Topological Vertex for SO(N) Gauge Theories and Beyond

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based on arXiv:2012.13303 (JHEP04(2021)292)
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and
work in progress with Satoshi Nawata (Fudan Univ.)

What we study:

5d $\mathcal{N}$=1 supersymmetric gauge theories (on $C^2_{q,t} \times S^1$)

gauge group: U(N) → SO(N) → G_2 → …

Why?

- **Exactly solvable (with localization method)**
- **non-perturbative physics**
- **dualities (S-duality, fiber-base duality, AGT (dual with CFTs))**
- **quantum integrability**
- **5d $\mathcal{N}$=1 → 4d $\mathcal{N}$=2**

…

Current situation:

many interesting properties discovered in the case of U(N)

- 4d/2d duality (with chiral algebra or non-unitary CFT)
- modular tensor category
- quantum integrability (~Calogero-Sutherland system)

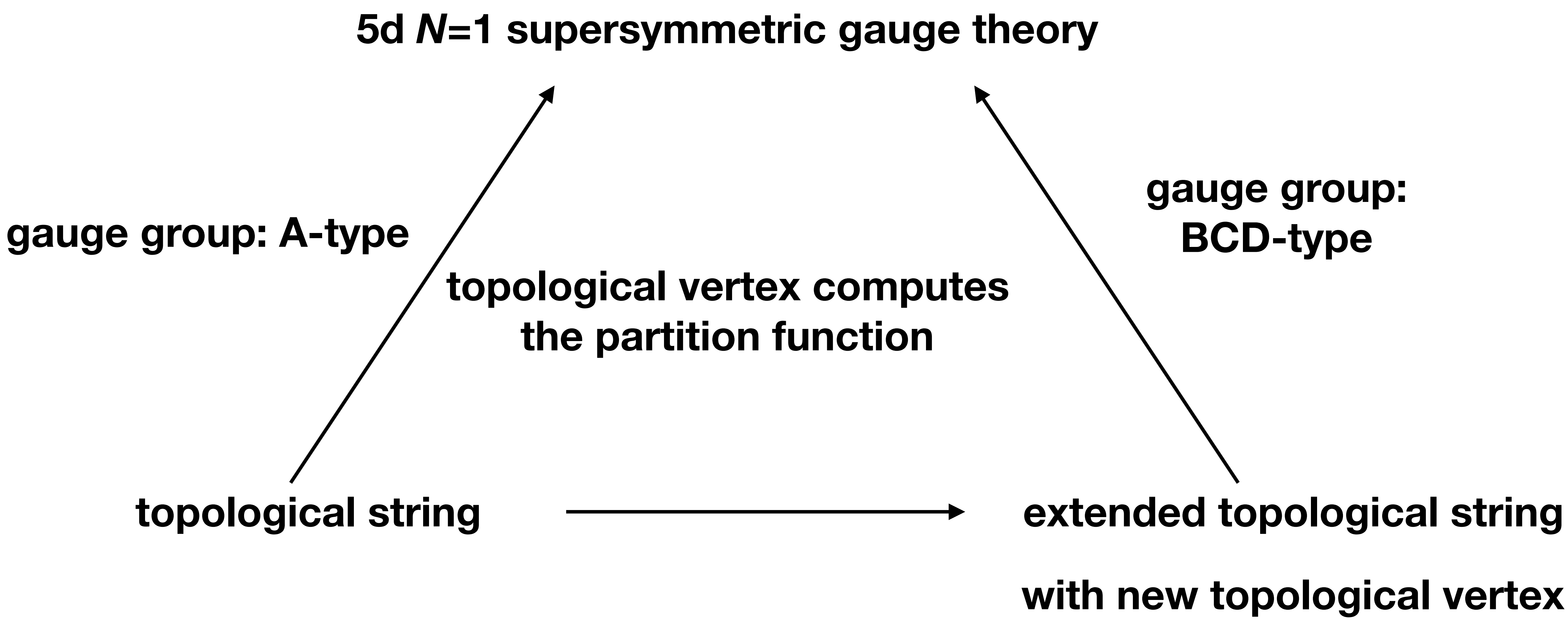
…

but not so many results known in SO(N) or Sp(N) theories

because of technical difficulties

- in the calculation of Nekrasov partition function, instanton counting is very difficult. ← what we try to solve
- in the study of chiral algebra, BCD-type Macdonald symmetric polynomial is difficult.

A sketch of what we did:



Plan of Talk:

1. Review 1: SU(N) gauge theories and localization
2. Review 2: brane construction and topological string
3. Review 3: AGT duality and DIM algebra
4. proposal: topological vertex formalism for SO(N) theory
5. consistency checks
6. Kim-Yagi's prescription and physical interpretation

5d N=1 gauge theory on $C^2_{q,t}$ x S^1

Ω-background with two deformation parameters

$q_1 = e^{R\epsilon_1} = t,$

$q_2 = e^{R\epsilon_2} = q^{-1}.$

Its partition function can be found via localization method.

R**: radius of S^1**

$$Z = Z_{cl} Z_{one\ loop} Z_{instanton}.$$

$Z_{one\ loop}$ → **perturbative part**

$Z_{instanton}$ → **non-perturbative part**

perturbative part is completely determined by the root system of the gauge group G.

$$Z_{one\ loop} = Z^G_{Cartan} Z^G_{root},$$

expression of the perturbative part:

$$Z^G_{\mathrm{root}} = P.E. \left(\left(\frac{q}{(1-q)(1-t)} + \frac{t}{(1-q)(1-t)} \right) \sum_{\alpha \in \Delta_+} e^{-\alpha \cdot a} \right),$$

$$Z^G_{\mathrm{Cartan}} = P.E. \left(\frac{\mathrm{rank}(G)}{2} \left(\frac{q}{(1-q)(1-t)} + \frac{t}{(1-q)(1-t)} \right) \right).$$

where Δ_+ is the set of all positive roots,

$$P.E. \left(f(x_1, x_2, \ldots, x_n) \right) := \exp \left(\sum_{k=1}^{\infty} \frac{1}{k} f(x_1^k, x_2^k, \ldots, x_n^k) \right).$$

non-perturbative part:

instanton counting

in 4d $\quad *F = F.$ $\quad$ **(codimension 4 object)**

in 5d: particle-like

in string theory: D(p-4) branes on Dp brane.

instanton counting is not easy

ADHM construction → Nekrasov partition function

For U(N) or SU(N) theory:

$$Z^{\mathrm{SU}(N)}_{\mathrm{loc,\ inst}} = \sum_{k=0}^{\infty} \mathfrak{q}^k \frac{1}{|W(\mathrm{SU}(k))|} \oint \left(\prod_{i=1}^{k} \frac{\mathrm{d}\phi_i}{2\pi i} \right) Z^{\mathrm{SU}(N)}_k ,$$

where

$$|W(G)| = \begin{cases} n! & G = \mathrm{SU}(n) \\ 2^{n-1+\delta} n! & G = O(2n+\delta) \\ 2^n n! & G = \mathrm{Sp}(n) \end{cases} ,$$

$$Z^{\mathrm{SU}(N)}_k = \frac{[2\epsilon_+]^k}{[\epsilon_{1,2}]^k} \prod_{i=1}^{k} \prod_{j=1}^{N} [\phi_i - a_j \pm \epsilon_+]^{-1} \prod_{\substack{i,j=1 \\ i<j}}^{k} \frac{[\phi_{ij}]^2 [\phi_{ij} \pm 2\epsilon_+]}{[\phi_{ij} \pm \epsilon_1][\phi_{ij} \pm \epsilon_2]} ,$$

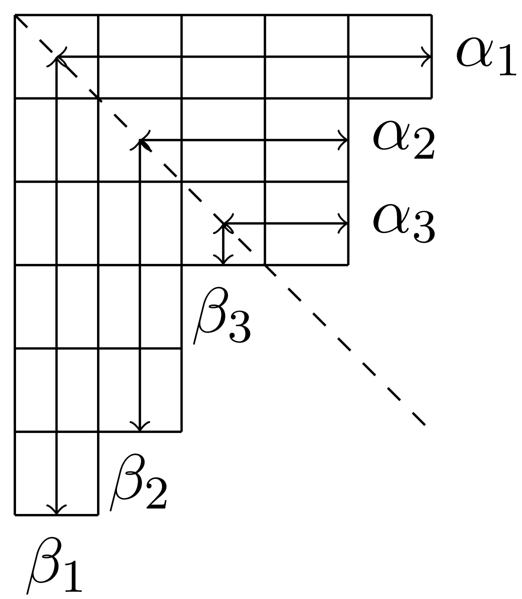
$$\phi_{ij} = \phi_i - \phi_j , \qquad [x] := 2 \sinh \frac{x}{2} = e^{\frac{x}{2}} - e^{-\frac{x}{2}} , \quad \epsilon_\pm = \frac{\epsilon_1 \pm \epsilon_2}{2} .$$

a_i: Coulomb branch parameters

Jeffrey-Kirwan (JK) residue in U(N) case

poles are labeled by a set of N partitions (Young diagram)

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n), \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.$$



pole

$$\phi_i = a_j + (r-1)\epsilon_1 + (s-1)\epsilon_2. \quad \longleftrightarrow \quad \textbf{box in Young diagram}$$

*** Frobenius basis of Young diagram**

$$\lambda = (\alpha_1, \alpha_2, \ldots | \beta_1, \beta_2, \ldots).$$

Analytic expression

the partition function can be written in terms of Nekrasov factors

[Nekrasov (2002)]

$$N_{\lambda\nu}(Q;q_1,q_2) := \prod_{(i,j)\in\lambda}\left(1 - Q q_2^{-\nu_i+j-1} q_1^{\lambda_j^t - i}\right) \prod_{(i,j)\in\nu}\left(1 - Q q_2^{\lambda_i - j} q_1^{-\nu_j^t+i-1}\right),$$

for example, pure U(N) gauge theories can be expressed as

$$Z = \sum_{\{\lambda_1,\lambda_2,\ldots,\lambda_N\}} \prod_{i=1}^{N}\left(\mathfrak{q}^{|\lambda_i|} N^{-1}_{\lambda_i\lambda_i}(1)\right) \prod_{i<j} N^{-1}_{\lambda_i\lambda_j}(Q_i/Q_j) N^{-1}_{\lambda_j\lambda_i}(Q_j/Q_i).$$

….. (*)

matter:

$$\sim N_{\lambda\emptyset}(m_f)$$

Chern-Simons term:

$$\sim \prod_{(i,j)\in\lambda} q^i t^{-j}.$$

c.f. SO(N) instanton partition function

$$Z_k^{\mathrm{SO}(2N+\delta)}$$
$$= (-1)^k \frac{[2\epsilon_+]^k}{[\epsilon_{1,2}]^k} \prod_{i<j} \mathcal{S}(\pm\phi_i \pm \phi_j - \epsilon_+)^{-1} \prod_{i=1}^{k} \prod_{j=1}^{N} [\pm\phi_i \pm a_j - \epsilon_+]^{-1} \prod_{i=1}^{k} \frac{[\pm 2\phi_i][\pm 2\phi_i + 2\epsilon_+]}{[\pm\phi_i - \epsilon_+]^\delta},$$

where

$$\mathcal{S}(\phi) := \frac{[\phi \pm \epsilon_-]}{[\phi \pm \epsilon_+]}.$$

$$Z_{\text{loc, inst}}^{\mathrm{SO}(2N+\delta)} = \sum_{k=0}^{\infty} \mathfrak{q}^k \frac{1}{|W(\mathrm{Sp}(k))|} \oint \left(\prod_{i=1}^{k} \frac{\mathrm{d}\phi_i}{2\pi i} \right) Z_k^{\mathrm{SO}(2N+\delta)}.$$

much more difficult! not just a more complicated integrand.

We do not know how to label the JK poles…

Sp(N) theories are even more difficult!…   Yet we want to work on them!

brane construction in string theory

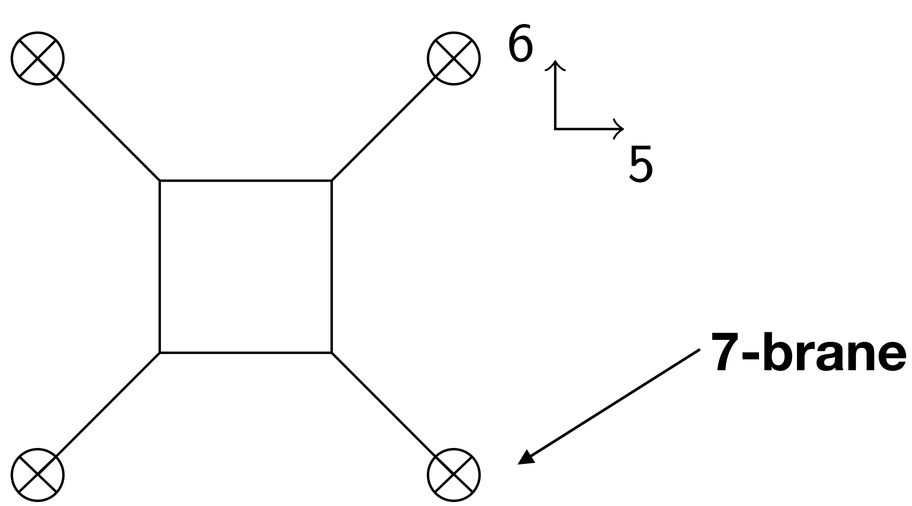
[Aharony, Hanany, Kol (1997)]

	0	1	2	3	4	5	6	7	8	9
D5	●	●	●	●	●	●	—	—	—	—
NS5	●	●	●	●	●	—	●	—	—	—
7-brane	●	●	●	●	●	—	—	●	●	●

all non-trivial information contained in this 2d plane

We draw a web diagram on this plane. (balance of tension ⇒ various kind of (p,q) 5-branes stretching along the vector (p,q))

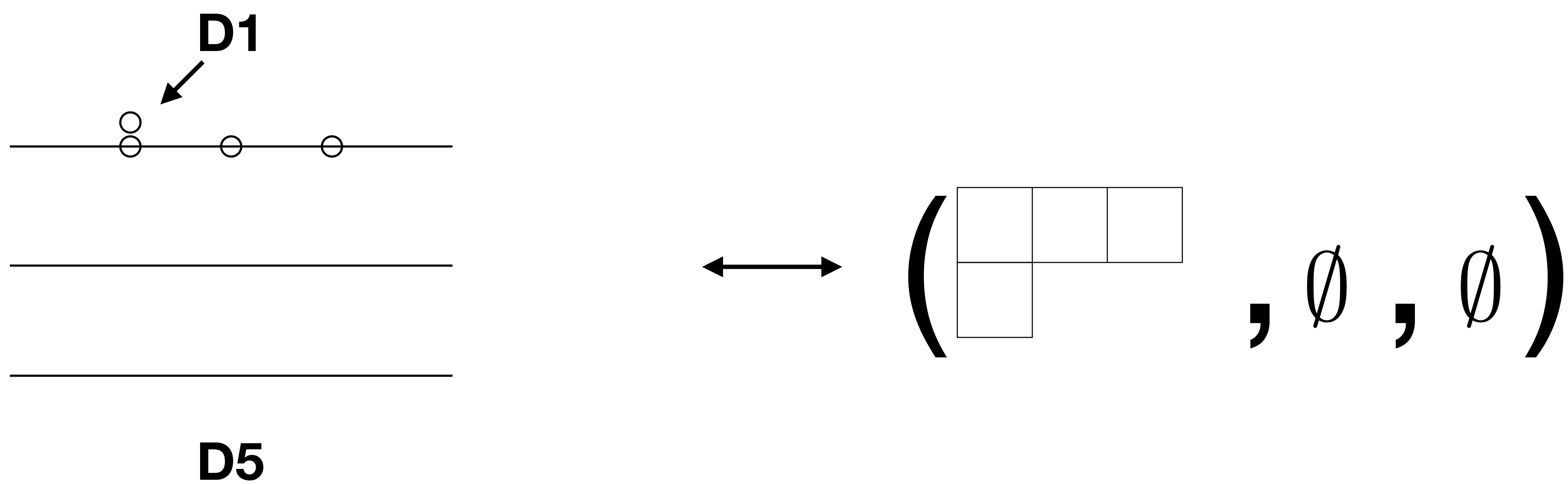
→ axio-dilaton charge



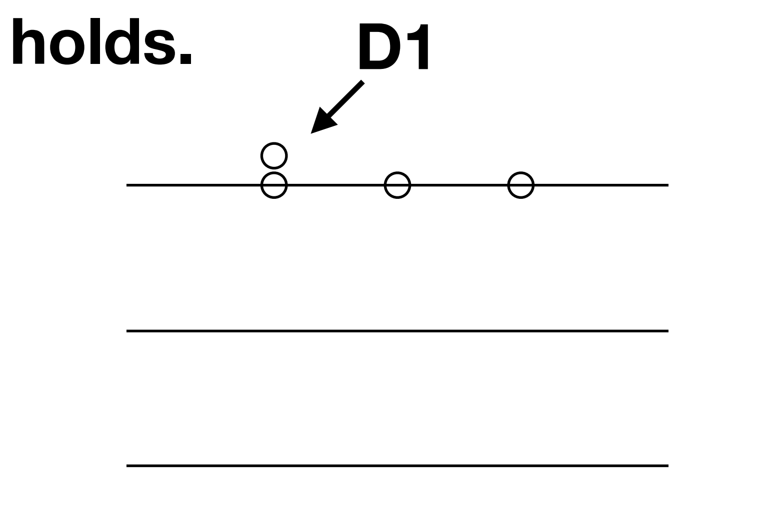
N D5 branes ⇒ U(N) gauge theory

In U(N) theory

instanton solutions labeled by N Young diagrams



Our results suggest that in the unrefined limit of SO(N) theories, the above picture still holds.



string duality with topological string theory

topological string (A-model) on toric Calabi-Yau

[Leung, Vafa (1997)]

captures the BPS spectrum

[Aganagic, Klemm, Marino, Vafa (2003)]

M-theory on toric Calabi-Yau ⟷ type IIB on Taub-NUT

=

(p,q) brane web

on S^1

The topological string is a convenient way to compute the index (partition function on S^1) or the instanton partition function of 5d N=1 gauge theories.

Topological String?

*The concrete definition etc. are not useful in this talk.

- It is a topologically twisted N=(2,2) sigma model.

[Witten, (1988)] [Vafa, (1991)] …

- Due to different ways of topological twist, we have A- and B-models.
  They are connected through the mirror symmetry.

[Candelas et al., (1985), Dixon, (1987), Lerche et al., (1989) ….]

- There are certainly the open and closed version of the string theory,
  and there is a open/closed duality.
  The open theory is deeply related to Chern-Simons theories.

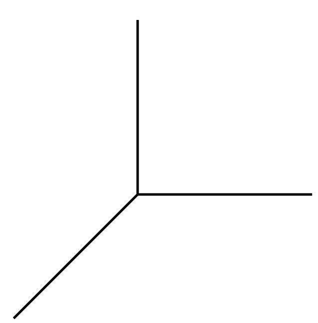
[Witten, (1992)] …

In the Calabi-Yau language, the web diagram corresponds to the toric diagram, in which each line denotes degenerate locus of the torus fiber (of toric Calabi-Yau).

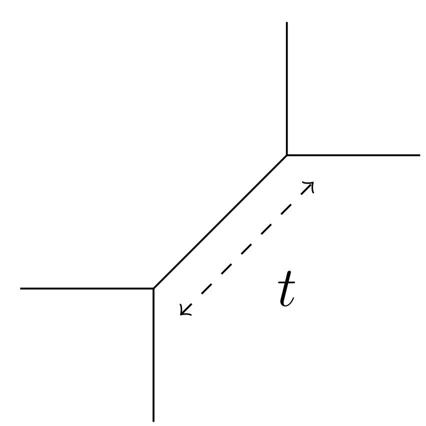
Toric Calabi-Yau

$$\begin{array}{c} T^2 \times R \\ \downarrow \\ R^3 \end{array}$$

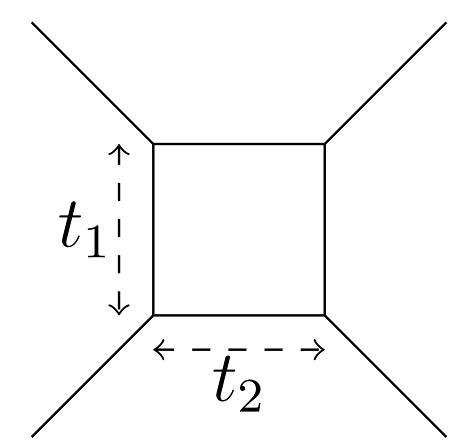
Simplest examples:



$\mathbb{C}^3$



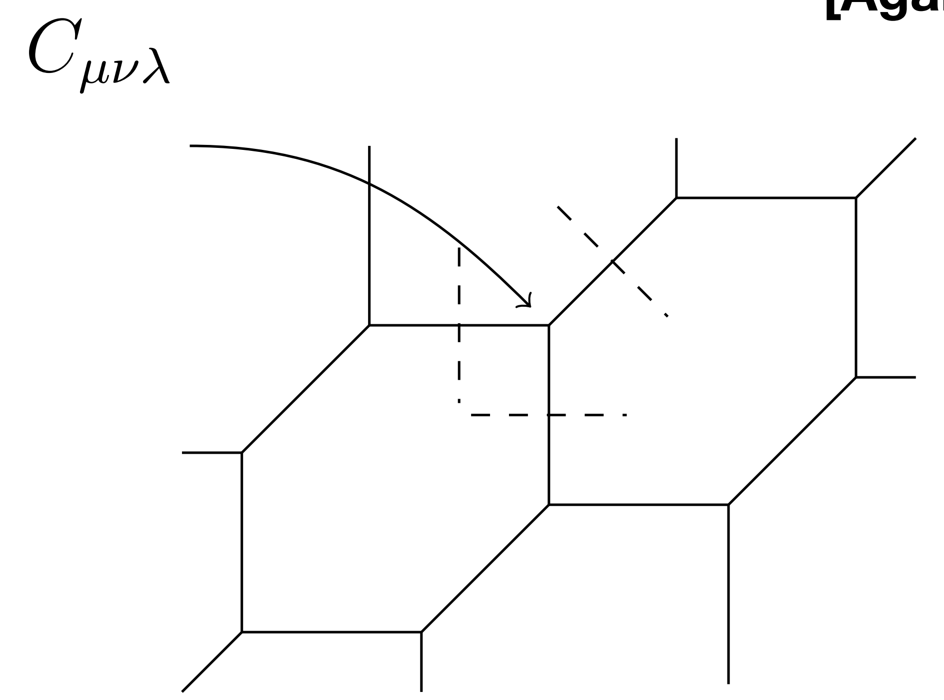
$\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$



$\mathcal{O}(-2,-2) \to \mathbb{CP}^1 \times \mathbb{CP}^1$

The A-model partition function can be computed with the topological vertex.

[Aganagic, Klemm, Marino, Vafa, (2003)]



It can be expressed in terms of (skew) Schur functions.

$$C_{\mu,\nu,\lambda} = q^{-\frac{\kappa(\mu)}{2}} s_\lambda(q^{-\rho}) \sum_\eta s_{\mu^t/\eta}(q^{-\lambda-\rho}) s_{\nu/\eta}(q^{-\lambda^t-\rho}).$$

It is similar to the Feynman diagram to compute the partition function of topological vertex,

$$Z_{top} = \sum_{\lambda,\mu,\nu,\sigma,\tau,\ldots} (-Q_1)^{|\lambda|}(-Q_2)^{|\lambda|} \ldots C_{\mu\nu\lambda} C_{\mu\sigma\tau} \ldots$$

Q_i: Kahler parameters

It reproduces part of the full Nekrasov partition function of the corresponding gauge theory.

$$Z_{top} = Z^{G}_{root} Z_{instanton}.$$

***Remark:**

following from the pole cancellation (or blow-up equation), one can determine the classical piece and the Cartan part of the full partition function.

[Grassi, Hatsuda, Marino (2014)]

The original topological string is dual to the "self-dual" point, with two omega-background parameters

$$\epsilon_1 + \epsilon_2 = 0.$$

We call it an unrefined setup, and we mainly focus on this special limit in this talk.

The refined version corresponding to a general omega-background was soon proposed.

[Awata, Kanno, (2005)] [Iqbal, Kozcaz, Vafa, (2007)]

$$\boxed{C_{\mu,\nu,\lambda}(t,q) = q^{\frac{||\mu^t||^2}{2}} t^{-\frac{||\mu||^2}{2}} P_\lambda(t^{-\rho}, q, t) \sum_\eta \left(\frac{q}{t}\right)^{\frac{|\eta|+|\mu|-|\nu|}{2}} s_{\mu^t/\eta}(q^{-\lambda} t^{-\rho}) s_{\nu/\eta}(t^{-\lambda^t} q^{-\rho}).}$$

$$q_1 = e^{R\epsilon_1} = t, \quad q_2 = e^{R\epsilon_2} = q^{-1}.$$

It is again expressed in terms of (skew) Schur function.

***The refined topological string has no world sheet description, and is based on the melting crystal model picture. There is a special leg, usually named the preferred direction of the vertex.**

Let us have a look at the details of Schur functions.

It can be expressed as an expectation value of a vertex operator.

$$s_{\lambda/\mu}(\vec{x}) = \langle\mu|\, V_+(\vec{x})\,|\lambda\rangle = \langle\lambda|\, V_-(\vec{x})\,|\mu\rangle\,,$$

where

$$V_\pm(\vec{x}) = \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\sum_i x_i^n J_{\pm n}\right),$$

$$|\lambda\rangle = (-1)^{\beta_1+\beta_2+\cdots+\beta_s+\frac{s}{2}}\psi^*_{-\beta_1}\psi^*_{-\beta_2}\ldots\psi^*_{-\beta_s}\psi_{-\alpha_s}\psi_{-\alpha_{(s-1)}}\ldots\psi_{-\alpha_1}\,|\mathrm{vac}\rangle\,,$$

$$\{\psi_n,\psi_m\} = \{\psi^*_n,\psi^*_m\} = 0, \quad \{\psi_n,\psi^*_m\} = \delta_{n+m,0}, \quad J_n := \sum_{j\in\mathbb{Z}+1/2}\psi_{-j}\psi^*_{j+n},$$

$$[J_n,\psi_k] = \psi_{n+k}, \quad [J_n,\psi^*_k] = -\psi^*_{n+k}, \quad [J_n,J_m] = n\delta_{n+m,0}.$$

Then we have

$$C_{\mu,\nu,\lambda}(t,q) \propto \sum_\eta s_{\mu^t/\eta}(t^{-\lambda}q^{-\rho+\{1/2\}})s_{\nu/\eta}(q^{-\lambda^t}t^{-\rho-\{1/2\}})$$
$$= \langle\mu^t|\, V_-(t^{-\lambda}q^{-\rho+\{1/2\}})V_+(q^{-\lambda^t}t^{-\rho-\{1/2\}})\,|\nu\rangle\,.$$

Cauchy identities of Schur functions revisited

e.g.

$$\sum_\lambda s_{\lambda/\mu}(x)s_{\lambda/\nu}(y) = \prod_{i,j}(1-x_iy_j)^{-1}\sum_\eta s_{\nu/\eta}(x)s_{\mu/\eta}(y),$$

can be derived as

complete basis (of Fock space)

$$\sum_\lambda s_{\lambda/\mu}(x)s_{\lambda/\nu}(y) = \sum_\lambda \langle\mu| V_+(\vec{x}) |\lambda\rangle\langle\lambda| V_-(\vec{y})|\nu\rangle = \langle\mu| V_+(\vec{x})V_-(\vec{y})|\nu\rangle$$

$$= \prod_{i,j}(1-x_iy_j)^{-1}\langle\mu| V_-(\vec{y})V_+(\vec{x})|\nu\rangle$$

$$= \prod_{i,j}(1-x_iy_j)^{-1}\sum_\eta \langle\mu| V_-(\vec{y})|\eta\rangle\langle\eta| V_+(\vec{x})|\nu\rangle$$

$$= \prod_{i,j}(1-x_iy_j)^{-1}\sum_\eta s_{\nu/\eta}(x)s_{\mu/\eta}(y),$$

with the commutation relation (Baker-Campbell-Hausdorff formula)

$$V_+(\vec{x})V_-(\vec{y}) = \prod_{i,j}\frac{1}{1-x_iy_j}V_-(\vec{y})V_+(\vec{x})$$

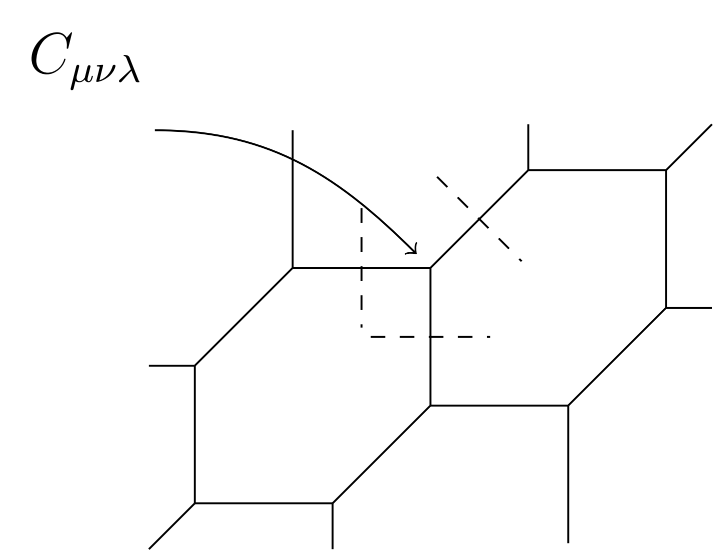
In the topological vertex formalism, these Cauchy identities lead to the appearance of Nekrasov factors in the partition function.

a typical summation in the computation:

$$\sum_{\lambda} Q^{|\lambda|} s_{\lambda/\mu}(q^{-\rho-\sigma}) s_{\lambda/\nu}(q^{-\rho-\tau})$$

$$= \underbrace{P.E.\left(\frac{q}{(1-q)^2}Q\right)}_{\text{one-loop factor}} \underbrace{N^{-1}_{\sigma^t\tau}(Q,q)}_{\text{Nekrasov factor}} \sum_{\eta} Q^{|\mu|+|\nu|-|\eta|} s_{\nu/\eta}(q^{-\rho-\sigma}) s_{\mu/\eta}(q^{-\rho-\tau})$$

Remark: all summations over Young diagrams in non-horizontal directions can be taken in this way.



Then we can see that the exact matching with Nekrasov's formula for U(N) theories,

i.e. eqn. (*).

Rewriting the vertex operators

[Awata, Feigin, Shiraishi (2011)]

$$C_{\mu,\nu,\lambda} = \langle \nu | \, \Phi \, | \mu \rangle \otimes | \lambda \rangle$$

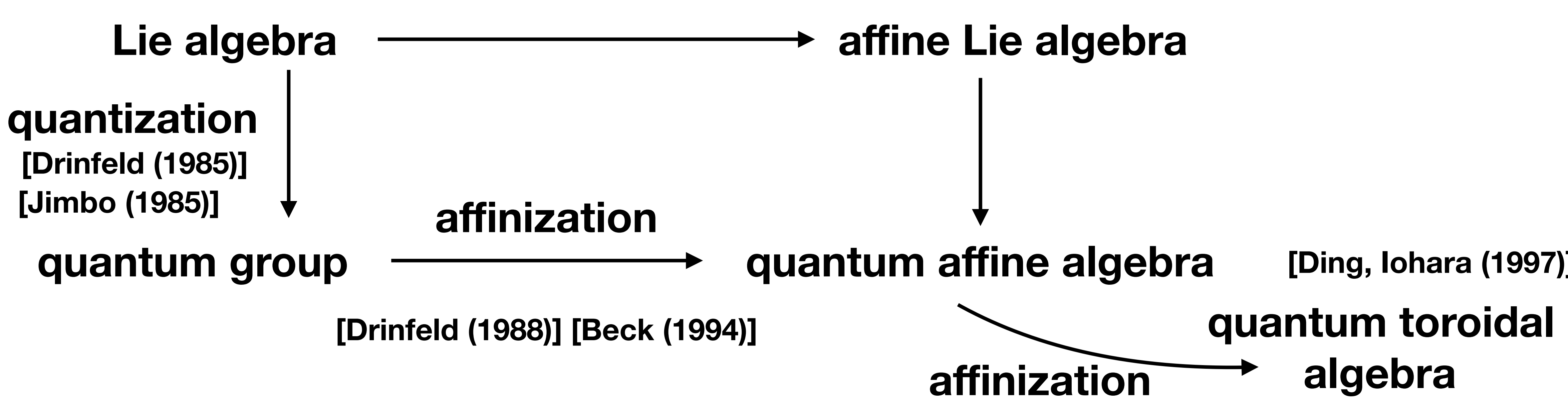
where

$$\Phi^{(n)}[u,v] : \; (1,n)_u \otimes (0,1)_v \to (1,n+1)_{-uv},$$

is the intertwiner in the Ding-Iohara-Miki algebra (quantum toroidal algebra of affine gl_1),

$$\Phi^{(n)}[u,v](\rho_v^{(0,1)} \otimes \rho_u^{(1,n)})\Delta(g(z)) = \rho_{-uv}^{(1,n+1)}(g(z))\Phi^{(n)}[u,v]$$

What is a quantum toroidal algebra?



quantum toroidal algebra (of gl₁) $U_{q,t}(\widehat{\widehat{\mathfrak{gl}}}_1)$

(Ding-Iohara-Miki algebra)

[Ding, Iohara (1997)]

[Miki (2007)]

$$
\begin{aligned}
[\psi^{\pm}(z),\psi^{\pm}(w)] &= 0,\\
\psi^{+}(z)\,\psi^{-}(w) &= \frac{g(\hat{\gamma}z/w)}{g(\hat{\gamma}^{-1}z/w)}\psi^{-}(w)\,\psi^{+}(z)\\
\psi^{\pm}(z)\,x^{+}(w) &= g\left(\hat{\gamma}^{\pm\frac{1}{2}}z/w\right)x^{+}(w)\,\psi^{\pm}(z)\\
\psi^{\pm}(z)\,x^{-}(w) &= g\left(\hat{\gamma}^{\mp\frac{1}{2}}z/w\right)^{-1}x^{-}(w)\,\psi^{\pm}(z)\\
x^{\pm}(z)\,x^{\pm}(w) &= g(z/w)^{\pm 1}\,x^{\pm}(w)\,x^{\pm}(z)\\
\left[x^{+}(z),x^{-}(w)\right] &= \frac{(1-q_1)(1-q_2)}{(1-q_3^{-1})}\left(\delta(\hat{\gamma}w/z)\;\psi^{+}\left(\hat{\gamma}^{\frac{1}{2}}w\right)-\delta(\hat{\gamma}^{-1}w/z)\;\psi^{-}\left(\hat{\gamma}^{-\frac{1}{2}}w\right)\right),
\end{aligned}
$$

where

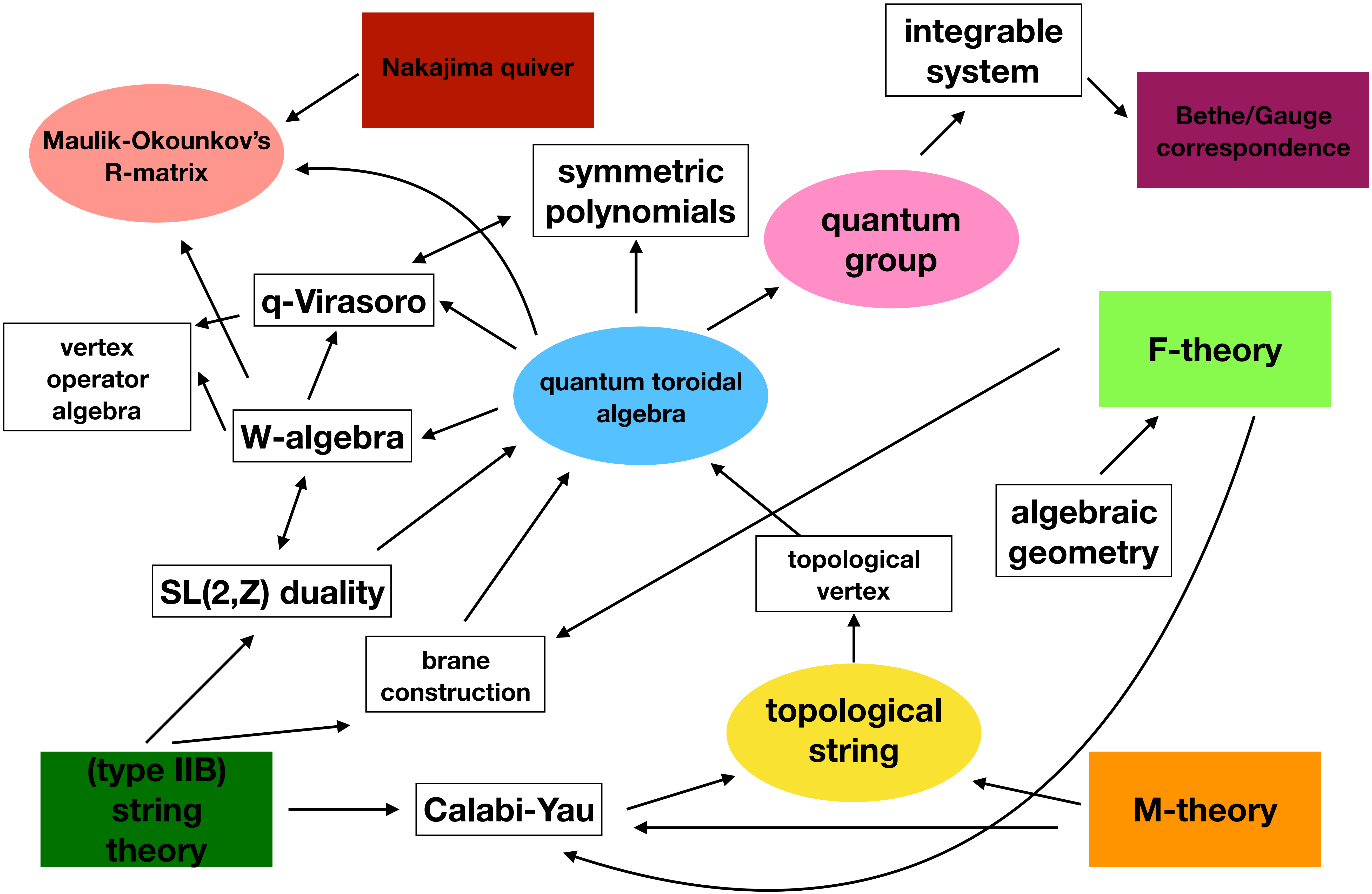
$$g(z)=\frac{(1-q_1z)(1-q_2z)(1-q_3z)}{(1-q_1^{-1}z)(1-q_2^{-1}z)(1-q_3^{-1}z)} \qquad q_1q_2q_3=1$$

$(q_1=t, q_2=q^{-1})$

$\hat{\gamma}$ **:center of the group**

ψ_0^{+}/ψ_0^{-} **is also a center**

A unifying framework of dualities



Relation with W-algebras

$U_{q,t}(\widehat{\widehat{\mathfrak{gl}}}_1)$ ⟷ **q-deformed** $\mathcal{W}_{1+\infty}[\mu]$

↓ $q \to 1, \; t = q^{\beta}$ **(4d limit)** | ↓

$\widehat{\mathfrak{gl}}_1$ **Yangian/SH$_c$ algebra** ⟷ **(A-type)** $\mathcal{W}_{1+\infty}[\mu]$

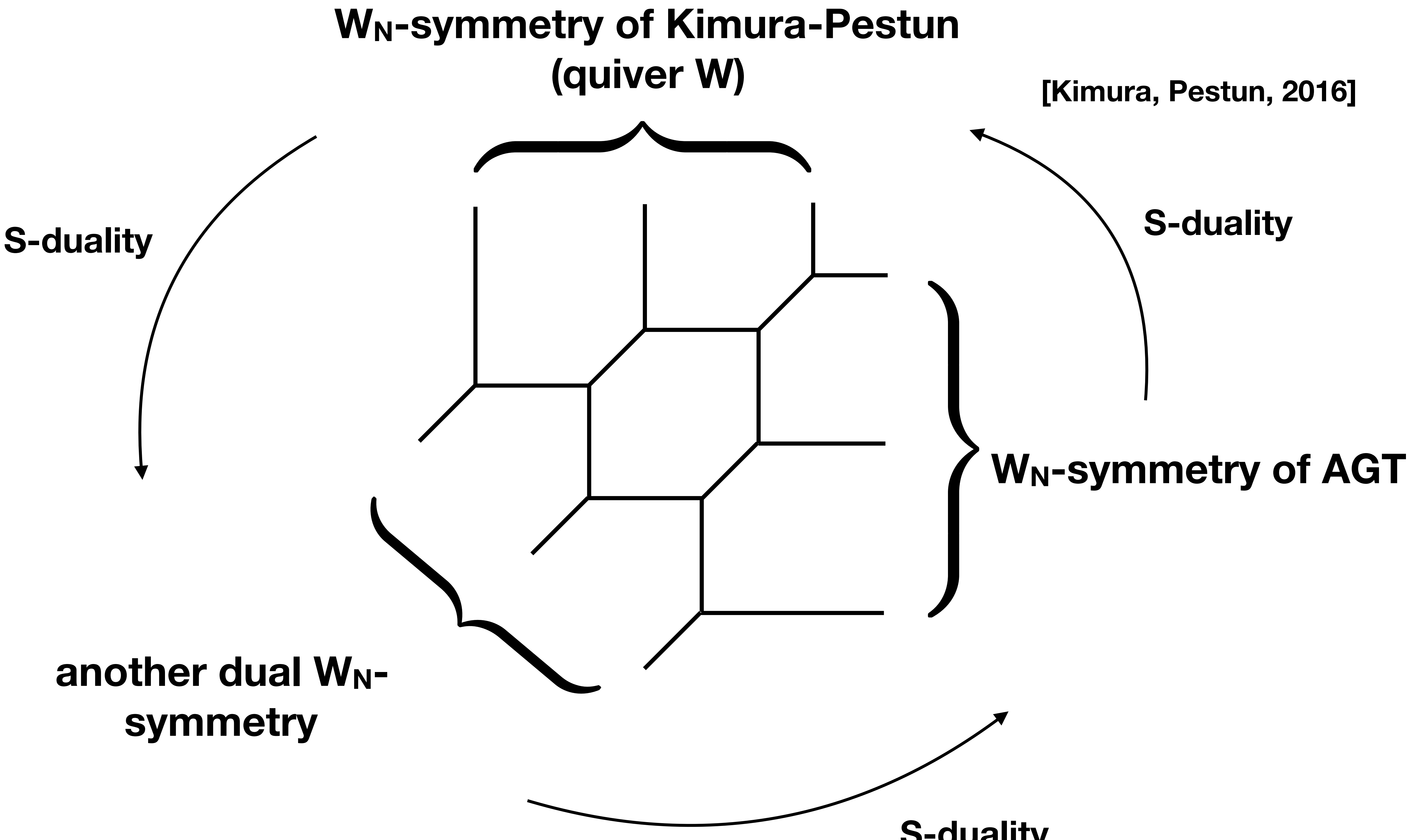
… [Prochazka, 2015]

[Shiffmann, Vasserot, 2012]

[Maulik, Okounkov, 2012]

- **Ding-Iohara-Miki algebra on the tensor product of N Fock spaces contains a U(1)x (q-deformed) W_N algebra.**

[Feigin, Hoshino, Shibahara, Shiraishi, Yanagida, 2010]



W_N-symmetry of Kimura-Pestun
(quiver W)
[Kimura, Pestun, 2016]
S-duality
S-duality
W_N-symmetry of AGT
another dual W_N-
symmetry
S-duality

After all, what we want to study is SO(N) theory.

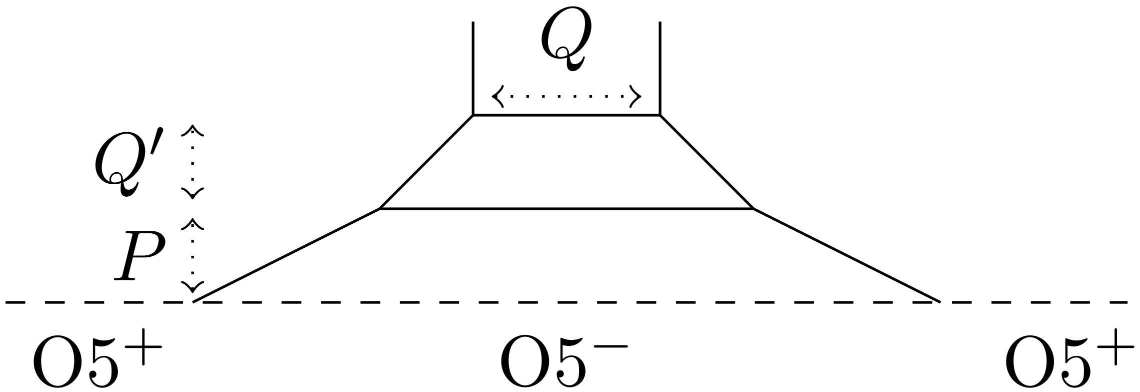
As we mentioned before, the integral is difficult to perform for SO(N) and Sp(N) theories.

However, the brane construction is simple.

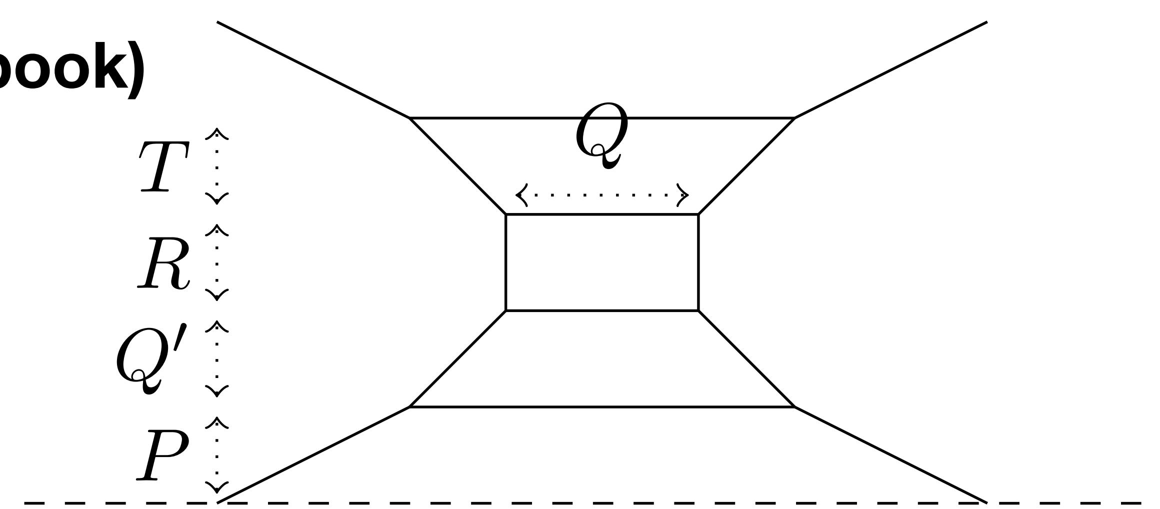
	0	1	2	3	4	5	6	7	8	9
D5	●	●	●	●	●	●	—	—	—	—
NS5	●	●	●	●	●	—	●	—	—	—
7-brane	●	●	●	●	●	—	—	●	●	●
$O5^{\pm}$	●	●	●	●	●	●	—	—	—	—

According to SO or Sp gauge group, we add $O5^+$ or $O5^-$ orientifold (O-plane).

(Well-known story in string theory text book)



SO(4) theory

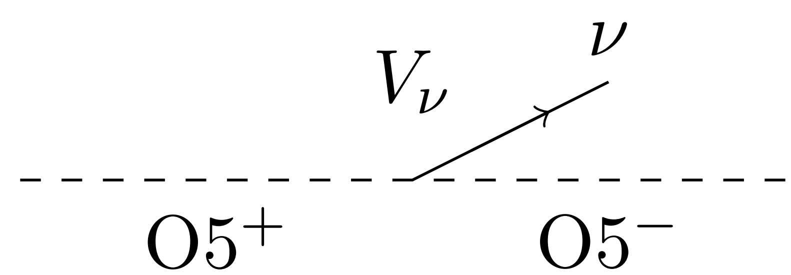


SO(8) theory

Proposal

[Hayashi, RZ (2020)]
[Nawata, RZ (wip)]

The topological vertex seems to be applicable to brane webs with O-plane, except for the intersection point of 5-brane and orientifold.



We propose a new topological vertex (O-vertex) for this intersection point.

$$V_\nu (-P)^{|\nu|} = \langle 0 | \, \mathbb{O}(P,q) \, | \nu \rangle \, .$$

Expectation value of a vertex-operator,

$$\mathbb{O}(P,q) = \exp \left(\sum_{n=1}^{\infty} \left(- \frac{P^{2n}(1+q^n)}{2n(1-q^n)} J_{2n} + \frac{P^{2n}}{2n} J_n J_n \right) \right) .$$

It directly follows that V_ν vanishes for odd-size Young diagram.

We can compute the explicit expression of the O-vertex.

$$V_\nu = \frac{P_\nu(q)}{(q;q)_{|\nu|/2}},$$

and it seems that P_ν is a polynomial.

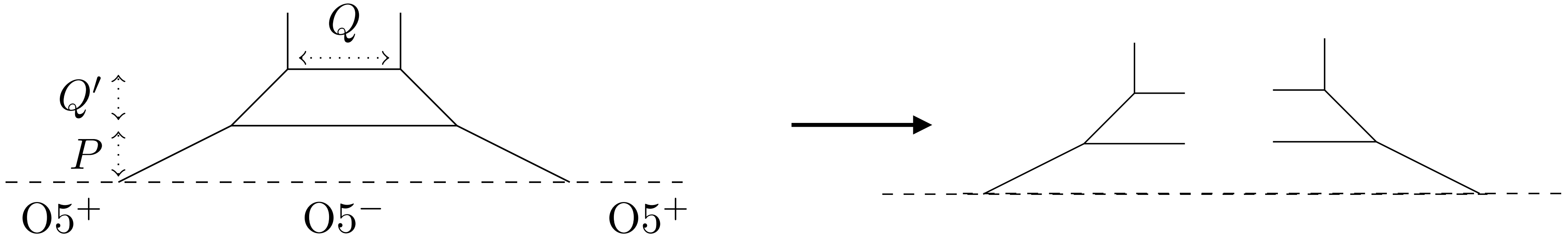
$$P_{(2)} = -q, \quad P_{(1,1)} = 1,$$
$$P_{(4)} = q^3, \quad P_{(3,1)} = -q, \quad P_{(2,2)} = 1 + q^3, \quad P_{(2,1,1)} = -q^2, \quad P_{(1,1,1,1)} = 1,$$

$$P_{(6)} = -q^6, \quad P_{(5,1)} = q^3, \quad P_{(4,2)} = -(q+q^5+q^6),$$
$$P_{(4,1,1)} = q^4+q^5, \quad P_{(3,3)} = 1+q^4+q^5, \quad P_{(3,2,1)} = 0,$$
$$P_{(3,1,1,1)} = -(q+q^2), \quad P_{(2,2,2)} = -(q+q^2+q^6), \quad P_{(2,2,1,1)} = 1+q+q^5,$$
$$P_{(2,1,1,1,1)} = -q^3, \quad P_{(1,1,1,1,1,1)} = 1$$

We do not have a closed-form formula for them.

Reasoning behind this proposal:

e.g. in SO(4) theory



full brane diagram **perturbative part**

To reproduce the one-loop part, we need to require

$$P.E.\left(\frac{P^2 q}{2(1-q)^2}\left(-\left(1+\sum_{i=1}^{N}\tilde{Q}_i^2\right)+\left(1+\sum_{i=1}^{N}\tilde{Q}_i\right)^2\right)\right)$$
$$=\left\langle 0\middle|\mathbb{O}(P,q)\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\frac{(1+\sum_{i=1}^{N}\tilde{Q}_i^n)q^{\frac{n}{2}}}{1-q^n}J_{-n}\right)\middle|0\right\rangle$$

this determines the O-vertex (up to terms annihilated by the vacuum state).

However, in the computation of partition function, the vertex operator is enough.

The part that involves O-vertex in the calculation:

$$\sum_{\nu,\eta_1,\eta_2,\ldots\eta_N} V_\nu(t,q)(-P)^{|\nu|}\left(\prod_{i=1}^{N} Q_i^{|\eta_i|}\right) s_{\nu/\eta_1}(q^{-\rho})s_{\eta_1/\eta_2}(q^{-\rho})\ldots s_{\eta_{N-1}/\eta_N}(q^{-\rho})s_{\eta_N}(q^{-\rho})$$
$$= \left\langle 0 \middle| \mathbb{O}(P,t,q)\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\frac{(1+\sum_{i=1}^{N}\tilde{Q}_i^n)q^{\frac{n}{2}}}{1-q^n}J_{-n}\right)\middle| 0\right\rangle,$$

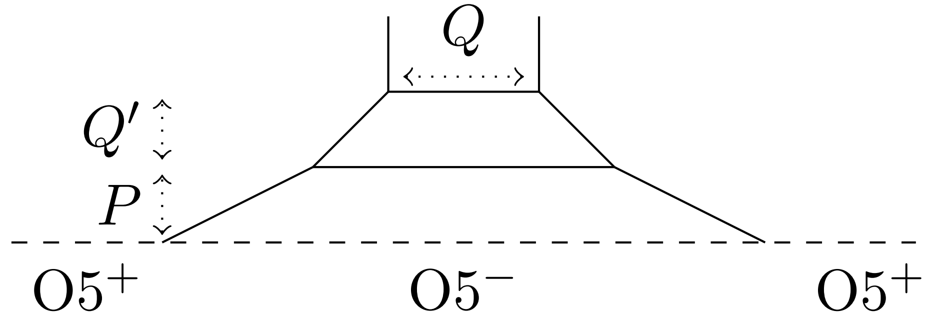
and it is not hard to evaluate with BCH formula.

We define an M-factor

$$M_{\vec{\lambda}}(\vec{A}) = \frac{\langle 0|\,\mathbb{O}(A_1,q)\prod_{s=1}^{N} V_-(A_sA_1^{-1}q^{-\rho-\lambda^{(s)}})\,|0\rangle}{\langle 0|\,\mathbb{O}(A_1,q)\prod_{s=1}^{N} V_-(A_sA_1^{-1}q^{-\rho})\,|0\rangle}\ .$$

We have a closed-form formula for M-factor, and the partition function of SO(N) theories are written in terms Nekrasov factor and M-factor.

More precisely

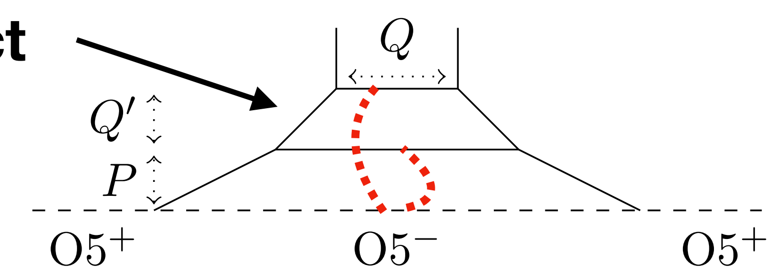


instanton part of SO(4) theory

$$Z^{instanton}_{SO(4)} = \sum_{\lambda,\sigma} Q^{|\lambda|+|\sigma|} Q'^{2|\sigma|} q^{\frac{\kappa(\lambda)}{2} - \frac{3\kappa(\sigma)}{2}} N^{-1}_{\lambda\lambda}(1,q) N^{-1}_{\sigma\sigma}(1,q) \times N^{-2}_{\sigma^t\lambda}(Q',q) M_{\sigma,\lambda}(P^2,P^2,P^2Q'^2)^2.$$

with

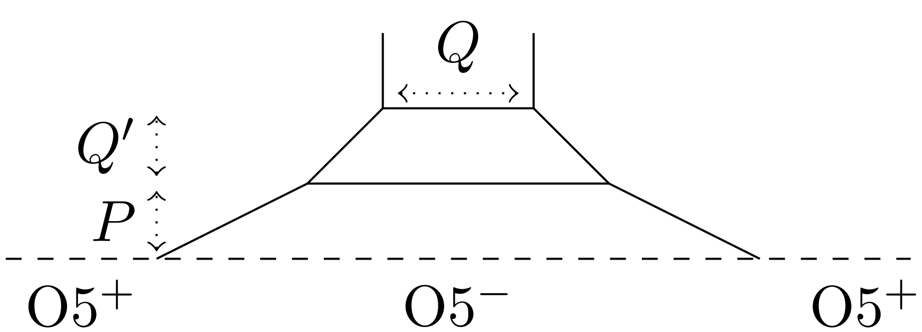
representing the effect of this kind of string

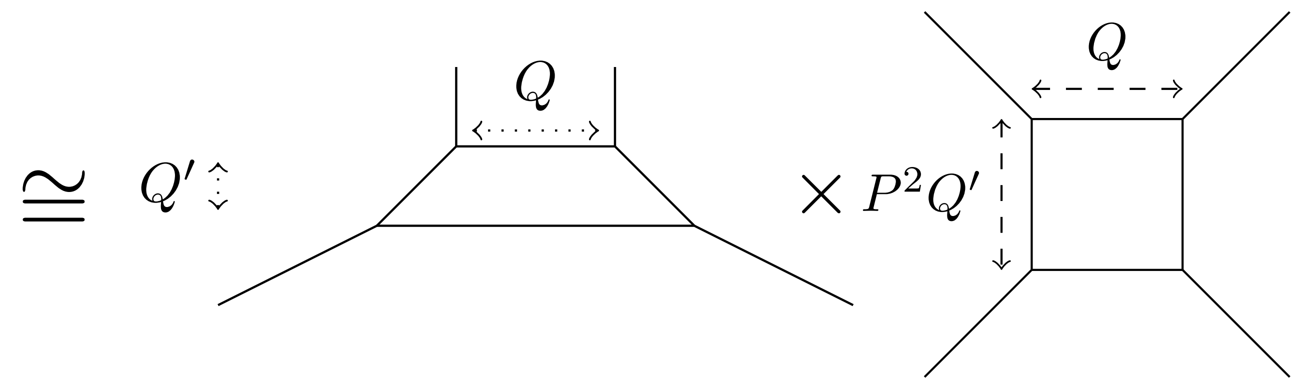


$$M_{\vec{\lambda}}(\vec{A}) = \frac{\prod_{s=1}^{N} \prod_{(i,j)\in\lambda^{(s)}} \left(1 - A_s^2 q^{i-j+(\lambda^\vee)^{(s)}_j - \lambda^{(s)}_i}\right)}{\prod_{1\le t<u\le N} \prod_{(i,j)\in\lambda^{(t)}} \left(1 - A_t A_u q^{i+j-1-\lambda^{(t)}_i - \lambda^{(u)}_j}\right) \prod_{(m,n)\in\lambda^{(u)}} \left(1 - A_t A_u q^{1-m-n+(\lambda^{(t)})^\vee_n + (\lambda^{(u)})^\vee_m}\right)}.$$

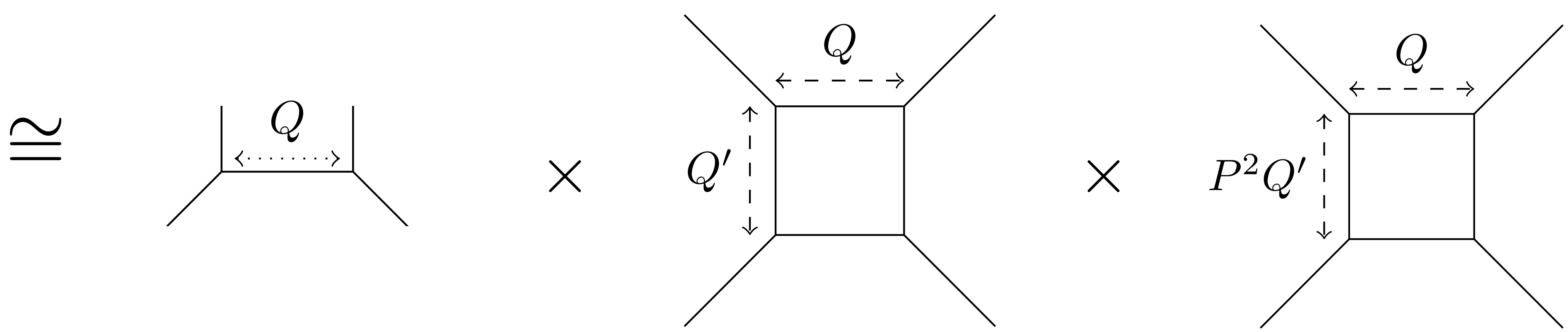
Consistency checks:

SO(4) theory





SU(2) with θ=2π **pure SU(2)**

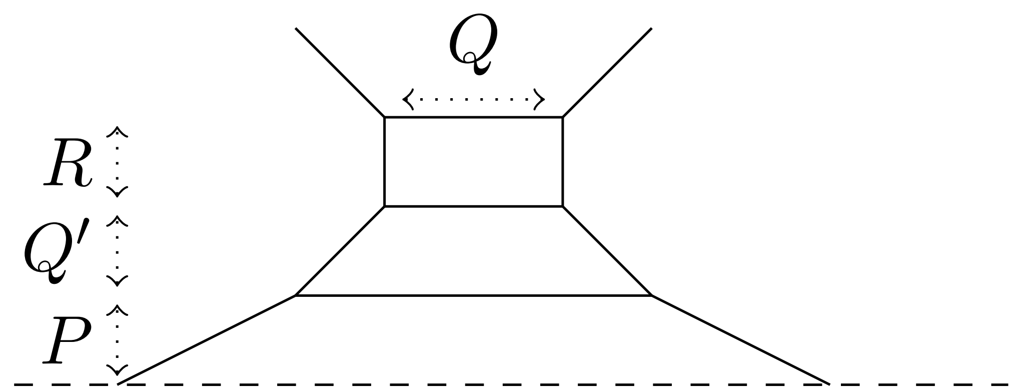


U(1) factor **pure SU(2)** **pure SU(2)’**

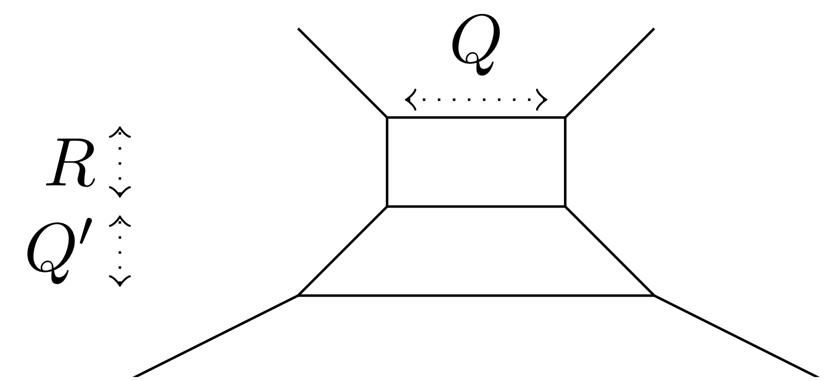
this factor vanishes in the 4d limit

one instanton ✅
two instantons ✅
three instantons ✅

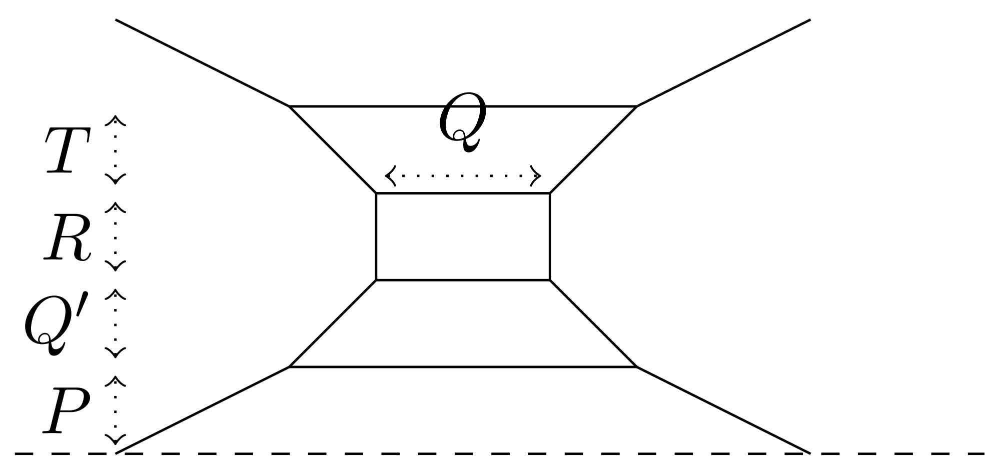
- **SO(6) ≅ SU(4)**



hint: SU(3) subgroup



- **SO(8)**

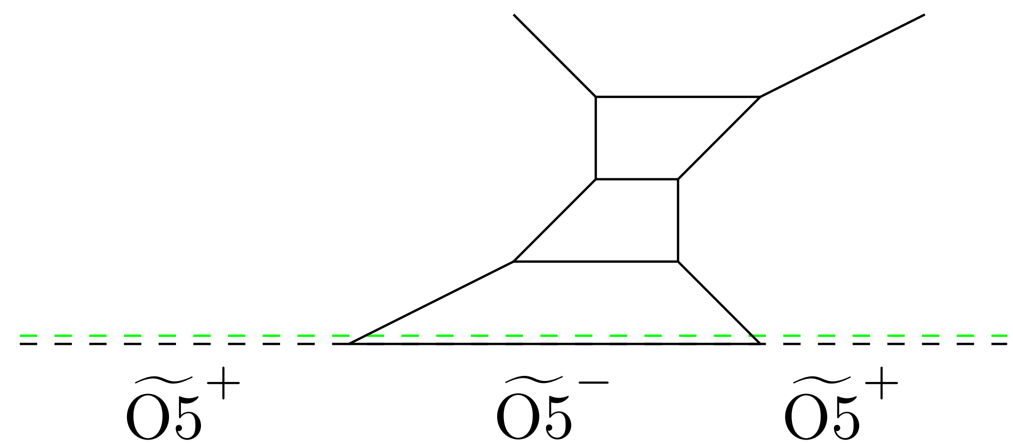


We checked that our results match with those known in the literature.

one instanton ☑
two instantons ☑
three instantons ☑

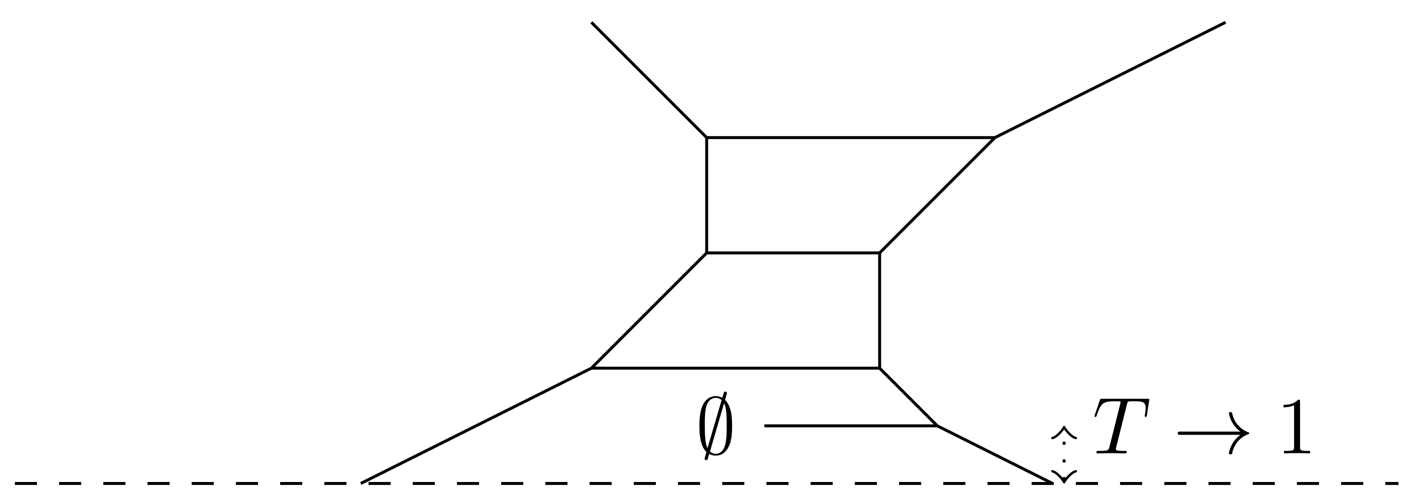
SO(2N+1) theories

the brane construction involves $\widetilde{\mathrm{O5}}^+$ and $\widetilde{\mathrm{O5}}^-$.



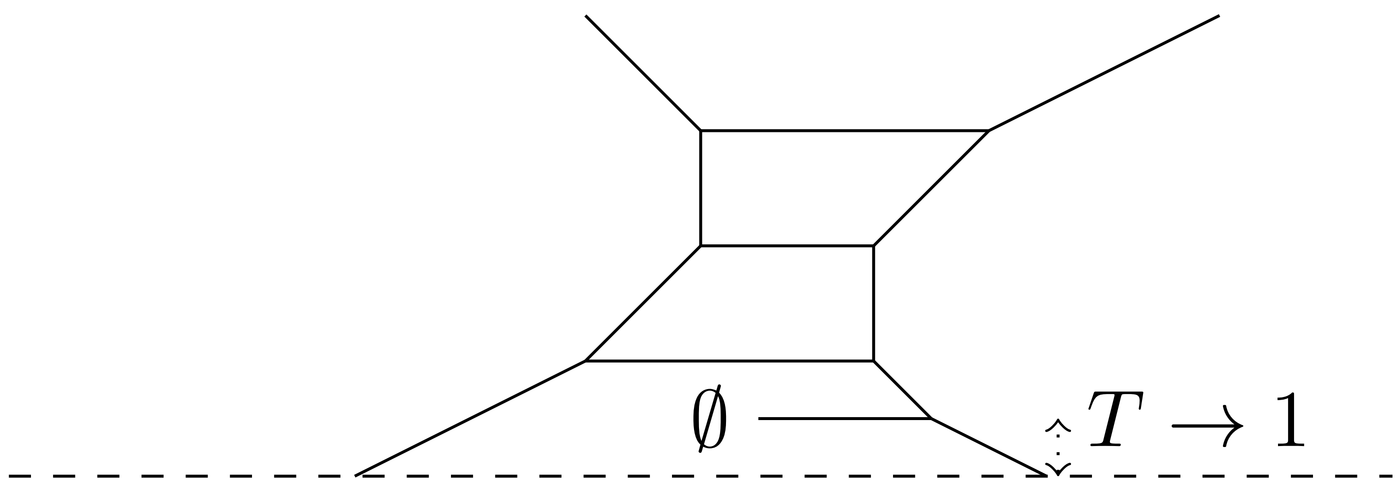
SO(7)

there is a Higgsing prescription to write down an effective web diagram that can be used to do computations.



We checked SO(5) and SO(7) theories matching with the known results in the literature up to three instantons.

Remark:



We need to take a special limit $T \to 1$ at the end. It appears to be difficult to do, since the computation involves

$$\sum_{\lambda} T^{|\lambda|} \ldots$$

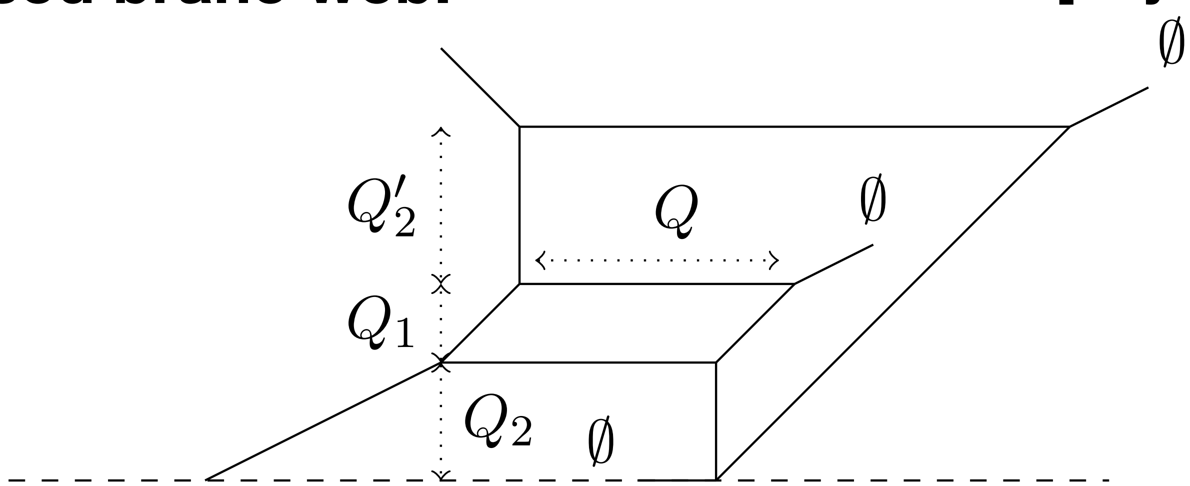
but the summation converges with finite non-zero contributions, and furthermore, in the vertex-operator formalism we proposed, we have a closed-form expression and the limit is straightforward to take.

G_2 gauge theory

No ADHM construction known, but there is brane construction proposed, and there is also a proposal for its blow-up equation to determine the Nekrasov partition function recursively.

Two proposed brane web: **[Hayashi, Kim, Lee, Yagi (2018)]**

①



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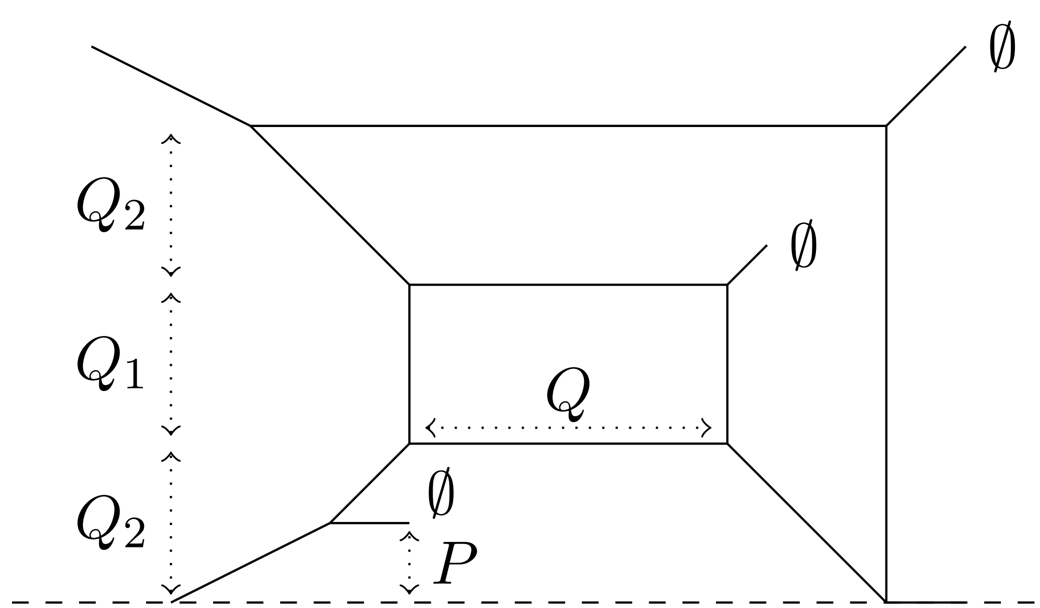
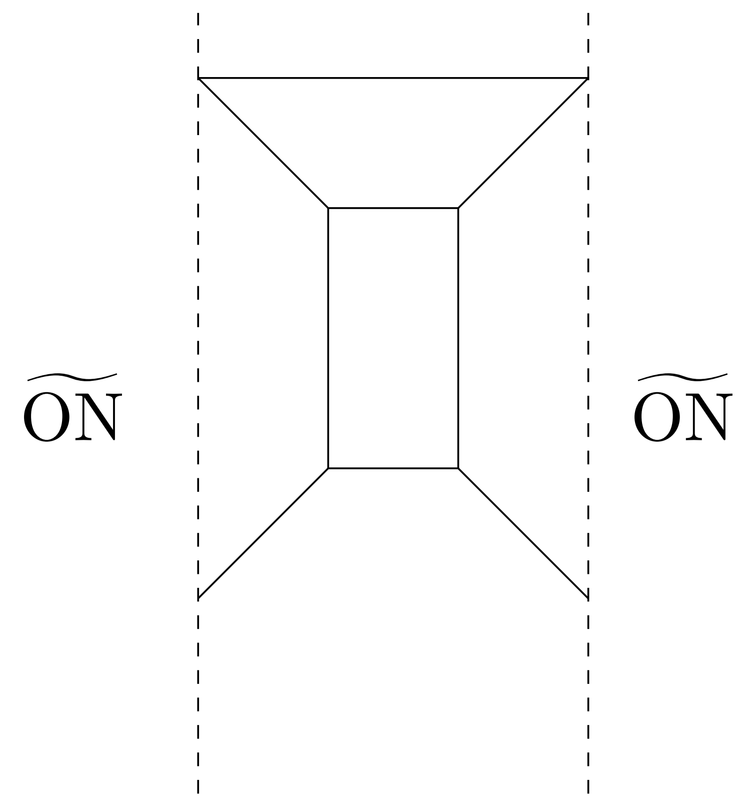


one instanton ☑
two instantons ☑

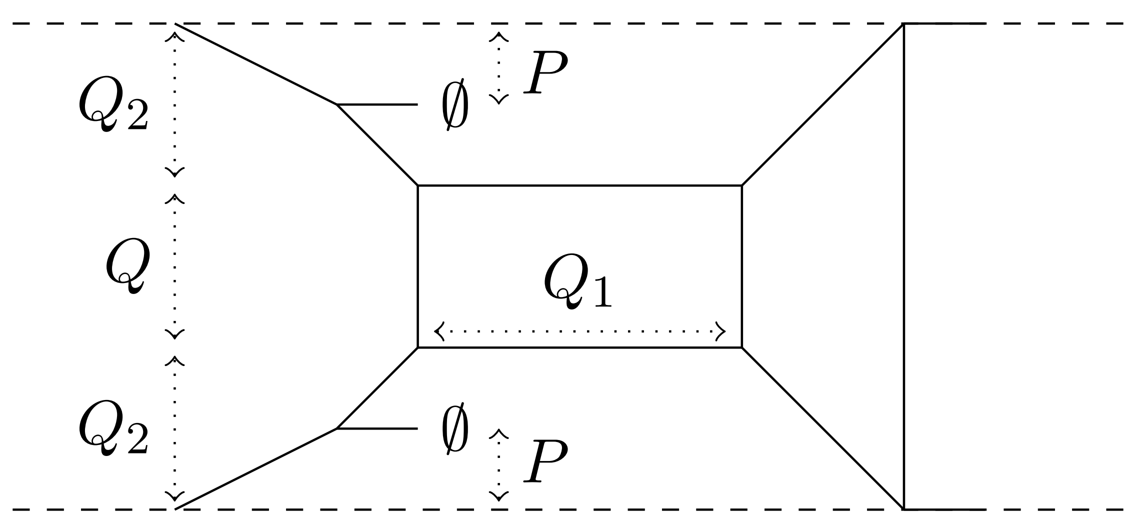
Potential application:

proposed brane diagram for 5d SU(3) theory at Chern-Simons level 9.



“non-traditional” theory

an S-dual effective web diagram can be computed with our O-vertex



one-loop ☑
Gopakumar-Vafa

Comment:

As far as we computed, our prescription literally reproduces each contribution in the localization integral.

In the unrefined limit, all JK poles are labeled by Young diagrams.

For example, we can effectively choose

$$\phi_i = \pm a_j + (r-1)\epsilon_1 + (s-1)\epsilon_2 + \frac{\epsilon_1 + \epsilon_2}{2}.$$

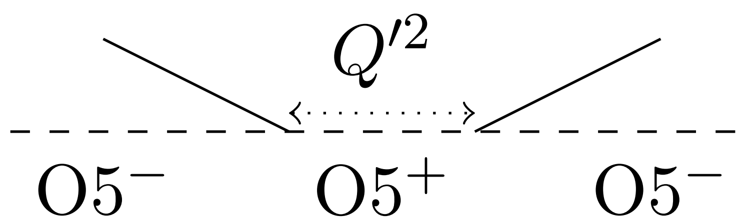
So the refinement is not straightforward to do in this framework.

On the other hand, Sp(N) theories are more complicated.

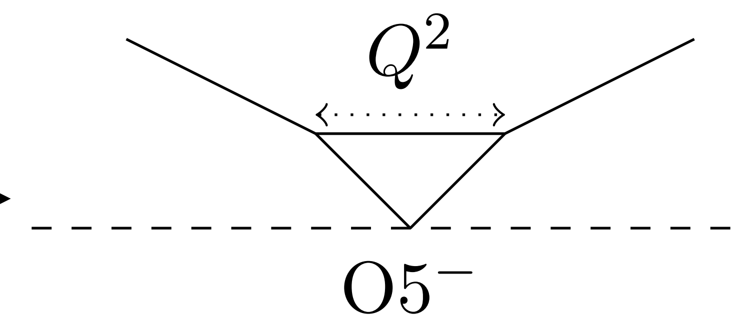
Kim-Yagi's prescription

[Kim, Yagi (2017)]

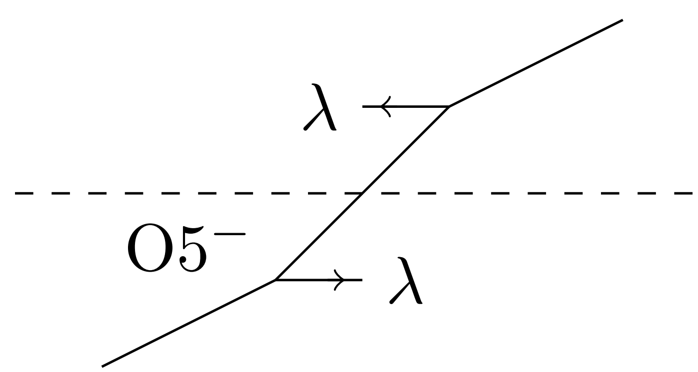
Sp(0) theory



"microscopic" picture
(geometric transition)



way to calculate

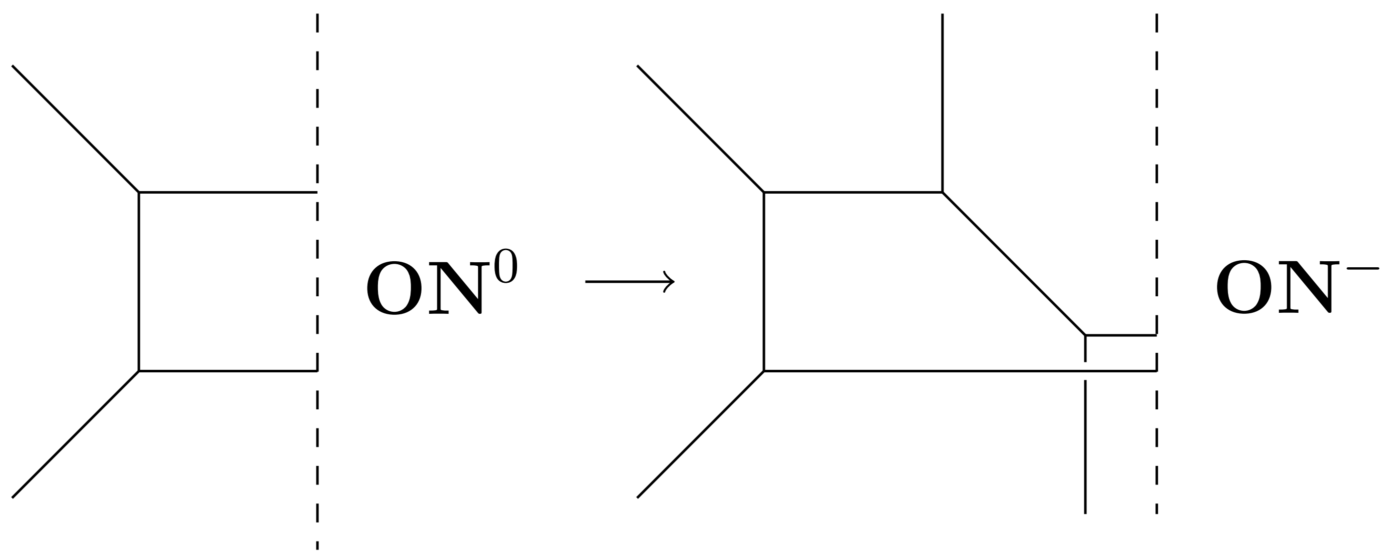


use a mirror image and sum over λ.

S-dual description

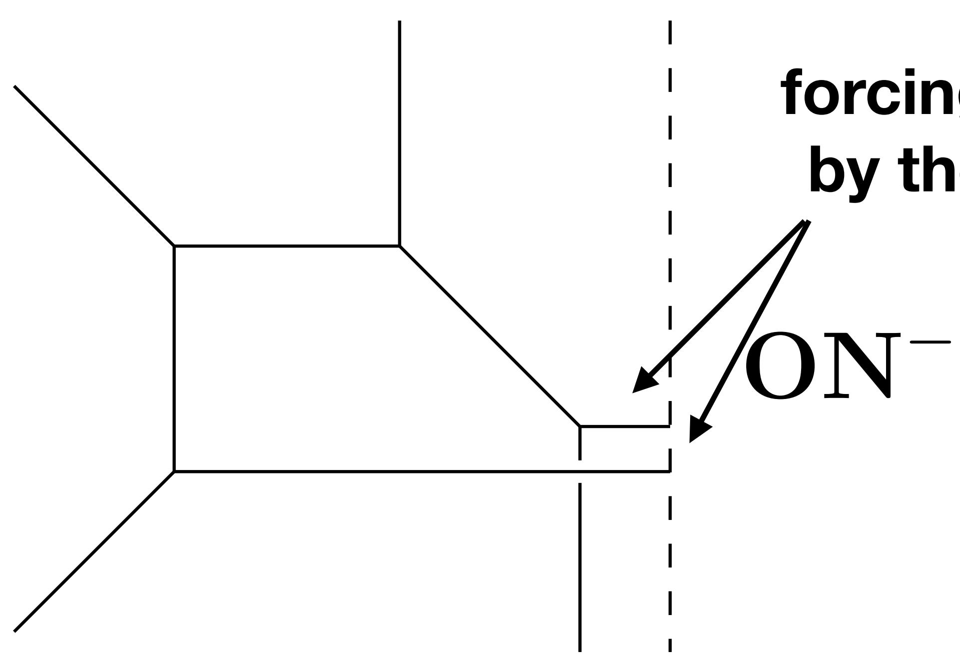
[Bourgine, Fukuda, Matsuo, RZ (2017)]

We again use a resolved brane diagram.



[Hayashi, Kim, Lee, Taki, Yagi, 2015]

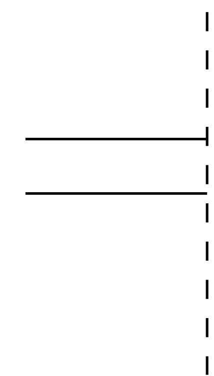
prescription:



forcing two legs to be labeled by the same Young diagram

Reflection state in quantum toroidal algebra

$$\sum_\lambda |v,\lambda\rangle \otimes |v\gamma,\lambda\rangle \quad \longleftrightarrow$$



$$\gamma = \sqrt{t/q}$$

We can use a reflection state in the quantum toroidal algebra

$$\left(x^{\pm}(z)\otimes 1 + 1\otimes x^{\mp}(z)\right)|\Omega\rangle\rangle = 0$$

where we showed the property in the unrefined limit (for simplicity)

$$|\Omega\rangle\rangle := \sum_\lambda |v,\lambda\rangle \otimes |v,\lambda\rangle$$

- **One intriguing observation here is that the reflection state above reduces to the boundary state in the 4d limit** $q\to 1,\ t=q^\beta$ **.**

$$(L_n\otimes 1 - 1\otimes L_{-n})|\Omega\rangle = 0.$$

“2d” picture of this construction?

***Sp(N) theory in ADHM basis**

In the ADHM construction, we have O(k) theory on D1 branes.

The Nekrasov partition function differs in the pieces of $O(k)^+$ and $O(k)^-$.

$$Z^{(k)}_{Sp(N),\ \theta=0,\pi} = \frac{Z_{O(k)^+} \pm Z_{O(k)^-}}{2}.$$

The integral expression of the partition function differs for instanton number k odd and even.

In total, we have four different contributions.

We gave an analytic expression for each piece of the contribution in terms of the M-factor.

[Nawata, RZ (wip)]

But what is the corresponding brane-web realization?

Future directions:

- **adding matter** — **fundamental matter ☑ (straightforward)**
  spinor?
- **topological vertex for Sp(N) in ADHM basis?**
- **qq-characters**

  } **[work in progress with S. Nawata]**
- **meaning in the context of topological string?**
- **O7-plane**
- **algebraic description of 3d N=2* web?**
- **AGT for SO/Sp gauge theories (algebraic structure, integrability)**

...

Summary

- beautiful results known for U(N) theories, but not so many things known for other gauge groups.

- The main reason is the difficulty to perform the localization integral.

- We used an extended version of the topological vertex formalism to write down an analytic expression of SO(N) (and some other) gauge theories.

- We expect our new formula to be useful in the analysis of properties, e.g. algebraic structure, in gauge theories beyond U(N).