Holographic correlators on higher genus Riemann surface

何 松

Jilin University

第五届"场论与弦论"学术研讨会@彭桓武高能基础理论研究中心

Based on arXiv: JHEP 06 (2023) 116, JHEP 03 (2024) 02, 2311.09636, 2405.01255, 2406.04042

Collaborators:

Yi Li (Jilin U.)
Yun-Ze Li (Jilin U.)
Yun-Fei Xie (Jilin U.)
Yunda Zhang (Jilin U.)

Outline

- ☐ Motivations
- ☐ Holographic prescriptions

1. Holographic torus correlators
2. High genus hCFT correlators
3. High genus correlators in Cutoff AdS3

- ☐ Summary and perspectives

Motivations

To understand holographic nature of Quantum Gravity proposed by t' Hooft and Susskind

AdS/CFT correspondence, Maldacena 1997

$$AdS_5 \times S^5 \longleftrightarrow N = 4 \text{ SYM theory}$$

Maldacena

I. AdS5/CFT4
II. AdS4/CFT3 (ABJM)
III. AdS3/CFT2
IV. nAdS2/nCFT1, nAdS2/SYK4
V. Non-AdS/CFT (Celestial Holography)
VI. DS/CFT
VII. ...

Symmetry, field contents

Partion function

Lower point correlation function, & application

Higher-point correlation Function (more concrete)...

AdS/CFT correspondence, Maldacena 1997

Dictionary: GKPW

S. S. Gubser, I. R. Klebanov and A. M. Polyakov，9802109
E. Witten，9802150

$$Z_{\mathrm{CFT}}[g_{ij}, J] = \int_{G_{\mu\nu}|_{\mathrm{bdy}}=g_{ij}, \Phi|_{\mathrm{bdy}}=J} [dG_{\mu\nu}][d\Phi] e^{-S_{\mathrm{grav}}[G_{\mu\nu}, \Phi]}$$

To check ("Prove") the AdS3/CFT2 correspondence:

$$\langle O \rangle = -i \frac{\delta Z[\phi_0]}{\delta \phi_0} = \frac{\delta S[\phi_0]}{\delta \phi_0}$$

Partition functions, generic correlation functions, etc.

$$\langle O(x_1) \ldots O(x_n) \rangle_{CFT} \sim \frac{\delta^n I_{grav}}{\delta \psi_0(x_1) \ldots \delta \psi_0(x_n)}$$

Recent Progress on Holographic correlators

- **Most previous research focuses on holographic correlators in pure AdS**
- **Holographic correlators from Minkowski AdS planar blackhole**

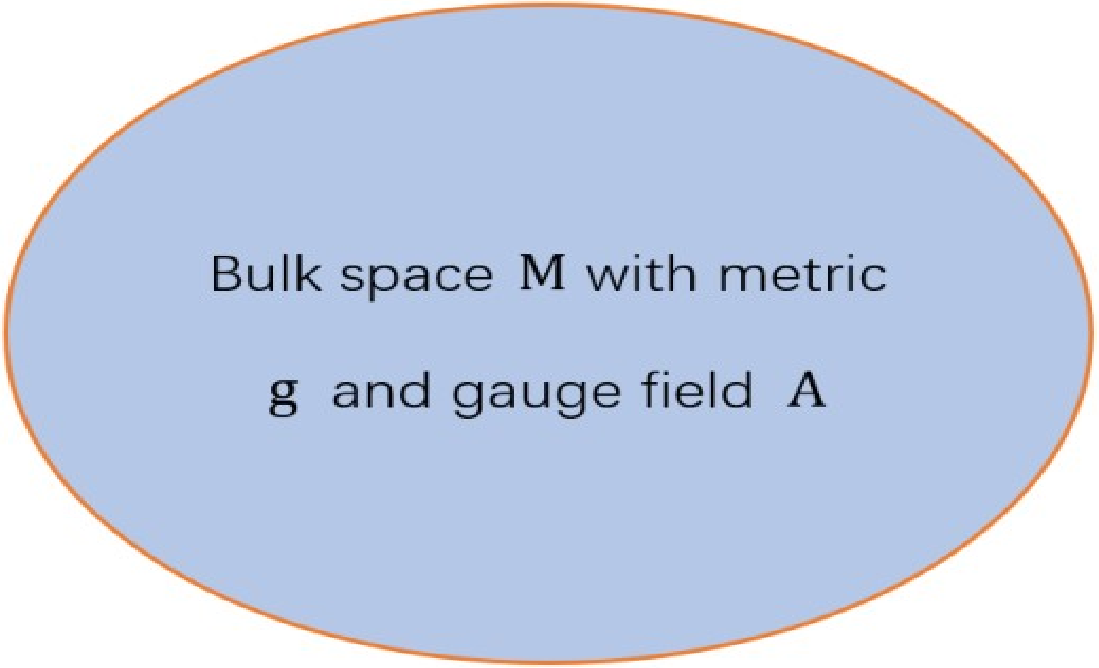
  holographic transport coefficients (specify B.C. on the horizon, ingoing)
- **Holographic correlators from Euclidean AdS planar blackhole**

  Scalar operator correlators worked out (arXiv 2206.07720),
  Only near-boundary analysis for stress tensor correlators (JHEP 09 (2022), 234)

We focus on correlators in the Euclidean spacetime with nontrivial topology.

$$\langle T_{i_1 j_1}(x_1)\dots T_{i_n j_n}(x_n)\rangle_{CFT} \sim \frac{\delta^n I_{grav}}{\delta\gamma^{i_1 j_1}(x_1)\dots\delta\gamma^{i_n j_n}(x_n)}$$

Boundary Value Problem



Conformal boundary ∂M
with boundary metric γ
and gauge field $\mathcal{A}$

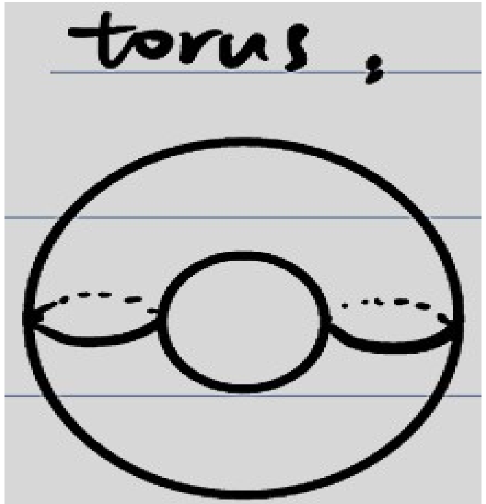
$$\gamma = r^2 g|_{r=0}, \mathcal{A} = A|_{r=0}$$

- **In general, for the given conformal boundary e.g., torus, we need to consider all gravity saddles with different bulk topology and metric.**

- **Near boundary geometry is well-understood [Charles Fefferman, C. Robin Graham, arXiv: 0710.0919, Commun. Math. Phys. 217 (2001) 595-622)]**

- **The global boundary value problem is much more difficult.**

Lower Dimensional: AdS3/CFT2
Warm up



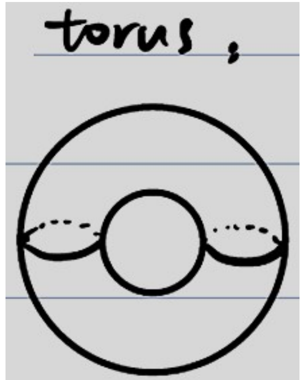
AdS3/CFT2

In AdS3/CFT2, the partition function Alexander Maloney, Edward Witten, 0712.0155

$$\sum_{\alpha} e^{-I^{(\alpha)}_{on-shell}} = Z_{CFT}$$

α labels the saddle points.

- **Thermal AdS3** for low temperature
- ⇕ S transformation of moduli parameter
- **BTZ black hole** for high temperature



Handlebody & Non-Handlebody

$$I[g] = \frac{-1}{16\pi G}\left[\int_{\mathcal{M}} \mathrm{d}^3x\sqrt{g}(R+2) + 2\int_{\partial\mathcal{M}} \mathrm{d}^2x\sqrt{\gamma}(\kappa-1) - 2\pi\chi\left(1+\log\frac{4R_0^2}{\epsilon^2}\right)\right]$$

Holographic stress tensor correlators:

$$ds^2 = \frac{dr^2}{r^2} + \frac{1}{r^2}\left[dz d\bar{z} - r^2\pi^2(dz^2 + d\bar{z}^2) + r^4\pi^4 dz d\bar{z}\right]$$

Variation of boundary metric

Variation of bulk metric

$$\delta\gamma_{ij} dx^i dx^j = \epsilon f_{ij}(z, \bar{z}) dx^i dx^j .$$

Ensures solution is well-behaved

$$\langle T_{i_1 j_1}(z_1) \dots T_{i_n j_n}(z_n)\rangle$$

$$= -\frac{(-2)^n \delta^n I[\gamma]}{\sqrt{\det(\gamma(z_1))} \dots \sqrt{\det(\gamma(z_n))}\,\delta\gamma^{i_1 j_1}(z_1) \dots \delta\gamma^{i_n j_n}(z_n)}$$

AdS3 gravity

$$S_E = -\frac{1}{16\pi G}\int \sqrt{g}(R-2\Lambda) - \frac{1}{8\pi G}\int \sqrt{\gamma}K + \frac{1}{8\pi G l}\int \sqrt{\gamma}.$$

Fefferman-Graham series truncates as Banados space-time

$$ds^2 = \frac{dr^2}{r^2} + \frac{1}{r^2} g_{ij}(x,r)dx^i dx^j .$$

truncates

$$g_{ij}(x,r) = g^{(0)}_{ij}(x) + g^{(2)}_{ij}(x)r^2 + g^{(4)}_{ij}(x)r^4 .$$

$$\langle T_{ij}\rangle = \frac{1}{8\pi G}\left(g^{(2)}_{ij} - g^{(0)kl}g^{(2)}_{kl}g^{(0)}_{ij}\right)$$

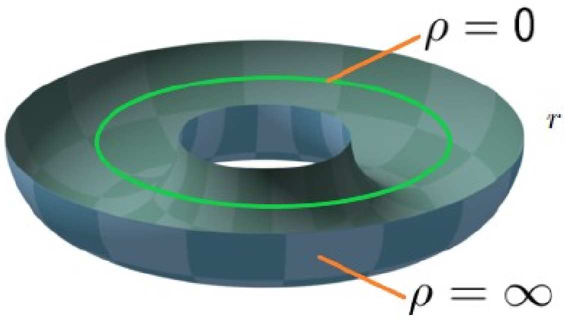
$$g^{(4)}_{ij} = \frac{1}{4} g^{(2)}_{ik} g^{(0)kl} g^{(2)}_{lj} .$$

Thermal AdS3：

(z, zbar) ~ (z + 1, zbar + 1) ~ (z + τ, zbar + τbar)

$$ds^2 = \frac{dz^2 + ds_X^2}{z^2} + \cdots$$

$$ds^2 = d\rho^2 + \cosh^2\rho dt^2 + \sinh^2\rho d\phi^2$$



$$r = \frac{1}{\pi e^{\rho}},\quad z = \frac{\phi + it}{2\pi},\quad \bar{z} = \frac{\phi - it}{2\pi}.$$

$$ds^2 = \frac{dr^2}{r^2} + \frac{1}{r^2}\left[dz d\bar{z} - r^2\pi^2(dz^2 + d\bar{z}^2) + r^4\pi^4 dz d\bar{z}\right]$$

conformal boundary at rho = ∞ or r = 0

Top-down approach

To the first order $ds^2 = (1+\epsilon\mathcal{L}_{V^{[1]}})(d\rho^2 + \cosh^2\rho dt^2 + \sinh^2\rho d\phi^2) + \epsilon g_{ij}^{FG[1]} dx^i dx^j.$

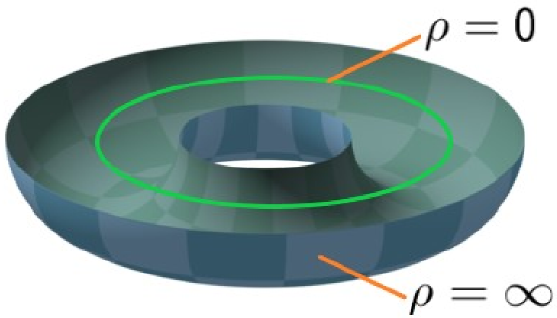
C. Fefferman and C. R. Graham, Ann. Math. Stud. 178, 1 (2011), arXiv:0710.0919 [math.DG]

boundary preserving diffeomorphism

$$V = \sum_{n=1}^{\infty} \epsilon^n V^{[n]} \qquad \langle T_{ij}^{\epsilon}\rangle = \sum_{n=1}^{\infty} \epsilon^n T_{ij}^{[n]}.$$

ensures solution is well-behaved

Thermal AdS3：



Regularity boundary conditions: bulk metric at rho=0 be regular.

$$\int_{\mathrm{T}^2} d^2z\, g_{0t\phi}^{FG[1]} = 0. \qquad \int_{\mathrm{T}^2} d^2z\, g_{2\phi\phi}^{FG[1]} = 0.$$

Two-point function:

SH, Yi Li, Yun-Ze Li, Yunda Zhang, JHEP 06 (2023) 116

$$T^{[1]}_{zz}(z) = \int d^2w \mathcal{G}(z-w)(2T^{[0]}_{zz}\partial_w - \frac{1}{16\pi G}\partial_w^3)f_{\bar{z}\bar{z}}(w)$$
$$+ \frac{1}{16\pi G}(-\partial_z\partial_{\bar{z}}f_{zz} + 2\partial_z^2 f_{z\bar{z}})(z) + C^{[1]}(z),$$

$$\frac{1}{\pi}\partial_{\bar{z}}G_\tau(z,w) = \delta(z,w) - \frac{1}{\mathrm{Im}\tau}$$

$$\frac{1}{\pi}\partial_z\partial_{\bar{z}}\tilde{G}_\tau(z,w) = \delta(z,w) - \frac{1}{\mathrm{Im}\tau},$$

Regularity at rho=0

vary $T^{[1]}_{zz}$ with respect to $f_{\bar{z}\bar{z}}$

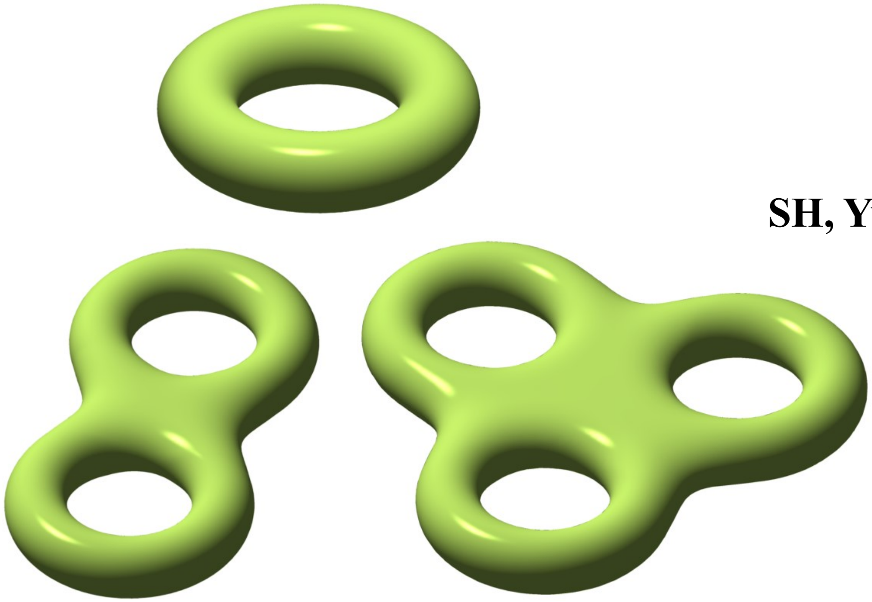
$$C^{[1]} = \frac{\pi}{4G\mathrm{Im}\tau}\int_{T^2} d^2z f_{\bar{z}\bar{z}}$$

$$\langle T(z)T(w)\rangle = \frac{c}{12}[\wp''_\tau(z-w) + 4\pi^2\wp_\tau(z-w) + 8\pi^2\zeta_\tau(\frac{1}{2})]$$

Consistent with:
T. Eguchi and H. Ooguri, Nucl. Phys. B 282, 308 (1987)
SH and Y. Sun, arXiv:2004.07486

Hard to obtain the higher point correlation function by using top-down approach !!!
Alternative way to obtain higher point correlation function, Please refer to …

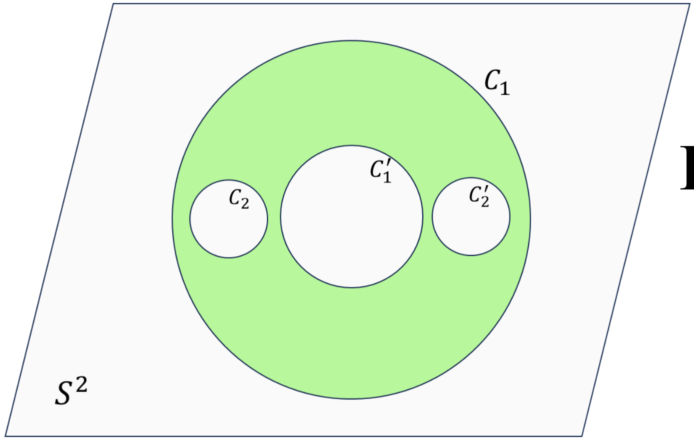
Higher genus surface: AdS3/CFT2 Without X



SH, Yun-Ze Li, Yunfei Xie, 2406.04042

$$I[g] = \frac{-1}{16\pi G}\left[\int_{\mathcal{M}} \mathrm{d}^3x\sqrt{g}(R+2) + 2\int_{\partial\mathcal{M}} \mathrm{d}^2x\sqrt{\gamma}(\kappa-1) - 2\pi\chi\left(1+\log\frac{4R_0^2}{\epsilon^2}\right)\right]$$

Construction of boundary space：Schottky Uniformization

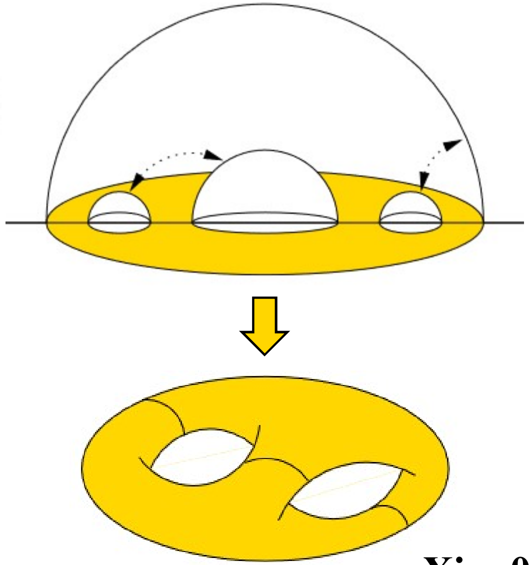


arXiv: 2406.04042

$$M_g = AdS_3/\Gamma_g$$

Kleinian group $\supset$ Schottky group

$$\Gamma_g \subset \mathrm{PSL}(2,\mathbb{C})$$



arXiv: 0912.2090

$$\Gamma_g \quad \frac{L_i(z)-a_i}{L_i(z)-b_i} = \lambda_i \frac{z-a_i}{z-b_i}, \quad 0<|\lambda_i|<1$$

⇩

Loxodromic operator

Holographic correlators

Brown-York tensor one pt: —GKPW→ Boundary CFT

Fefferman-Graham gauge in AdS3:

$$\mathrm{d}s^2 = \frac{\mathrm{d}\rho^2}{4\rho^2} + \frac{1}{\rho} g_{ij}(x,\rho)\,\mathrm{d}x^i\,\mathrm{d}x^j$$

$$g_{ij}(x,\rho) = g_{ij}^{(0)}(x) + g_{ij}^{(2)}(x)\rho + g_{ij}^{(4)}(x)\rho^2$$

Einstein equation:

$$g_{ij}^{(4)} = \frac{1}{4} g_{ik}^{(2)} g^{(0)kl} g_{lj}^{(2)},$$

$$\nabla^{(0)i} g_{ij}^{(2)} = \nabla_j^{(0)} g^{(2)i}{}_i,$$

$$g^{(2)i}{}_i = -\frac{1}{2} R[g^{(0)}].$$

$$\langle T_{ij} \rangle = -\frac{1}{8\pi G}(K_{ij} - K h_{ij} + h_{ij})$$

$$\langle T_{ij} \rangle = \frac{1}{8\pi G}\left(g_{ij}^{(2)} - g^{(0)kl} g_{kl}^{(2)} g_{ij}^{(0)}\right)$$

Conservation equation & Weyl anomaly equation:

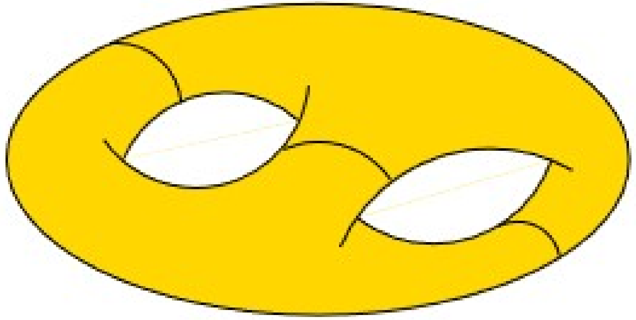
$$\nabla^i \langle T_{ij} \rangle = 0,$$

$$\langle T_i^i \rangle = \frac{1}{16\pi G} R[g^{(0)}].$$

Holographic correlators

$$ds^2 = \frac{\mathrm{d}z^2 + \mathrm{d}s_X^2}{z^2} + \cdots$$

Higher genus Rienmman surface：



$$\mathrm{d}s_X^2 = e^{2\phi(z,\bar{z})}\,\mathrm{d}z\,\mathrm{d}\bar{z}$$

$R = -1$

Single valued = Automorphic form

$$8\partial_z\partial_{\bar{z}}\phi = e^{2\phi}$$

$$\phi(\gamma(z), \overline{\gamma(z)}) = \phi(z, \bar{z}) - \frac{1}{2}\ln|\gamma'(z)|^2, \quad \forall\gamma \in \Gamma_g$$

Quasi-periodic boundary conditions

Holographic correlators

$$ds^2 = e^{2\phi(z,\bar{z})}\,dz\,d\bar{z}$$

$$ds^2 = \frac{d\xi^2}{4\xi^2} + \frac{dy\,d\bar{y}}{\xi},$$

$$\xi = \frac{\rho e^{-2\phi}}{(1+\rho e^{-2\phi}|\partial_z\phi|^2)^2}, \qquad y = z + \partial_{\bar{z}}\phi \frac{\rho e^{-2\phi}}{1+\rho e^{-2\phi}|\partial_w\phi|^2}$$

K. Krasnov, hep-th/0109198

$$ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}e^{2\phi}\,dz\,d\bar{z} + \mathcal{T}^\phi\,dz^2 + \bar{\mathcal{T}}^\phi\,d\bar{z}^2 + 2\mathcal{R}\,dz\,d\bar{z} + \rho e^{-2\phi}(\mathcal{T}^\phi\,dz + \mathcal{R}\,d\bar{z})(\bar{\mathcal{T}}^\phi\,d\bar{z} + \mathcal{R}\,dz)$$

$$\mathcal{T}^\phi = \partial_z^2\phi - (\partial_z\phi)^2, \quad \mathcal{R} = \partial_z\partial_{\bar{z}}\phi$$

$$g^{(0)} = \begin{pmatrix} 0 & \frac{1}{2}e^{2\phi} \\ \frac{1}{2}e^{2\phi} & 0 \end{pmatrix}, \quad g^{(2)} = \begin{pmatrix} \mathcal{T}^\phi & \mathcal{R} \\ \mathcal{R} & \bar{\mathcal{T}}^\phi \end{pmatrix},$$

$$g^{(4)} = e^{-2\phi}\begin{pmatrix} \mathcal{T}^\phi\mathcal{R} & \frac{1}{2}(\mathcal{T}^\phi\bar{\mathcal{T}}^\phi + \mathcal{R}^2) \\ \frac{1}{2}(\mathcal{T}^\phi\bar{\mathcal{T}}^\phi + \mathcal{R}^2) & \mathcal{R}\bar{\mathcal{T}}^\phi \end{pmatrix}.$$

$$\langle T_{zz}\rangle = \frac{1}{8\pi G}\mathcal{T}^\phi = \frac{1}{8\pi G}\left(\partial_z^2\phi - (\partial_z\phi)^2\right),$$

$$\langle T_{z\bar{z}}\rangle = \langle T_{\bar{z}z}\rangle = -\frac{\mathcal{R}}{8\pi G} = -\frac{1}{8\pi G}\partial_z\partial_{\bar{z}}\phi = -\frac{e^{2\phi}}{64\pi G},$$

$$\langle T_{\bar{z}\bar{z}}\rangle = \frac{1}{8\pi G}\bar{\mathcal{T}}^\phi = \frac{1}{8\pi G}\left(\partial_{\bar{z}}^2\phi - (\partial_{\bar{z}}\phi)^2\right).$$

$$\langle T_{ij}\rangle = \frac{1}{8\pi G}\left(g_{ij}^{(2)} - g^{(0)kl}g_{kl}^{(2)}g_{ij}^{(0)}\right)$$

Metric variation and functional of higher point correlation

$$\delta g_{ij}^{(0)}\,\mathrm{d}x^i\,\mathrm{d}x^j = \epsilon\chi_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$$

$$\delta\left\langle T_{ij}\right\rangle = \sum_{i=1}^{n}\epsilon^n\left\langle T_{ij}\right\rangle^{[n]}$$

$$\nabla^i\left\langle T_{ij}\right\rangle = 0,$$

$$\left\langle T_i^i\right\rangle = \frac{1}{16\pi G}R[g^{(0)}].$$

$$\delta g_{ij}^{(0)}\mathrm{d}z^i\mathrm{d}z^j = \sum_{\alpha=1}^{3g-3}\left(\phi_{\alpha zz}\delta\bar{\tau}_\alpha(\mathrm{d}z)^2 + \overline{\phi_{\alpha zz}}\delta\tau_\alpha(\mathrm{d}\bar{z})^2\right).$$

⇩

First order variation

$$\partial_{\bar{z}}\frac{\delta\left\langle T_{zz}\right\rangle^{[1]}(z)}{\delta\chi_{\bar{w}\bar{w}}(w)} + \frac{1}{16\pi G}e^{-2\phi(z)}\left(12(\partial_z\phi)^2\partial_z + 12\partial_z\phi\partial_z^2\phi - 8(\partial_z\phi)^3 - 6\partial_z^2\phi\partial_z - 6\partial_z\phi\partial_z^2 - 2\partial_z^3\phi + \partial_z^3\right)\delta^{(2)}(z-w) = 0.$$

$$\langle T_{zz}(z)T_{ww}(w)\rangle$$

Derivation of two point correlation function

$$\partial_{\bar{z}}\frac{\delta\left\langle T_{zz}\right\rangle^{[1]}(z)}{\delta\chi_{\bar{w}\bar{w}}(w)}+\frac{1}{16\pi G}e^{-2\phi(z)}\left(12(\partial_z\phi)^2\partial_z+12\partial_z\phi\partial_z^2\phi-8(\partial_z\phi)^3-6\partial_z^2\phi\partial_z-6\partial_z\phi\partial_z^2\right.$$
$$\left.-2\partial_z^3\phi+\partial_z^3\right)\delta^{(2)}(z-w)=0.$$

Green function： $\frac{1}{\pi}\partial_{\bar{z}}G^{z}{}_{ww}(z,\bar{z};w,\bar{w})=\delta^{(2)}(z-w)-p_2(z,\bar{z};w)$ $\qquad p_2(z,\bar{z};w)=\sum_{\alpha=1}^{3g-3}\mu^{z}_{\alpha\bar{z}}(z,\bar{z})\phi_{\alpha ww}(w)$

μ: Beltrami differential

ϕ: holomorphic quadratic differential

$$\frac{\delta\left\langle T_{zz}\right\rangle^{[1]}(z)}{\delta\chi_{\bar{w}\bar{w}}(w)}=\int_{\mathcal{D}}\mathrm{d}^2z_0\,\delta^{(2)}(z_0-z)\frac{\delta\left\langle T_{zz}\right\rangle^{[1]}(z_0)}{\delta\chi_{\bar{w}\bar{w}}(w)}$$
$$=\int_{\mathcal{D}}\mathrm{d}^2z_0\left(\frac{1}{\pi}\partial_{\bar{z}_0}G^{z_0}{}_{zz}(z_0,\bar{z}_0;z,\bar{z})+p_2(z_0,\bar{z}_0;z)\right)\frac{\delta\left\langle T_{zz}\right\rangle^{[1]}(z_0)}{\delta\chi_{\bar{w}\bar{w}}(w)}$$

Derivation of two point correlation function

Green function: $\frac{1}{\pi}\partial_{\bar{z}} G^{z}{}_{ww}(z,\bar{z};w,\bar{w}) = \delta^{(2)}(z-w) - p_2(z,\bar{z};w)$ $\quad p_2(z,\bar{z};w) = \sum_{\alpha=1}^{3g-3} \mu^{z}_{\alpha\bar{z}}(z,\bar{z})\phi_{\alpha ww}(w)$

μ: Beltrami differential

ϕ: holomorphic quadratic differential

$$\frac{\delta \langle T_{zz}\rangle^{[1]}(z)}{\delta \chi_{\bar{w}\bar{w}}(w)} = \int_{\mathcal{D}} \mathrm{d}^2 z_0\, \delta^{(2)}(z_0 - z) \frac{\delta \langle T_{zz}\rangle^{[1]}(z_0)}{\delta \chi_{\bar{w}\bar{w}}(w)}$$
$$= \int_{\mathcal{D}} \mathrm{d}^2 z_0 \left(\frac{1}{\pi}\partial_{\bar{z}_0} G^{z_0}{}_{zz}(z_0,\bar{z}_0;z,\bar{z}) + p_2(z_0,\bar{z}_0;z) \right) \frac{\delta \langle T_{zz}\rangle^{[1]}(z_0)}{\delta \chi_{\bar{w}\bar{w}}(w)}$$

$$\int_{\mathcal{D}} \mathrm{d}^2 z_0\, \overline{\phi_{\alpha z_0 z_0}} \frac{\delta \langle T_{ww}(w)\rangle}{\delta g^{(0)}_{\bar{z}_0\bar{z}_0}(z_0)} = \frac{\partial}{\partial \tau_\alpha} \langle T_{ww}(w)\rangle,$$

$$= \int_{\mathcal{D}} \mathrm{d}^2 z_0 \frac{1}{\pi}\partial_{z_0} \left[G^{z_0}{}_{zz}(z_0,\bar{z}_0;z,\bar{z}) \frac{\delta \langle T_{zz}\rangle^{[1]}(z_0)}{\delta \chi_{\bar{w}\bar{w}}(w)} \right]$$
$$- \int_{\mathcal{D}} \mathrm{d}^2 z_0\, G^{z_0}{}_{zz}(z_0,\bar{z}_0;z,\bar{z}) \Big[\frac{-1}{16\pi^2 G} e^{-2\phi(z_0,\bar{z}_0)} \big(12(\partial_{z_0}\phi)^2 \partial_{z_0} + 12\partial_{z_0}\phi \partial^2_{z_0}\phi - 8(\partial_{z_0}\phi)^3$$
$$- 6\partial^2_{z_0}\phi\partial_{z_0} - 6\partial_{z_0}\phi\partial^2_{z_0} - 2\partial^3_{z_0}\phi + \partial^3_{z_0}\big)\delta^{(2)}(z_0 - w)\Big]$$
$$= -\frac{i}{2}\oint_{\partial \mathcal{D}} \mathrm{d}z_0 \frac{1}{\pi} G^{z_0}{}_{zz}(z_0,\bar{z}_0;z,\bar{z}) \frac{\delta \langle T_{zz}\rangle^{[1]}(z_0)}{\delta \chi_{\bar{w}\bar{w}}(w)} - \frac{1}{16\pi^2 G} e^{-2\phi(w,\bar{w})} \partial^3_w G^{w}{}_{zz}(w,\bar{w};z,\bar{z})$$

$$\int_{\mathcal{D}} \mathrm{d}^2 z_0\, p_2(z_0,\bar{z}_0;z) \frac{\delta \langle T_{zz}\rangle^{[1]}(z_0)}{\delta \chi_{\bar{w}\bar{w}}(w)} = e^{-2\phi(w,\bar{w})} \int_{\mathcal{D}} \mathrm{d}^2 z_0\, p_2(z_0,\bar{z}_0;z) \langle T_{z_0 z_0}(z_0) T_{ww}(w)\rangle$$
$$= \sum_{\alpha=1}^{3g-3} \phi_{\alpha zz} e^{-2\phi(w,\bar{w})} \int_{\mathcal{D}} \mathrm{d}^2 z_0\, \overline{\phi_{\alpha z_0 z_0}} \frac{\delta \langle T_{ww}(w)\rangle}{\delta g^{(0)}_{\bar{z}_0\bar{z}_0}(z_0)}.$$

$$\oint_{\partial D} = \sum_{i=1}^{g} \left(\oint_{C_i} - \oint_{C_i'} \right)$$

$$\langle T_{zz}(z) T_{ww}(w)\rangle = -\frac{1}{16\pi^2 G} \partial^3_w G^{w}{}_{zz}(w,\bar{w};z,\bar{z}) + \sum_{\alpha=1}^{3g-3} \phi_{\alpha zz} \frac{\partial}{\partial \tau_\alpha} \langle T_{ww}(w)\rangle$$

T. Eguchi and H. Ooguri, Nucl. Phys. B 282, 308 (1987)

Consistent with CFT data!!!

Holographic two-point correlation function

$$\langle T_{zz}(z)T_{ww}(w)\rangle = -\frac{1}{16\pi^2 G}\partial_w^3 G^w{}_{zz}(w,\bar{w};z,\bar{z}) + \sum_{\alpha=1}^{3g-3}\phi_{\alpha zz}\frac{\partial}{\partial\tau_\alpha}\langle T_{ww}\rangle\,,$$

$$\langle T_{zz}(z)T_{\bar{w}\bar{w}}(w)\rangle = \frac{1}{16\pi G}(4\partial_w\phi\partial_{\bar{w}}\phi - \partial_w\phi\partial_{\bar{w}} + 2\partial_{\bar{w}}\phi\partial_w - 8\partial_w\partial_{\bar{w}}\phi - \partial_w\partial_{\bar{w}})\delta^{(2)}(w-z)$$
$$+\frac{3}{8\pi G}\partial_w\partial_{\bar{w}}\phi\, p_2(w,\bar{w};z) + \sum_{\alpha=1}^{3g-3}\phi_{\alpha zz}\frac{\partial}{\partial\tau_\alpha}\langle T_{\bar{w}\bar{w}}\rangle\,,$$

$$\langle T_{z\bar{z}}(z)T_{ww}(w)\rangle = \frac{1}{16\pi G}\left(2\partial_z^2\phi - 2(\partial_z\phi)^2 - 2\partial_z\phi\partial_z + \partial_z^2\right)\delta^{(2)}(z-w),$$

$$\langle T_{z\bar{z}}(z)T_{w\bar{w}}(w)\rangle = \frac{1}{16\pi G}\left(2\partial_z\phi\partial_{\bar{z}} + 2\partial_{\bar{z}}\phi\partial_z - 4\partial_z\phi\partial_{\bar{z}}\phi - \partial_z\partial_{\bar{z}}\right)\delta^{(2)}(z-w).$$

Holographic recurrence relations n-point correlation function

$$\langle T_{zz}(z)T_{\bar{z}\bar{z}}(z_1)\cdots T_{\bar{z}\bar{z}}(z_n)\rangle = \frac{1}{\pi}\sum_{i=1}^{n} G^{z_i}{}_{zz}(z_i,\bar{z}_i;z,\bar{z})\partial_{\bar{z}_i}\langle T_{\bar{z}\bar{z}}(z_1)\cdots T_{\bar{z}\bar{z}}(z_n)\rangle$$

$$-\frac{2}{(n-2)!\pi}\sum_{\sigma\in S_n} G^{z_{\sigma(1)}}{}_{zz}(z_{\sigma(1)},\bar{z}_{\sigma(1)};z,\bar{z})\Big(2\langle T_{\bar{z}\bar{z}}(z_{\sigma(1)})\cdots T_{\bar{z}\bar{z}}(z_{\sigma(n-1)})\rangle\partial_{\bar{z}_{\sigma(1)}}$$

$$+\partial_{\bar{z}_{\sigma(1)}}\langle T_{\bar{z}\bar{z}}(z_{\sigma(1)})\cdots T_{\bar{z}\bar{z}}(z_{\sigma(n-1)})\rangle + 4\langle T_{\bar{z}\bar{z}}(z_{\sigma(1)})\cdots T_{\bar{z}\bar{z}}(z_{\sigma(n-1)})\rangle\partial_{\bar{z}_{\sigma(1)}}\phi\Big)\delta^{(2)}(z_{\sigma(1)}-z_{\sigma(n)})$$

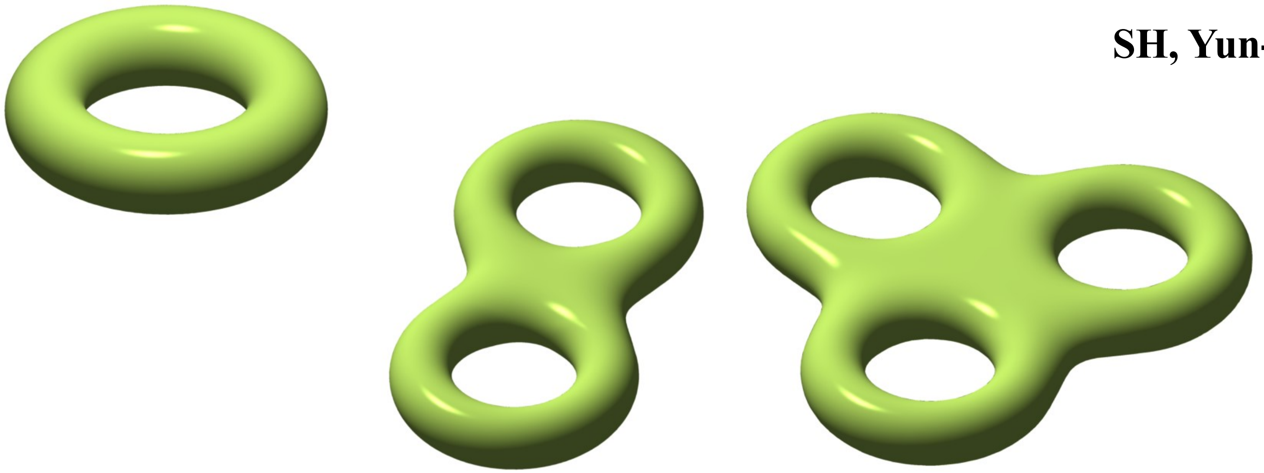
$$+\sum_{\alpha=1}^{3g-3}\phi_{\alpha zz}(z)\frac{\partial}{\partial\tau_\alpha}\langle T_{\bar{z}\bar{z}}(z_1)\cdots T_{\bar{z}\bar{z}}(z_n)\rangle\,, \tag{56}$$

Consistent with VOA construction: Zhu relation!!

Yongchang Zhu： Journal of the American Mathematical Society
Vol. 9, No. 1 (Jan., 1996), pp. 237-302

Higher genus surface on Cutoff surface: AdS3/CFT2:Without X

SH, Yun-Ze Li, Yunfei Xie, 2406.04042



TTbar deformation

$$Z_G[g^{(c)}_{ij}] = \Big\langle \exp\Big[-\frac{1}{2}\int \mathrm{d}^2 z \sqrt{g^{(c)}} g^{(c)ij} T_{ij}\Big]\Big\rangle_{\mathrm{EFT}},$$

Hard radial cutoff AdS3

cutoff-AdS_3 / $T\bar{T}$-deformed CFT 对偶 $\lambda = 16\pi G \rho_c$

$$\frac{\mathrm{d}S_\lambda}{\mathrm{d}\lambda} = -\frac{1}{4}\int \mathrm{d}^2 z \,\det[T_\lambda]$$

$$\langle T_{ij}\rangle_{\rho_c} = -\frac{1}{8\pi G}(K^{(c)}_{ij} - K^{(c)} h^{(c)}_{ij} + h^{(c)}_{ij})$$

Brown-York Energy Momentum tensor

$$\nabla^i \langle T_{ij}\rangle_{\rho_c} = 0,$$

Conserved equation,

$$\langle T^i_i\rangle_{\rho_c} = \frac{1}{16\pi G} R^{(c)} - 8\pi G \rho_c \det[T]_{\rho_c},$$

$$\partial_{\rho_c}\langle T_{ij}\rangle_{\rho_c} = 4\pi G[2\langle T^k_i\rangle_{\rho_c}\langle T_{kj}\rangle_{\rho_c} - \langle T^k_k\rangle_{\rho_c}\langle T_{ij}\rangle_{\rho_c} - \det[T]_{\rho_c} g^{(c)}_{ij}]$$

Wely anomaly equation

Radial flow effect of the stress tensor within the same FG coordinate system

Variation of energy momentum tensor

Choose Conformal gauge on the cutoff surface:

$$g_{ij}(\rho, z, \bar{z})\mathrm{d}x^i\mathrm{d}x^j = e^{2\omega_\rho(z,\bar{z})}\mathrm{d}z\mathrm{d}\bar{z}$$

$$\delta_\rho g_{ij} = (g_{ij}^{(2)} + 2\rho g_{ij}^{(4)})\delta\rho = 8\pi G(\langle T_{ij}\rangle_\rho - g^{kl}\langle T_{kl}\rangle_\rho g_{ij})\delta\rho.$$

$$\delta g_{ij} = \delta_\rho g_{ij} + \mathcal{L}_\epsilon g_{ij} = 2\delta\omega_\rho g_{ij}.$$

$$\partial_{\bar{z}}\epsilon_\rho^z = -8\pi G e^{-2\omega_\rho}\langle T_{\bar{z}\bar{z}}\rangle_\rho\delta\rho, \quad \partial_z\epsilon_\rho^{\bar{z}} = -8\pi G e^{-2\omega_\rho}\langle T_{zz}\rangle_\rho\delta\rho,$$

$$\delta\omega_\rho = \frac{1}{2}e^{-2\omega_\rho}[\partial_z(e^{2\omega_\rho}\epsilon_\rho^z) + \partial_{\bar{z}}(e^{2\omega_\rho}\epsilon_\rho^{\bar{z}}) - 16\pi G\langle T_{z\bar{z}}\rangle_\rho\delta\rho].$$

$$\delta\langle T_{zz}\rangle_\rho = (\epsilon^k\partial_k + 2\partial_z\epsilon^z)\langle T_{zz}\rangle_\rho,$$

$$\delta\langle T_{\bar{z}\bar{z}}\rangle_\rho = (\epsilon^k\partial_k + 2\partial_{\bar{z}}\epsilon^{\bar{z}})\langle T_{zz}\rangle_\rho,$$

$$\delta\langle T_{z\bar{z}}\rangle_\rho = (\epsilon^k\partial_k + \partial_k\epsilon^k)\langle T_{z\bar{z}}\rangle_\rho - 2\pi G e^{2\omega_\rho}\det[T]_\rho\delta\rho.$$

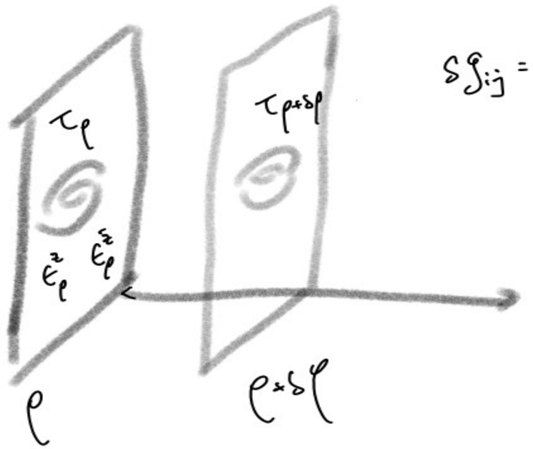
$$\epsilon_\rho^z, \quad \epsilon_\rho^{\bar{z}}$$

Ensure the invariance of the line element under the action of Schottky group $\Gamma_g \subset \mathrm{PSL}(2,\mathbb{C})$

$$\omega_\rho(\gamma_\rho(z), \overline{\gamma_\rho(z)}) = \omega_\rho(z,\bar{z}) - \frac{1}{2}\ln|\gamma_\rho'(z)|^2,$$

Keep the conformal gauge during the radial flow

$$\tau_\alpha = \tau_\alpha(\rho), \quad \alpha = 1, 2, ..., 3g-3.$$



$\delta g_{ij} = \delta_\rho g_{ij} + \mathcal{L}_\epsilon g_{ij}$

diff

One-Point function with deformation

Apply Green function & Single valued to guess

$$\partial_{\bar{z}}\epsilon^z_\rho = -8\pi G e^{-2\omega_\rho}\langle T_{\bar{z}\bar{z}}\rangle_\rho \delta\rho, \quad \partial_z \epsilon^{\bar{z}}_\rho = -8\pi G e^{-2\omega_\rho}\langle T_{zz}\rangle_\rho \delta\rho$$

$$\tau_\alpha = \tau_\alpha(\rho), \quad \alpha = 1, 2, ..., 3g-3.$$

$$\epsilon^z_\rho = -8G\delta\rho \int_{\mathcal{D}_\rho} e^{-2\omega_\rho(w,\bar{w})}\langle T_{\bar{w}\bar{w}}\rangle_\rho \Big[G^z_{ww} + \sum_{\alpha=1}^{3g-3} f^z_\alpha \phi_{\alpha ww}\Big]_{\Gamma_\rho} \mathrm{d}^2 w$$

$$\epsilon^{\bar{z}}_\rho = -8G\delta\rho \int_{\mathcal{D}_\rho} e^{-2\omega_\rho(w,\bar{w})}\langle T_{ww}\rangle_\rho \Big[\overline{G^z_{ww}} + \sum_{\alpha=1}^{3g-3} \overline{f^z_\alpha \phi_{\alpha ww}}\Big]_{\Gamma_\rho} \mathrm{d}^2 w$$

f^z_α **: Bers potential**

$$\frac{1}{\pi}\partial_{\bar{z}} f^z_\alpha = \mu^z_{\alpha\bar{z}}, \quad \alpha = 1, 2, ..., 3g-3$$

$$\frac{\mathrm{d}\tau_\alpha}{\mathrm{d}\rho} = 8\pi G \int_{\mathcal{D}_\rho} e^{-2\omega_\rho(w,\bar{w})}\langle T_{\bar{w}\bar{w}}\rangle_\rho \phi_{\alpha ww}\mathrm{d}^2 w, \quad \alpha = 1, 2, ..., 3g-3.$$

★ **Modular flow equation with respect to radial cutoff**

Plug into

$$\delta\omega_\rho = \frac{1}{2} e^{-2\omega_\rho}[\partial_z(e^{2\omega_\rho}\epsilon^z_\rho) + \partial_{\bar{z}}(e^{2\omega_\rho}\epsilon^{\bar{z}}_\rho) - 16\pi G\langle T_{z\bar{z}}\rangle_\rho \delta\rho]$$

$$\delta\langle T_{zz}\rangle_\rho = (\epsilon^k\partial_k + 2\partial_z\epsilon^z)\langle T_{zz}\rangle_\rho,$$

$$\delta\langle T_{\bar{z}\bar{z}}\rangle_\rho = (\epsilon^k\partial_k + 2\partial_{\bar{z}}\epsilon^{\bar{z}})\langle T_{zz}\rangle_\rho,$$

$$\delta\langle T_{z\bar{z}}\rangle_\rho = (\epsilon^k\partial_k + \partial_k\epsilon^k)\langle T_{z\bar{z}}\rangle_\rho - 2\pi G e^{2\omega_\rho}\det[T]_\rho \delta\rho.$$

→ **One point functional Wely factor**

One-Point function with deformation(Perturbative)

$$\omega_0(z,\bar z)=\phi(z,\bar z)+\sum_{n=1}^{\infty}\rho_c^n\phi_n(z,\bar z),\quad \langle T_{ij}(z,\bar z)\rangle_{\rho_c}=\sum_{n=0}^{\infty}\rho_c^n\langle T_{ij}(z,\bar z)\rangle_n$$

$$\omega_{\rho_c}(z,\bar z)=\phi(z,\bar z),\quad \Gamma_{\rho_c}=\Gamma.$$

$$\delta\langle T_{zz}\rangle_\rho=(\epsilon^k\partial_k+2\partial_z\epsilon^z)\langle T_{zz}\rangle_\rho,$$

$$\delta\langle T_{\bar z\bar z}\rangle_\rho=(\epsilon^k\partial_k+2\partial_{\bar z}\epsilon^{\bar z})\langle T_{zz}\rangle_\rho,$$

$$\delta\langle T_{z\bar z}\rangle_\rho=(\epsilon^k\partial_k+\partial_k\epsilon^k)\langle T_{z\bar z}\rangle_\rho-2\pi G e^{2\omega_\rho}\mathrm{det}[T]_\rho\delta\rho.$$

$$\epsilon^z_\rho=-8G\delta\rho\int_{\mathcal{D}_\rho}e^{-2\omega_\rho(w,\bar w)}\langle T_{\bar w\bar w}\rangle_\rho\Big[G^z_{ww}+\sum_{\alpha=1}^{3g-3}f^z_\alpha\phi_{\alpha ww}\Big]_{\Gamma_\rho}\mathrm{d}^2w,$$

$$\epsilon^{\bar z}_\rho=-8G\delta\rho\int_{\mathcal{D}_\rho}e^{-2\omega_\rho(w,\bar w)}\langle T_{ww}\rangle_\rho\Big[\overline{G^z_{ww}}+\sum_{\alpha=1}^{3g-3}\overline{f^z_\alpha\phi_{\alpha ww}}\Big]_{\Gamma_\rho}\mathrm{d}^2w,$$

$$\phi_1(z,\bar z)=\frac{1}{2\pi}\Big[-\frac{\pi}{4}+\int_{\mathcal{D}}e^{-2\phi(w,\bar w)}\Big((\partial_{\bar w}^2\phi-(\partial_{\bar w}\phi)^2)(\partial_z+2\partial_z\phi)(G^z_{ww}+\sum_{\alpha=1}^{3g-3}f^z_\alpha\phi_{\alpha ww})$$
$$+(\partial_w^2\phi-(\partial_w\phi)^2)(\partial_{\bar z}+2\partial_{\bar z}\phi)(\overline{G^z_{ww}}+\sum_{\alpha=1}^{3g-3}\overline{f^z_\alpha\phi_{\alpha ww}})\Big)\mathrm{d}^2w\Big],$$

$$\langle T_{zz}\rangle_1=\frac{1}{16\pi^2G}\Big[\frac{\pi}{4}(\partial_z^2\phi-(\partial_z\phi)^2)+\int_{\mathcal{D}}e^{-2\phi(w,\bar w)}(\partial_{\bar w}^2\phi-(\partial_{\bar w}\phi)^2)\partial_z^3(G^z_{ww}+\sum_{\alpha=1}^{3g-3}f^z_\alpha\phi_{\alpha ww})\mathrm{d}^2w\Big],$$

$$\langle T_{\bar z\bar z}\rangle_1=\frac{1}{16\pi^2G}\Big[\frac{\pi}{4}(\partial_{\bar z}^2\phi-(\partial_{\bar z}\phi)^2)+\int_{\mathcal{D}}e^{-2\phi(w,\bar w)}(\partial_w^2\phi-(\partial_w\phi)^2)\partial_{\bar z}^3(\overline{G^z_{ww}}+\sum_{\alpha=1}^{3g-3}\overline{f^z_\alpha\phi_{\alpha ww}})\mathrm{d}^2w\Big],$$

$$\langle T_{z\bar z}\rangle_1=-\frac{1}{8\pi G}\Big[\frac{1}{64}e^{2\phi(z,\bar z)}-e^{-2\phi(z,\bar z)}|\partial_z^2\phi-(\partial_z\phi)^2|^2\Big].$$

One-point function up to first-order deformation

k-Point functions up to first order

To obtain the k-Point function：

$$g_{ij}^{(c)}(z,\bar{z}) = e^{2\phi(z,\bar{z})}\eta_{ij} + \epsilon\chi_{ij}(z,\bar{z}) \qquad \langle T_{ij}\rangle_{\rho_c}^{[\epsilon]} = \sum_{n=0}^{\infty}\epsilon^n \langle T_{ij}\rangle_{\rho_c}^{[n]}$$

$$\nabla^i \langle T_{ij}\rangle_{\rho_c} = 0,$$

$$\langle T_i^i\rangle_{\rho_c} = \frac{1}{16\pi G}R^{(c)} - 8\pi G\rho_c \det[T]_{\rho_c},$$

Impose the conserved equation & anomaly equation

$$\langle T_{z\bar{z}}\rangle_{\rho_c}^{[k]} = \bar{A}_{\rho_c}\langle T_{zz}\rangle_{\rho_c}^{[k]} + A_{\rho_c}\langle T_{\bar{z}\bar{z}}\rangle_{\rho_c}^{[k]} + \mathcal{F}_{z\bar{z}\rho_c}^{[k]},$$

$$\partial_{\bar{z}}\langle T_{zz}\rangle_{\rho_c}^{[k]} = -e^{2\phi}\partial_z(e^{-2\phi}\langle T_{z\bar{z}}\rangle_{\rho_c}^{[k]}) + \mathcal{F}_{zz\rho_c}^{[k]},$$

$$\partial_z\langle T_{\bar{z}\bar{z}}\rangle_{\rho_c}^{[k]} = -e^{2\phi}\partial_{\bar{z}}(e^{-2\phi}\langle T_{z\bar{z}}\rangle_{\rho_c}^{[k]}) + \mathcal{F}_{\bar{z}\bar{z}\rho_c}^{[k]}.$$

$$A_{\rho_c}(z,\bar{z}) = \frac{8\pi G\rho_c\langle T_{zz}(z,\bar{z})\rangle_{\rho_c}^{[0]}}{e^{2\phi(z,\bar{z})} + 16\pi G\rho_c\langle T_{z\bar{z}}(z,\bar{z})\rangle_{\rho_c}^{[0]}}$$

$$\bar{A}_{\rho_c}(z,\bar{z}) = \frac{8\pi G\rho_c\langle T_{\bar{z}\bar{z}}(z,\bar{z})\rangle_{\rho_c}^{[0]}}{e^{2\phi(z,\bar{z})} + 16\pi G\rho_c\langle T_{z\bar{z}}(z,\bar{z})\rangle_{\rho_c}^{[0]}}$$

k-Point functions up to first order deformation

$$\langle T_{z\bar{z}}\rangle_1^{[k]} = 8\pi G e^{-2\phi}\left(\langle T_{\bar{z}\bar{z}}\rangle_0^{[0]}\langle T_{zz}\rangle_0^{[k]} + \langle T_{zz}\rangle_0^{[0]}\langle T_{\bar{z}\bar{z}}\rangle_0^{[k]}\right) + \mathcal{F}_{z\bar{z}1}^{[k]},$$

$$\begin{aligned}\partial_{\bar{z}}\langle T_{zz}\rangle_1^{[k]} = &-8\pi G e^{2\phi}\partial_z[e^{-4\phi}(\langle T_{\bar{z}\bar{z}}\rangle_0^{[0]}\langle T_{zz}\rangle_0^{[k]} + \langle T_{zz}\rangle_0^{[0]}\langle T_{\bar{z}\bar{z}}\rangle_0^{[k]})] \\ &+ \mathcal{F}_{zz1}^{[k]} - e^{2\phi}\partial_z[e^{-2\phi}\mathcal{F}_{z\bar{z}1}^{[k]}],\end{aligned}$$

$$\begin{aligned}\partial_{z}\langle T_{\bar{z}\bar{z}}\rangle_1^{[k]} = &-8\pi G e^{2\phi}\partial_{\bar{z}}[e^{-4\phi}(\langle T_{\bar{z}\bar{z}}\rangle_0^{[0]}\langle T_{zz}\rangle_0^{[k]} + \langle T_{zz}\rangle_0^{[0]}\langle T_{\bar{z}\bar{z}}\rangle_0^{[k]})] \\ &+ \mathcal{F}_{\bar{z}\bar{z}1}^{[k]} - e^{2\phi}\partial_{\bar{z}}[e^{-2\phi}\mathcal{F}_{z\bar{z}1}^{[k]}].\end{aligned}$$

Summary

- **Proposed prescription to study Holographic torus/higher genus stress tensor correlator， which are consistent with CFTs data.**
- **Offer a precise a check (Proof) AdS3/CFT2.**
- **TTbar deformed holographic correlators up to deformation first-order.**
- **Other topologies (X: cross cap), Mixing operators, etc.**

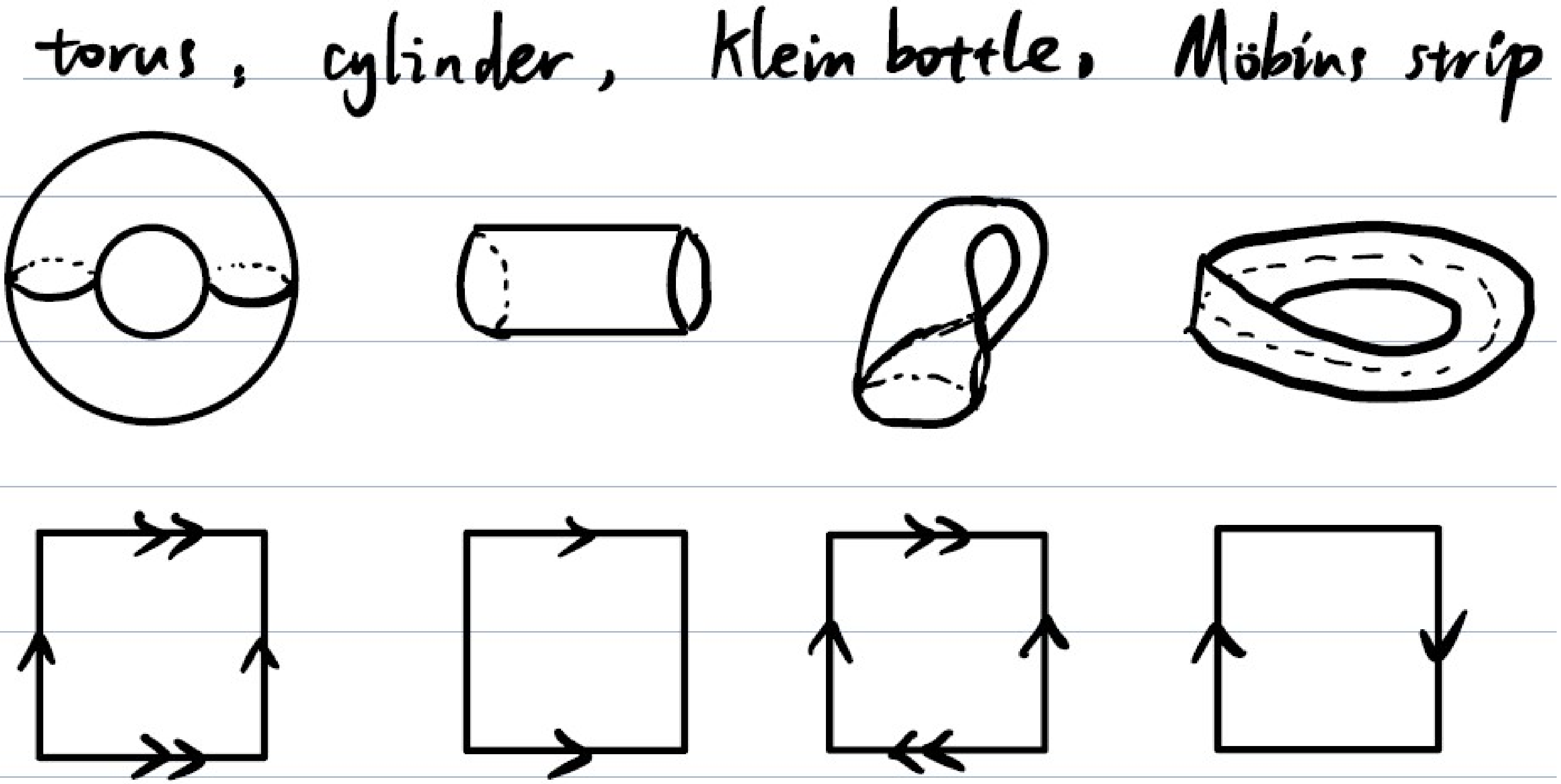
AdS3/CFT2

$$S_E = -\frac{1}{16\pi G}\int\sqrt{g}(R-2\Lambda) - \frac{1}{8\pi G}\int\sqrt{\gamma}K + \frac{1}{8\pi G l}\int\sqrt{\gamma}.$$

Holography for generic 2D Riemann surface

Euclidean AdS3 by discrete identifications

Kirill Krasnov, 0005106



Thermal AdS3

Partition functions, correlation functions, etc.

Thanks