Defects and Seiberg-Witten Curves in 6d SCFTs of Class $\mathcal{S}_k$

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based on an upcoming work jointly with Marcus Sperling

Motivation: What is a (quantum) Seiberg-Witten curve

- A classical algebraic curve

$$\mathcal{C} = \{(x,y) \in \mathbb{C}\times\mathbb{C} | \, \mathcal{H}(x,y) = 0\} \,, \quad \lambda_{\mathrm{SW}} = y\mathrm{d}x$$

  arise in the context of 4d $\mathcal{N}=2$ Supersymmetric Yang-Mills gauge theories, when Seiberg and Witten study the BPS spectra of their low-energy physics. Especially, the masses of dyons are captured by the cycles of the algebraic curve $\mathcal{C}$, now known as the Seiberg-Witten curve. An typical example: the 4d pure $\mathcal{N}=2$ $\mathrm{SU}(N)$ gauge theory has SW-curve

$$y + \frac{\Lambda^{2N}}{y} - \mathcal{W}(x,u_i) = 0$$

  where $W(x,u_i)$ is a polynomial of degree N in x, with coefficients dependent on u_i, the Coulomb moduli parameters.

- Nekrasov and Okounkov later developed powerful localization methods to compute the so-called Nekrasov instanton partition function under $\Omega_{\epsilon_{1,2}}$-background,

$$\mathcal{Z}^{\mathrm{4d}}_{\mathrm{inst.}}(\epsilon_{1,2};\mathfrak{q}) = \sum_{k=0}^{\infty} \mathfrak{q}^k \oint_{\widetilde{\mathcal{M}_k(\epsilon_{1,2})}} 1 \,, \quad \text{and} \quad \mathcal{F} = \epsilon_1\epsilon_2 \log \mathcal{Z}^{\mathrm{4d}}_{\mathrm{inst.}}$$

  The SW-curve can be extracted from the saddle point eq. of the instanton integral by taking $\epsilon_{1,2}\to 0$.

- In this picture, before taking the limit $\epsilon_{1,2} \to 0$, the Seiberg-Witten curve is ϵ-deformed, e.g.

$$\mathcal{W}(x, u_i, \epsilon_{1,2}) = \mathcal{Y}(x) + \frac{\Lambda^{2N}}{\mathcal{Y}(x + \epsilon_1 + \epsilon_2)}$$

  where $\mathcal{W}(x, u_i, \epsilon_{1,2})$ is known as *qq*-characters of the chiral ring, and the operator $\mathcal{Y}$ assigns certain sum rules to compute the instanton partition function $\mathcal{Z}^{\mathrm{4d}}_{\mathrm{inst.}}$, that can be regarded as an operator version of the chiral ring generating function.

- The physical meaning of $\mathcal{Y}$ is more clear, when we take the Nekrasov and Shatashvili limit (NS-limit) $\epsilon_2 \to 0$, while keeping $\epsilon_1 \equiv \hbar$. The $\mathcal{Y}$-operator can be interpreted as: One Inserts a mesonic codimensional two defect into the theory, computes the *normalized* 4d/2d coupled partition function, and $\mathcal{Y}(x)$ is given by taking the ratio

$$\Psi(x) \equiv \lim_{\epsilon_2 \to 0} \frac{\mathcal{Z}^{\mathrm{4d/2d}}_{\mathrm{inst.}}(x)}{\mathcal{Z}^{\mathrm{4d}}_{\mathrm{inst.}}}\,, \quad \text{and} \quad \mathcal{Y}(x) = \frac{\Psi(x-\hbar)}{\Psi(x)}\,,$$

  where x is now regarded as the mass of the defect.

- Now the ϵ-deformed curve can be recast as

$$\left(\hat{y} + \Lambda^{2N}\hat{y}^{-1}\right) \cdot \Psi(x) = \mathcal{W}(x; u_i, \hbar) \cdot \Psi(x)\,,$$

where $\hat{y} \equiv e^{-\hbar\partial_x}$ is understood as a shift operator satisfying non-trivial commutation relation with x,

$$[x, -\hbar\partial_x] = \hbar\,, \qquad \hat{y} \cdot x = x - \hbar$$

- The SW-curve $\mathcal{C}$ is thus quantized to a Hamiltonian operator

$$\hat{H}(x, \hat{y}) \equiv \hat{y} + \Lambda^{2N}\hat{y}^{-1}\,,$$

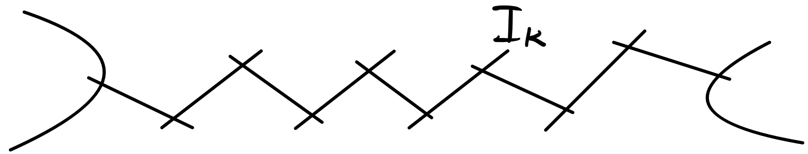
acting on a codim two defect $\Psi(x)$, with eigenvalue of the q-character $\mathcal{W}(x, \hbar)$,

$$\mathcal{H}(x, y) \longrightarrow \hat{\mathcal{H}}(x, \hat{y}) \equiv \hat{H}(x, \hat{y}) - \mathcal{W}(x)\,, \quad \hat{\mathcal{H}}(x, \hat{y}) \cdot \Psi(x) = 0\,.$$

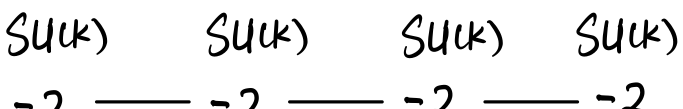
- Along this line, one can study the SW-curves in 5d and 6d SCFTs, from their instanton PFs on $\mathbb{R}^4 \times S^1$ for 5d, or $\mathbb{R}^4 \times T^2$ for 6d, to understand the moduli space of the vacua in these theories at strongly coupled regime.

- In this setup, the SW-curves naturally arise in the context of geometrical compactification of M/F-theory on a non-compact elliptic Calabi-Yau manifold.

  As the main example of this talk, consider the following elliptic CY 3-folds $\mathcal{M}$,



  where the discriminant locus is a linear chain of N "-2" curves, above which the fiber degenerates according to Kodaira I_k. Compactifying F-theory on it gives the 6d $\mathcal{N} = (1,0)$ SCFT of S_k class,



  where each "-2"-curve gives a tensor moduli, on it the singular fiber supports $\mathrm{SU}(k)$ gauge group.

- The (classical) SW-curve $\mathcal{C}$ of the $\mathcal{S}_k$ class is captured by the mirror Calabi-Yau $\widetilde{\mathcal{M}}$: a cone over the Riemann surface [Haghighat, Yan, Yau '17]

$$\mathcal{H}_{N,k}(x,y) = y^N + v_1(x) y^{N-1} + \cdots + v_N(x) = 0\,,$$

where the v_i are Jacobi forms in elliptic parameter x, giving

$$\mathcal{C} = \left\{ (x,y) \in \mathbb{T}^2 \times \mathbb{C}^* \,|\, \mathcal{H}(x,y) = 0 \right\}\,.$$

- Once again, quantization of the SW-curve can be done by introducing codimen 2 defects under NS-limit. The curve $\mathcal{H}(x,y)$ enhances to a difference operator,

$$\mathcal{H}(x,y) \longrightarrow \hat{\mathcal{H}}(x,\hat{y})$$

where the coefficients $v_i(x)$'s become $\hbar$-deformed q-characters $v_i(x,\hbar)$'s, and the *normalized* 6d/4d PF, $\Psi(x)$, gets annihilated by the quantum curve,

$$\hat{\mathcal{H}}(x,\hat{y}) \cdot \Psi(x) = 0$$

- Geometrically, the codim 2 defect corresponds to a special Lagrangian brane in the mirror CY $\widetilde{\mathcal{M}}$ intersecting the curve $\mathcal{C}$ at a point.

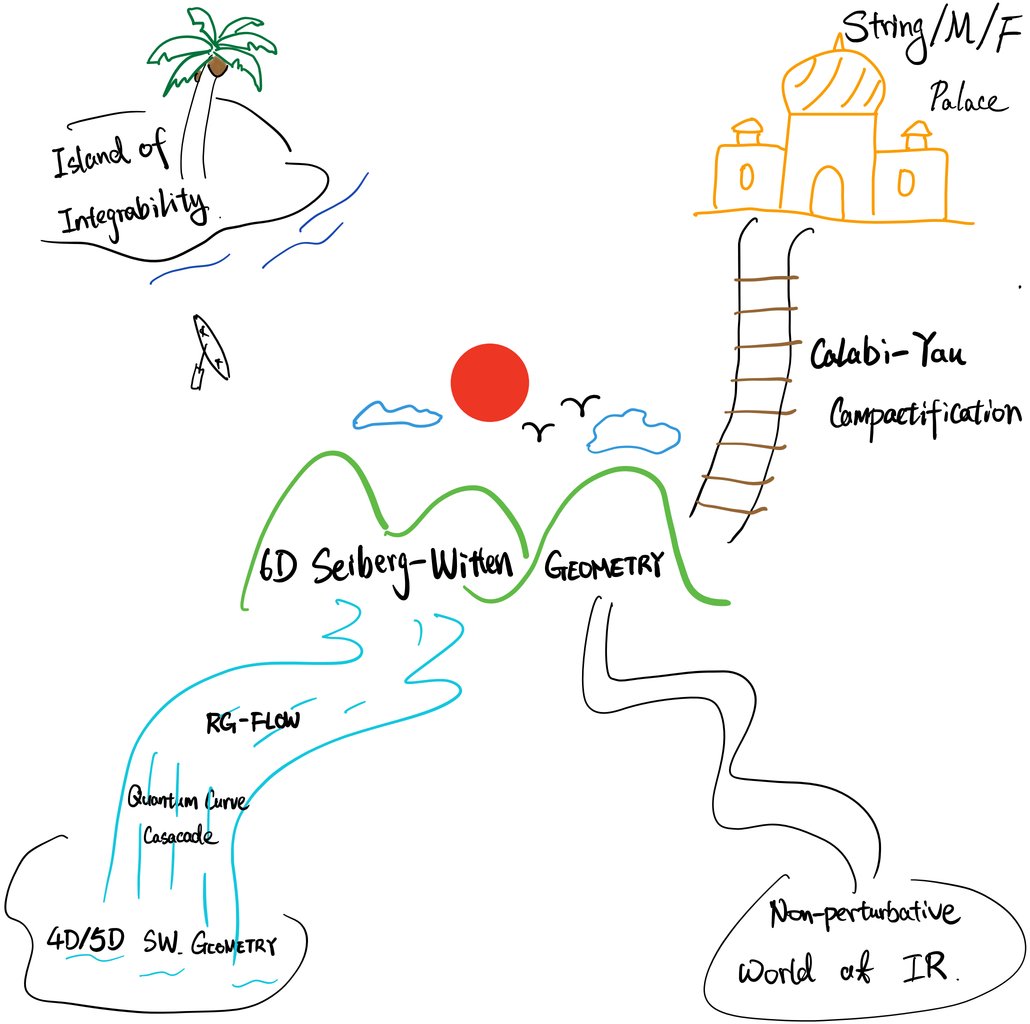
Motivation: Elliptic Integrable Systems from SW-curves

- On the other hand, the SW-curves $\mathcal{C}$ are remarkably identified to the spectral curves of various classical integrable systems. It therefore bridges two completely different research communities, the SYM gauge theories and integrable systems. (e.g. the SW-curve of 4d pure $\mathcal{N}=2$ $\mathrm{SU}(N)$ corresponds to the closed Toda system.)

- The quantization of the SW-curves therefore naturally define a procedure to quantize the associated integrable systems.

- The dictionary between the two fields are summarized,

Gauge/Integrable	Spectral curve	Wave function	Curve coefficients
$\mathbb{R}^4$	Rational	4d/2d defect PF	Chiral ring operators
$\mathbb{R}^4 \times \mathbb{S}^1$	Trigonometric	5d/3d defect PF	Wilson loops
$\mathbb{R}^4 \times \mathbb{T}^2$	Elliptic	6d/4d defect PF	Wilson surfaces


String/M/F
Palace
Island of
Integrability
Calabi-Yau
Compactification
6D Seiberg-Witten
GEOMETRY
RG-FLOW
Quantum Curve
Casacade
4D/5D SW GEOMETRY
Non-perturbative
World of IR.

- In the past years, we verified various ideas around 6d SW-curves:

  $\mathcal{S}_k$ with a single tensor: the SW-curve is established and identified with 2-body Ruijsennars-Schneider model; [JC, Haghighat, Kim, Sperling '20]

  E-string: the SW-curve is established identified with van Diejen model; [JC, Haghighat, Kim, Sperling, Wang '21]

  $\mathrm{SO}(N)$ on "-4"-curve: the SW-curve is established and RG-flow to its 5d circle compactification with $\mathbb{Z}_2$ twist; [JC, Haghighat, Kim, Lee, Sperling, Wang '21]

  $\mathrm{Sp}(N)$ on "-1"-curve: the SW-curve is established and identified with elliptic Garnier model, and RG-flow to various 5d SCFTs. [JC, Lv, Wang '23]

- **Questions:**

  Establish SW-curves for a generic 6d SCFT of quiver-like + classical gauge group;

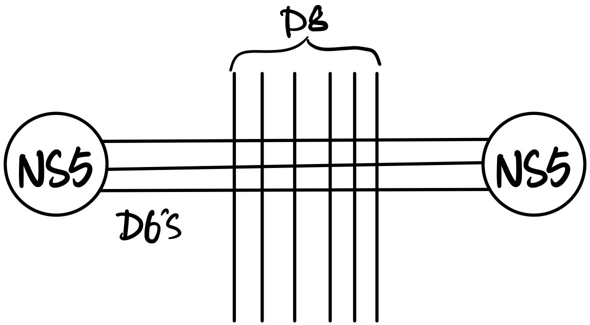
  RG-flow to various descendants from a known 6d SW-curves;

  Can a generic mesonic codim 2 defect engineer a SW-curve?

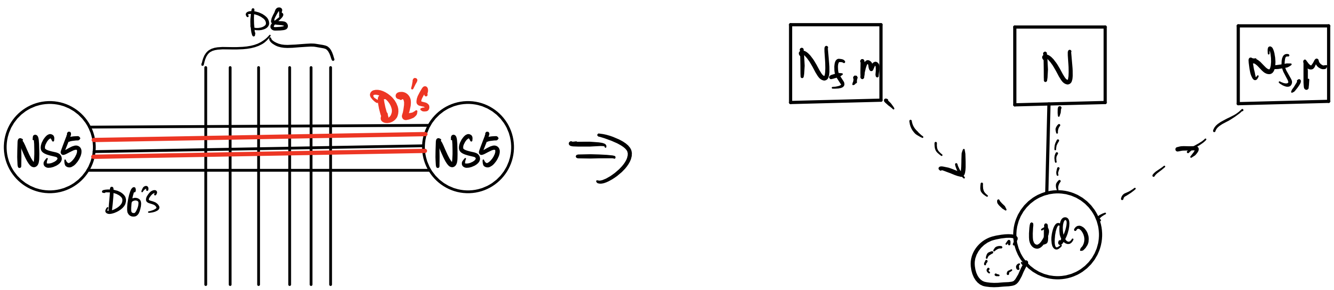
- Motivated by the proposed questions, I will explain how to establish the SW-curves for a generic SCFT of $\mathcal{S}_k$ class (A_{N-1}-type quiver+$\mathrm{SU}(k)$ gauge group) and its various higgsing hierarchies

- I will also discuss the associated SW-curves when there are *multiple* codim 2 defects introduced in $\mathcal{S}_k$ class theories.

- To study the instanton PF, we go to the frame of IIA-theory, where class $\mathcal{S}_k$ describes the low-energy dynamics of a NS5/D6/D8 brane system, e.g. a single tensor moduli,



- An l-th instanton string is realized by additional D2-branes, whose contributions to the PF are modeled by the ADHM construction via a 2d $\mathcal{N}=(0,4)$ quiver GLSM,

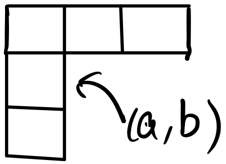


- Its elliptic genus reads [Benini, Eager, Hori, Tachikawa '13]

$$\mathcal{Z}_l = \oint [\mathrm{d}\vec{u}]\ Z_{\mathrm{vec.}}(\vec{u}, \vec{a}, \epsilon_{1,2}) \cdot Z_{\mathrm{mat.}}(\vec{u}, \vec{a}, \vec{m}, \vec{\mu}, \epsilon_{1,2})\,,$$
$$= \oint [\mathrm{d}\vec{u}]\ {\prod_{i,j}}' \frac{\theta_1(u_{ij} + 2\epsilon_+)\theta_1(u_{ij})}{\theta_1(u_{ij} + \epsilon_{1,2})} \prod_{\alpha,f,i} \frac{\theta_1(u_i - m_f)\theta_1(\mu_f - u_i)}{\theta_1(\epsilon_+ \pm (u_i - a_\alpha))}\,,$$

where the poles of above integrand are taken by the method of JK-residue. In the case of $U(l)$ group, the poles are collected by k-tuple of Young tablueax, $\vec{Y} = \{Y_\alpha\}_{\alpha=1}^{k}$, e.g. in Y_α, all poles are given by

$$u_i = a_\alpha - \epsilon_+ - (a-1)\epsilon_1 - (b-1)\epsilon_2$$



with (a, b) labeling the position of the boxes in Y_α.

- Overall, the instanton PF is given by

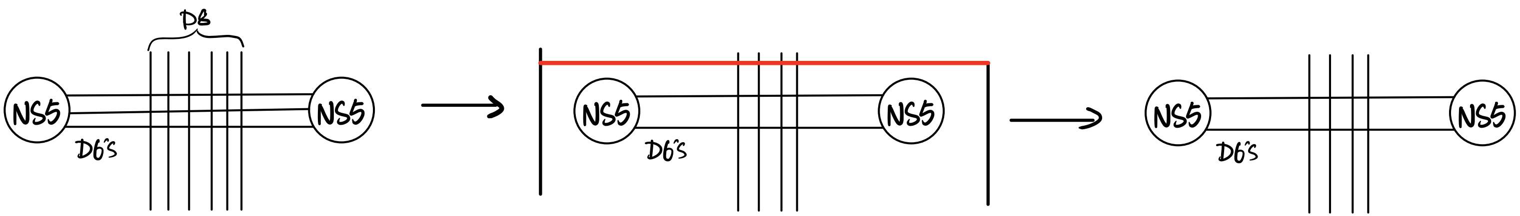
$$\mathcal{Z}_{\mathrm{6d}} = \sum_{l=0}^{\infty} \mathfrak{q}^l \mathcal{Z}_l = \sum_{|\vec{Y}|} \mathfrak{q}^{|\vec{Y}|} \sum_{\vec{Y}} \mathcal{Z}_{\vec{Y}}\,.$$

where $\mathcal{Z}_{\vec{Y}}$ denotes the partition function picking the residues in a possible configuration $\vec{Y}$ with a fixed instanton number $|\vec{Y}|$.

- Now the $\frac{1}{2}$-BPS codimension 2 defect can be introduced by higgsing a meson operator. For a higgsible $\mathcal{S}_k$ class theory, one assign a vev to a meson $M = Q\tilde{Q}$, [Gaiotto, Rastelli, Razamat '12; Gaiotto, Kim '14; ...]

$$\langle M \rangle = const.$$

The vev triggers a RG flow, along which part of the gauge multiplets acquire masses. In the end one gets new SCFT with lower rank, e.g.



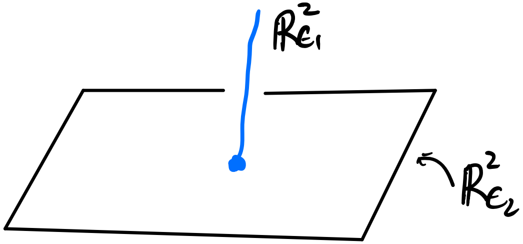
- On the level of PF, the higgsing assigns eqs.

$$\begin{cases} m_f = a_\alpha + \epsilon_+ \\ \mu_f = a_\alpha - \epsilon_+ \end{cases} \implies \frac{\theta_1(u_i - m_f)\theta_1(\mu_f - u_i)}{\theta_1(\epsilon_+ \pm (u_i - a_\alpha))} = 1\,.$$

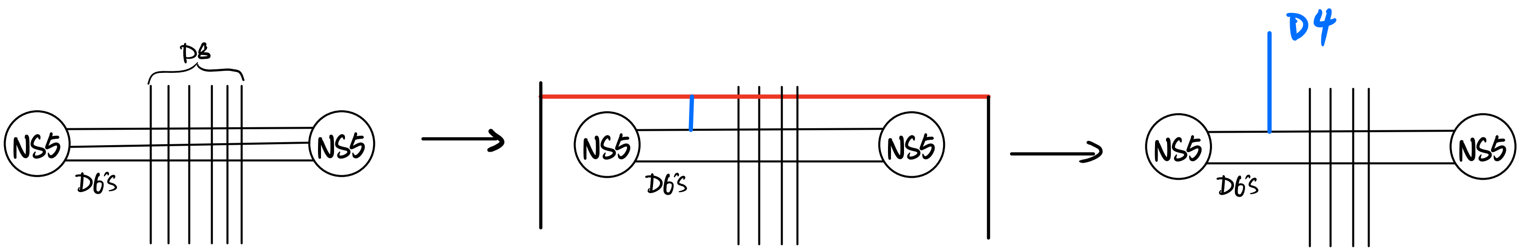
- If the vev of M is not constant, but taking s unit of non-zero $\mathrm{SO}(2)_{\epsilon_2}$ flux instead,

$$\langle M \rangle = e^{-s\epsilon_2}$$

it will trigger out a RG-flow similarly, but in the end of the flow, there is an immobilized "vortex" like defect $\mathcal{D}_s$, extending along the direction $\mathbb{R}^2_{\epsilon_1}$, but point-like on $\mathbb{R}^2_{\epsilon_2}$



- In the brane picture, it is realized by an additional D4 brane suspending between higgsed and preserved D6 branes,



- On the level of the PF, the defect higgsing eqs for a defect $\mathcal{D}_1$,

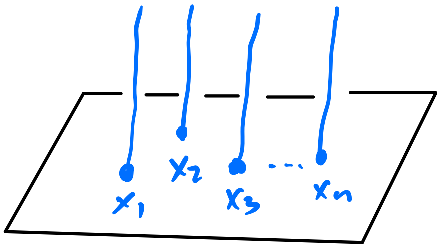
$$\begin{cases} m_f = a_\alpha + \epsilon_+ - \epsilon_+ \\ \mu_f = a_\alpha - \epsilon_+ - \epsilon_- \end{cases} \implies \frac{\theta_1(u_i - m_f)\theta_1(\mu_f - u_i)}{\theta_1(\epsilon_+ \pm (u_i - a_\alpha))} = \frac{\theta_1(u_i - x + \epsilon_+)}{\theta_1(u_i - x + \epsilon_-)} .$$

where x labels the mass of the defect $\mathcal{D}_1$. We thus have

$$\mathcal{Z}_{\mathrm{6d/4d}}(x) = \oint [\mathrm{d}\vec{u}]\ Z_{\mathrm{vec.}} \cdot Z_{\mathrm{mat.}} \cdot Z_{4d}(x; \mathcal{D}_1) \,,$$

$$\text{with} \quad Z_{\mathrm{4d}}(x; \mathcal{D}_1) \sim \frac{\theta_1(u_i - x + \epsilon_+)}{\theta_1(u_i - x + \epsilon_-)} \,,$$

- As many as such defects can be introduced via such a procedure, e.g.

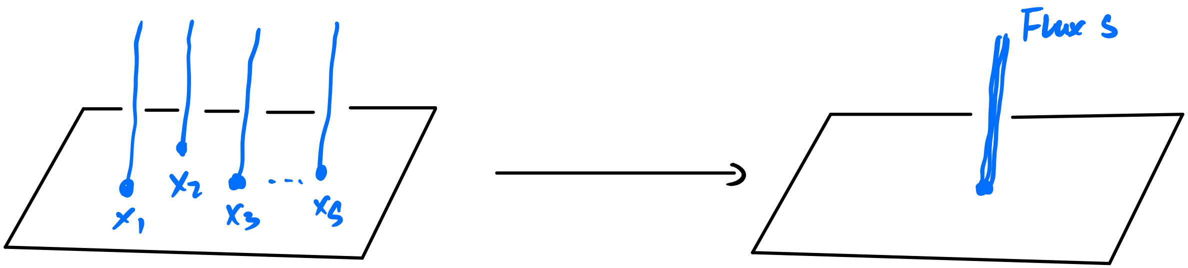


contributing the PF

$$Z_{\mathrm{4d}}(\{x_n\}, \mathcal{D}_{1,\dots,1}) = \prod_{n=1}^{s} \frac{\theta_1(u_i - x_n + \epsilon_+)}{\theta_1(u_i - x_n + \epsilon_-)}$$

- Flux s defects $\mathcal{D}_s$ can be obtained by merging s number of $\mathcal{D}_1$ defects via fine-tuning their fugacities $\{x_n\}$,

$$x_n = x_{n-1} - \epsilon_2$$



and it contributs to the PF as

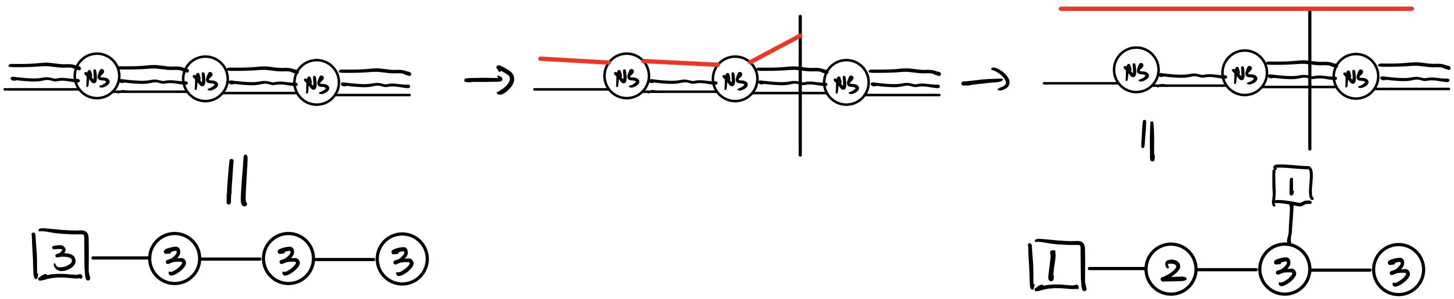
$$Z_{\rm 4d}(x, \mathcal{D}_s) = \frac{\theta_1(u_i - x + s\epsilon_+)}{\theta_1(u_i - x + s\epsilon_-)}$$

Based on this observation, we call $\mathcal{D}_1$ the *minimal defect*.

Codimension 2 defect for a generic quiver of class $\mathcal{S}_k$

- Now starting from a class $\mathcal{S}_k$ theory of N tensor moduli, one can use the mesonic higgsing operation to consecutively obtain a hierarchy of lower rank *SCFTs*, e.g.

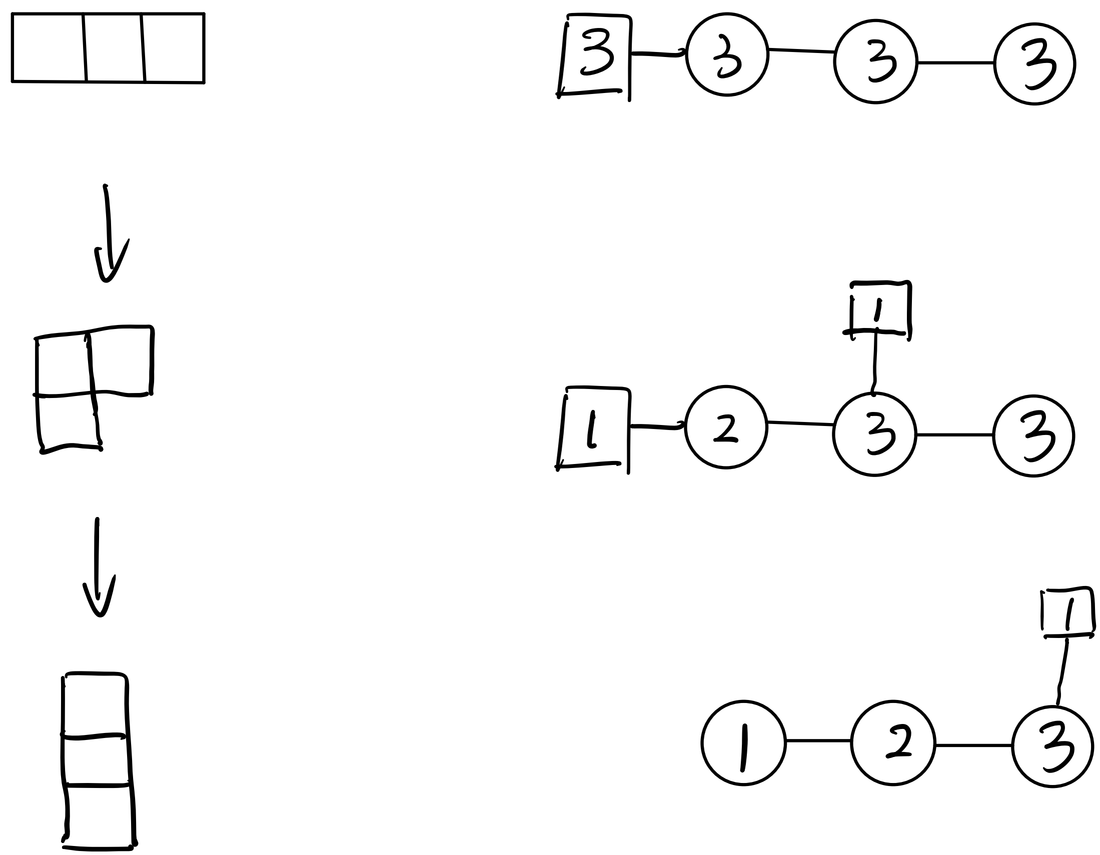
  [Heckman, Rudelius, Tomasiello '16; Hassler, Heckman '20]



- The generic higgsing procedure is controlled by the nilpotent orbits of the left and right flavor group $\mathrm{SU}(k)$: Given the integer k, every lower rank descendant of the $\mathcal{S}_k$ theory is determined by a partition $\mu = (\mu_1, \ldots, \mu_m)$ of k. The higgsed gauge groups with ranks N_i are determined by

$$N_i = \mu_i^T - \mu_{i-1}^T\,, \quad \text{with} \quad N_0 = 0\,,$$

where μ^T is the transpose of the partition μ, e.g.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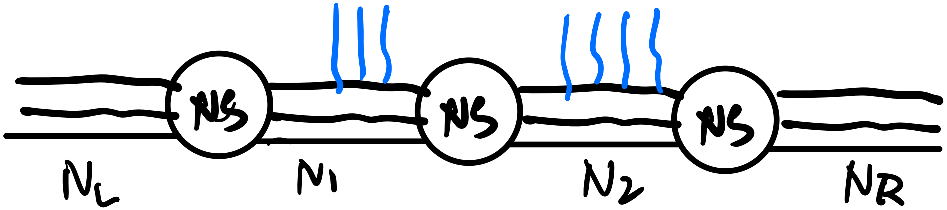
- Our codim 2 defects can be equally introduced for a general higgsing of class $\mathcal{S}_k$ theory, controlled by a pair of partitions (μ_L, μ_R) corresponding to the left and right $\mathrm{SU}(k)$ flavors.

- Therefore, one can compute the 6d/4d coupled instanton PF, in prescence of a bunch of generic codimension two defects $\mathcal{D}(\{x^{(n)}_{i_n,s_i}\})$, of a 6d SCFT higgsed from class $\mathcal{S}_k$ theory $\mathcal{T}[\mathrm{SU}(k), \mu_L, \mu_R, N]$, where $1 \leq n \leq N$ labels the defects in which tensor moduli, and $\{x^{(n)}_{i_n,s_i}\}$ label these defects' $\mathrm{SO}(2)_{\epsilon_2}$ fugacities with fluxs s_i, e.g.



- Having this PF at hand, $\mathcal{Z}_{6d/4d}\left(\mathcal{D}(\{x^{(n)}_{i_n,s_i}\}); \{\mu_L, \mu_R, N\}\right)$, one may ask, under NS-limit, what is the quantum curve the defect PF can engineer, and what about the corresponding q-characters?

- **Partial Answers:**

  For $n = 1$ or N, say the defects localized at the first or the last tensor moduli, and there is only one minimal defect, the curve can be establish umambiguously.

  For $n = 1$ or N, and there are multiple minimal defects, the corresponding quantum curves factorized as the product of the curves of a single minimal defect.

  For generic n, the engineered q-charater are factorized as products of the q-charaters engineered by minimal defects.

Codimension 4 defects (Wilson surfaces)

[Wong, Tong '14; Nekrasov '15; Bullimore, Kim, Koroteev '14; Agrawal, Kim, Kim, Sciarappa '18]

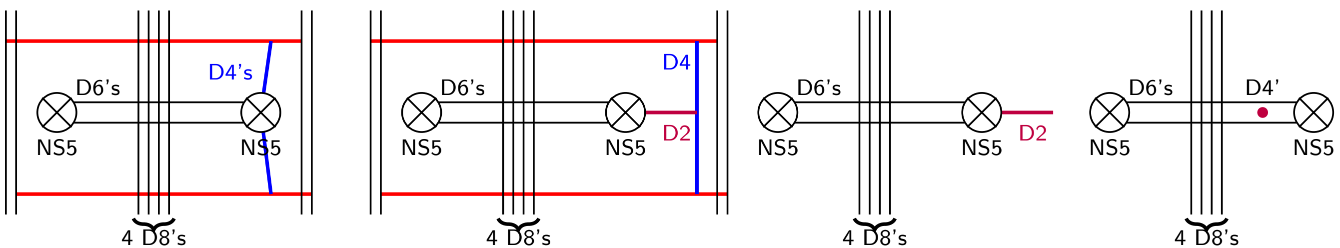
- The last piece of jigsaw puzzle in a SW-curve is the q-character observables. In the context of 6d SCFTs, they are another type of defect of codimension 4, known as Wilson surfaces.

- They are charged surface defects wrapping on the torus of $\mathbb{R}^4 \times \mathbb{T}^2$ by introducing a heavy probe string whose worldsheet is on the $\mathbb{T}^2$. The picture is more clear, when the 6d SCFT is compactified onto $\mathbb{S}^1$, the heavy probed string reduces to a heavy quark localized at the origin of $\mathbb{R}^4$, and the Wilson surface is thus reduced to a Wilson loop in $\mathbb{R}^4 \times \mathbb{S}^1$ (see more details in Xin's talk).

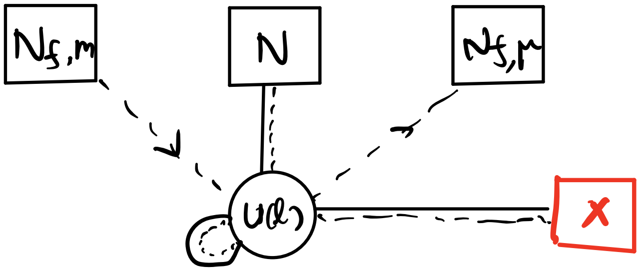
- In our setup, they can be constructed by a "double higgsing" from two codim 2 defects, and realized as another D4' brane.

- The corresponding brane configuration and higgsing procedure are given below

IIA	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
NS5	×	×	×	×	×	×				
D6	×	×	×	×	×	×	×			
D8	×	×	×	×	×	×		×	×	×
D2	×	×					×			
D4	×	×	×	×				×		
D4′	×	×						×	×	×



- In presence of the Wilson surface, the brane system gives the following 2d GLSM quivers



and thus the PF receives extra contributions as

$$\mathcal{Z}_{\mathrm{6d/2d}}(x) = \oint [\mathrm{d}\vec{u}] \; Z_{\mathrm{vec.}} \cdot Z_{\mathrm{mat.}} \cdot Z_{2d}(x) \,,$$

$$\text{with} \quad Z_{\mathrm{2d}}(x) \sim \frac{\theta_1(\epsilon_- \pm (u_i - x))}{\theta_1(-\epsilon_+ \pm (u_i - x))} \,,$$

from which, one can compute $\mathcal{Z}_{\mathrm{6d/2d}}(x)$ by instanton orders.

SW-curves: finite poles sector

- When there are extra contributions from the codimension two/four defects, one needs to pick poles in $\mathcal{Z}_{\mathrm{6d/4d}}(x)$ or $\mathcal{Z}_{\mathrm{6d/2d}}(x)$. All poles resp. to x are also collected in various Young tableaux, Y_x, with poles

$$u_i = x - \epsilon_- - (a-1)\epsilon_1 - (b-1)\epsilon_2\,, \quad \text{for} \quad \text{codim two}$$
$$u_i = x + \epsilon_+ - (a-1)\epsilon_1 - (b-1)\epsilon_2\,, \quad \text{for} \quad \text{codim four}$$

  where (a, b) labels the positions of boxes in Y_x. We denote $\mathcal{Z}_{Y_x}$ the resulted PF after taking residues.

- When taking NS-limit, $\epsilon_2 \to 0$, the Young tablueax Y_x collapses along the ϵ_2-direction. It results in the divergence of $\mathcal{Z}_{Y_x}$. We thus have to define the *normalized* defect partition function

$$\mathcal{Z}^{\mathrm{norm}}_{\mathrm{Defect}}(x) \equiv \lim_{\epsilon_2 \to 0} \frac{\mathcal{Z}_{\mathrm{Defect}}(x)}{\mathcal{Z}_{\mathrm{6d}}}$$

  However, for the partition $Y_x = (n)$, there is no poles in ϵ_2-direction, there the defect PF in this sector is *finite*.

- Since the quantum curve holds for each independent configurations of Young tableaux, one can only focus on the finite poles sector, and help establish the curve.

Example 1: class $\mathcal{S}_k$ of a single tensor moduli

- For the class S_k theory of a single tensor moduli, when inserting a minimal $\mathcal{D}_1$ defect, the finite poles sector can be computed in closed form,

$$\mathcal{Z}^{\mathrm{f}}_{\mathcal{D}_1}(x) = \sum_{i=0}^{\infty} \mathfrak{q}^i \prod_{n=1}^{i} Q\left(x - (n-1)\,\hbar\right)$$

$$\mathcal{Z}^{\mathrm{f}}_{\mathcal{W}_1}(x) = 1 + \mathfrak{q} Q(x + \hbar)$$

where $Q(x)$ is a Jacobi form given by

$$Q(x) = \frac{\prod_f \theta_1\left(x - m_f - \frac{\hbar}{2}\right)\theta_1\left(x - \mu_f - \frac{\hbar}{2}\right)}{\prod_\alpha \theta_1(x - a_\alpha)\theta_1(x - a_\alpha - \hbar)}$$

- The curve is easily checked to satisfy

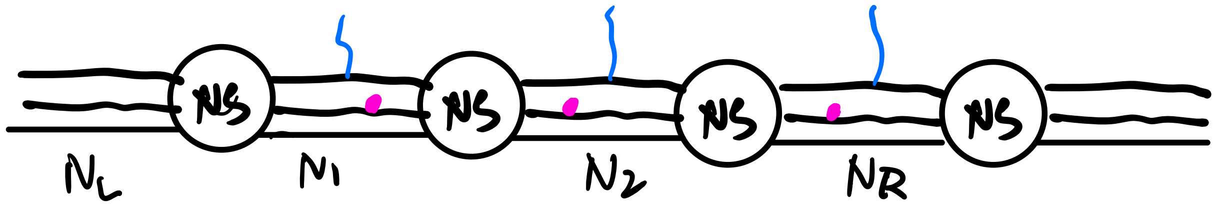
$$\left(\hat{y} + \mathfrak{q} Q(x)\hat{y}^{-1}\right) \cdot \mathcal{Z}^{\mathrm{f}}_{\mathcal{D}_1}(x) = \mathcal{Z}^{\mathrm{f}}_{\mathcal{W}_1}(x) \cdot \mathcal{Z}^{\mathrm{f}}_{\mathcal{D}_1}(x)\,,$$

or say, the quantum curve is

$$\hat{\mathcal{H}}(x, \hat{y}) = \hat{y} + \mathfrak{q} Q(x)\hat{y}^{-1} - \mathcal{Z}^{\mathrm{f}}_{\mathcal{W}_1}(x)$$

Example 2: generic class $\mathcal{S}_k$ and its higgsing hierarchies (3 tensor moduli)

- Using similar stradegy, for a generic class $\mathcal{S}_k$ theory, insert minimal defect $\mathcal{D}_{1,n}$ at the n-th tensor moduli, and possible minimal Wilson defect candidates $\mathcal{W}_{1,n}$, e.g.



- By matching the poles in $\mathcal{W}_{1,n}$, one finds

$$\mathcal{W}_{1,1}(x) = \mathcal{Y}_1(x) + \frac{Q_1(x)\,\mathcal{Y}_2(x+\frac{\hbar}{2})}{\mathcal{Y}_1(x+\hbar)} + \frac{Q_1(x)Q_2(x+\frac{\hbar}{2})\,\mathcal{Y}_3(x+\hbar)}{\mathcal{Y}_2(x+\frac{3\hbar}{2})} + \frac{Q_1(x)Q_2(x+\frac{\hbar}{2})Q_3(x+\hbar)}{\mathcal{Y}_3(x+2\hbar)}$$

$$\mathcal{W}_{1,2}(x) = \mathcal{Y}_2(x) + \frac{Q_2(x)\,\mathcal{Y}_1(x+\frac{\hbar}{2})\,\mathcal{Y}_3(x+\frac{\hbar}{2})}{\mathcal{Y}_2(x+\hbar)} + \frac{Q_2(x)Q_3(x+\frac{\hbar}{2})\,\mathcal{Y}_1(x+\frac{\hbar}{2})}{\mathcal{Y}_3(x+\frac{3\hbar}{2})} + \frac{Q_1(x+\frac{\hbar}{2})Q_2(x)\,\mathcal{Y}_3(x+\frac{\hbar}{2})}{\mathcal{Y}_1(x+\frac{3\hbar}{2})}$$

$$+ \frac{Q_1(x+\frac{\hbar}{2})Q_2(x)Q_3(x+\frac{\hbar}{2})\,\mathcal{Y}_2(x+\hbar)}{\mathcal{Y}_1(x+\frac{3\hbar}{2})y_3(x+\frac{3\hbar}{2})} + \frac{Q_1(x+\frac{\hbar}{2})Q_2(x)Q_2(x+\hbar)Q_3(x+\frac{\hbar}{2})}{\mathcal{Y}_2(x+2\hbar)},$$

$$\mathcal{W}_{1,3}(x) = \mathcal{Y}_3(x) + \frac{Q_3(x)\,\mathcal{Y}_2(x+\frac{\hbar}{2})}{\mathcal{Y}_3(x+\hbar)} + \frac{Q_2(x+\frac{\hbar}{2})Q_3(x)\,\mathcal{Y}_1(x+\hbar)}{\mathcal{Y}_2(x+\frac{3\hbar}{2})} + \frac{Q_1(x+\hbar)Q_2(x+\frac{\hbar}{2})Q_3(x)}{\mathcal{Y}_1(x+2\hbar)}.$$

where $Q_i(x)$ are the 1-instanton order contribution from finite sector of defect $\mathcal{D}_{1,i}$, and $\mathcal{Y}_i(x)$ are the $\mathcal{Y}$-operator in NS-limit,

$$\mathcal{Y}_i(x) \equiv \lim_{\epsilon_2\to 0} \frac{\mathcal{Z}_{D_{1,i}}(x-\hbar)}{\mathcal{Z}_{D_{1,i}}(x)}$$

- The result perfectly match with the q-deformed iWeyl s_i reflections of A_3-type quiver, [Nekrasov, Shatashvili '12; Fucito, Morales, Pacifici '12; Nekrasov, Pestun, Shatashvili '14]

$$s_1 : \begin{cases} \mathcal{Y}_1(x) \to \frac{Q_1(x)\,\mathcal{Y}_2(x+\frac{\hbar}{2})}{\mathcal{Y}_1(x+\hbar)} \\ \mathcal{Y}_i(x) \to \mathcal{Y}_i(x)\,, \quad \text{for} \quad i \neq 1\,, \end{cases}$$

$$s_2 : \begin{cases} \mathcal{Y}_2(x) \to \frac{Q_2(x)\,\mathcal{Y}_1(x+\frac{\hbar}{2})\,\mathcal{Y}_3(x+\frac{\hbar}{2})}{\mathcal{Y}_2(x+\hbar)} \\ \mathcal{Y}_i(x) \to \mathcal{Y}_i(x)\,, \quad \text{for} \quad i \neq 2\,, \end{cases}$$

$$s_3 : \begin{cases} \mathcal{Y}_3(x) \to \frac{Q_3(x)\,\mathcal{Y}_2(x+\frac{\hbar}{2})}{\mathcal{Y}_3(x+\hbar)} \\ \mathcal{Y}_i(x) \to \mathcal{Y}_i(x)\,, \quad \text{for} \quad i \neq 3\,, \end{cases}$$

these orbits determins the Wilson surface defect $\mathcal{W}_{1,n}$,

$$\mathcal{W}_{1,1}: \quad \mathcal{Y}_1 \xrightarrow{s_1} \frac{Q_1\mathcal{Y}_2}{\mathcal{Y}_1} \xrightarrow{s_2} \frac{Q_1Q_2\mathcal{Y}_3}{\mathcal{Y}_2} \xrightarrow{s_3} \frac{Q_1Q_2Q_3}{\mathcal{Y}_3}$$

$$\mathcal{W}_{1,2}: \quad \mathcal{Y}_2 \xrightarrow{s_2} \frac{Q_2\mathcal{Y}_1\mathcal{Y}_3}{\mathcal{Y}_2} \begin{array}{l} \xrightarrow{s_1} \frac{Q_1Q_2\mathcal{Y}_3}{\mathcal{Y}_1} \xrightarrow{s_3} \\ \xrightarrow{s_3} \frac{Q_2Q_3\mathcal{Y}_1}{\mathcal{Y}_3} \xrightarrow{s_1} \end{array} \frac{Q_1Q_2Q_3\mathcal{Y}_2}{\mathcal{Y}_1\mathcal{Y}_3} \xrightarrow{s_2} \frac{Q_1Q_2^2Q_3}{\mathcal{Y}_2}$$

$$\mathcal{W}_{1,3}: \quad \mathcal{Y}_3 \xrightarrow{s_3} \frac{Q_3\mathcal{Y}_2}{\mathcal{Y}_3} \xrightarrow{s_2} \frac{Q_2Q_3\mathcal{Y}_1}{\mathcal{Y}_2} \xrightarrow{s_1} \frac{Q_1Q_2Q_3}{\mathcal{Y}_1}$$

- From these eqs, one can reconstruct the quntum curve associated to the first defect PF $\mathcal{Z}_{D_{1,1}}(x)$ as

$$\hat{\mathcal{H}}_1(x,\hat{y}) = \hat{y} - P_1(x) + P_2(x)\,\hat{y}^{-1} - P_3(x)\,\hat{y}^{-2} + P_4(x)\,\hat{y}^{-3}$$

satisfying

$$\hat{\mathcal{H}}_1(x,\hat{y})\cdot\mathcal{Z}_{D_{1,1}} = 0$$

where Jacobi form $P_i(x)$ are given by

$$\begin{cases} P_1(x) = \mathcal{W}_1(x) \\ P_2(x) = Q_1(x)\,\mathcal{W}_2(x+\frac{\hbar}{2}) \\ P_3(x) = Q_1(x)Q_1(x+\hbar)Q_2(x+\frac{\hbar}{2})\,\mathcal{W}_3(x+\hbar) \\ P_4(x) = Q_1(x)Q_1(x+\hbar)Q_1(x+2\hbar)Q_2(x+\frac{\hbar}{2})Q_2(x+\frac{3\hbar}{2})Q_3(x+\hbar) \end{cases}$$

Examples 3: multiple minimal defects $\mathcal{D}_{1^n}$

- At last, we look at the curve engineered by multiple minimal defects $\mathcal{D}_{1^n}$ in the class S_k of one tensor moduli.

- Under NS-limit, the defect partition function $\mathcal{Z}_{\mathcal{D}_{1^n}}(x_1, \ldots, x_n)$ is simply factorized as

$$\mathcal{Z}^{\mathrm{norm}}_{\mathcal{D}_{1^n}}(x_1, \ldots, x_n) = \prod_{i=1}^{n} \mathcal{Z}^{\mathrm{norm}}_{\mathcal{D}_1}(x_i)$$

- It implies that the Wilson surface defects as well as the the curve have to be factorized accordingly, i.e. [cf. Wang, 22]

$$\mathcal{Z}^{\mathrm{norm}}_{\mathcal{W}_{1^n}}(x_1, \ldots, x_n) = \prod_{i=1}^{n} \mathcal{Z}^{\mathrm{norm}}_{\mathcal{W}_1}(x_i)$$

and

$$\prod_{i}^{n} \left(y_i - Q(x_i) \hat{y}_i^{-1} \right) \cdot \mathcal{Z}^{\mathrm{norm}}_{\mathcal{D}_{1^n}}(x_1, \ldots, x_n) = \mathcal{Z}^{\mathrm{norm}}_{W_{1^n}}(x_1, \ldots, x_n) \cdot \mathcal{Z}^{\mathrm{norm}}_{\mathcal{D}_{1^n}}(x_1, \ldots, x_n)$$

- By taking the merging limit,

$$x_n = x_{n-1} - \epsilon_2 \,,$$

one can merge these defects to an single one with higher flux, $\mathcal{D}_{1^n} \to \mathcal{D}_n$

- On the RHS of the curve, the Wilson surface becomes

$$\mathcal{Z}^{\mathrm{norm}}_{W_{1^n}}(x_1, \ldots, x_n) \longrightarrow \left(\mathcal{Z}^{\mathrm{norm}}_{W_1}(x)\right)^n$$

From a 5d perspective, it contains Wilson loop in higher Reps. of $\mathrm{SU}(2)$

- However, on the LHS of the curve, there is at least no umambiguous difference operator that can be extracted out under the merging limit. It would be an interesting question to investigate further.

Summary and Outlook

- The (non-minimal) codimension two defects, and corresponding Wilson surface defects and curves engineered by them are discussed in the 6d SCFTs of class $\mathcal{S}_k$.

- Several unsolved questions: e.g.

  The minimal defects $\mathcal{D}_{1,n}$, for $1 < n < N$, will engineer a quantum curve or not;

  The nonminial defects $\mathcal{D}_s$ will engineer a unambiguous quantum curve or not.

- It is tempting to generalize the discussion on quantum curves to other 6d SCFTs of the A-type quiver, e.g. the "-1"−"-4"−"-1" quiver dressed with Sp and SO gauge groups;

- It would be interesting to study the quantum curves of affine A-type little string theories;

- It is also interesting but challenging to study the quantum curves on D-type quiver SCFTs, as their instanton ADHM constructions are not known.

THANKS FOR YOUR ATTENTIONS!