The number of the set of vectors N equal the dimension of $\mathcal{W}_{n,m}$

$$\begin{aligned}
&\dim\ \mathcal{W}_{n,m} \\
&= \sum_{s=0}^{m} \sum_{i=0}^{\lfloor \frac{m-s}{2} \rfloor} (-)^s \binom{m}{s} \binom{m-s}{2i} \frac{(2i)!}{2^i\ i!} (n-2)^{m-s-2i} \\
&= \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{s=0}^{m-2i} (-)^s \frac{m!}{s!(m-s-2i)!\ 2^i\ i!} (n-2)^{m-s-2i} \\
&= \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \frac{m!}{i!\ 2^i\ (m-2i)!} (n-3)^{m-2i} = N
\end{aligned}$$

Content

Expansion of EYM in BCJ basis

We change to expand EYM amplitudes in gauge invariant basis of $\mathcal{W}_{n+m,m}$,

$$A^{\text{EYM}}_{n,m} = \sum \widetilde{\mathcal{C}}_i(\tilde{\epsilon})\mathcal{B}_i(\epsilon)$$

the new coefficients of the basis will be linear combinations of YM amplitudes.

- Calculating the coefficients $\mathcal{C}(\epsilon)$ is difficult,
- gauge invariant structures of $\mathcal{C}$ or $\mathcal{B}_i$ are known,
- differential operators act on coefficients $\mathcal{B}_i(\epsilon)$.

The exact basis is chosen as

$$\mathcal{B}(\alpha,\beta,\gamma) := \prod_{i=1}^{p} \mathsf{F}_{h_{\alpha_{2i-1}} h_{\alpha_{2i}}} \prod_{i=1}^{q} \mathsf{F}_{h_{\beta_i}}^{h_{\beta'_i}} \prod_{i=1}^{r} \mathsf{F}_{h_{\gamma_i}}^{a_{\gamma_i}} \quad ,$$

with $2p+q+r=m$ and

$$\mathsf{F}_{h_i h_j} := \frac{k_1 \cdot f_{h_i} \cdot f_{h_j} \cdot k_1}{(k_1 \cdot k_{h_i})(k_1 \cdot k_{h_j})}, \; \mathsf{F}_{h_i}^{h_j} := \frac{k_1 \cdot f_{h_i} \cdot k_{h_j}}{k_1 \cdot k_{h_i}}, \; \mathsf{F}_{h_i}^{a} := \frac{k_1 \cdot f_{h_i} \cdot K_a}{k_1 \cdot k_{h_i}}.$$

Applying a differential operator as a multiplication of m insertion operators for ϵ to the expansion,

$$\mathcal{T}^m A_{n,m}^{\text{EYM}} = \sum \mathcal{C}(\alpha,\beta,\gamma)\left(\mathcal{T}^m \mathcal{B}(\alpha,\beta,\gamma)\right) = \sum A_{n+m}^{\text{YM}}.$$

Finding enough many differential operators, we get enough linear equations to solve the coefficients.

$$\begin{pmatrix} \mathcal{T}_1^m\mathcal{B}_1 & \mathcal{T}_1^m\mathcal{B}_2 & \dots & \mathcal{T}_1^m\mathcal{B}_N \\ \mathcal{T}_2^m\mathcal{B}_1 & \mathcal{T}_2^m\mathcal{B}_2 & \dots & \mathcal{T}_2^m\mathcal{B}_N \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{T}_N^m\mathcal{B}_1 & \mathcal{T}_N^m\mathcal{B}_2 & \dots & \mathcal{T}_N^m\mathcal{B}_N \end{pmatrix} \begin{pmatrix} \mathcal{C}_1 \\ \mathcal{C}_2 \\ \vdots \\ \mathcal{C}_N \end{pmatrix} = \begin{pmatrix} \mathcal{T}_1^m A_{n,m}^{\text{EYM}} \\ \mathcal{T}_2^m A_{n,m}^{\text{EYM}} \\ \vdots \\ \mathcal{T}_N^m A_{n,m}^{\text{EYM}} \end{pmatrix}$$

A simple example

To illustrate the idea, consider a simple example $A_{n,2}(1,\cdots,n;h_1,h_2)$. dim $\mathcal{W}_{n+2,2}=(n-1)^2+1$ and the basis is

$$\mathsf{F}^{a_1}_{h_1}\mathsf{F}^{a_2}_{h_2},\ \mathsf{F}^{h_1}_{h_2}\mathsf{F}^{a_1}_{h_1},\ \mathsf{F}^{h_2}_{h_1}\mathsf{F}^{a_2}_{h_2},\ \mathsf{F}_{h_1h_2},\ \mathsf{F}^{h_2}_{h_1}\mathsf{F}^{h_1}_{h_2}.$$

with $2\leq a_1,a_2\leq n-1$.

The expansion of $A_{n,2}(1,\cdots,n;h_1,h_2)$ is

$$A_{n,2}=\sum_{a_1,a_2=2}^{n-1}\mathcal{C}[\mathsf{F}^{a_1}_{h_1}\mathsf{F}^{a_2}_{h_2}]\mathsf{F}^{a_1}_{h_1}\mathsf{F}^{a_2}_{h_2}+\sum_{a=2}^{n-1}\left(\mathcal{C}[\mathsf{F}^{h_1}_{h_2}\mathsf{F}^{a}_{h_1}]\mathsf{F}^{h_1}_{h_2}\mathsf{F}^{a}_{h_1}+\mathcal{C}[\mathsf{F}^{h_2}_{h_1}\mathsf{F}^{a}_{h_2}]\mathsf{F}^{h_2}_{h_1}\mathsf{F}^{a}_{h_2}\right)$$
$$+\mathcal{C}[\mathsf{F}_{h_1h_2}]\mathsf{F}_{h_1h_2}.$$

There are four types of basis, then we apply four types differential operators,

$$\begin{aligned}
\mathcal{T}_{a_1h_1(a_1+1)}\mathcal{T}_{a_2h_2(a_2+1)} &\Rightarrow \mathcal{C}[\mathsf{F}^{a_1}_{h_1}\mathsf{F}^{a_2}_{h_2}] \\
\mathcal{T}_{h_2h_1n}\mathcal{T}_{a_2h_2(a_2+1)} &\Rightarrow \mathcal{C}[\mathsf{F}^{h_2}_{h_1}\mathsf{F}^{a_2}_{h_2}] \\
\mathcal{T}_{h_1h_2n}\mathcal{T}_{a_1h_1(a_1+1)} &\Rightarrow \mathcal{C}[\mathsf{F}^{h_1}_{h_2}\mathsf{F}^{a_1}_{h_1}] \\
(k_1\cdot k_{h_2})\mathcal{T}_{h_2h_1n}\mathcal{T}_{1h_22} &\Rightarrow \mathcal{C}[\mathsf{F}_{h_1h_2}].
\end{aligned}$$

There are totally $(n-1)^2+1$ differential operators.

Directly solving the system of linear equations is difficult. Properly choosing differential operators, we can simplify the process of solving the system of linear equations.

Strategy

Construct a good differential operator as a multiplication of m properly chosen insertion operators, under its action, there is only one unknown coefficient appearing in the equation, while other appearing coefficients have been calculated.

The problem is reduced to construct good differential operators, which only select specific $\mathcal{B}$'s.

Constructing good differential operators require us to know more about the structures of vectors in the gauge invariant basis.

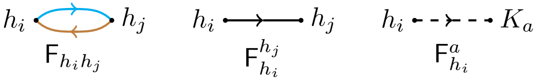
$$\begin{pmatrix} \mathcal{T}_1^m\mathcal{B}_1 & 0 & 0 & 0 & \cdots \\ 0 & \mathcal{T}_{N_1}^m\mathcal{B}_{N_1} & 0 & 0 & \cdots \\ \mathcal{T}_{N_1+1}^m\mathcal{B}_1 & \mathcal{T}_{N_1+1}^m\mathcal{B}_{N_1} & \mathcal{T}_{N_1+1}^m\mathcal{B}_{N_1+1} & 0 & 0 \\ \mathcal{T}_{N_1+N_2}^m\mathcal{B}_1 & \mathcal{T}_{N_1+N_2}^m\mathcal{B}_{N_1} & \mathcal{T}_{N_1+N_2}^m\mathcal{B}_{N_1+1} & \mathcal{T}_{N_1+N_2}^m\mathcal{B}_{N_1+N_2} & 0 \\ \mathcal{T}_{N_1+N_2+1}^m\mathcal{B}_1 & \mathcal{T}_{N_1+N_2+1}^m\mathcal{B}_{N_1} & \mathcal{T}_{N_1+N_2+1}^m\mathcal{B}_{N_1+1} & \mathcal{T}_{N_1+N_2+1}^m\mathcal{B}_{N_1+N_2} & \mathcal{T}_{N_1+N_2+1}^m\mathcal{B}_{N_1+N_2+1} \end{pmatrix}$$

Quivers: fundamental f-terms

The structures of the gauge invariant basis can be depicted by some quivers.

- $(\epsilon_h \cdot k)$'s in gauge invariant vectors are important.

- The quiver representation of fundamental f-terms are

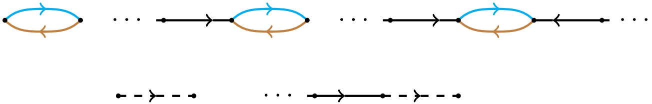


The colour loop of $\mathsf{F}_{h_i h_j}$ is a *pseudo-loop*. Real loops are dropped for the index circles.

Structures of gauge invariant basis

- The quiver of a vector of the basis has many disconnected components.
  All pseudo-loops, and these points labelled by K_a are topological disconnected from each other.

- Every component of the quiver of a vector in the basis has one of the following structures



For example: $\mathsf{F}_{h_1h_2}\mathsf{F}^{h_1}_{h_3}\mathsf{F}^{h_5}_{h_4}\mathsf{F}^{2}_{h_5}\mathsf{F}^{7}_{h_6}$ and $\mathsf{F}^{h_4}_{h_1}\mathsf{F}^{h_4}_{h_2}\mathsf{F}^{h_3}_{h_4}\mathsf{F}^{4}_{h_3}\mathsf{F}^{6}_{h_5}\mathsf{F}^{4}_{h_6}$

Constructing differential operators

The vector in the gauge invariant basis is the multiplication of three types of fundamental f-terms, the constructed differential operators should distinguish them.

- First, $\mathcal{T}_{ah_i(a+1)}$ can only select $\mathsf{F}^{a}_{h_i}$ uniquely,

$$\mathcal{T}_{ah_i(a+1)}\ \mathsf{F}^{b}_{h_j} = \delta_{ij}\delta_{ab},\quad \mathcal{T}_{ah_i(a+1)}\ \mathsf{F}_{h_{i'}h_{j'}} = 0,\quad \mathcal{T}_{ah_i(a+1)}\ \mathsf{F}^{h_{j'}}_{h_{i'}} = 0.$$

- Second, $\mathcal{T}_{h_j h_i n}$ only selects $\mathsf{F}^{h_j}_{h_i}$, the $\mathsf{F}_{h_{i'}h_{j'}}$ is left in the next step,

$$\mathcal{T}_{h_j h_i n}\ \mathsf{F}^{h_{j'}}_{h_{i'}} = \delta_{ii'}\delta_{jj'},\quad \mathcal{T}_{h_j h_i n}\ \mathsf{F}^{a_{i'}}_{h_{i'}} = 0,$$

$$\mathcal{T}_{h_j h_i n}\ \mathsf{F}_{h_{i'}h_{j'}} = \frac{\epsilon_{h_j}\cdot k_1}{k_1\cdot k_{h_j}}(\delta_{ii'}\delta_{jj'} + \delta ij'\delta_{ji'}).$$

- Third, $(k_1 \cdot k_{h_j})\mathcal{T}_{1h_j2}\mathcal{T}_{h_jh_in}$ only selects the term $\mathsf{F}_{h_ih_j}$,

$$(k_1 \cdot k_{h_j})\mathcal{T}_{1h_j2}\mathcal{T}_{h_jh_in}\ \mathsf{F}_{h_{i'}h_{j'}} = \delta_{ii'}\delta_{jj'},$$

$$(k_1 \cdot k_{h_j})\mathcal{T}_{1h_j2}\mathcal{T}_{h_jh_in}\mathsf{F}_{h_i}^{h_j}\mathsf{F}_{h_j}^{a_t} = -k_{h_j} \cdot (k_1 + K_{a_t}),$$

$$(k_1 \cdot k_{h_j})\mathcal{T}_{1h_j2}\mathcal{T}_{h_jh_in}\mathsf{F}_{h_i}^{h_j}\mathsf{F}_{h_j}^{h_p} = -k_{h_j} \cdot (k_1 + K_{h_p}).$$

The map from a given gauge invariant vector to a corresponding differential operator is

Method of constructing differential operators

$$\mathcal{B}_{\alpha\beta\gamma}=\prod_{i=1}^{p}\mathsf{F}_{h_{\alpha_{2i-1}}h_{\alpha_{2i}}}\prod_{i=1}^{q}\mathsf{F}_{h_{\beta_i}}^{h_{\beta'_i}}\prod_{i=1}^{r}\mathsf{F}_{h_{\gamma_i}}^{a_{\gamma_i}}\Longrightarrow$$

$$\mathcal{D}_{\alpha\beta\gamma}=\prod_{i=1}^{p}(k_1\cdot k_{h_{\alpha_{2i}}})\mathcal{T}_{h_{\alpha_{2i}}h_{\alpha_{2i-1}}n}\mathcal{T}_{1h_{\alpha_{2i}}2}\prod_{i=1}^{q}\mathcal{T}_{h_{\beta'_i}h_{\beta_i}n}\prod_{i=1}^{r}\mathcal{T}_{a_{\gamma_i}h_{\gamma_i}(a_{\gamma_i}+1)}.$$

Algorithm

The algorithm is implemented order by order, starting from $p = 0$ to the largest value p and for a given p, we start from the largest r to $r = 0$.

- First, calculating these vectors with 0 pseudo-loop. Apply these differential operators

$$\left(\prod_{i=1}^{q} \mathcal{T}_{h_{\beta_i'} h_{\beta_i} n}\right)\left(\prod_{i=1}^{r} \mathcal{T}_{a_{\gamma_i} h_{\gamma_i} (a_{\gamma_i}+1)}\right)$$

  to the expansion equation, each operator gives one linear equation of one unknown coefficients.

- Second, calculating these vectors with 1 pseudo-loop. Substitute the solutions in first step back to the expansion, apply

$$\Big((k_1 \cdot k_{\alpha_2})\mathcal{T}_{h_{\alpha_2} h_{\alpha_1} n}\mathcal{T}_{1 h_{\alpha_2} 2}\Big)\left(\prod_{i=1}^{q}\mathcal{T}_{h_{\beta'_i} h_{\beta_i} n}\right)\left(\prod_{i=1}^{r}\mathcal{T}_{a_{\gamma_i} h_{\gamma_i} (a_{\gamma_i}+1)}\right),$$

each operator gives one linear equation with one unknown coefficient.

- ...

- Repeat the procedure until these vectors with $[m/2]$ pseudo-loops.

Example: $A_{n,3}^{\mathrm{EYM}}$

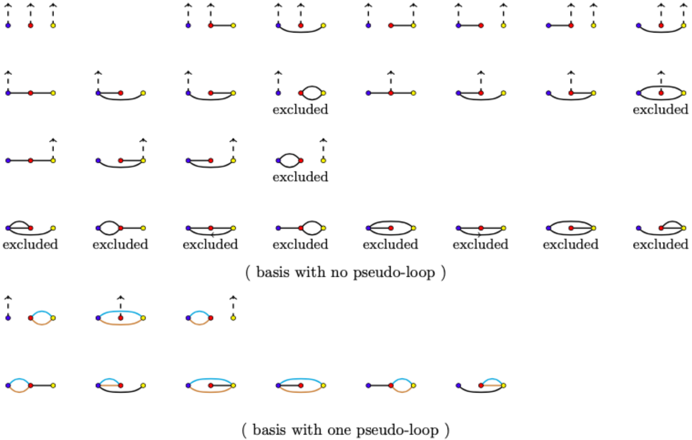


Figure 3. Quiver representation of gauge invariant basis for $A_{n,3}^{\mathrm{EYM}}$. For simplicity, h_1, h_2 and h_3 are denoted as blue, red and yellow dots respectively. Arrows always flow from starting points of solid line toward pseudo-loops or the ending points of dashed line, so they are omitted unless causing confusion. The ending point of dashed line is K_{a_i} depending on the h_i it connects, and $2 \leq a_1, a_2, a_3 \leq n-1$. Quivers with real loops are excluded.

There are totally $\dim\mathcal{W}_{n,3} - 3(n-2) - 8 = n^3 - 2$ terms contributing.

1. 0 pseudo-loop.
   - 3 K: $\mathcal{T}_{a_1 h_1 (a_1+1)}\mathcal{T}_{a_2 h_2 (a_2+1)}\mathcal{T}_{a_3 h_3 (a_3+1)} \Rightarrow \mathcal{C}[\mathsf{F}^{a_1}_{h_1}\mathsf{F}^{a_2}_{h_2}\mathsf{F}^{a_3}_{h_3}]$.
   - 2 K: $\mathcal{T}_{h_{\beta_1'} h_{\beta_1} n}\mathcal{T}_{a_{\gamma_1} h_{\gamma_1} (a_{\gamma_1}+1)}\mathcal{T}_{a_{\gamma_2} h_{\gamma_2} (a_{\gamma_2}+1)} \Rightarrow \mathcal{C}[\mathsf{F}^{h_{\beta_1'}}_{h_{\beta_1}}\mathsf{F}^{a_{\gamma_1}}_{h_{\gamma_1}}\mathsf{F}^{a_{\gamma_2}}_{h_{\gamma_2}}]$.
   - 1 K: $\mathcal{T}_{h_{\beta_1'} h_{\beta_1} n}\mathcal{T}_{h_{\beta_2'} h_{\beta_2} n}\mathcal{T}_{a_{\gamma_1} h_{\gamma_1} (a_{\gamma_1}+1)} \Rightarrow \mathcal{C}[\mathsf{F}^{h_{\beta_1'}}_{h_{\beta_1}}\mathsf{F}^{h_{\beta_2'}}_{h_{\beta_2}}\mathsf{F}^{a_{\gamma_1}}_{h_{\gamma_1}}]$.
2. 1 pseudo-loop.
   - 1 K: $\mathcal{T}_{h_3 h_2 n}\mathcal{T}_{1 h_3 2}\mathcal{T}_{a_1 h_1 (a_1+1)} \Rightarrow \mathcal{C}[\mathsf{F}_{h_2 h_3}\mathsf{F}^{a_1}_{h_1}]$.
   - 0 K: $\mathcal{T}_{h_3 h_2 n}\mathcal{T}_{1 h_3 2}\mathcal{T}_{h_2 h_1 n} \Rightarrow \mathcal{C}[\mathsf{F}_{h_2 h_3}\mathsf{F}^{h_2}_{h_1}]$.

Thanks for you attentions!

Proof of Proposition III

Proof:

- Inductively, consider $\mathfrak{h}_{n,1}(k_1, \cdots, k_n, \epsilon_1)$

$$\mathfrak{h}_{n,1} = \sum_{i=1}^{n-1} \alpha_i (\epsilon_1 \cdot k_i).$$

Solving the condition of gauge invariance of ϵ_1 in $\mathfrak{h}_{n,1}$,

$$\sum_{i=1}^{n-1} \alpha_i (k_1 \cdot k_i) = 0 \Rightarrow \alpha_{n-1} = -\sum_{i=1}^{n-2} \alpha_i \frac{(k_1 \cdot k_i)}{(k_{n-1} \cdot k_1)}.$$

Replacing α_{n-1},

$$\mathfrak{h}_{n,1} = \sum_{i=1}^{n-2} \alpha_i \frac{(k_{n-1} f_1 k_i)}{(k_{n-1} k_1)}.$$

Proof of Proposition III

Proof:

- (1) $\mathfrak{h}_{n,m}(k_1, \cdots, k_n, \epsilon_1, \cdots, \epsilon_m)$ always has the form

$$\mathfrak{h}_{n,m} = \sum_{i=2}^{m}(\epsilon_1 \cdot \epsilon_i)T_{1i} + \sum_{i=2}^{m}(\epsilon_1 \cdot k_i)(\epsilon_i \cdot T'_{1i}) + \sum_{i=m+1}^{n-1}(\epsilon_1 \cdot k_i)T''_{1i},$$

and $\mathcal{G}_a \mathfrak{h}_{n,m} = 0, 1 \leq a \leq m$.
(2) Applying $[\mathcal{T}_{a1n}, \mathcal{G}_a] = \mathcal{T}_{a1}$ with $2 \leq a \leq m$,

$$T_{1a} = -(k_a \cdot T'_{1a}).$$

Then

$$\mathfrak{h}_{n,m} = \sum_{i_1=2}^{m}(\epsilon_1 \cdot f_{i_1} \cdot T'_{1i_1}) + \sum_{i_1=m+1}^{n-1}(\epsilon_1 \cdot k_{i_1})T''_{1i_1}.$$

Proof of Proposition III

Proof:

- (3) Solving the condition of gauge invariance of ϵ_1 of $\mathfrak{h}_{n,m}$, like $\mathfrak{h}_{n,1}$, get

$$\mathfrak{h}_{n,m} = \sum_{i_1=2}^{m} \frac{(k_{n-1} \cdot k_1 \cdot f_{i_1} \cdot T'_{1i_1})}{(k_1 \cdot k_{n-1})} + \sum_{i_1=m+1}^{n-1} \frac{(k_{n-1} f_1 \cdot k_{i_1})}{(k_1 \cdot k_{n-1})} T''_{1i_1}.$$

- (4) T''_{1i_1} has already been the desired form, then expand T'_{1i_1} as before, and apply the operator equations. Continuing the procedure to the end, finally get

$$\mathfrak{h}_{n,m} = \sum_{i_1=m+1}^{n-1} \frac{(k_{n-1} f_1 \cdot k_{i_1})}{(k_1 \cdot k_{n-1})} T''_{1i_1} + \sum_{s=2}^{m} \tilde{\mathfrak{h}}_{m,s}$$

Proof of Proposition III

Proof:

$$\widetilde{\mathfrak{h}}_{n,s} = \sum_{i_1=2}^{m} \sum_{\substack{i_2=2\\ i_2\neq i_1}}^{m} \cdots \sum_{\substack{i_{s-1}=2\\ i_{s-1}\neq i_1,i_2,\ldots,i_{s-2}}}^{m} \sum_{\substack{i_s=m+1\\ i_s=1,i_1,i_2,\ldots,i_{s-2}}}^{n-1} \frac{k_{n-1}\cdot f_1 \cdot f_{i_1}\cdots f_{i_{s-1}}\cdot k_{i_s}}{k_1\cdot k_{n-1}} T''_{(1 i_1\cdots i_{s-1}) i_s}$$

- (5) Applying the following identity, all "long" f-terms can be split into fundamental f-terms,

$$(B\cdot f_p\cdot A)(C\cdot k_p) = (B\cdot f_p\cdot C)(A\cdot k_p) + (C\cdot f_p\cdot A)(B\cdot k_p).$$

The proof is finished.

Terms with Index Circle (1)

- *Terms with index circles* are those the expansion of them contain such factors $(\epsilon_{i_1}\cdot k_{i_2})(\epsilon_{i_2}\cdot k_{i_3})\cdots(\epsilon_{i_{s-1}}\cdot k_{i_1})$.
- Consider applying a differential operator $\mathcal{T}_{ah_{i_1}h_{i_2}}\mathcal{T}_{ah_{i_2}h_{i_3}}\cdots\mathcal{T}_{ah_{i_s}h_{i_1}}$ to the EYM amplitude in CHY,

$$\begin{aligned}&\mathcal{T}_{ah_{i_1}h_{i_2}}\mathcal{T}_{ah_{i_2}h_{i_3}}\cdots\mathcal{T}_{ah_{i_s}h_{i_1}}A^{\mathrm{EYM}}_{n,m}\\ &=\int d\mu\mathrm{PT}(1,2,\cdots,n)(\mathcal{T}_{ah_{i_1}h_{i_2}}\mathcal{T}_{ah_{i_2}h_{i_3}}\cdots\mathcal{T}_{ah_{i_s}h_{i_1}}\mathrm{Pf}\Psi_{H_m})\mathrm{Pf}'\Psi.\end{aligned}$$

- $\mathrm{Pf}\Psi_{H_m}$ can be expanded as the sum of all permutations like

$$\mathrm{Pf}\Psi_{H_m}=\sum_{\substack{1\le i_1\le i_2\le\cdots\le i_m\le n\\ i_1+i_2+\cdots+i_m=n}}(-1)^{n-m}P_{i_1i_2\cdots i_m},$$

which is organized by the unique cycle decomposition of these permutations. When the length of the cycle is one, it is denoted by $\Psi_{(h_i)}$, which is $\Psi_{(h_i)}=-\sum_{b\neq h_i}\frac{\epsilon_{h_i}\cdot k_b}{\sigma_{h_ib}}$. When the length of the cycle is bigger than one, it's given by $\Psi_{(h_{i_1}\cdots h_{i_r})}=\frac{\mathrm{tr}(f_{h_{i_1}}\cdots f_{h_{i_r}})}{2\sigma_{h_{i_1}h_{i_2}}\cdots\sigma_{h_{i_r}h_{i_1}}}$

Terms with Index Circle (2)

- Take the $s = 2$ as an example

$$
\begin{aligned}
&\mathcal{T}_{ah_1h_2}\mathcal{T}_{ah_2h_1}\mathrm{Pf}\Psi_{H_m} \\
&= \mathcal{T}_{ah_1h_2}\mathcal{T}_{ah_2h_1} \sum_{\substack{1\leq i_1\leq i_2\leq\cdots\leq i_m\leq n \\ i_1+i_2+\cdots+i_m=n}} (-1)^{n-m} P_{i_1 i_2\cdots i_m} \\
&= \mathcal{T}_{ah_1h_2}\mathcal{T}_{ah_2h_1}\Big\{\Psi_{(1)}\Psi_{(2)}\Psi_{H_{m-2}} - \Psi_{(12)}\Psi_{H_{m-2}} + \Psi_{(1)}\Psi_{(2\cdots)}(\cdots) \\
&\quad + \Psi_{(2)}\Psi_{(1\cdots)}(\cdots) + \Psi_{(1..2...)}(\cdots)\Big\} \\
&= \mathcal{T}_{ah_1h_2}\mathcal{T}_{ah_2h_1}\Big\{\Psi_{(1)}\Psi_{(2)} - \Psi_{(12)}\Big\}\Psi_{H_{m-2}}.
\end{aligned}
$$

Among all cycle structures of permutations, only the first two give nonzero contributions. Carrying it out explicitly, we get

$$
\begin{aligned}
\mathcal{T}_{ah_1h_2}\mathcal{T}_{ah_2h_1}\mathrm{Pf}\Psi_{H_m} &= \mathcal{T}_{ah_1h_2}\mathcal{T}_{ah_2h_1}\Big\{\Psi_{(1)}\Psi_{(2)} - \Psi_{(12)}\Big\}\Psi_{H_{m-2}} \\
&= \Big\{\frac{\sigma_{h_2a}}{\sigma_{h_1a}\sigma_{h_1h_2}}\frac{\sigma_{h_1a}}{\sigma_{h_2h_1}\sigma_{h_2a}} - \frac{1}{\sigma_{h_1h_2}\sigma_{h_2h_1}}\Big\}\mathrm{Pf}\Psi_{H_m} = 0.
\end{aligned}
$$

- The proof is easy to generalize to the general case.