1D repulsive Hubbard model: From quantum liquid to transport

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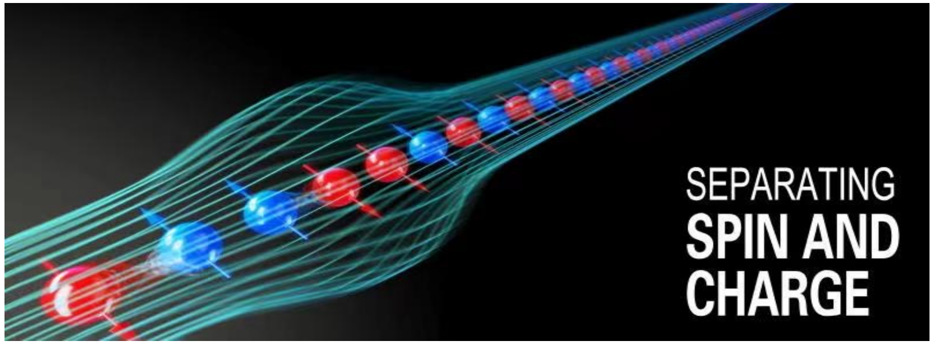
[#]Innovation Academy for Precision Measurement Science and Technology
Chinese Academy of Science



Collaborators:

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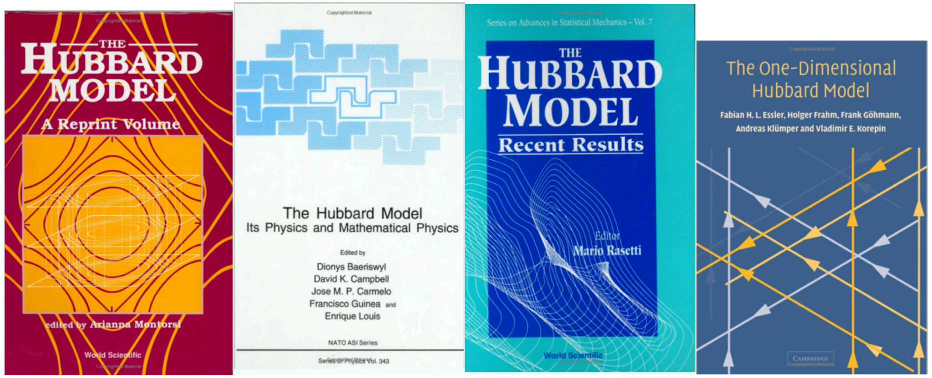
University of Science and Technology of China,
Hefei, China, April 23, 2024

Hubbard model with cold atoms



(a) (b) (c) (d)

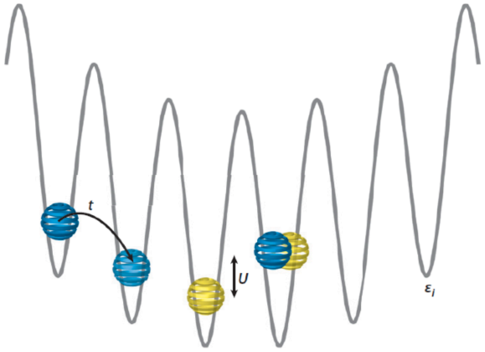
A paradigm of physics in condensed matter:

- **Electronic properties of solids with narrow bands**
- **Band magnetism**
- **Metal-Mott insulator transition,**
- **Fractional excitations, FFLO pairing**
- **…**

The Hubbard model has also become increasingly important in

- cold atoms
- quantum metrology
- quantum information

Nichols et. al., Science 363, 383 (2019)
Brown, et. al. Science 363,379 (2019)
Shao, et. al. ArXiv:2402.14605 (2024)
…



Hart, et al. Nature 519, 211 (2015)
Boll et al. Science 353, 1257 (2016)
Parsons et al. Science 353, 1253 (2016)
Cheuk, et al. Science 353, 1260 (2016)
Cheuk, et al. PRL 116, 235301 (2016)
Hilker, et al. Science 357, 484 (2017)
Cocchi, et al, Phys. Rev. X, 7, 031025 (2017)
Chiu, et al, Science 365, 251(2019)}
Hart, et al. Nature 565, 56 (2019)
Vijayan, et. al., Science, 367, 186 (2020)

Outline

I. 1D Hubbard model

Ground state, phase diagram at zero and finite T, critical scaling functions

II. Interaction driven criticality and Contact

III. Quantum transport

Spin and charge Drude weights at zero and finite temperature

IV. Conclusion and discussion

I. 1D Hubbard model: A prototypical integrable model

$$H_0 = -\sum_{j=1}^{L}\sum_{a=\uparrow\downarrow}\left(c_{j,a}^{+}\, c_{j+1,a} + c_{j+1,a}^{+} c_{j,a}\right)$$
$$+u\sum_{j=1}^{L}(1-2n_{j\uparrow})(1-2n_{j\downarrow})$$
$$H_{GE} = H_0 - \mu\hat{N} - 2B\hat{S}^z$$

- C_{ja} and C_{ja}^{+} : annihilation and creation operators of electrons with spin a at site j
- $n_{ja} = C_{ja}^{+}C_{ja}$
- $\hat{N} = \sum_{j=1}^{L} n_{j\uparrow} + n_{j\downarrow}$
- $u < 0$ ($u > 0$): on-site attractive (repulsive) interaction

Lieb, Wu PRL 20 , 1445 (1968)

Spin SU(2) symmetry

$$S^{\alpha} = \frac{1}{2}\sum_{j=1}^{L}\sum_{a,b=1}^{2} c_{j,a}^{\dagger}\,(\sigma^{\alpha})_b^a\, c_{j,b}, \alpha = x, y, z$$

Eta-pairing Symmetry

$$\eta^{+} = \sum_{j=1}^{L}(-1)^{j+1}\, c_{j,\uparrow}^{\dagger}\, c_{j,\downarrow}^{\dagger}, \qquad \eta^{z} = \frac{1}{2}(\hat{N} - L)$$
$$\eta^{-} = \sum_{j=1}^{L}(-1)^{j+1}\, c_{j,\uparrow}c_{j,\downarrow},$$

The model has been realized with ultracold atoms in lab

Hart, et al. Nature 565, 56 (2019)
Vijayan, et. al., Science, 367, 186 (2020)

V. Korepin, A. A. Ovchinnikov, F. Essler, H. Frham, P. Schlottmann, N. Kawakami, J. M. Carmelo, N.M. Bogoliubov, F. Woy- narovich, B.S. Shastry

Exact Lieb-Wu Equations:

$$e^{\mathrm{i}k_j L} = \prod_{\ell=1}^{M} \frac{\lambda_\ell - \sin k_j - \mathrm{i}u}{\lambda_\ell - \sin k_j + \mathrm{i}u}, \qquad j = 1, \ldots, N,$$

$$\prod_{j=1}^{N} \frac{\lambda_\ell - \sin k_j - \mathrm{i}u}{\lambda_\ell - \sin k_j + \mathrm{i}u} = \prod_{\substack{m=1 \\ m\neq\ell}}^{M} \frac{\lambda_\ell - \lambda_m - 2\mathrm{i}u}{\lambda_\ell - \lambda_m + 2\mathrm{i}u}, \qquad \ell = 1, \ldots, M.$$

Energy and Momentum

$$E = -2\sum_{j=1}^{N} \cos k_j + u(L - 2N)$$

$$P = \left[\sum_{j=1}^{N} k_j\right] \mathrm{mod}\ 2\pi\ .$$

String hypothesis for $u > 0$:

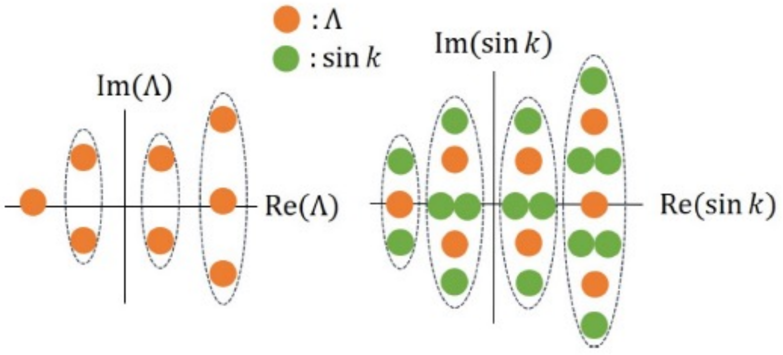
k: real quasimomentum root M_e

Λ: spin wave bound state M_n

$k - \Lambda$:charge bound state M'_n

$$N = \mathcal{M}_e + \sum_{n=1}^{\infty} 2nM'_n$$

$$M = \sum_{n=1}^{\infty} n\left(M_n + M'_n\right)$$



Fundamental concepts

Fractional quasiparticles, Universal thermodynamics,

Luttinger liquid, Quantum criticality,

Magnetism, Caloric effect,

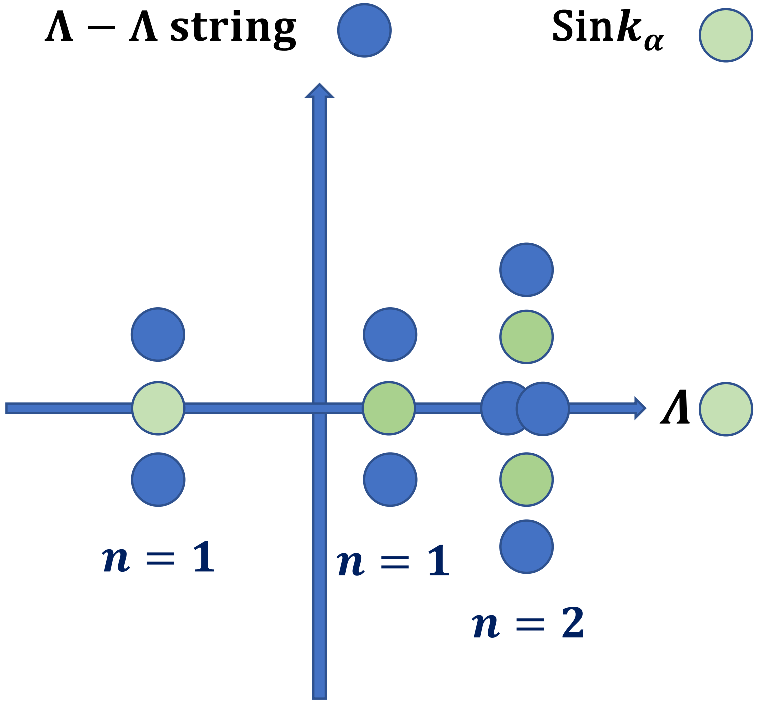
Hydrodynamics, Transport

Length-n Λ strings(Orange dots):
n-magnons form a bound state

Length-n $k - \Lambda$ strings (Green dots):
2n electrons form a bound state

$K-\Lambda$ strings

$$
\begin{aligned}
k_\alpha^1 &= \pi - \arcsin(\Lambda'^m_\alpha + miu),\\
k_\alpha^2 &= \arcsin(\Lambda'^m_\alpha + (m-2)iu),\\
k_\alpha^3 &= \pi - k_\alpha^2,\\
&\vdots\\
k_\alpha^{2m-2} &= \arcsin(\Lambda'^m_\alpha - (m-2)iu),\\
k_\alpha^{2m-1} &= \pi - k_\alpha^{2m-2},\\
k_\alpha^{2m} &= \pi - \arcsin(\Lambda'^m_\alpha - miu),
\end{aligned}
$$



$\Lambda-\Lambda$ strings

$$\Lambda'^{m,j}_\alpha = \Lambda'^m_\alpha + (m-2j+1)iu\,, \quad j = 1, \ldots m.$$

Thermodynamics Bethe ansatz equations

Equation of state

$$f = -T\int_{-\pi}^{\pi}\frac{\mathrm{d}k}{2\pi}\ln\left(1+\mathrm{e}^{-\frac{\kappa(k)}{T}}\right)+u$$
$$-T\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}\Lambda}{\pi}\mathrm{Re}\frac{1}{\sqrt{1-(\Lambda-inu)^2}}\ln\left(1+\mathrm{e}^{-\frac{\varepsilon_n'(\Lambda)}{T}}\right)$$

- quantum many body systems
- microscopic state energy E_i
- partition function $Z=\sum_{i=1}^{\infty}W_i \mathrm{e}_i^{-E_i/(k_BT)}$
- free energy $F=-k_BTlnZ$
- challenge: finding new physics

$$\kappa(k) = -2\cos k-\mu-2u-B+\sum_{n=1}^{\infty}\int d\Lambda a_n(\sin k-\Lambda)T\ln\left(1+e^{-\frac{\varepsilon_n'(\Lambda)}{T}}\right)$$
$$-\sum_{n=1}^{\infty}\int \mathrm{d}\Lambda a_n(\sin k-\Lambda)\ln\left(1+e^{-\frac{\varepsilon_n(\Lambda)}{T}}\right)$$

Charge particle dispersion

Real k

$$\varepsilon_n(\Lambda) = 2nB-\int dk\cos k a_n(\sin k-\Lambda)T\ln\left(1+e^{-\frac{\kappa(k)}{T}}\right)+\sum_{m=1}^{\infty}A_{nm}*T\ln\left(1+e^{-\frac{\varepsilon_m(\Lambda)}{T}}\right)$$

Spin wave bound states

Length-n spin strings

$$\varepsilon_n'(\Lambda) = 4Re\sqrt{1-(\Lambda-inu)^2}-2n\mu-4nu-\int dk\cos k a_n(\sin k-\Lambda)T\ln\left(1+e^{-\frac{\kappa(k)}{T}}\right)$$
$$+\sum_{m=1}^{\infty}A_{nm}*T\ln\left(1+e^{-\frac{\varepsilon_m'(\Lambda)}{T}}\right)$$

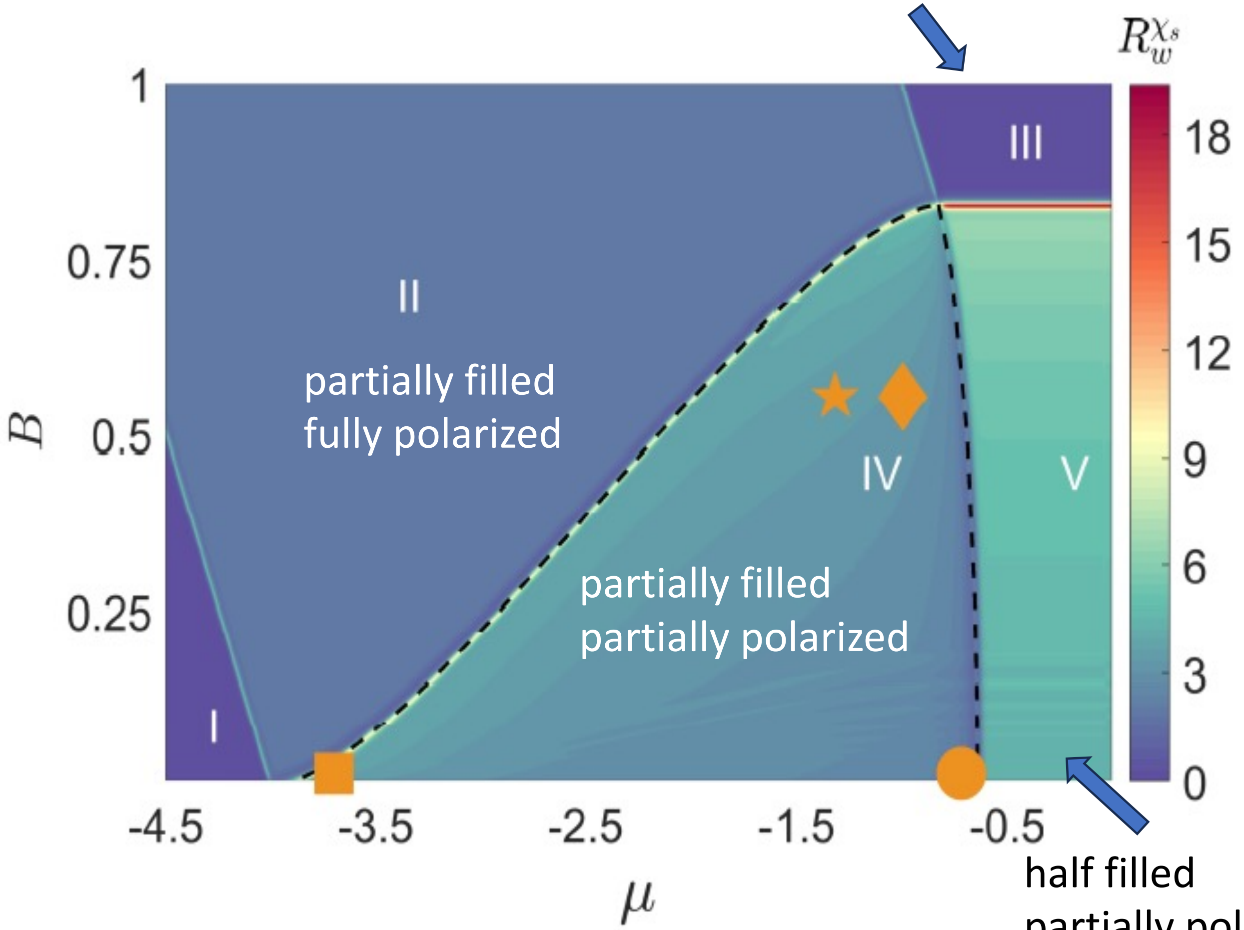
Charge particle bound states

Length-n electron BS

M. Takahashi One-dimensional Hubbard model at finite temperature, Progress of Theoretical Physics, 1972, 47(1): 69-82.

Wilson ratio maps out T=0 phase diagram

half filled
fully polarized



half filled
partially polarized

Wilson ratio: $$R_w^{\chi_s} = \frac{4}{3}\left(\frac{\pi k_B}{\mu_B g}\right)^2 \frac{\chi_s}{C_v/T}$$

χ -- susceptibility
c_v -- specific heat
T -- temperature

For Luttinger liquid phases at T=0

II: $R_w^{\chi_s} \approx 2$

IV: $R_w^{\chi_s} \approx 4(v_c K_s + v_s K_{sc})/(v_s + v_c)$

New result

V: $R_w^{\chi_s} \approx 8k_s$

I, III: $R_w^{\chi_s} \approx 0$

K_s -- spin Luttinger parameter

$v_{c,s}$ -- charge and spin velocities

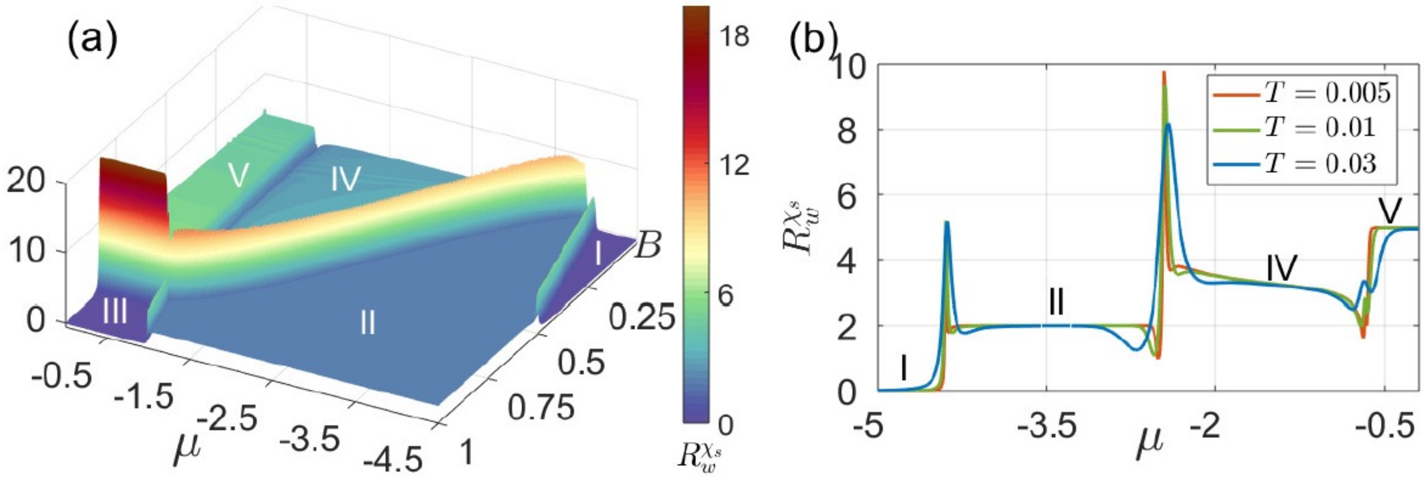
Luo, Pu, Guan, PRB **107**, L201103 (2023)
Luo, Pu, Guan, arXiv: 2307.00890

Wilson ratio maps out T=0 phase diagram

Wilson ratio

(a) $T = 0.005, u = 1$

(b) B$= 0.4, u = 1$



TBA equations at low temperature

$$\kappa(k) = -2\cos k - \mu - 2u - B + \sum_{n=1}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\Lambda a_n(\sin k - \Lambda) T \ln\left(1 + \mathrm{e}^{-\frac{\varepsilon'_n(\Lambda)}{T}}\right)$$

$$- \sum_{n=1}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\Lambda a_n(\sin k - \Lambda) T \ln\left(1 + \mathrm{e}^{-\frac{\varepsilon_n(\Lambda)}{T}}\right),$$

$$\varepsilon_n(\Lambda) = 2nB - \int_{-\pi}^{\pi} \mathrm{d}k \cos k a_n(\sin k - \Lambda) T \ln\left(1 + \mathrm{e}^{-\frac{\kappa(k)}{T}}\right) + \sum_{m=1}^{\infty} A_{nm} * T \ln\left(1 + \mathrm{e}^{-\frac{\varepsilon_m(\Lambda)}{T}}\right)$$

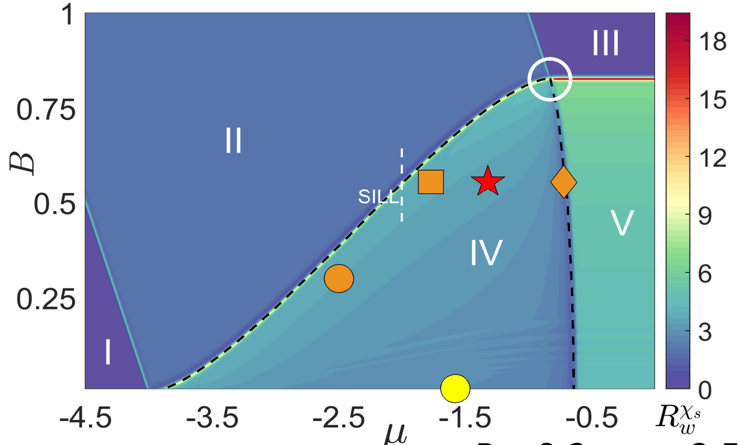
$$f = f_0 - \frac{\pi T^2}{6}\left(\frac{1}{v_c} + \frac{1}{v_s}\right) \quad \text{phase IV}$$

$$f = f_0 - \frac{\pi T^2}{6}\frac{1}{v_c} \quad \text{phase II}$$

$$f = f_0 - \frac{\pi T^2}{6}\frac{1}{v_s} \quad \text{phase V}$$

Free energy at low energy

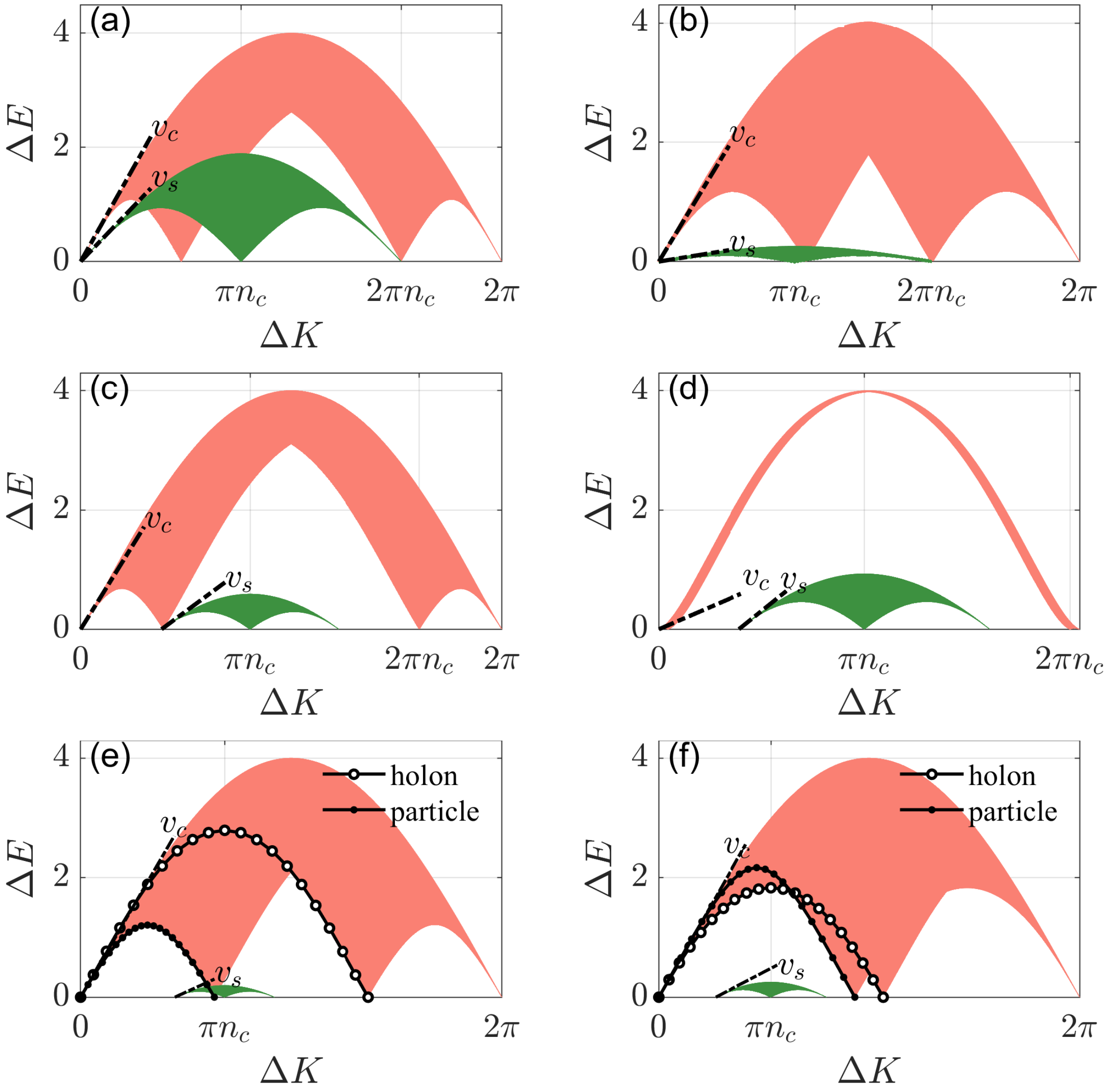
Fractional Excitations



B = 0.3, μ = −2.5

All figures are drawn in the first Brillouin zone

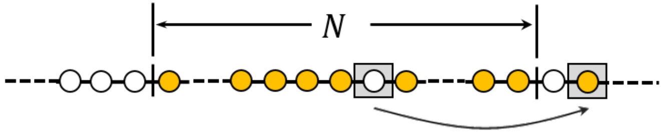
(a): Yellow circle	$B = 0, \mu = -1.5765$
(b): Strong coupling	$n_c = 0.6496, u = 10$
(c): Red star	$B = 0.555, \mu = -1.32$
(d): Diamond	$B = 0.555, \mu = -0.722$
(e): Square	$B = 0.555, \mu = -1.77$
(f): Orange circle	$B = 0.3, \mu = -2.5$



Elemental Fractional Excitations at ●

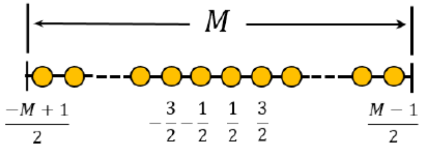
Particle-hole

Charge

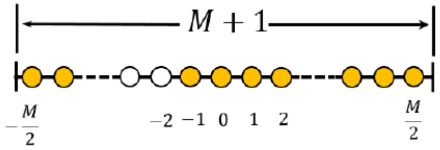


Fractional spinons

Length-1 Λ string
(Ground state)



Length-1 Λ string
(Two spinons)

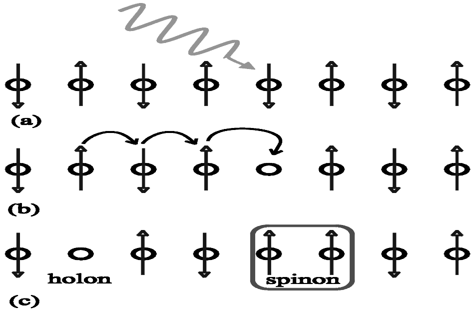


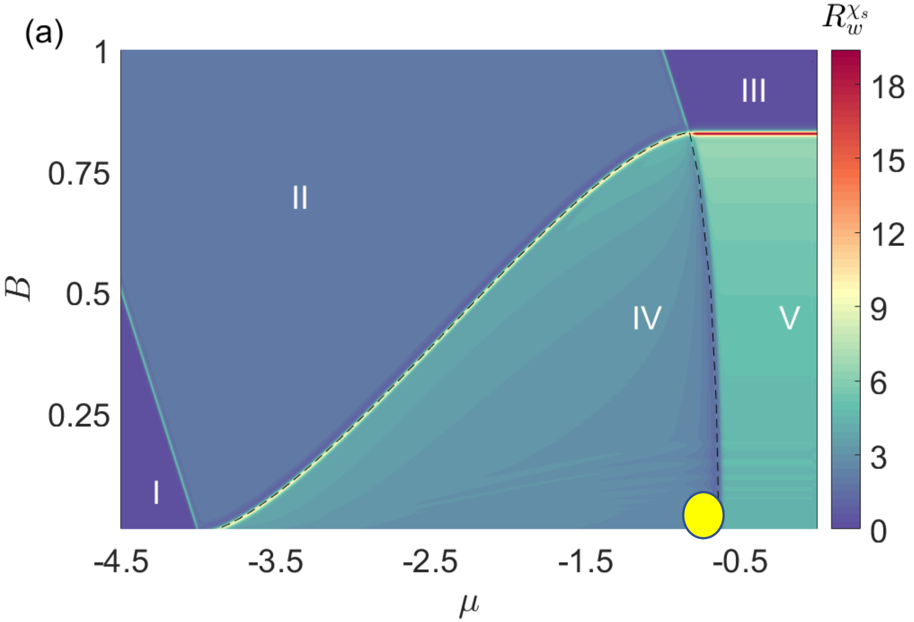
Two fractional spinons:

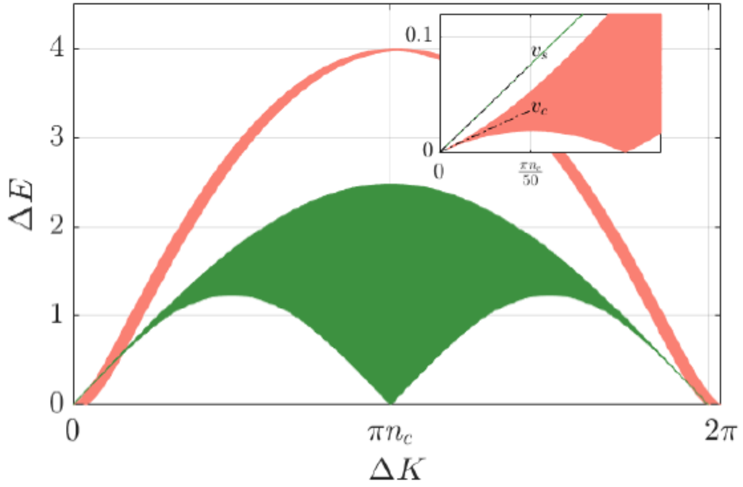
$$\Delta S^z=(\mathrm{N}-2\mathrm{M})/2=1$$

Fractional charge holons:

$$\Delta\eta^z=(\mathrm{N}-\mathrm{L})/2=0$$



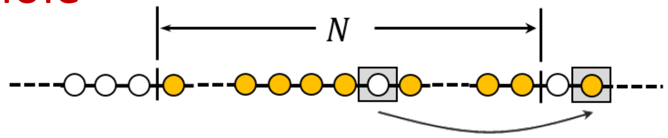




Elemental Fractional Excitations

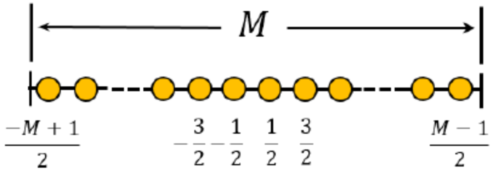
Particle-hole

Charge

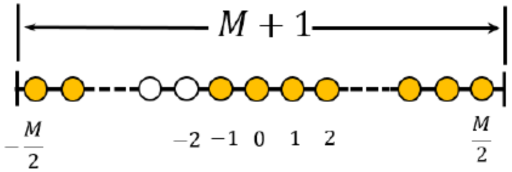


Fractional spinions

Length-1 Λ string
(Ground state)



Length-1 Λ string
(Two spinons)

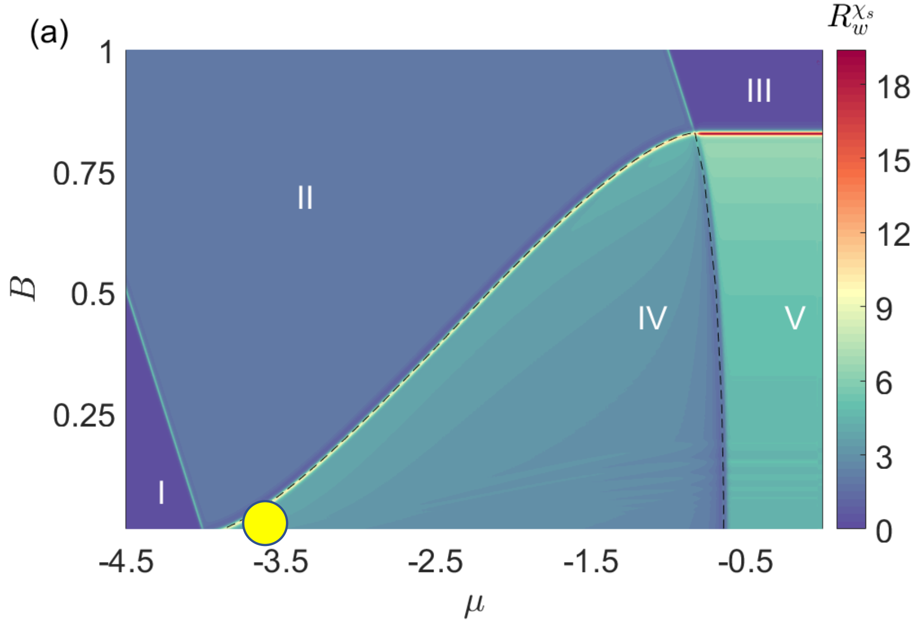


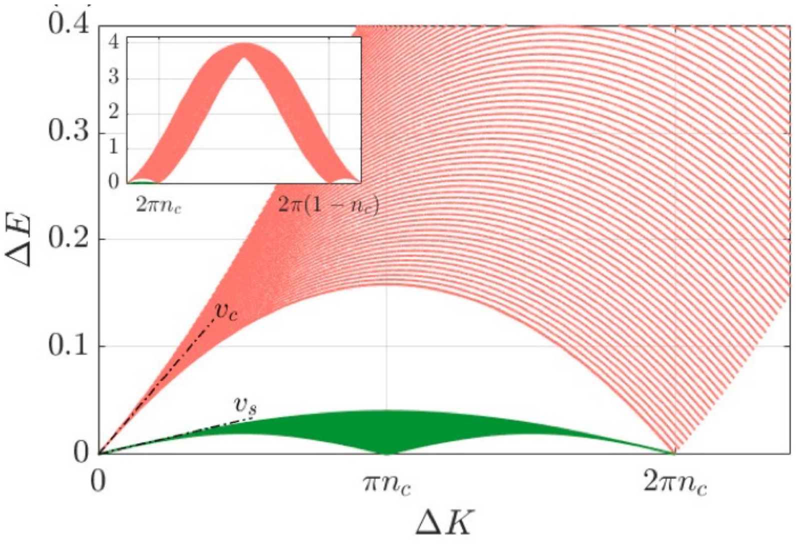
Two fractional spinons: ΔS^z=1

Fractional charge holon: $\Delta\eta^z = 0$

Spin incoherent liquid condition:

$$E_{spin} \ll K_B T \ll E_{charge}$$





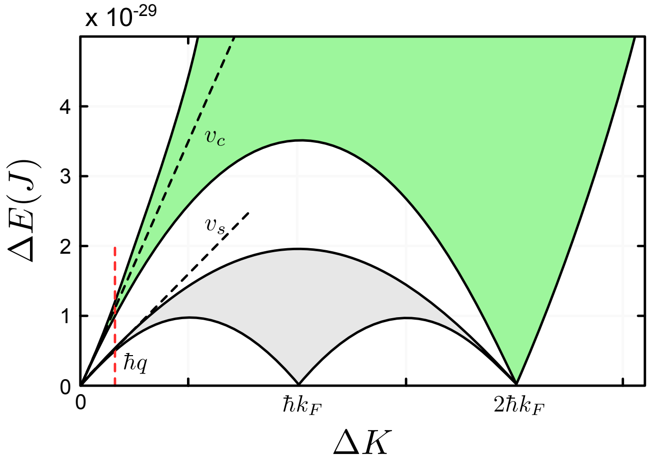
Spin-Charge Separation: originated from elementary excitations

Particle-hole spectrum (green) $q = \hbar\Delta K$

$$\omega_\pm = v_c q \pm \frac{1}{2m^*} q^2 \qquad \frac{m}{m^*} = \frac{\varepsilon_c''(k_0)}{2(2\pi\rho_c(k_0)^2} - \frac{\pi\rho_c'(k_0)\varepsilon_c'(k_0)}{(2\pi\rho_c(k_0))^3}$$

Two-spinons spectrum (grey):

$$\omega_{s+}(q) = v_s|q| - \frac{v_s q^3}{2k_s^2} + \cdots \qquad \omega_{s-}(q) = v_s|q| - \frac{2v_s q^3}{k_s^2} + \cdots$$



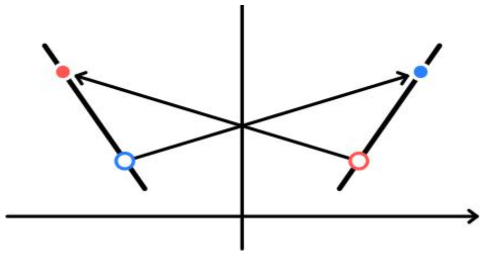
Separated pin and change excitations

Effective Field Theory: separated spin and charge TLLs

Charge: $$H_v = \frac{1}{2\pi}\int dx\left[u_v K_v\left(\pi\Pi_v(x)\right)^2 + \frac{u_v}{K_v}(\nabla\varphi_v(x))\text{^}2\right], v = c$$

Spin: $$H_\sigma = \frac{1}{2\pi}\int dx\left[u_\sigma K_\sigma\left(\pi\Pi_\sigma(x)\right)^2 + \frac{u_\sigma}{K_\sigma}(\nabla\varphi_\sigma(x))\text{^}2\right]$$

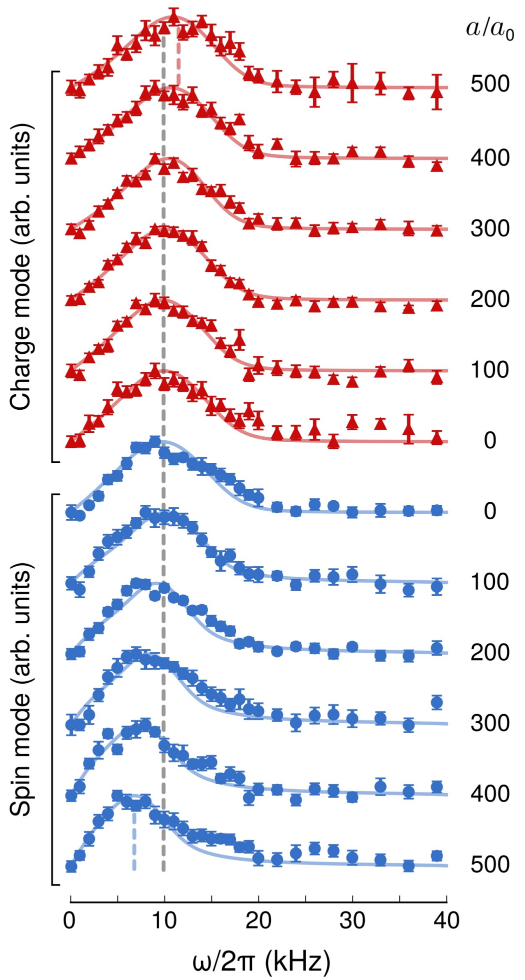
Backward scattering $$H_g = \frac{2g_1}{(2\pi\alpha)^2}\int dx cos\left(\sqrt{8}\varphi_\sigma\right)$$



Spin backward scattering

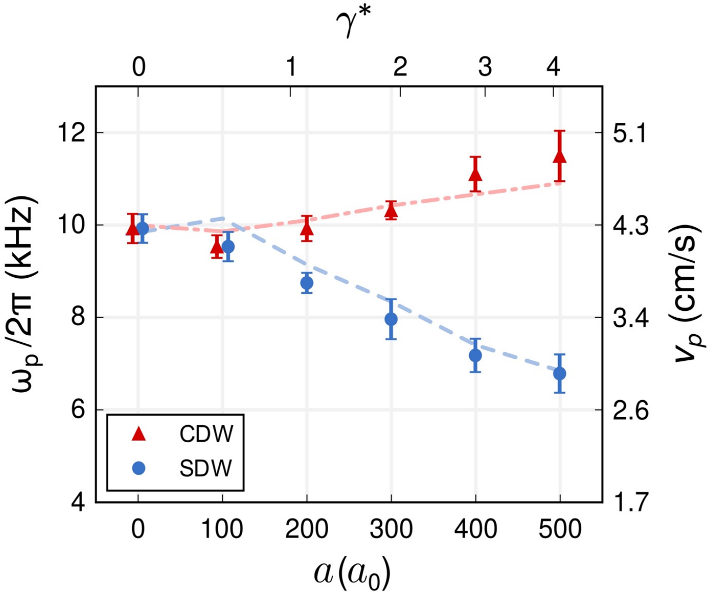
He, Jiang, Lin, Hulet, Pu, Guan, PRL 125, 190401 (2020)

Observation of Spin-coherent liquid: Spin-charge separation



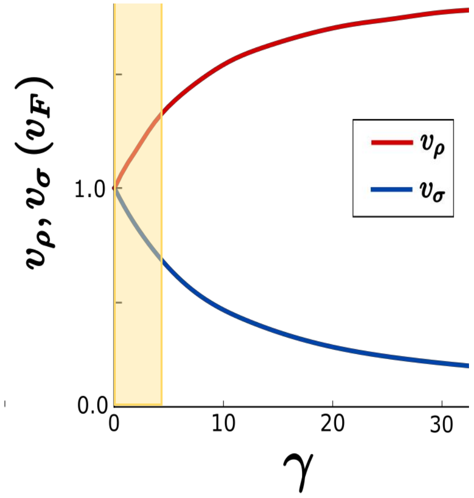
Charge(red) and spin(blue) dynamical structure factors

Encoding Nonlinear TLL Effect



Peak frequencies and velocities

$$v_p = \omega\ _p/q$$

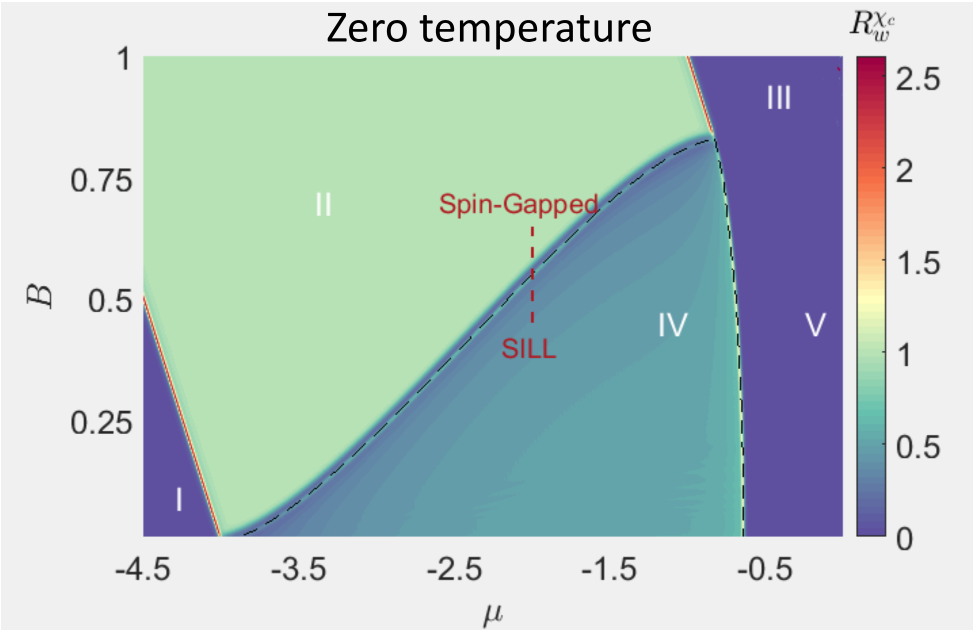


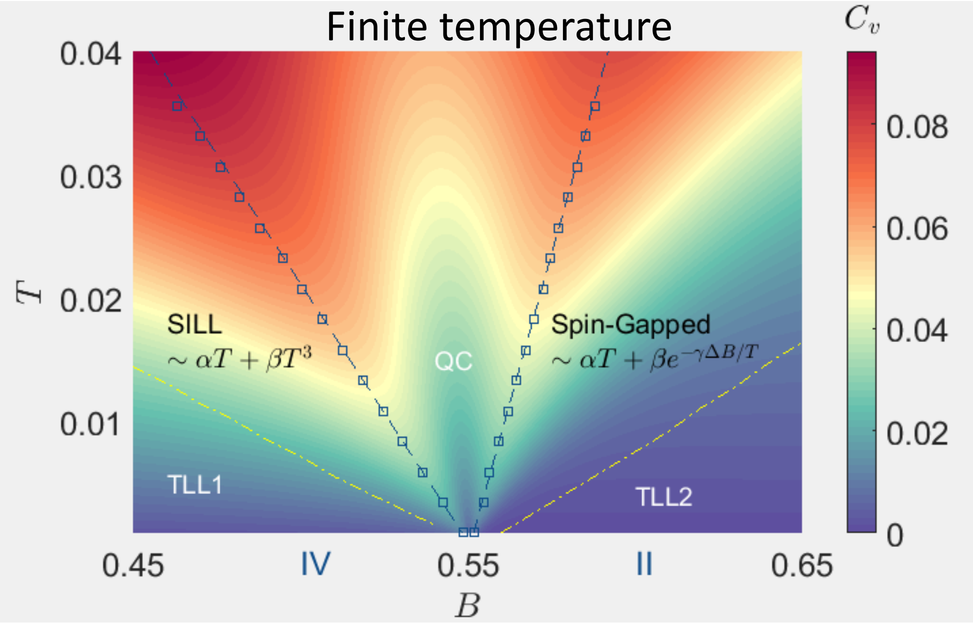
Velocities of spin and charge shift in opposite directions!

Senaratne, et. al., Pu, Guan, Hulet, Science 376, 1305 (2022)

Guan, Batchelor, Lee, Rev. Mod. Phys. 85, 163 (2013)

Finite temperature: spin-coherent and –incoherent Luttinger liquid





QC — Quantum criticality $|B - B_c| \ll K_B T$

$$\frac{C_v}{T} = C_v^0 + T^{\frac{d}{z}+1-\frac{2}{vz}} K\left(\frac{\mu-\mu_c}{T^{1/vz}}\right) \quad z = 2, v = 1/2$$

TLL—Tomonaga-Luttinger liquid $K_B T \ll E_{spin} \ll E_{charge}$

$$H_v = \int dx \left(\frac{\pi v_v K_v}{2} \Pi_v^2 + \frac{v_v}{2\pi K_v} (\partial_x \phi_v)^2\right), v = c, s$$

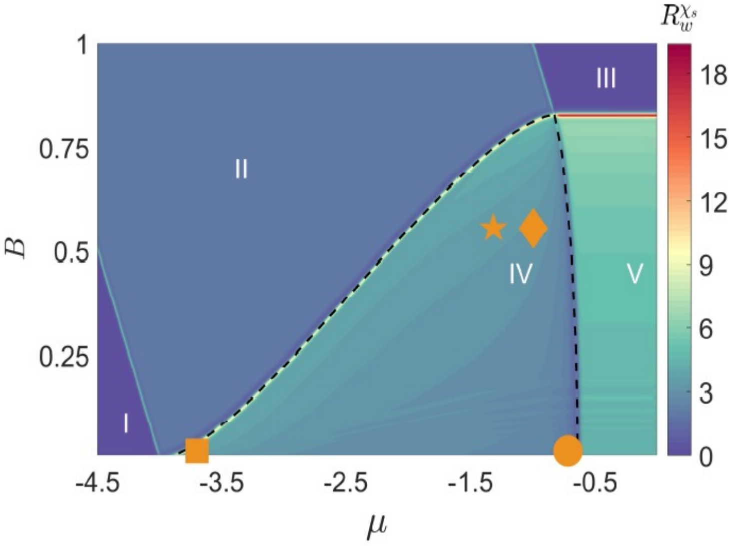
SILL — Spin incoherent TLL $E_{spin} \ll K_B T \ll E_{charge}$

$$C_v = \frac{\pi T}{3}\left(\frac{1}{v_c} + \frac{1}{v_s}\right) + \frac{7\pi^3 T^3}{40 v_s\left(-\varepsilon_1(0)\right)^2} + O(T^4)$$

$$G_\sigma^{SILL}(x,\tau) \sim \frac{\mathrm{e}^{-2k_F|x|(\ln 2/\pi)}}{(x^2 + v_c^2\tau^2)^{\Delta_{K_c}}} \frac{\mathrm{e}^{\mathrm{i}\left(2k_F x - \phi_{K_c}^+\right)}}{v_c \tau - \mathrm{i}x} + c.c.$$

New Result

Universal Scaling Functions near phase boundaries



$$\text{I-II:}\quad f = u + T^{\frac{3}{2}}\pi^{\frac{1}{2}}\rho(0)\left(\frac{\kappa''(0)}{2}\right)^{-\frac{1}{2}}\mathrm{Li}_{\frac{3}{2}}\left(-\mathrm{e}^{-\frac{\kappa(0)}{T}}\right),$$

$$\text{II-III:}\quad f = f_0 + T^{\frac{3}{2}}\pi^{\frac{1}{2}}\rho(\pi)\left(\frac{-\kappa''(\pi)}{2}\right)^{-\frac{1}{2}}\mathrm{Li}_{\frac{3}{2}}\left(-\mathrm{e}^{\frac{\kappa(\pi)}{T}}\right),$$

$$\text{V-III:}\quad f = f_0 + T^{\frac{3}{2}}\pi^{\frac{1}{2}}\sigma_1(0)\left(\frac{\varepsilon_1''(0)}{2}\right)^{-\frac{1}{2}}\mathrm{Li}_{\frac{3}{2}}\left(-\mathrm{e}^{-\frac{\varepsilon_1(0)}{T}}\right),$$

$$\text{II-IV:}\quad f = f_0 - \frac{\pi T^2}{6v_c} + T^{\frac{3}{2}}\pi^{\frac{1}{2}}\sigma_1(0)\left(\frac{\varepsilon_1''(0)}{2}\right)^{-\frac{1}{2}}\mathrm{Li}_{\frac{3}{2}}\left(-\mathrm{e}^{-\frac{\varepsilon_1(0)}{T}}\right),$$

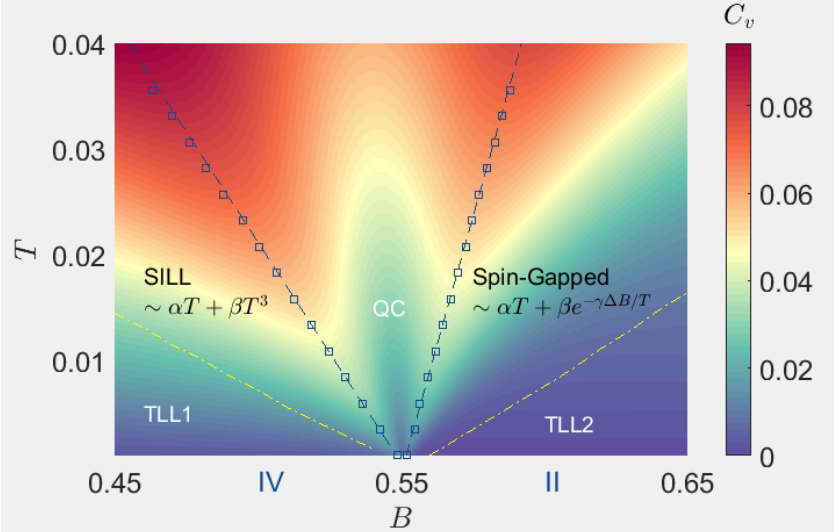
$$\text{V-IV:}\quad f = f_0 - \frac{\pi T^2}{6v_s} + T^{\frac{3}{2}}\pi^{\frac{1}{2}}\rho(\pi)\left(\frac{-\kappa''(\pi)}{2}\right)^{-\frac{1}{2}}\mathrm{Li}_{\frac{3}{2}}\left(-\mathrm{e}^{\frac{\kappa(\pi)}{T}}\right),$$

Universal Scaling Functions

$$m = m_0 + T^{d/z+1-1/(vz)}O_1\left[\frac{\mu-\mu_c}{T^{1/vz}}, \frac{B-B_c}{T^{1/vz}}, \frac{u-u_c}{T^{1/vz}}\right]$$

$$\chi_s = \chi_{s0} + T^{d/z+1-2/(vz)}O_2\left[\frac{\mu-\mu_c}{T^{1/vz}}, \frac{B-B_c}{T^{1/vz}}, \frac{u-u_c}{T^{1/vz}}\right]$$

Spin incoherent liquid in 1D Hubbard model



Distinguishing TLL and SILL:

$$G^{\uparrow} = \left\langle \phi_{\uparrow}(x,t)\phi_{\uparrow}^{\dagger}(0,0) \right\rangle$$

$$G^{p} = \left\langle \phi_{\downarrow}(x,t)\phi_{\uparrow}(x,t)\phi_{\uparrow}^{\dagger}(0,0)\phi_{\downarrow}^{\dagger}(0,0) \right\rangle$$

May be obtained near B_c from conformal field theory

$$\langle \phi(x,t)\phi(0,0)\rangle_0 = \sum A(D_c, D_s, N_c^{\pm}, N_s^{\pm}) \frac{\exp(-2iD_c k_{F,\uparrow}x)\exp(-2i(D_c+D_s)k_{F,\downarrow}x)}{(x-iv_ct)^{2\Delta_c^+}(x+iv_ct)^{2\Delta_c^-}(x-iv_st)^{2\Delta_s^+}(x+iv_st)^{2\Delta_s^-}}$$

$$\begin{aligned}\langle \phi(x,t)\phi(0,0)\rangle_T = &\sum A(D_c, D_s, N_c^{\pm}, N_s^{\pm})\exp(-2iD_c k_{F,\uparrow}x)\exp(-2i(D_c+D_s)k_{F,\downarrow}x) \\ &\times \left(\frac{\pi T}{v_c \sinh(\pi T(x-iv_ct)/v_c)}\right)^{2\Delta_c^+} \left(\frac{\pi T}{v_c \sinh(\pi T(x+iv_ct)/v_c)}\right)^{2\Delta_c^-} \\ &\times \left(\frac{\pi T}{v_s \sinh(\pi T(x-iv_st)/v_s)}\right)^{2\Delta_s^+} \left(\frac{\pi T}{v_s \sinh(\pi T(x+iv_st)/v_s)}\right)^{2\Delta_s^-}\end{aligned}$$

Essler, Frahm, Göhman, Klümper and Korepin, the one-dimensional Hubbard model, Cambridge University Press, 2010

1D Hubbard model near B_c: Spin and charge coherent liquid at T=0

$$G^{\uparrow}_{B\to B_c} \approx \exp\left(-ik_{F,\uparrow}x\right) S^{-}_{\uparrow}(x-iv_st)S^{+}_{\uparrow}(x+iv_st)C^{-}_{\uparrow}(x-iv_ct)+h.c.$$

$$S^{\pm}_{\uparrow}(Z)=\frac{const}{Z^{2\Delta_s^{\pm}}} \qquad C^{\pm}_{\uparrow}(Z)=\frac{const}{Z^{2\Delta_c^{\pm}}}$$

$$2\Delta_s^{+}=\frac{1}{4}-\frac{3}{2\pi}\sqrt{1-\frac{B}{B_c}}, 2\Delta_s^{-}=\frac{1}{4}+\frac{1}{2\pi}\sqrt{1-\frac{B}{B_c}}$$

$$2\Delta_c^{-}=1-\frac{2}{\pi}\sqrt{1-\frac{B}{B_c}}, 2\Delta_c^{+}=0$$

$$G^{p}_{B\to B_c} \approx \exp\left(-i(k_{F,\uparrow}+k_{F,\downarrow})x\right) S^{-}_{p2}(x-iv_st)S^{+}_{p2}(x+iv_st)C^{-}_{p2}(x-iv_ct)C^{+}_{p2}(x+iv_ct)+h.c.$$

$$S^{\pm}_{p2}(Z)=\frac{const}{Z^{\Delta_s^{\pm}}} \qquad C^{\pm}_{p2}(Z)=\frac{const}{Z^{\Delta_c^{\pm}}}$$

$$2\Delta_s^{+}=\Delta_s^{-}=\frac{1}{2}-\frac{3}{\pi}\sqrt{1-\frac{B}{B_c}}$$

$$2\Delta_c^{-}=\frac{9}{4}, 2\Delta_c^{+}=\frac{1}{4}$$

Correlation functions show a power law decay of distance!

Spin incoherent liquid: Exponential decay

$E_{spin} \ll K_B T \ll E_{charge}$

$$G^{\uparrow}_{B\to B_c} \approx e^{-ik_{F,\uparrow}x} C^{-}_{\uparrow}(x - iv_c t)\langle S^{+}_{R}(x,t) S_R(0,0)\rangle + h.c.$$

$$\langle S^{+}_{R}(x,t) S_R(0,0)\rangle \sim (2\pi\alpha k_F)^{\frac{1}{2}-\frac{1}{\pi}\sqrt{1-\frac{B}{B_c}}} \boldsymbol{e}^{-\pi\alpha\left(\frac{1}{2}-\frac{1}{\pi}\sqrt{1-\frac{B}{B_c}}\right)k_F x}$$

$$C^{-}_{\uparrow}(Z) = \frac{const}{Z^{2\Delta^{-}_{c}}}$$

$$2\Delta^{-}_{c} = 1 - \frac{2}{\pi}\sqrt{1-\frac{B}{B_c}}$$

$$G^{p}_{B\to B_c} \approx e^{-i(k_{F,\uparrow}+k_{F,\downarrow})x} C^{-}_{p2}(x - iv_c t) C^{+}_{p2}(x + iv_c t)\langle S^{+}_{R}(x,t) S_R(0,0)\rangle + h.c.$$

$$C^{\pm}_{p2}(Z) = \frac{const}{Z^{\Delta^{\pm}_{c}}}$$

$$\langle S^{+}_{R}(x,t) S_R(0,0)\rangle \sim (2\pi\alpha k_F)^{\frac{1}{2}-\frac{3}{\pi}\sqrt{1-\frac{B}{B_c}}} \boldsymbol{e}^{-\pi\alpha\left(\frac{1}{2}-\frac{3}{\pi}\sqrt{1-\frac{B}{B_c}}\right)k_F x}$$

$$2\Delta^{-}_{c} = \frac{9}{4}, 2\Delta^{+}_{c} = \frac{1}{4}$$

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To better capture the **interaction-driven** effects, we define:

New Result

Contact (interaction driven MIPT)

$$C = \frac{\partial f}{\partial u} = 4d - 2n_c + 1$$

$$d = \frac{1}{N}\sum_i \langle n_{i,\uparrow} n_{i,\downarrow} \rangle$$ double occupancy

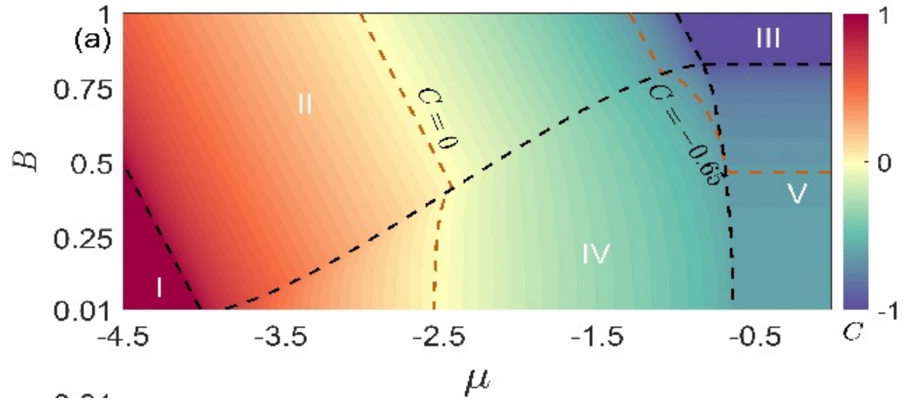
Contact Susceptibilities

$$f = e - \mu n_c - 2Bm - Ts - uC$$

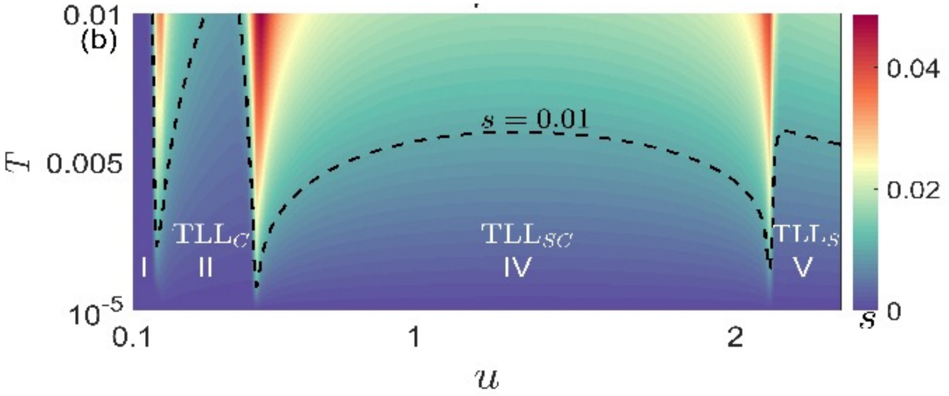
Maxwell relations

$$\frac{\partial n_c}{\partial u} = -\frac{\partial C}{\partial \mu} \qquad \frac{\partial m}{\partial u} = -\frac{\partial C}{\partial (2B)} \qquad \frac{\partial s}{\partial u} = -\frac{\partial C}{\partial T}$$

Contour plot of the Contact @ $T = 0.005$ and $u = 1$



Contour plot of the entropy @ $B = 0.15, \mu = -2.5$



- **Interaction-driven phase transitions (II-IV) and (V-IV)**

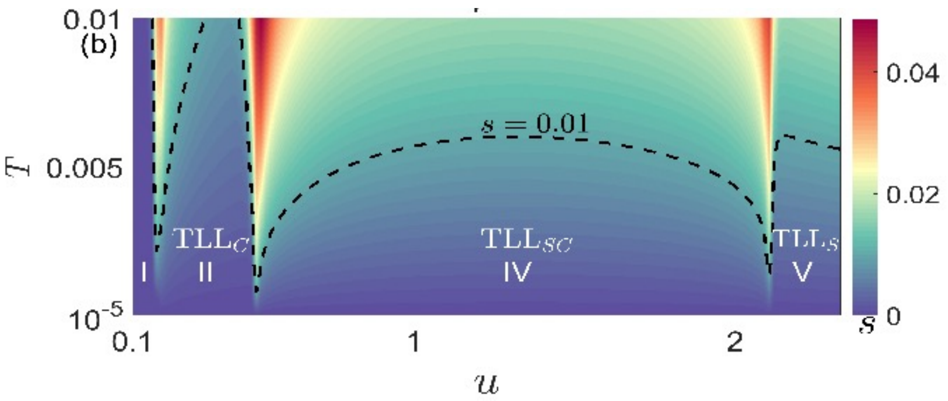
$$f = f_0 - \frac{\pi T^2}{6v_c} + T^{\frac{3}{2}}\pi^{\frac{1}{2}}\sigma_1(0)\left(\frac{\varepsilon_1''(0)}{2}\right)^{-\frac{1}{2}}\mathrm{Li}_{\frac{3}{2}}(-e^{-\frac{\varepsilon_1(0)}{T}})$$

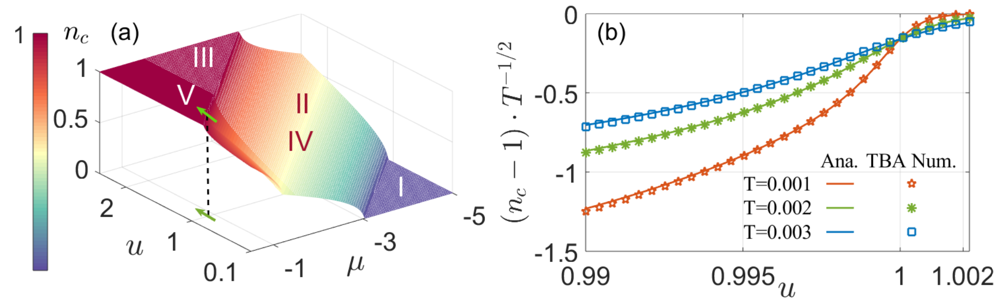
$$f = f_0 - \frac{\pi T^2}{6v_s} + T^{\frac{3}{2}}\pi^{\frac{1}{2}}\rho(\pi)\left(\frac{-\kappa''(\pi)}{2}\right)^{-\frac{1}{2}}\mathrm{Li}_{\frac{3}{2}}(-e^{\frac{\kappa(\pi)}{T}})$$

$$\boldsymbol{\varepsilon_1(0), \kappa(\pi) = \alpha_B \Delta B + \alpha_\mu \Delta\mu + \alpha_u \Delta u}$$

$$\frac{\alpha_u}{\alpha_B} = -\frac{\partial B}{\partial u}, \qquad \frac{\alpha_u}{\alpha_\mu} = -\frac{\partial \mu}{\partial u}, \qquad \frac{\alpha_B}{\alpha_\mu} = -\frac{\partial \mu}{\partial B}$$

Entropy accumulation at phase transitions!





Upper: Contour plot of the entropy in T-u plane for $B = 0.15, \mu = -2.5$, a maximum entropy at QC.
Lower: IV-V phase transition: density shows universal scaling behaviour driven by interaction.

- **Contact susceptibilities and applications**

$$\frac{\partial s}{\partial u} = -\frac{\partial C}{\partial T}; \quad \frac{C_v}{T}\frac{\partial T}{\partial u} = \frac{\partial C}{\partial T}$$

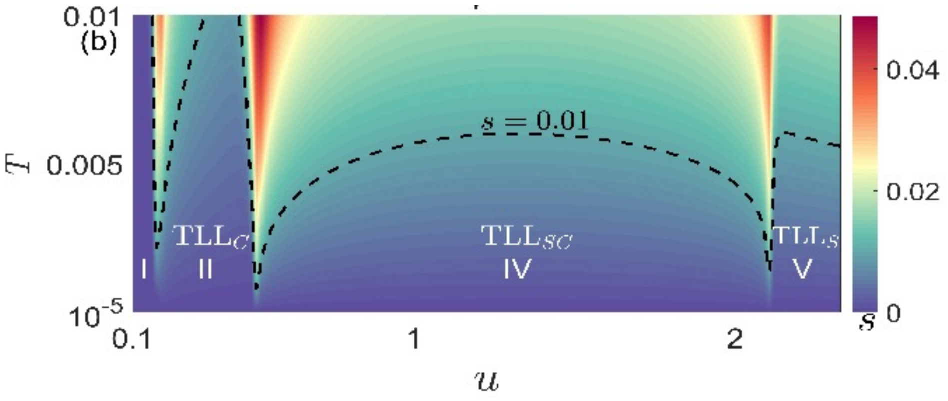
$$\frac{\partial C}{\partial u}|_{s,N,V,H} = \frac{T}{u}\Gamma_{int}, \qquad \Gamma_{int} = \frac{\partial C}{\partial T}\frac{u}{c_v}$$

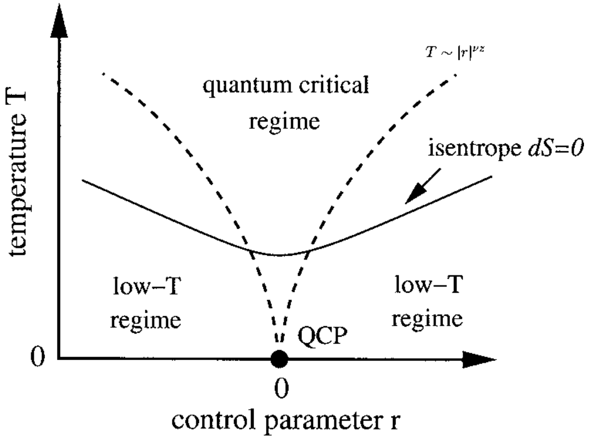
For quantum cooling

Grüneisen parameter

Also a large change of $\boldsymbol{\Gamma_{int}}$

$\frac{\partial s}{\partial u}$, $\frac{\partial C}{\partial T}$ change rapidly





Isentrope:

$$ds = \frac{\partial s}{\partial u}\mathrm{du} + \frac{\partial s}{\partial T}\mathrm{dT} = 0$$

Quantum Cooling

- Entropy peaks near phase boundaries.
- Isentropic process:
  maximum entropy → minimum T

A potentially novel way of cooling quantum gases in lattice!

Adiabatic interaction ramping cooling!

Also see Adiabatic demagnetization cooling:

Wolf et. al. PNAS, 108, 6862 (2011)

• Contact susceptibilities and applications

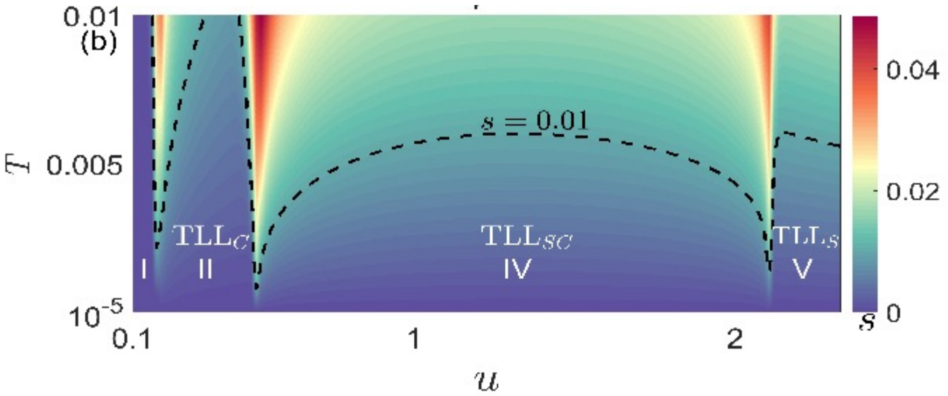
$$\frac{\partial s}{\partial u} = -\frac{\partial C}{\partial T}; \quad \frac{C_v}{T}\frac{\partial T}{\partial u} = \frac{\partial C}{\partial T}$$

$$\frac{\partial C}{\partial u}|_{s,N,V,H} = \frac{T}{u}\Gamma_{int}, \qquad \Gamma_{int} = \frac{\partial C}{\partial T}\frac{u}{c_v}$$

For quantum cooling

Grüneisen parameter

$\frac{\partial s}{\partial u}, \frac{\partial C}{\partial T}$ change rapidly !

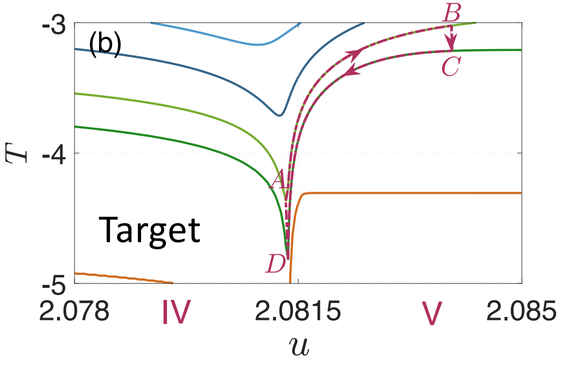


Quantum refrigeration

1) A->B: adiabatically ramp up;
2) B->C: hot isochore process;
3) C->D: adiabatically ramp down;
4) D->A: cold isochore process.

Target material T_{tar}
Substance: lattice model
Hot Ambient T_C

Ambient



Isentropic lines

Quantum Cooling

- Entropy peaks near phase boundaries.
- Isentropic process:
  maximum entropy → minimum T

A potentially novel way of cooling quantum gases in lattice!

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Transport in integrable systems:

Spin transport--magnetic field gradient

Heat transport--temperature gradient

Transport coefficients—dynamical correlation

Kubo formulas for conductivities

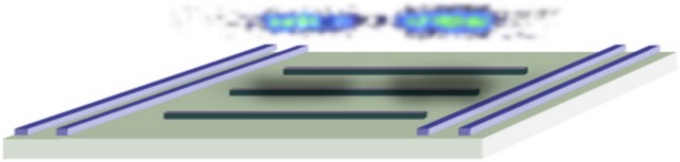
Generalized hydrodynamics

Bosonization

Drude weight can be obtained from real-time equilibrium current-current correlation function:

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t dt' < J(t')J(0) >$$

$$\begin{cases} D \neq 0 & \text{conductor ballistic} \\ D = 0 & \text{insulator} \end{cases}$$



Transport Coefficients in spin chain

$$\begin{pmatrix} \mathcal{J}^{\rm th} \\ \mathcal{J}^{s} \end{pmatrix} = \begin{pmatrix} \kappa_{\rm th} & C_s^{\rm th} \\ C_{\rm th}^{s} & \sigma_s \end{pmatrix} \begin{pmatrix} -\nabla T \\ \nabla h \end{pmatrix}$$

Thermal and Spin conductivities:

κ_{th} σ_s

$$\sigma_s'(k=0,\omega) = 2\pi D_s \delta(\omega) + \sigma_s^{\rm reg}(\omega)$$

Thermal current: $\mathcal{J}^{th} = \mathcal{J}^{E} - \mathcal{J}^{s}$

Energy and spin currents

Bertini, et. al. Rev. Mod. Phys. **93**, 025003 (2021)
Sirker, SciPost Phys. Lect. Notes 17, 2020

Linear response theory

$$\sigma_s(\omega) = \frac{\mathrm{i}}{\omega}\left[\frac{\langle H_{\mathrm{kin}}\rangle}{N} - \frac{\mathrm{i}}{N}\int_0^\infty dt\,\mathrm{e}^{\mathrm{i}\omega t}\langle[\mathcal{J}^s(t),\mathcal{J}^s(0)]\rangle\right]$$

$$\sigma'_s(\omega) = -\frac{\pi}{N}\sum_{n,m}\frac{p_n-p_m}{E_n-E_m}|\langle n|\mathcal{J}^2|m\rangle|^2\delta(\omega-(E_m-E_n))$$

$$= \frac{\beta\pi}{N}\sum_{E_n=E_m}p_n|\langle n|\mathcal{J}^2|m\rangle|^2\delta(\omega) + \frac{\pi}{N}\sum_{E_n\neq E_m}\frac{p_n-p_m}{E_m-E_n}|\langle n|\mathcal{J}^2|m\rangle|^2\delta(\omega-(E_m-E_n))$$

Drude weight D

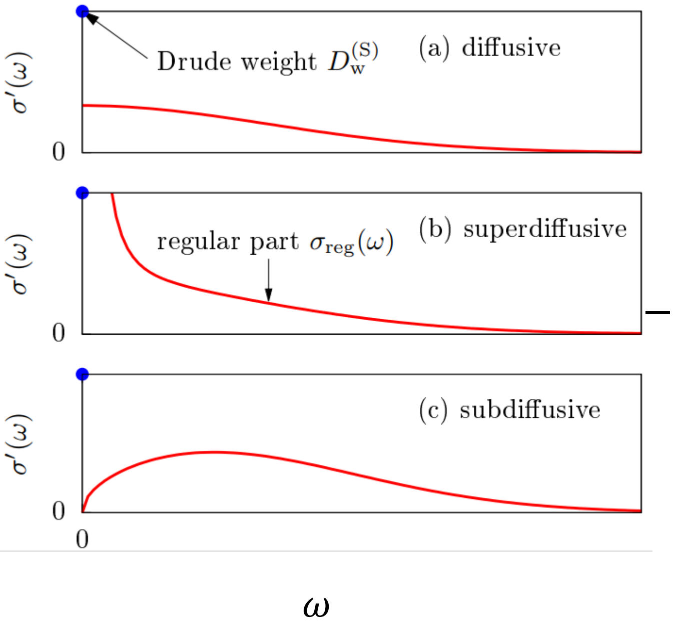
Regular part: Diffusion constant $\mathcal{D}$

finite at $t \to \infty$

vanish at $t \to \infty$

$$\begin{aligned}\sigma'_s(\omega) &= \frac{1-\mathrm{e}^{-\beta\omega}}{2\omega}\int_{-\infty}^{\infty} dt\,\mathrm{e}^{\mathrm{i}\omega t}\,[(\mathcal{J}^s\mathcal{J}^s)_\infty + C_s^{\mathrm{reg}}(t)] \\ &= 2\pi\frac{(\mathcal{J}\mathcal{J})_\infty}{2T}\delta(\omega) + \frac{1-\mathrm{e}^{-\beta\omega}}{2\omega}C_s^{\mathrm{reg}}(\omega)\,.\end{aligned}$$

$\sigma'(\omega \to 0) \sim |\omega|^{\alpha}$ **Conductor: $\alpha = -1$**



$\alpha \to 0$

$-1 < \alpha < 0$

$\alpha > 0$

Theoretical challenging!

$$D_s = \frac{(\mathcal{J}^s\mathcal{J}^s)_\infty}{2T} = \lim_{t\to\infty}\lim_{N\to\infty}\frac{1}{2NT}\langle\mathcal{J}^s(t)\mathcal{J}^s(0)\rangle$$

$$\sigma_s^{\mathrm{reg}}(\omega\to 0) = \beta\int_0^\infty dt\,C_s^{\mathrm{reg}}(t) = \chi_s(\beta)\mathcal{D}_s$$

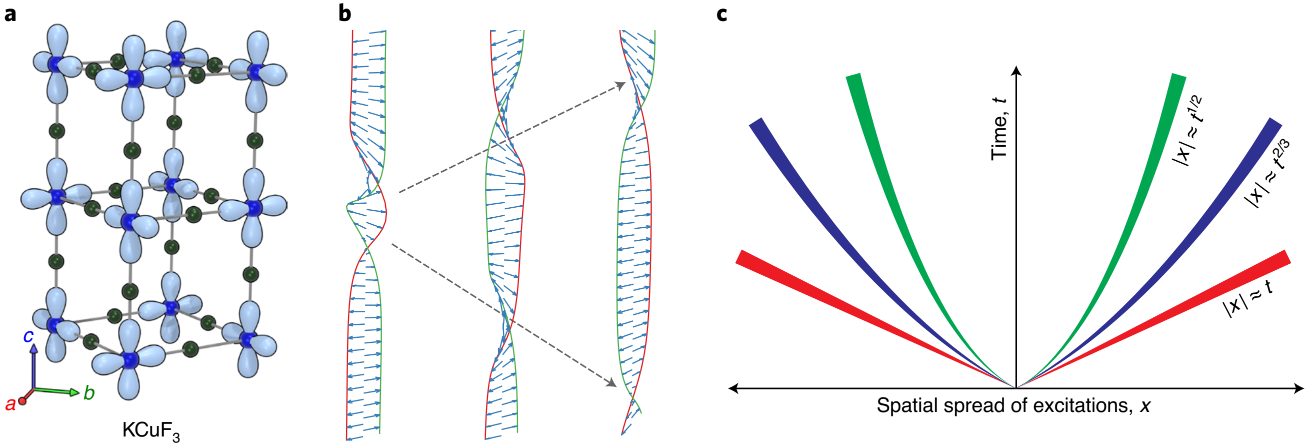
Bertini, et. al. Rev. Mod. Phys. **93**, 025003 (2021)
Nardis, Bernard, Doyon, SciPost Phys. 6, 049 (2019)

Ballistic, super diffusive and diffusive spin transport

Spin chain

$$H=-J\sum_{j=1}^{L}(S_j^xS_{j+1}^x+S_j^yS_{j+1}^y+\Delta S_j^zS_{j+1}^z)$$

Easy plane $\Delta < 1$: DW $D_{DW} > 0$, ballistic transport
Easy axis $\Delta > 1$: DW $D_{DW} = 0$, sub-ballistic transport
Isotropic point $\Delta = 1$: super diffusive transport



(a) Potassium copper tri-fluorine 1D spin chain;
(b) Spinon exitiation;
(c) different spin transports:
Ballistic z=1, supper diffusive z=3/2; diffusive z=2

Theoretical challenging!

$$D_s=\frac{(\mathcal{J}^s\mathcal{J}^s)_\infty}{2T}=\lim_{t\to\infty}\lim_{N\to\infty}\frac{1}{2NT}\langle\mathcal{J}^s(t)\mathcal{J}^s(0)\rangle$$

$$\sigma_s^{\mathrm{reg}}(\omega\to 0)=\beta\int_0^\infty dt\, C_s^{\mathrm{reg}}(t)=\chi_s(\beta)\mathcal{D}_s$$

$$\boldsymbol{\sigma'(\omega\to 0)\sim|\omega|^\alpha}$$

$\boldsymbol{\alpha=-1}$	**Conductor**
$\boldsymbol{-1<\alpha<0}$	**Super diffusive**
$\boldsymbol{\alpha\to 0}$	**Diffusive**
$\boldsymbol{\alpha>0}$	**Subdiffusive**

Kardar-Parisi-Zhang hydrodynamics!

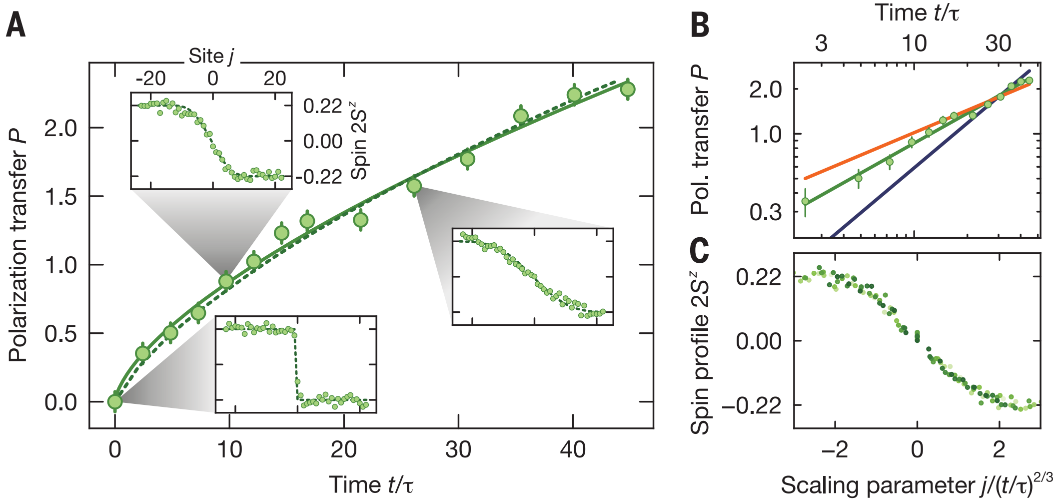
$$D_s(t)\sim t^z,\qquad z=?$$

Wei, et. al. Science 376, 716 (2024)
Scheie, et. al. Nat. Phys. 17, 726 (2021);
Gopalakrishman, et. al., PRL 122, 1272020 (2019)

Super diffusive spin transport

Quantum gas $H = -J \sum_{j=1}^{L} (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z)$

Ferromagnetic $J = \frac{4\tilde{t}^2}{U};\ \Delta \approx 1$



Super diffusive transport in Heisenberg chain at high T

Kardar-Parisi-Zhang hydrodynamics!

Polarization

$$P(t) = (P_L(t) - P_R(t))/2 \propto t^{1/z}$$

$$P_{L,R}(t) = 2 \sum_{i=L,R} \left(S_i^z(t) - S_i^z(0)\right)$$

KPZ dynamics: z=3/2

(A) The polarization transfer for a domain wall initial state with a contrast $\eta = 0.22$.
The insets show spin profiles $2S^z(t)$ at t=0, 10, 26 J/h
(B)Polarization transfer in log-log plot
(C)Spatial spin profiles at times t=5-35 j/h

Wei, et. al. Science 376, 716 (2024)

Quantum transport in 1D Hubbard model

Kubo formula for interacting electrons

$$\text{response}\quad \begin{pmatrix} J^c \\ J^s \\ J^e \end{pmatrix} = \begin{pmatrix} \sigma^c \ldots\ldots \\ \ldots\sigma^s\ \ldots \\ \ldots\ldots\sigma^e \end{pmatrix} \begin{pmatrix} \nabla\mu \\ \nabla h \\ -\nabla T \end{pmatrix} \quad \text{perturbation}$$

J^c : charge current

J^s : spin current

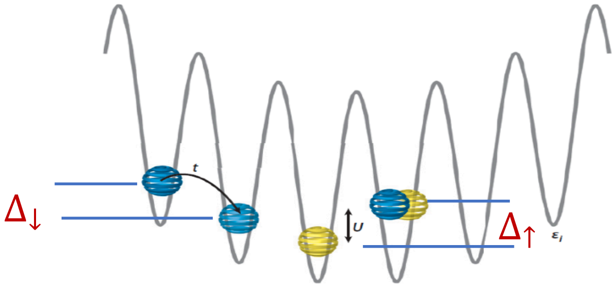
J^e : kinetic current

$$\mathrm{Re}(\sigma)(k=0,\omega) = \sigma' = 2\pi D\delta(\omega) + \sigma^{\mathrm{reg}}(\omega)$$

leading subleading

$$D = \sum\nolimits_k \frac{\langle J Q_k\rangle^2}{\langle Q_k^2\rangle}, D^c, D^s, D^e \ \ \ldots$$

dynamic process ⟷ equilibrium problem



$$H_0 = -\sum_{j=1}^{L}\sum_{a=\uparrow\downarrow}\left(c_{j,a}^{+}\, c_{j+1,a} + c_{j+1,a}^{+} c_{j,a}\right)$$

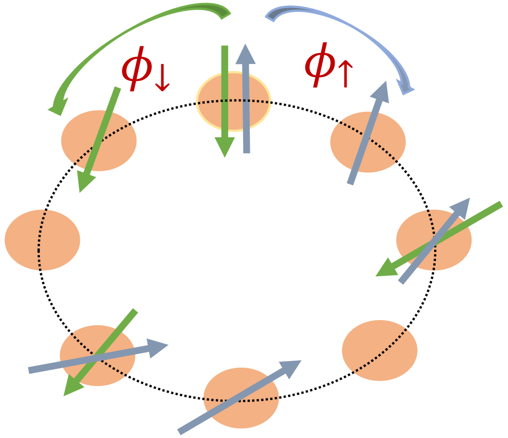
$$+u\sum_{j=1}^{L}(1-2n_{j\uparrow})(1-2n_{j\downarrow})$$

Breakdown integrability

$$+\Delta_\downarrow \sum_j j\, n_{j\downarrow} + \Delta_\uparrow \sum_j j\, n_{j\uparrow}$$

Nichols et. al., Science 363, 383 (2019)

Inducing flux for spin & charge: Two U(1) symmetries



$$H = -\sum_{j=1}^{L}\sum_{a=\uparrow\downarrow}\left(e^{i\phi_a/L}c^{+}_{j,a}\,c_{j+1,a} + \mathrm{H.c.}\right) + u\sum_{j=1}^{L}(1-2n_{j\uparrow})(1-2n_{j\downarrow})$$

Twisted boundary condition: $c^{\dagger}_{L+1,a} = e^{i\phi_a}c^{\dagger}_{1,a}$

$$e^{ik_jL} = \boxed{e^{i\phi_\uparrow}}\prod_{L=1}^{M}\frac{\lambda_l - \sin k_j - iu}{\lambda_l - \sin k_j + iu}$$

$$\boxed{e^{i(\phi_\uparrow-\phi_\downarrow)}}\prod_{j=1}^{N}\frac{\lambda_l - \sin k_j - iu}{\lambda_l - \sin k_j + iu} = -\prod_{m=1}^{M}\frac{\lambda_l - \lambda_m - 2iu}{\lambda_l - \lambda_m + 2iu}$$

$$k_j = k_j^{\infty} + \frac{x_1}{L} + \frac{x_2}{L^2} + \frac{x_3}{L^3} + \frac{x_4}{L^4}\ldots$$

$$\Lambda_\alpha^{n} = \Lambda_\alpha^{n\infty} + \frac{y_{1n}}{L} + \frac{y_{2n}}{L^2} + \frac{y_{3n}}{L^3} + \frac{y_{4n}}{L^4}\ldots$$

$$\Lambda_\alpha'^{n} = \Lambda_\alpha'^{n\infty} + \frac{z_{1n}}{L} + \frac{z_{2n}}{L^2} + \frac{z_{3n}}{L^3} + \frac{z_{4n}}{L^4}\ldots$$

$$\frac{\boldsymbol{E}}{\boldsymbol{L}} = \boldsymbol{E_0} + \frac{\boldsymbol{E_1}}{\boldsymbol{L}} + \frac{\boldsymbol{E_2}}{\boldsymbol{L}} + \frac{\boldsymbol{E_3}}{\boldsymbol{L}} + \ldots$$

$$\boldsymbol{D^{(l)}} = \frac{\boldsymbol{L^l}}{\boldsymbol{2Z}}\sum\nolimits_{l}\boldsymbol{e^{-\beta E_i}}\frac{\boldsymbol{\partial^{l+1}E_i}}{\boldsymbol{\partial\phi^{l+1}}}\bigg|_{\boldsymbol{\phi_{c,s}=0}} \quad \left\{\begin{array}{ll} (n_\downarrow + n_\uparrow) \rightarrow D^{c}\!: \phi_\uparrow = \phi_\downarrow = \phi_c & \text{for charge} \\ (n_\uparrow - n_\downarrow) \rightarrow D^{s}\!: \phi_\uparrow = -\phi_\downarrow & \text{for spin} \end{array}\right.$$

Linear $D^{(1)} \sim \langle J(t_1)J(t_2)\rangle$

Nonlinear $D^{(3)} \sim \langle J(t_1)J(t_2)J(t_3)J(t_4)\rangle$

$D^{(N)} \sim \langle J(t_1)\ldots\ldots J(t_{N+1})\rangle$

Luo, Pu, Guan, PRB **107**, L201103 (2023)
Luo, Pu, Guan, arXiv: 2307.00890
Guan, Yang, Nucl. Phys. B 512, 601 (1998)

Dressed charge: q_α^{dr}

$$\boxed{D^{c,s} = \frac{1}{2T}\sum_{\alpha=k,\Lambda,k-\Lambda}\int d\theta_\alpha \rho_\alpha (1-n_\alpha) v_\alpha^2 \underline{\left(2\pi(\rho_\alpha+\rho_\alpha^h)\frac{dx_\alpha}{d\phi}\right)}^2}$$

$$\boxed{\begin{aligned} x_1 &= g_1^x\phi;\ x_n = g_n^x\phi^n/n! \\ y_1 &= g_1^y\phi;\ x_n = g_n^y\phi^n/n! \\ z_1 &= g_1^z\phi;\ x_n = g_n^z\phi^n/n! \\ \theta_\gamma &= \theta_\gamma^\infty + \frac{x_{\gamma\,1}}{L} + \frac{x_{\gamma\,2}}{L^2} + \frac{x_{\gamma\,3}}{L^3} + \cdots \end{aligned}}$$

$$D = \int d\theta \rho(\theta)(1-n(\theta))v^{\text{eff}}(\theta)^2 q^{dr}(\theta)^2$$

bare charge q $\xrightarrow[\text{dressing}]{\text{interaction}}$ $\boxed{q_a^{\text{dr}} = \text{sign}(p_a')2\pi(\rho_a+\rho_a^h)\frac{dx_a}{d\phi}}$

New result

$$\text{sign}(p_a'(\theta)) = 1, 1, -1 \text{ for } k, \Lambda, k-\Lambda$$

numerically

$$q_a^{\text{dr}} = (\text{I} - \boldsymbol{B})_{ab}^{-1} * q_b$$

related to TBA kernels

$$\boldsymbol{B} = \begin{bmatrix} 0 & [a_n(\sin k-\Lambda)n_n]\,|_{1\times N} & -[a_n(\sin k-\Lambda)n_m']\,|_{1\times M} \\ \cos k\,[a_n(\sin k-\Lambda)n_k]\,|_{N\times 1} & \left[-\frac{1}{2\pi}\left(\frac{\partial}{\partial\Lambda}\Theta_{nm}\left(\frac{\Lambda-\Lambda'}{u}\right)\right)n_m\right]|_{N\times N} & 0|_{N\times M} \\ \cos k\,[a_n(\sin k-\Lambda)n_k]\,|_{M\times 1} & 0|_{M\times N} & \left[-\frac{1}{2\pi}\left(\frac{\partial}{\partial\Lambda}\Theta_{nm}\left(\frac{\Lambda-\Lambda'}{u}\right)\right)n_m'\right]|_{M\times M} \end{bmatrix}$$

Bare charges q

q_a^{bare}: particle number , magnetization number, energy

$$\begin{array}{llll}
k: & o_k = 1 & m_k = 1/2 & e_k = -2\cos k - \mu - 2u - B \\
\Lambda: & o_{n|\Lambda} = 0 & m_{n|\Lambda} = -n & e_{n|\Lambda} = 2nB \\
k-\Lambda: & o_{n|k-\Lambda} = 2n & m_{n|k-\Lambda} = 0 & e_{n|k\Lambda} = 4\mathrm{Re}\sqrt{1-(\Lambda - \mathrm{i}nu)^2} - 2n\mu - 4nu
\end{array}$$

$$q_a^{\text{dr}} = (\mathrm{I} - \boldsymbol{B})^{-1}_{ab} * q_a^{\text{bare}}$$

Dressed charges q^{dr} at T=0 ($k-\Lambda$ strings are gapped)

$$\begin{aligned}
q_k^{\text{dr}} &= 1 + \int_{-A}^{A} \mathrm{d}\Lambda a_1(\sin k - \Lambda) q_\Lambda^{\text{dr}} \\
q_\Lambda^{\text{dr}} &= \alpha + \int_{-Q}^{Q} \mathrm{d}k \cos k a_1(\Lambda - \sin k) q_k^{\text{dr}} - \int_{-A}^{A} \mathrm{d}\Lambda' a_2(\Lambda - \Lambda') q_\Lambda^{\text{dr}}
\end{aligned}$$

$\alpha = 0, -2$ for charge and spin transport

Beyond the bosonization result: finite magnetic field at T=0

New Result

Bosonization at $H = 0$

$$D^c = \frac{K_c v_c}{\pi}, \chi^c = \frac{2K_c}{\pi v_c}$$

$$D^s = \frac{K_s v_s}{\pi}, \chi^s = \frac{K_s}{2\pi v_s}$$

spin rotation symmetry $K_s = 1$

Drude weight

at $H \neq 0, \mu \neq 0$

for Phase IV

$$D^c = \frac{1}{2\pi} q_k^{c,\mathrm{dr}\,2} v_k|_Q + \frac{1}{2\pi} q_\Lambda^{c,\mathrm{dr}\,2} v_\Lambda|_A$$

$$D^s = \frac{1}{2\pi} q_k^{s,\mathrm{dr}\,2} v_k|_Q + \frac{1}{2\pi} q_\Lambda^{s,\mathrm{dr}\,2} v_\Lambda|_A$$

Contributions from another degrees of states

$\{Z_{\alpha\beta}\}$ **are the dressed charges**

$$\mathrm{Z} = \begin{pmatrix} \xi_{cc}(Q) & \xi_{cs}(A) \\ \xi_{sc}(Q) & \xi_{ss}(A) \end{pmatrix}$$

$$\xi_{ab}(x_b) = \delta_{ab} + \sum_d \int_{-X_d}^{X_d} \mathrm{d}x_d \xi_{ad}(x_d) K_{db}(x_d, x_b)$$

Susceptibility

at $H \neq 0$

$$\chi_c|_B = \frac{\mathrm{Z}_{cc}^2}{\pi v_c} + \frac{\mathrm{Z}_{cs}^2}{\pi v_s},$$

$$\chi_s|_\mu = \frac{(\mathrm{Z}_{cc} - 2\mathrm{Z}_{sc})^2}{4\pi v_c} + \frac{(\mathrm{Z}_{cs} - 2\mathrm{Z}_{ss})^2}{4\pi v_s}$$

General result:

arbitrary H, μ

For all phases

$$D^c = \frac{K_c v_c}{\pi} + \frac{K_{cs} v_s}{\pi}, \chi^c = \frac{2K_c}{\pi v_c} + \frac{2K_{cs}}{\pi v_s}$$

$$D^s = \frac{K_s v_s}{\pi} + \frac{K_{sc} v_c}{\pi}, \chi^s = \frac{K_s}{2\pi v_s} + \frac{K_{sc}}{2\pi v_c}$$

$q_k^{c,dr} = \xi_{cc}$, $q_\Lambda^{c,dr} = \xi_{cs}$

$q_k^{s,dr} = \xi_{cc} - 2\xi_{sc}$, $q_\Lambda^{s,dr} = \xi_{cs} - 2\xi_{ss}$

Crossing Luttinger parameters: K_{cs}, K_{sc}

Luttinger parameters v.s. Dressed charges

Cover the bosonization result at vanishing magnetic field.

Phase II $$K_c = {q_k^{c,dr}}^2 = \mathrm{Z}_{cc}^2 = 1$$ free lattice

Phase V $$K_s = \frac{{q_\Lambda^{s,dr}}^2}{4} = \mathrm{Z}_{ss}^2 \xrightarrow{h=0} \frac{1}{2}$$ spin chain

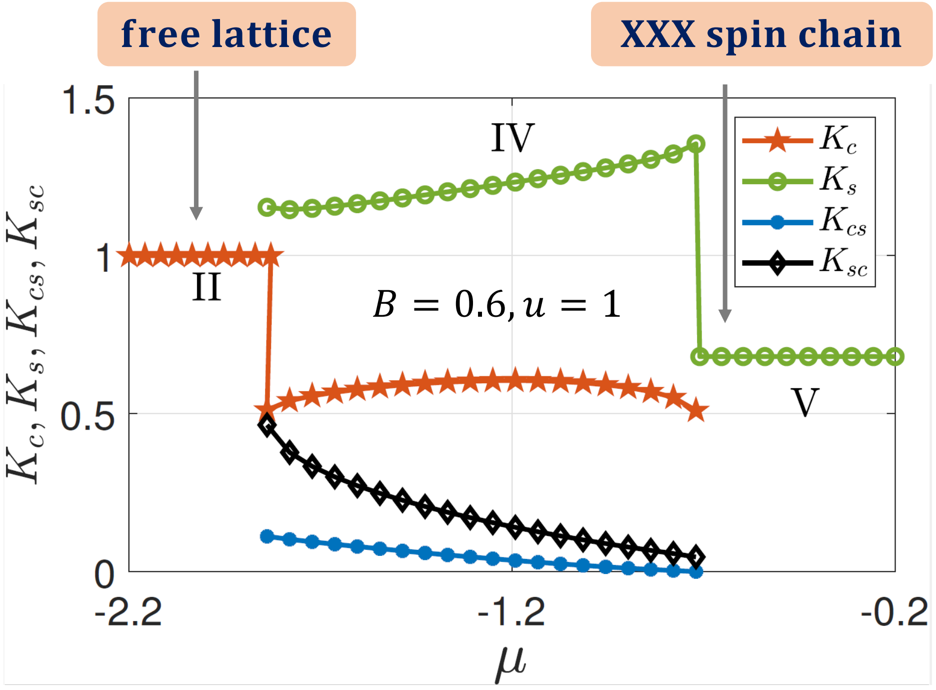
Phase IV $$K_c = \frac{{q_k^{c,dr}}^2}{2} = \frac{\mathrm{Z}_{cc}^2}{2}$$

$$K_s = \frac{{q_\Lambda^{s,dr}}^2}{2} = \frac{(\mathrm{Z}_{cs} - 2\mathrm{Z}_{ss})^2}{2} \xrightarrow{h=0} 1$$

$$K_{cs} = \frac{{q_\Lambda^{c,dr}}^2}{2} = \frac{\mathrm{Z}_{cs}^2}{2} \xrightarrow{h=0} 0$$

$$K_{sc} = \frac{{q_k^{s,dr}}^2}{2} = \frac{(\mathrm{Z}_{cc} - 2\mathrm{Z}_{sc})^2}{2} \xrightarrow{h=0} 0$$

$\{Z_{\alpha\beta}\}$ are the dressed charge



For fixed B and u

Dressed charges at infinite interaction

Drude Weights displays a feature of spin charge coupling!

$$D^c = \frac{K_c v_c}{\pi} + \cancel{\frac{K_{cs} v_s}{\pi}}$$

$$D^s = \cancel{\frac{K_s v_s}{\pi}} + \frac{K_{sc} v_c}{\pi}$$

$$v_s \xrightarrow{c=\infty} 0$$

$$v_c \xrightarrow{c=\infty} 2\sin(\pi n_c)$$

Phase IV

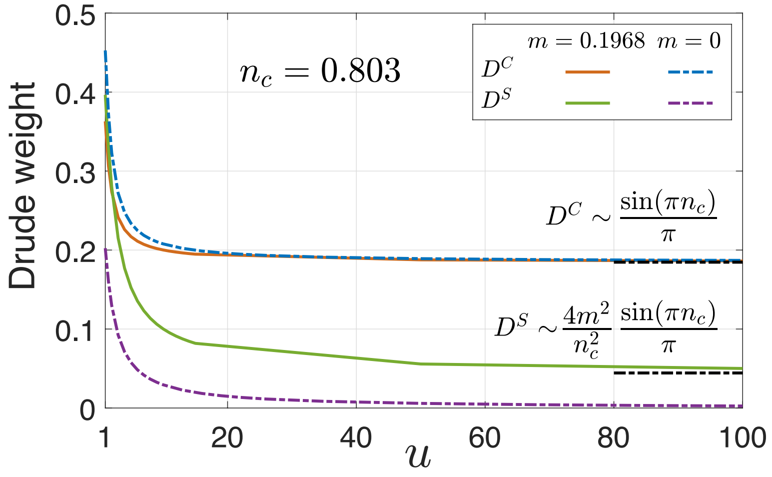
$$K_c = \frac{{q_k^{c,dr}}^2}{2} = \frac{Z_{cc}^2}{2} \xrightarrow{c=\infty} \frac{1}{2}$$

$$K_s = \frac{{q_\Lambda^{s,dr}}^2}{2} = \frac{(Z_{cs} - 2Z_{ss})^2}{2}$$

$$K_{cs} = \frac{{q_\Lambda^{c,dr}}^2}{2} = \frac{Z_{cs}^2}{2} \xrightarrow{c=\infty} 0$$

Subtle spin polarization!

$$K_{sc} = \frac{{q_k^{s,dr}}^2}{2} = \frac{(Z_{cc} - 2Z_{sc})^2}{2} \xrightarrow{c=\infty} \frac{2m^2}{n_c^2}$$



DWs v.s. interaction for fixed n and m

DWs essentially depends on polarization and filling factor!

Linear Drude weight at finite temperature

New Result

- **Dressed charges:**

$$q_a^{\mathrm{dr}} = \mathrm{sign}(p'_a)2\pi(\rho_a + \rho_a^h)\frac{dx_a}{d\phi}$$

Density, magnetization, energy

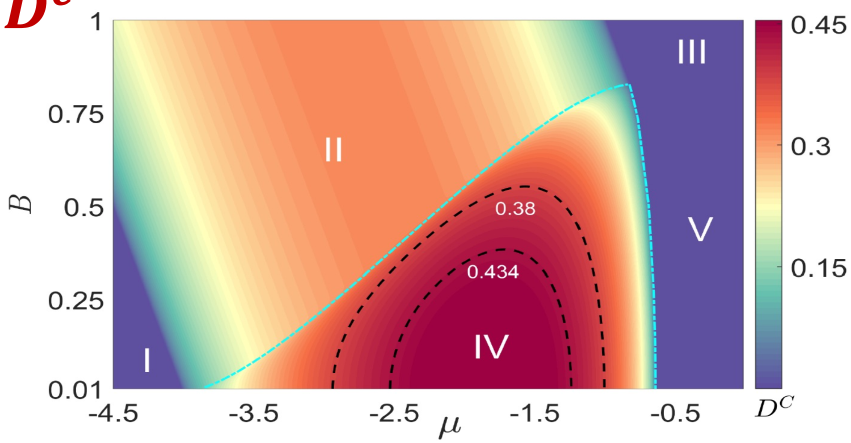
- **Universal laws in TLL**

$k - \Lambda$ **strings has no contribution**

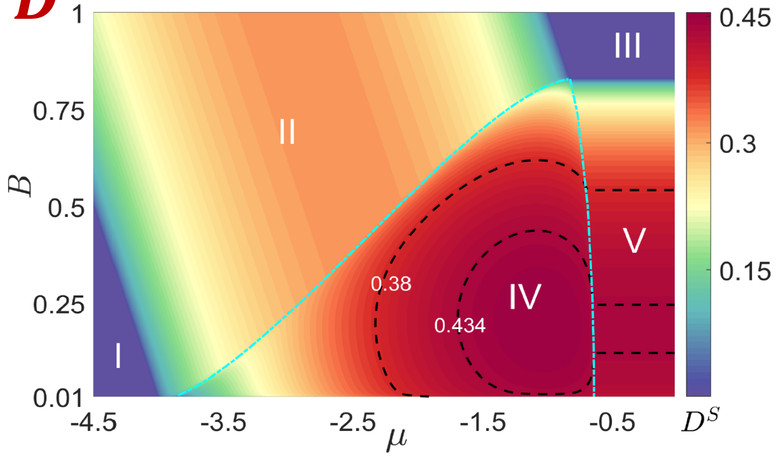
$$D^{c,s} = \sum_{a=k,\Lambda,k-\Lambda} \left\{ \frac{1}{2\pi} \left(q_a^{c,s\,\mathrm{dr}}\right)^2 v_a \Big|_Q + \frac{\pi T^2}{12} \frac{\partial^2}{\partial \epsilon^2} \left[\left(q_a^{c,s\;dr}\right)^2 v_a \right] \Big|_{\varepsilon(Q)=0} \right\}$$

- **Phase diagram: characteristic of Luttinger liquid**

D^c



D^s



Universal scaling laws at quantum criticality

New Result

- **II-IV** $D_\Lambda = \frac{2}{\sigma(0)}\left(\frac{q_\Lambda^{\mathrm{dr}}(0)}{2\pi}\right)^2\left(\frac{\varepsilon''(0)}{2}\right) \quad f_{1/2}$

  $D_k = D^0\left\{1 - f_{1/2}\frac{\sigma(0)q_\Lambda^{\mathrm{dr}\prime}(0) - 2\sigma'(0)q_\Lambda^{\mathrm{dr}}(0)}{\rho^0 q_\mathrm{k}^{\mathrm{dr}0}}\right\} + \frac{\pi^2 T^2}{6}\frac{\partial^2}{\partial\kappa^2}D^0\Big|_{\kappa(k)=0}$

- **IV-V** $D_\mathrm{k} = \frac{2}{\rho(\pi)}\left(\frac{q_\mathrm{k}^{\mathrm{dr}}(\pi)}{2\pi}\right)^2\left(-\frac{\kappa''(\pi)}{2}\right) \quad k_{1/2}$

  $D_\Lambda = D^0\left\{1 + k_{1/2}\frac{\rho(\pi)q_\mathrm{k}^{\mathrm{dr}\prime}(\pi) - 2\rho'(\pi)q_\mathrm{k}^{\mathrm{dr}}(\pi)}{\sigma^0 q_\Lambda^{\mathrm{dr}0}}\right\} + \frac{\pi^2 T^2}{6}\frac{\partial^2}{\partial\varepsilon^2}D^0\Big|_{\varepsilon(\Lambda)=0}$

General !

- **I-II** $D_\mathrm{k} = \frac{2}{\rho(0)}\left(\frac{q_\mathrm{k}^{\mathrm{dr}}(0)}{2\pi}\right)^2\left(\frac{\kappa''(0)}{2}\right) \quad \bar{k}_{1/2}$
- **II-III** $D_\mathrm{k} = \frac{2}{\rho(\pi)}\left(\frac{q_\mathrm{k}^{\mathrm{dr}}(\pi)}{2\pi}\right)^2\left(-\frac{\kappa''(\pi)}{2}\right) \quad k_{1/2}$
- **III-V** $D_\Lambda = \frac{2}{\sigma(0)}\left(\frac{q_\Lambda^{\mathrm{dr}}(0)}{2\pi}\right)^2\left(\frac{\varepsilon''(0)}{2}\right) \quad f_{1/2}$

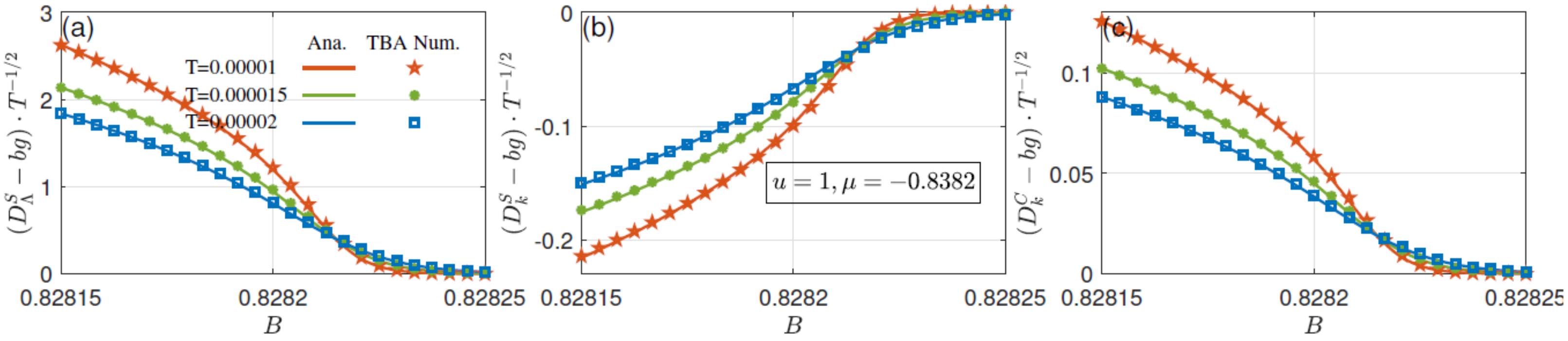
Also applied to other systems

$$f_{1/2} = -\frac{T^{1/2}}{2}\left(\frac{\varepsilon''(0)}{2}\right)^{-\frac{1}{2}}\pi^{\frac{1}{2}}\mathrm{Li}_{\frac{1}{2}}\left(-\mathrm{e}^{-\varepsilon(0)/T}\right)$$

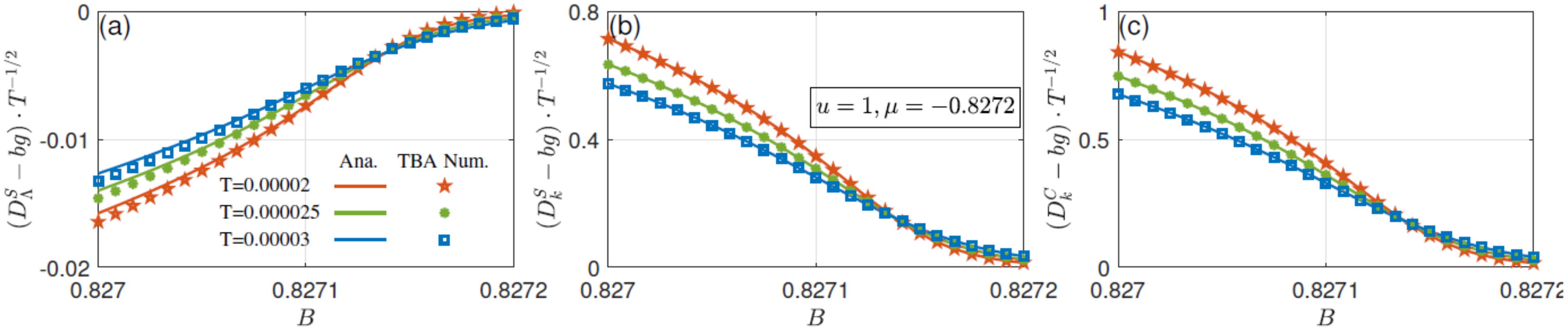
$$k_{1/2} = -\frac{T^{1/2}}{2}\left(-\frac{\kappa''(\pi)}{2}\right)^{-\frac{1}{2}}\pi^{\frac{1}{2}}\mathrm{Li}_{\frac{1}{2}}\left(-\mathrm{e}^{\kappa(\pi)/T}\right)$$

$$\bar{k}_{1/2} = -\frac{T^{1/2}}{2}\left(\frac{\kappa''(0)}{2}\right)^{-\frac{1}{2}}\pi^{\frac{1}{2}}\mathrm{Li}_{\frac{1}{2}}\left(-\mathrm{e}^{-\kappa(0)/T}\right)$$

Universal scaling for phase transition from II to IV



Universal scaling for phase transition from IV to V



Excellent agreement between numerical and analytical results!

Nonlinear Drude weight

- **Universal laws at ground state**

$$D^{(3)} = \sum \frac{\partial^2}{\partial \varepsilon^2} \underbrace{[2\rho^T g_1^4 \dot{\varepsilon}^3]}_{C} \Big|_{\varepsilon=0} - \frac{\partial}{\partial \varepsilon} \underbrace{[12\rho^T (g_1^4 \dot{\varepsilon}\ddot{\varepsilon} + 2 g_1^2 g_2 \dot{\varepsilon}^2)]}_{B} \Big|_{\varepsilon=0}$$
$$+\underbrace{2\rho^T[(12 g_2^2 + 24 g_1 g_3)\dot{\varepsilon} + 36 g_1^2 g_2 \ddot{\varepsilon} + 4 g_1^4 \dddot{\varepsilon} + 3 g_1^4 \ddot{\varepsilon}^2/\dot{\varepsilon}]}_{A} \Big|_{\varepsilon=0}$$

$$2\pi\rho^t g_1 = q^{dr},$$
$$g_n = g_1 \partial g_{n-1}/n$$

$x_1 = g_1\phi;\ x_n = g_n \phi^n/n!$

$v, m, \lambda \ldots q, \dot{q}, \ddot{q} \ldots \rho, \dot{\rho}$ ...for nonlinear DW

v: $d\varepsilon/dp$ velocity
m: $d^2\varepsilon/dp^2$ mass
λ: $d^3\varepsilon/dp^3$?

$$C = \frac{q^4 v^3}{\pi}, B = \frac{6 q^2 v}{\pi}\left(\frac{q^2}{m} + \frac{q\dot{q}v}{2\pi\rho}\right)$$

$$A = \frac{q^3}{\pi}\left(4q\lambda + \frac{3q}{m^2 v} + \frac{9\dot{q}}{\pi\rho m}\right) + \frac{q^2 v}{\pi(2\pi\rho)^2}\left(7\dot{q}^2 + 4q\ddot{q} - \frac{4q\dot{q}\dot{\rho}}{\rho}\right)$$

Linear DW only depends on *q* and *v*

$$D^{(1)} = \sum \frac{1}{2\pi}\left(q_a^{\mathrm{dr}}\right)^2 v_a \Big|_Q$$

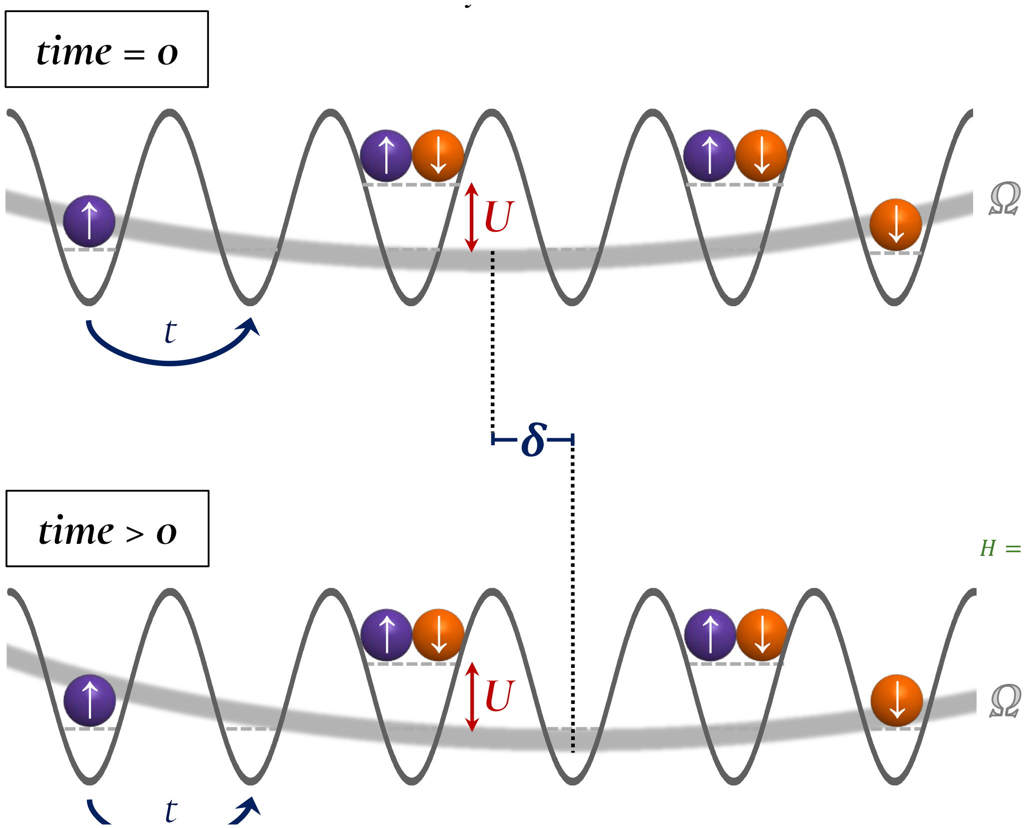
- **Universal laws in TLL area**

$$D^{(3)} = \underrightarrow{\left[A - \frac{\partial B}{\partial \varepsilon} + \frac{\partial^2 C}{\partial \varepsilon^2}\right]} \Big|_{\varepsilon=0} + \frac{\pi^2 T^2}{6} \frac{\partial^2}{\partial \epsilon^2} \underrightarrow{\left[A - \frac{\partial B}{\partial \varepsilon} + \frac{\partial^2 C}{\partial \varepsilon^2}\right]} \Big|_{\varepsilon=0}$$

The general features in linear and nonlinear Drude weight

Conjecture

	linear	three order nonlinear	higher order nonlinear
T=0 results	$D_0^{(1)}$	$D_0^{(3)}$	$D_0^{(l)}, l > 3$
parameters	q^{dr}, v	$\boldsymbol{q^{dr}}, \dot{q}^{dr}, \ddot{q}^{dr}; \boldsymbol{v}, m, \lambda; \rho, \dot{\rho}$	$\partial^{l-1}q^{dr}, d^l\varepsilon/dp^l, \partial^{l-2}\rho$
TLL areas	$D_0^{(1)} + \frac{\pi^2 T^2}{6}\frac{\partial^2}{\partial\epsilon^2}D_0^{(1)}$	$D_0^{(3)} + \frac{\pi^2 T^2}{6}\frac{\partial^2}{\partial\epsilon^2}D_0^{(3)}$	$D_0^{(l)} + \frac{\pi^2 T^2}{6}\frac{\partial^2}{\partial\epsilon^2}D_0^{(l)}$

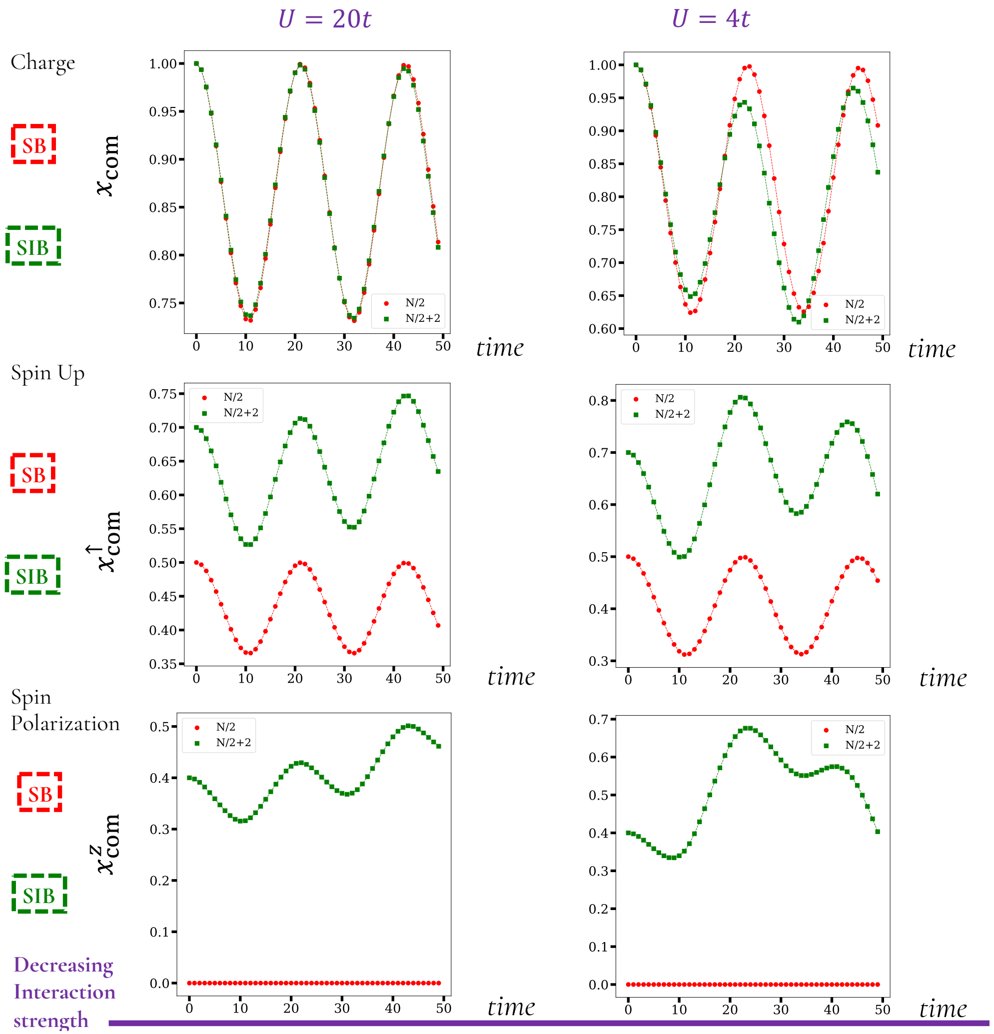


Motion of the center of the charge and spin

$$H_0 = -\sum_{j=1}^{L}\sum_{a=\uparrow\downarrow}\left(c^{+}_{j,a}\, c_{j+1,a} + c^{+}_{j+1,a} c_{j,a}\right)$$
$$+u\sum_{j=1}^{L}(1-2n_{j\uparrow})(1-2n_{j\downarrow})$$
$$+\Omega \textstyle\sum_{j=1}^{L} x_j^2\, n_j$$
$$x_j = \frac{L}{2} - j$$

Quench

$$H_t = -\sum_{j=1}^{L}\sum_{a=\uparrow\downarrow}\left(c^{+}_{j,a}\, c_{j+1,a} + c^{+}_{j+1,a} c_{j,a}\right)$$
$$+u\sum_{j=1}^{L}(1-2n_{j\uparrow})(1-2n_{j\downarrow})$$
$$+\Omega \textstyle\sum_{j=1}^{L}\left(x_j - \delta\right)^2 n_j$$
$$x_j = \frac{L}{2} - j$$

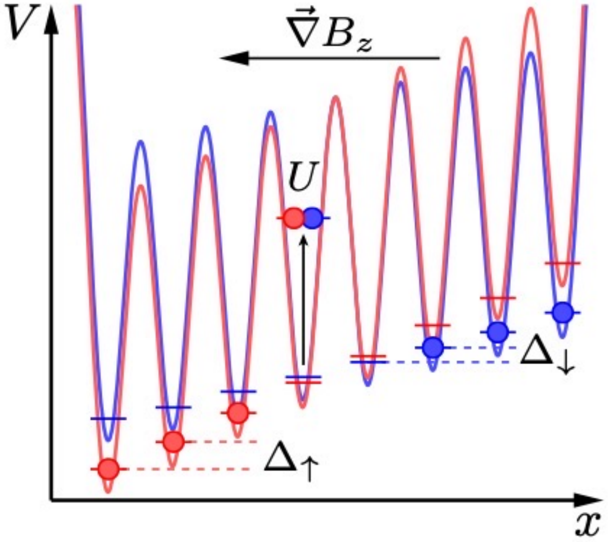


Motion of the center of the charge and spin

- $x_{\text{com}} = \sum_j \frac{x_j n_j}{N};$
- $x^{\uparrow}_{\text{com}} = \sum_j \frac{x_j n_j^{\uparrow}}{N};$
- $x^{z}_{\text{com}} = \sum_j \frac{x_j (n_j^{\uparrow} - n_j^{\downarrow})}{N}$
- where: $x_j = {}^{L}/_{2} - j$

Spin transport in a Mott insulator in Hubbard model

$$\hat{H} = -t \sum_{\langle i,j \rangle,\sigma} \left(\hat{c}^{\dagger}_{\sigma,i} \hat{c}_{\sigma,j} + h.c. \right) + U \sum_{i} \hat{n}_{\uparrow,i} \hat{n}_{\downarrow,i}$$
$$-\mu_{\uparrow} \sum_{i} \hat{n}_{\uparrow,i} - \mu_{\downarrow} \sum_{i} \hat{n}_{\downarrow,i}$$
$$+\Delta_{\uparrow} \sum_{i} i_x \hat{n}_{\uparrow,i} + \Delta_{\downarrow} \sum_{i} i_x \hat{n}_{\downarrow,i}.$$



Observation of spin conduction and spin diffusion

$\Delta_{\uparrow\downarrow}$ present spin dependent tilt of the potential along x-direction

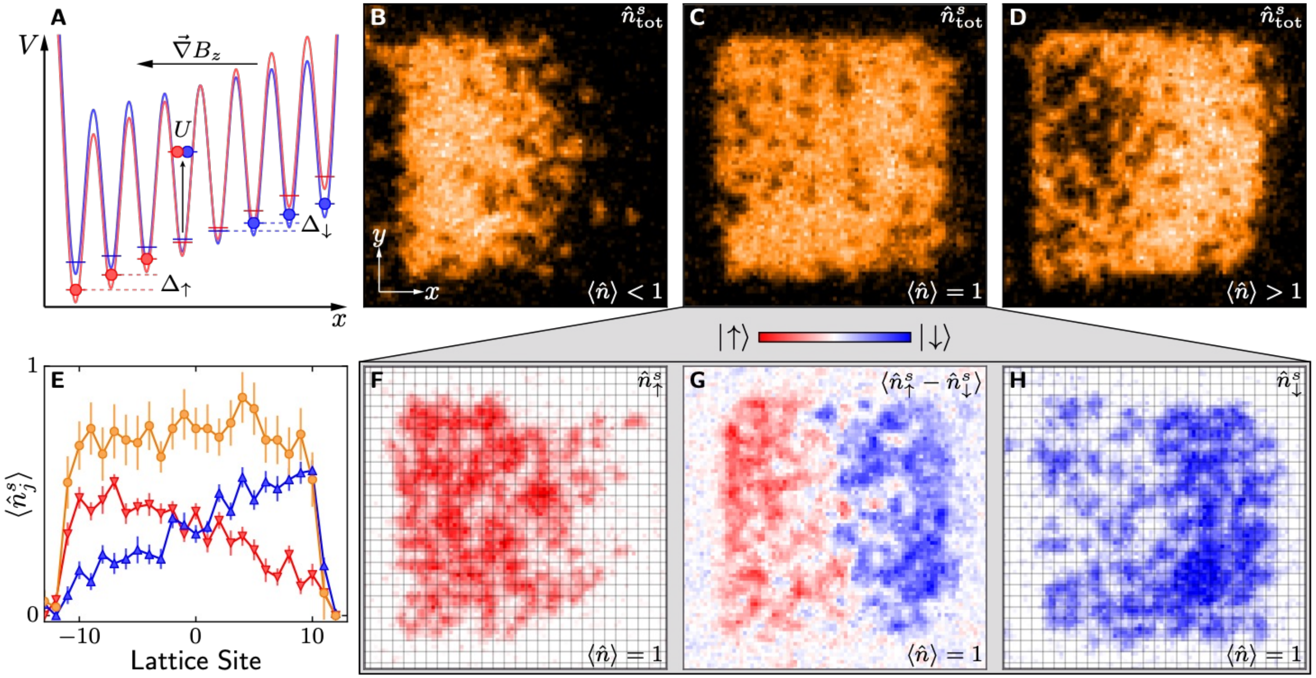
$$\frac{\Delta_{\uparrow}}{h} = \sim 41.1Hz; \ \frac{\Delta_{\downarrow}}{h} = \sim 15.4\ Hz; \frac{t}{h} \sim 100Hz; \frac{U}{h} \sim 1kHz$$

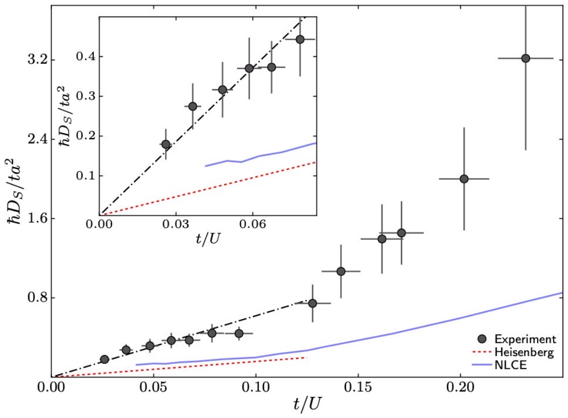
Nichols et. al., Science 363, 383 (2019)

Aims:

Spin diffusion coefficients $\boldsymbol{D_s}$

Spin conducting $\boldsymbol{\sigma_s}$





$\Delta_{\uparrow\downarrow}$ present spin dependent tilt

$$\frac{\Delta_\uparrow}{h} = \sim 41.1Hz; \quad \frac{\Delta_\downarrow}{h} = \sim 15.4\ Hz;$$

$$\frac{t}{h} \sim 100Hz; \quad \frac{U}{h} \sim 1kHz$$

$$\hat{H} = -t \sum_{\langle i,j \rangle, \sigma} \left(\hat{c}^\dagger_{\sigma,i} \hat{c}_{\sigma,j} + h.c. \right) + U \sum_i \hat{n}_{\uparrow,i} \hat{n}_{\downarrow,i}$$

$$-\mu_\uparrow \sum_i \hat{n}_{\uparrow,i} - \mu_\downarrow \sum_i \hat{n}_{\downarrow,i}$$

$$+\Delta_\uparrow \sum_i i_x \hat{n}_{\uparrow,i} + \Delta_\downarrow \sum_i i_x \hat{n}_{\downarrow,i}.$$

$$\mathcal{I}(\tau) = \sum_L \left\langle \hat{S}_{z,j}(\tau) \right\rangle - \sum_R \left\langle \hat{S}_{z,j}(\tau) \right\rangle$$

$$J_S = D_S \nabla \left\langle \hat{S}_{z,j=0} \right\rangle$$

Einstein relation: $\sigma_S = D_S \chi$

Nichols et al. 363, 383 (2019)

Conclusion and discussion

1. The 1D repulsive Hubbard model exhibits novel phases of Luttinger liquids and phase transitions driven by either external potentials or interaction.
2. The spin and charge Drude weights at low temperature have been analytically obtained, showing universal ballistic transport with spin polarization.
3. We have built up exact relations between Luttinger parameters and dressed charges.
4. The universal scaling laws of the Drude weight at quantum criticality obtained shed light on non-Fermi liquid behaviour.

The decade-old 1D Hubbard model continues to yield new and exciting physics!

Thanks for your listening!