第五届全国“场论与弦论”学术研讨会（中国科学技术大学）

西安交通大学

# Exact solution for open G2 vertex model


李广良

合作者：曹俊鹏、杨文力、石康杰、王玉鹏

CONTENT

# 目录

# 01

# 量子可积模型

# 01 量子可积模型

量子可积：

N自由度的量子定态系统，具有N个独立且相互对易的力学量。

$$[L_i, L_j] = 0, (i, j = 1..., N)$$

$$R_{12}(k_1 - k_2)R_{13}(k_1 - k_3)R_{23}(k_2 - k_3) = R_{23}(k_2 - k_3)R_{13}(k_1 - k_3)R_{12}(k_1 - k_2)$$

$$T(u) = R_{01}(u)R_{02}\cdots R_{0N}(u) \quad t(u) = tr_0(T_0(u))$$

$$\mathcal{A}_{n-1}^{(1)}, \mathcal{B}_n^{(1)}, \mathcal{C}_n^{(1)}, \mathcal{D}_n^{(1)}, \mathcal{A}_{2n}^{(2)}, \mathcal{A}_{2n-1}^{(2)}, \text{ and } \mathcal{D}_{n+1}^{(2)}$$

$$[t(u), t(v)] = 0$$

$$t(u) = \sum_{n=0}^{\infty} t^{(n)} u^n$$

$$[t^m, t^n] = 0 \quad [H, t^n] = 0 \quad H = \frac{1}{2}\sum_{j=1}^{N} \sigma_j \cdot \sigma_{j+1} = \frac{\partial \ln t(u)}{\partial u}\bigg|_{u=0,\{\theta_j=0\}} - \frac{1}{2}N,$$

Yang C N  P.R.L, 1967, 19(23): 1312; Baxter R J  Academic Press, 1982

# 可积模型的应用

- 经典和量子统计力学
- 量子场论
- 量子多体和凝聚态物理
- 低维纳米物理
- 量子光学
- 弦理论(AdS/CFT)

典型模型

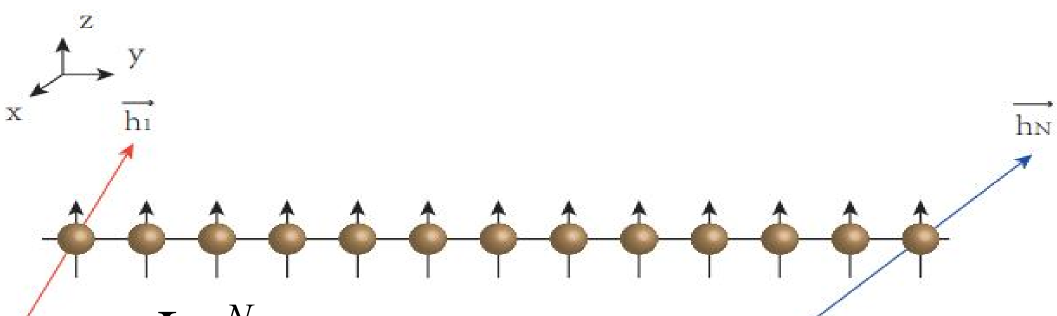
- 中子散射中Heisenberg量子自旋链模型
- 高温超导的Hubbard模型
- 可控量子点中的Kondo模型
- 光晶格中的具有相互作用Bose和Fermi气体

01

# 一维自旋链模型

$$H=-\sum_{n=1}^{N}(\sigma_n^x\sigma_{n+1}^x+\sigma_n^y\sigma_{n+1}^y+\cosh\eta\,\sigma_n^z\sigma_{n+1}^z),$$

$$\sigma_{N+1}^x=\sigma_1^x,\quad \sigma_{N+1}^y=-\sigma_1^y,\quad \sigma_{N+1}^z=-\sigma_1^z.$$




$$H=\frac{J}{2}\sum_{j=1}^{N}\sigma_j\bullet\sigma_{j+1}+h_1\sigma_1^z+h_{Nz}\sigma_N^z+h_{Nx}\sigma_N^x$$

# 02

# ODBA方法

# 精确解

(1) 坐标Bethe Ansatz,

*Bethe H ,Zeitschrift fur Physik 1931,71(205):13*

(2) 代数Bethe Ansatz ,

*Faddeev L et al. Soviet Physics Doklady:1978,243:902-904*

(3) 解析Bethe Ansatz,

*Baxter R J, Annals of Physics 1972,281:187-222； N. Yu. Reshetikhin,Theor. Math. Phys. 63 (1985) 555–569.*

(4) 非对角Bethe Ansatz,

*Y. Wang, W. -L. Yang, J. Cao and K. Shi, Off-Diagonal Bethe Ansatz for Exactly Solvable Models, Springer Press (2015)*

02

# 非对角Bethe-Ansatz(ODBA)

2013年，王玉鹏研究团队提出ODBA（摒弃参考态）：

N阶多项式需要N+1个条件来确定，如果可以找到足够的独立的定解条件，则可以完全确定这个多项式。

$$\Lambda(u) = \Lambda_0\{\prod_{j=1}^{N} f(u-u_j)\}$$

$$T_0(u) \quad = \quad R_{0,N}(u-\theta_N)\cdots R_{0,1}(u-\theta_1)$$

$$t(u) = tr_0 T_0(u) \qquad H = J\frac{\partial \ln t(u)}{\partial u}|_{u=0,\{\theta_j=0\}} - \frac{NJ}{2}$$

$$t(\theta_j)t(\theta_j-1) = a(\theta_j)d(\theta_j-1), \quad j=1,\cdots,N$$

$$a(u) = \prod_{j=1}^{N}(u-\theta_j+1), \quad d(u) = \prod_{j=1}^{N}(u-\theta_j)$$

$$a(\theta_j-1)=0, \quad d(\theta_j)=0$$

# 非对角Bethe-Ansatz(ODBA)

$$\Lambda(u)=a(u)\frac{Q(u-1)}{Q(u)}+d(u)\frac{Q(u+1)}{Q(u)}$$

$$Q(u)=\prod_{j=1}^{N}(u-u_j)$$

非齐次T-Q 关系：

$$t(u)|_{u\to\infty}=2u^N\times id+\cdots$$

$$\Lambda(u)=e^{i\phi}a(u)\frac{Q(u-1)}{Q(u)}+e^{-i\phi}d(u)\frac{Q(u+1)}{Q(u)}+2(1-cos\phi)\frac{a(u)d(u)}{Q(u)}$$

第三项的加入是很重要的突破！不仅仅可以解决U（1）对称破缺的模型，对于U（1）对称性没有破缺的模型也适用！

$$e^{i\phi}a(u_j)Q(u_j-1)+e^{-i\phi}d(u_j)Q(u_j+1)+2(1-cos\phi)a(u_j)d(u_j)=0$$

## 02 取得的成果以及待解决的问题

（1）反周期边界条件的XXZ模型被精确求解；

Cao J, Yang W L, Shi K, Wang Y. Off-diagonal Bethe Ansatz and exact solution of a topological spin ring[J]. Physical Review Letters, 2013, 111: 137201-137205.

（2）任意边界的自旋-1/2XXX模型；

Cao J, Yang W L, Shi K, Wang Y. Off-diagonal Bethe Ansatz solution of the XXX spin chain with arbitrary boundary conditions[J]. Nuclear Physics B, 2013, 875(1): 152-165.

（3）任意边界的Su(n)自旋链模型被精确求解；

Cao J, Yang W L, Shi K, Wang Y. Nested off-diagonal Bethe Ansatz and exact solutions of the su(n) spin chain with generic integrable boundaries[J]. Journal of High Energy Physics, 2014, 04: 143-171.

（4）任意边界的Hubbard模型被精确求解；

Li Y Y, Cao J, Yang W L, Shi K, Wang Y. Exact solution of the one-dimensional Hubbard model with arbitrary boundary magnetic fields[J]. Nuclear Physics B, 2014, 879: 98-109.

（5）奇数格点的XYZ自旋链模型被精确求解；

Cao J, Cui S, Yang W L, Shi K, Wang Y. Spin-1/2 XYZ model revisit: general solutions via off-diagonal Bethe Ansatz[J]. Nuclear Physics B, 2014, 886: 185-201.

待解决的问题：（1）对于B、C、D可积模型，ODBA方法适不适用？封闭关系有什么样规律？（2）怎么寻找更多的定解条件？

03

# G2的求解

# ODBA方法应用的最新进展

01 对于Bn可积模型，为了得到算子封闭关系，需要引入旋量表示，B1模型封闭关系需要3个，B2模型封闭关系需要7个；4n-1

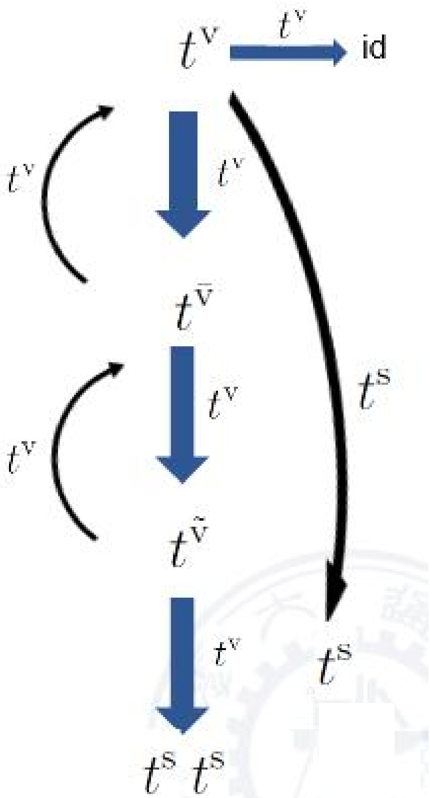
02 对于Cn可积模型，不需要引入旋量表示，C2模型封闭关系需要3个； C3模型封闭关系需要5个；2n-1

03 对于Dn可积模型，需要引入两种不等价的旋量表示，D3模型封闭关系需要4个；2n-2

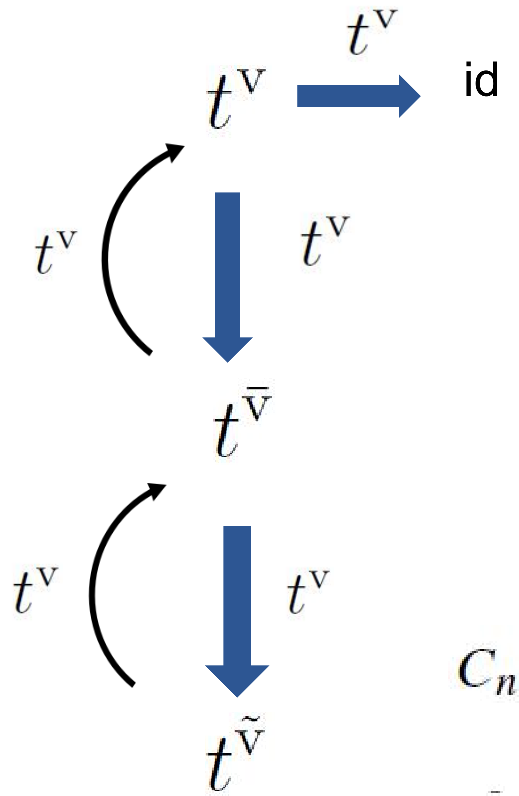
04 D2模型退化点简并，无法给出足够封闭关系，需要分解

03

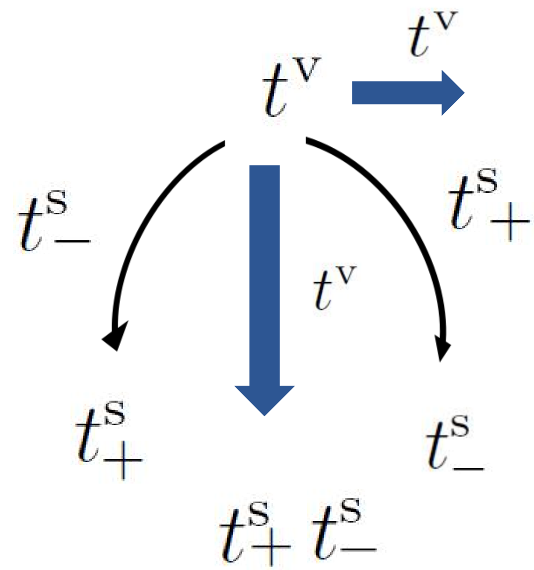
# C2,C3,D3开边界条件下的算子封闭关系



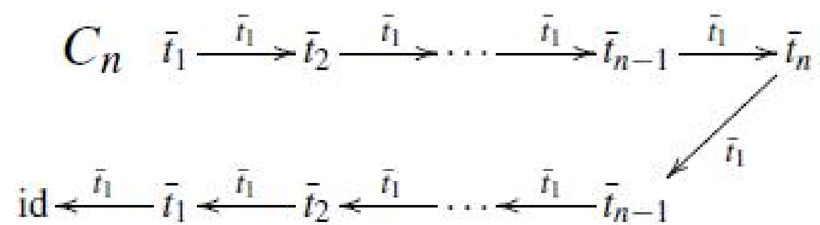

**B2**




**C3**




**D3**




G-L. Li et al, B2 NPB 946 (2019) 114719

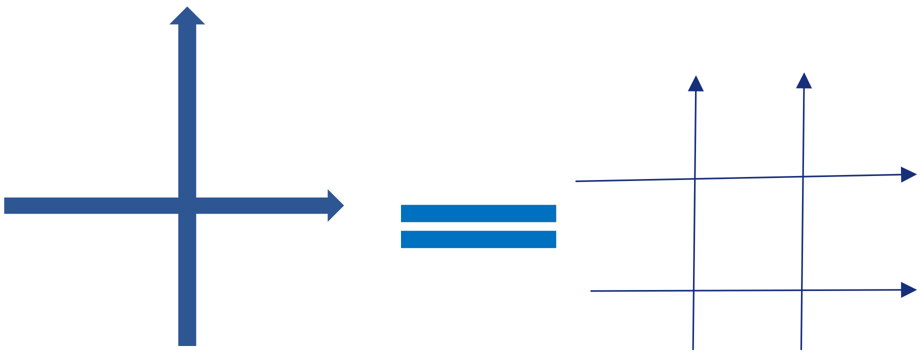
C2 JHEP 05 (2019) 067; Cn NPB 965 (2021) 115333

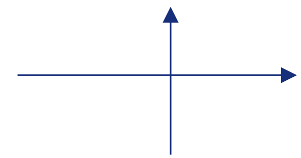
D3 JHEP 12 (2019) 051

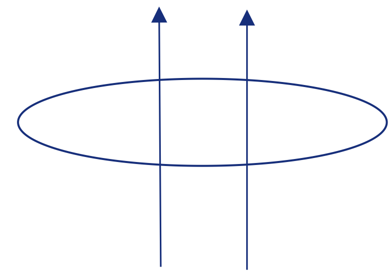
$D_3^{(2)}$ JHEP 03 (2022) 175 $A_3^{(2)}$ CMP 399 (2023) 651

# $D_2^{(2)}$ 模型的分解 G-L. Li et al, JHEP 04（2022）101

$$t(u) = tr_{0'}\{\bar{K}_{0'}(u)T_{0'}(u)K_{0'}(u)\hat{T}_{0'}(u)\}$$



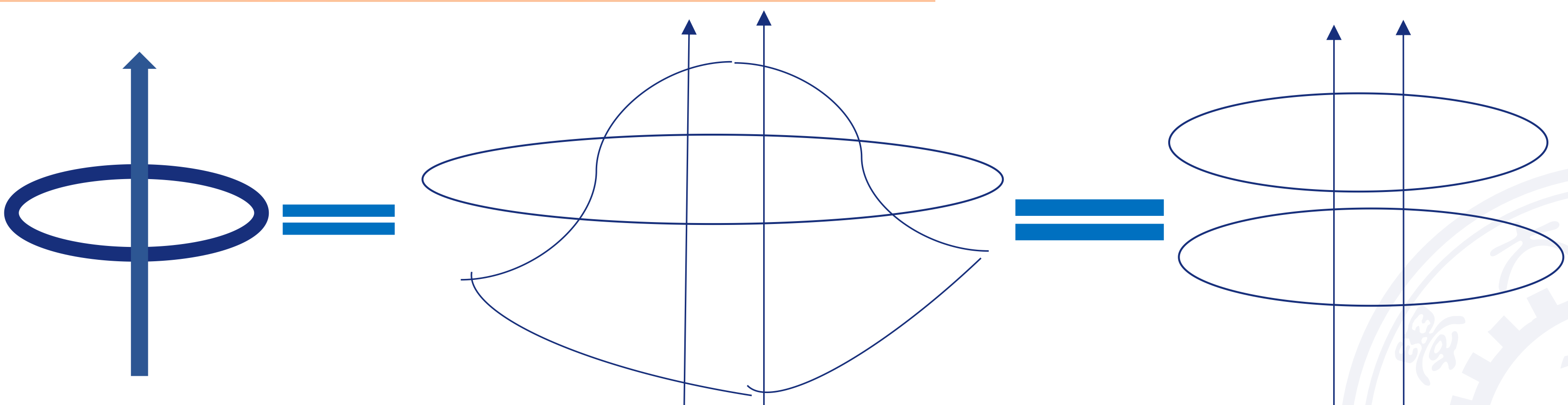




$$R^d_{12}(u) = 2^4 S_{1'2'}S_{3'4'}R_{1'4'}(u+i\pi)R_{1'3'}(u)R_{2'4'}(u)R_{2'3'}(u-i\pi)S^{-1}_{1'2'}S^{-1}_{3'4'}$$

$$K_1^+(u) = [\rho_s(i\pi)]^{-\frac{1}{2}} S\tilde{R}_{2'1'}(i\pi)\tilde{K}^+_{2'}(u)\tilde{M}^{-1}_{2'}\tilde{R}_{1'2'}(-2u+4\eta-i\pi)\tilde{M}_{2'}\tilde{K}^+_{1'}(u)S^{-1},$$

$$K_1^-(u) = [\rho_s(i\pi)]^{-\frac{1}{2}} S\tilde{K}^-_{1'}(u+i\pi)\tilde{R}_{2'1'}(2u+i\pi)\tilde{K}^-_{2'}(u)\tilde{R}_{1'2'}(-i\pi)S^{-1},$$



$$t^d(u) = tr_0\{\bar{K}^d_0(u)T^d_0(u)K^d_0(u)\hat{T}^d_0(u)\}.$$

$$\bar{t}(u+i\pi)\bar{t}(u) = 2^{-8N}[\rho_s(2u+2i\pi-2\eta)]^{-1}\mathcal{S}^{-1}t^d(u)\mathcal{S},$$

Solved $A_n, B_2, C_n, D_3, A_2^{(2)}, A_3^{(2)}, D_2^{(1)}, D_2^{(2)}, D_3^{(2)}$

Unsolved $G_2^{(1)}, D_4^{(3)}, E_6^{(1)}, E_6^{(2)}$ 超对称模型

---

## G2模型

$$\{H_1, H_2, E_1, \cdots, E_6, E_1^\dagger, \cdots, E_6^\dagger\}$$

$$H_1 = \widehat{H}_1 + \widehat{H}_2, \quad H_2 = \frac{1}{\sqrt{3}}(\widehat{H}_1 - \widehat{H}_2)$$

$$\widehat{H}_1 = \mathrm{diag}\{1, 0, 1, 0, -1, 0, -1\}, \quad \widehat{H}_2 = \mathrm{diag}\{0, 1, -1, 0, 1, -1, 0\},$$

$$E_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$E_3 = \frac{1}{\sqrt{2}}[E_1, E_2], \quad E_4 = \sqrt{\frac{3}{8}}[E_1, E_3], \quad E_5 = \frac{1}{\sqrt{2}}[E_1, E_4], \quad E_6 = \frac{1}{\sqrt{2}}[E_2, E_5]$$

$$C_{i,i+1} = H_1^{(i)} \otimes H_1^{(i+1)} + H_2^{(i)} \otimes H_2^{(i+1)} + \sum_{l=1}^{6}\left(E_l^{(i)} \otimes [E_l^{(i+1)}]^\dagger + [E_l^{(i)}]^\dagger \otimes E_l^{(i+1)}\right)$$

$$\rho_0(\lambda) = (1+\lambda)(4+\lambda)(6+\lambda)$$

$$P_{12}^{(0)} = \frac{1}{56}C_{12} - \frac{1}{112}C_{12}^2 - \frac{3}{896}C_{12}^3,$$

$$P_{12}^{(7)} = -\frac{1}{8}C_{12} + \frac{5}{64}C_{12}^2 + \frac{3}{256}C_{12}^3,$$

$$P_{12}^{(14)} = \mathrm{Id}_1 \otimes \mathrm{Id}_2 - \frac{3}{8}C_{12} - \frac{1}{4}C_{12}^2 - \frac{3}{128}C_{12}^3,$$

$$P_{12}^{(27)} = \frac{27}{56}C_{12} + \frac{81}{448}C_{12}^2 + \frac{27}{1792}C_{12}^3$$

$$7 \otimes 7 = 1 \oplus 7 \oplus 14 \oplus 27.$$

$$R_{12}(\lambda) = \rho_0(\lambda)\left(P_{12}^{(27)} + \frac{\lambda-1}{\lambda+1}P_{12}^{(14)} + \frac{\lambda-4}{\lambda+4}P_{12}^{(7)} + \frac{(\lambda-1)(\lambda-6)}{(\lambda+1)(\lambda+6)}P_{12}^{(0)}\right)$$

# G2 R和K矩阵

$$\text{regularity} \;:\; R_{12}(0) = \rho_1(0)^{\frac{1}{2}}\mathcal{P}_{12},$$

$$\text{unitarity} \;:\; R_{12}(u)R_{21}(-u) = \rho_1(u) = a(u)a(-u),$$

$$V^i_j = (-1)^{i-1}\delta_{i,\bar{j}}.$$

$$\text{crossing}-\text{symmetry} \;:\; R_{12}(u) = -V_1\left\{R_{12}(-u-6)\right\}^{t_2}V_1 = -V_2\left\{R_{12}(-u-6)\right\}^{t_1}V_2,$$

$$R_{12}(u-v)K_1^-(u)R_{21}(u+v)K_2^-(v) = K_2^-(v)R_{12}(u+v)K_1^-(u)R_{21}(u-v).$$

$$K^-(u) = 1 + Mu,$$

$$M = \begin{pmatrix} c_{11} & 0 & 0 & 0 & c_1 & c_2 & 0 \\ 0 & c_{22} & c_3 & 0 & 0 & 0 & -c_2 \\ 0 & c_3 & c_{33} & 0 & 0 & 0 & c_1 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ c_1 & 0 & 0 & 0 & c_{33} & -c_3 & 0 \\ c_2 & 0 & 0 & 0 & -c_3 & c_{22} & 0 \\ 0 & -c_2 & c_1 & 0 & 0 & 0 & c_{11} \end{pmatrix},$$

$$c_{11} = \frac{c_1c_3}{c_2} + \frac{c_2c_3}{c_1} - 2,$$

$$c_{22} = 2 - \frac{c_2c_3}{c_1},$$

$$c_{33} = 2 - \frac{c_1c_3}{c_2}.$$

$$\frac{c_1c_3}{c_2} + \frac{c_2c_3}{c_1} + \frac{c_1c_2}{c_3} = 4.$$

# 投影算子性质

$$R_{12}(\delta) = P_{12}^{(d)} \times S,$$

$$V_1 \otimes V_2 \to V_{[d]}$$

$$P_{12}^{(d)} R_{23}(u) R_{13}(u+\delta) P_{12}^{(d)} = R_{23}(u) R_{13}(u+\delta) P_{12}^{(d)}, \Rightarrow R_{[d]3}(u)$$

$$P_{21}^{(d)} R_{32}(u) R_{31}(u+\delta) P_{21}^{(d)} = R_{32}(u) R_{31}(u+\delta) P_{21}^{(d)}, \Rightarrow R_{3[d]}(u)$$

$$P_{12}^{(d)} K_2^-(u) R_{12}(2u+\delta) K_1^-(u+\delta) P_{21}^{(d)}$$

$$= K_2^-(u) R_{12}(2u+\delta) K_1^-(u+\delta) P_{21}^{(d)}, \Rightarrow K^-_{[d]}(u)$$

$$P_{21}^{(d)} K_1^+(u+\delta) R_{21}(-2u-2\kappa-\delta) K_2^+(u) P_{12}^{(d)}$$

$$= K_1^+(u+\delta) R_{21}(-2u-2\kappa-\delta) K_2^+(u) P_{12}^{(d)}. \Rightarrow K^+_{[d]}(u)$$

# 投影算子的产生

$$T(u) = R_{01}(u)R_{02}\cdots R_{0N}(u) \qquad t(u) = tr_0 T_0(u)$$

$$\begin{aligned}
T_a(\theta_j)T_b(\theta_j+\delta) &= R_{a1}(\theta_j-\theta_1)\cdots R_{aj-1}(\theta_j-\theta_{j-1})R_{aj}(0)R_{aj+1}(\theta_j-\theta_{j+1})\cdots \\
&\quad\times R_{aN}(\theta_j-\theta_N)R_{b1}(\theta_j-\theta_1+\delta)\cdots R_{bj-1}(\theta_j-\theta_{j-1}+\delta)R_{bj}(\delta) \\
&\quad\times R_{aj}(0)R_{ja}(0)\rho_{ab}(0)^{-1}R_{bj+1}(\theta_j-\theta_{j+1}+\delta)\cdots R_{bN}(\theta_j-\theta_N+\delta) \\
&= R_{jj+1}(\theta_j-\theta_{j+1})\cdots R_{jN}(\theta_j-\theta_N)R_{a1}(\theta_j-\theta_1)\cdots R_{aj-1}(\theta_j-\theta_{j-1}) \\
&\quad\times R_{b1}(\theta_j-\theta_1+\delta)\cdots R_{bj-1}(\theta_j-\theta_{j-1}+\delta) \\
&\quad\times P_{ba}^{(d)}S_d R_{ja}(0)R_{bj+1}(\theta_j-\theta_{j+1}+\delta)\cdots R_{bN}(\theta_j-\theta_N+\delta) \\
&= P_{ba}^{(d)}R_{a1}(\theta_j-\theta_1)\cdots R_{aj-1}(\theta_j-\theta_{j-1})R_{aj}(0)R_{ja}(0)\rho_{ab}(0)^{-1}R_{jj+1}(\theta_j-\theta_{j+1})\cdots \\
&\quad\times R_{jN}(\theta_j-\theta_N)R_{b1}(\theta_j-\theta_1+\delta)\cdots R_{bj-1}(\theta_j-\theta_{j-1}+\delta) \\
&\quad\times R_{ba}(\delta)R_{ja}(0)R_{bj+1}(\theta_j-\theta_{j+1}+\delta)\cdots R_{bN}(\theta_j-\theta_N+\delta) \\
&= P_{ba}^{(d)}T_a(\theta_j)T_b(\theta_j+\delta).
\end{aligned}$$

$$\hat{T}_a(-\theta_j)\hat{T}_b(-\theta_j+\delta) = P_{ab}^{(d)}\hat{T}_a(-\theta_j)\hat{T}_b(-\theta_j+\delta),$$

## 算子恒等式

$$t^{(p)}(u)|_{u\to\pm\infty} = 7u^{3N}\times \mathrm{id} + \cdots\,,\quad \bar{t}^{(p)}(u)|_{u\to\pm\infty} = 15u^{2N}\times \mathrm{id} + \cdots\,,$$

$$\tilde{t}^{(p)}(u)|_{u\to\pm\infty} = 34u^{4N}\times \mathrm{id} + \cdots\,.$$

(1) $$t^{(p)}(\theta_j)\,t^{(p)}(\theta_j-6) = \prod_{i=1}^{N} a(\theta_j-\theta_i)e(\theta_j-\theta_i-6)\times \mathrm{id},$$

(2) $$t^{(p)}(\theta_j)\,t^{(p)}(\theta_j-4) = \prod_{i=1}^{N}(\theta_j-\theta_i+1)(\theta_j-\theta_i-4)(\theta_j-\theta_i-6)\,t^{(p)}(\theta_j-2),$$

(3) $$t^{(p)}(\theta_j)\,t^{(p)}(\theta_j-1) = \prod_{i=1}^{N}(\theta_j-\theta_i-1)a(\theta_j-\theta_i)\,\bar{t}^{(p)}(\theta_j-\frac{1}{2}),$$

(4) $$t^{(p)}(\theta_j)\,\bar{t}^{(p)}(\theta_j-\frac{11}{2}) = \prod_{i=1}^{N}(\theta_j-\theta_i+4)(\theta_j-\theta_i+6)\,t^{(p)}(\theta_j-5),$$

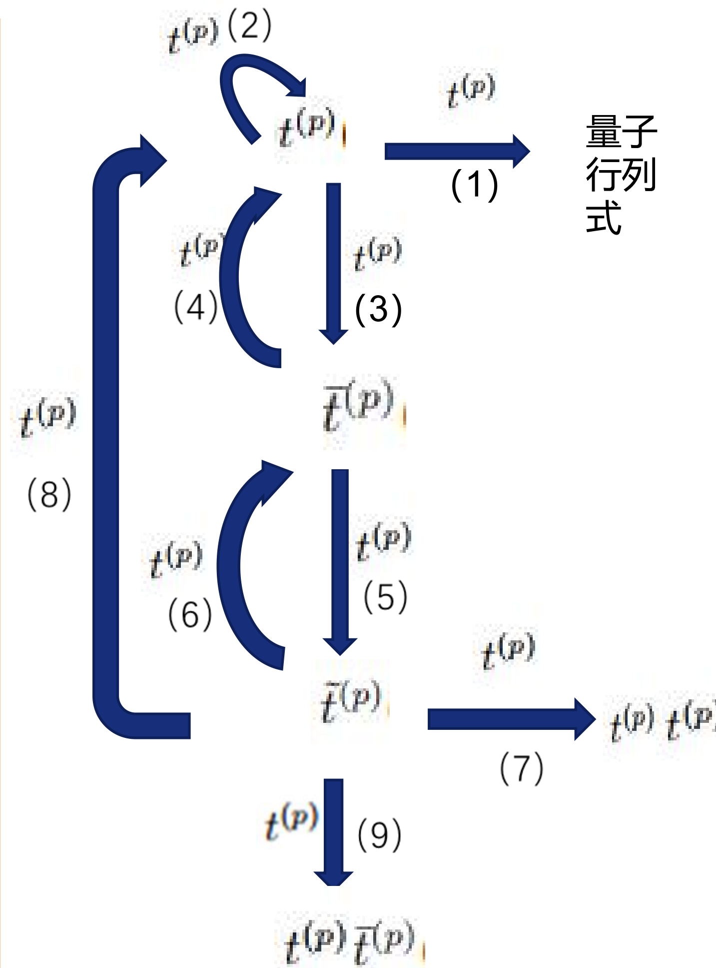
(5) $$t^{(p)}(\theta_j)\,\bar{t}^{(p)}(\theta_j-\frac{7}{2}) = \prod_{i=1}^{N}(\theta_j-\theta_i+6)\,\tilde{t}^{(p)}(\theta_j-\frac{5}{2}),$$

(6) $$t^{(p)}(\theta_j)\,\tilde{t}^{(p)}(\theta_j-\frac{7}{2}) = \prod_{i=1}^{N}(\theta_j-\theta_i-1)(\theta_j-\theta_i-4)a(\theta_j-\theta_i)\,\bar{t}^{(p)}(\theta_j-\frac{5}{2}),$$

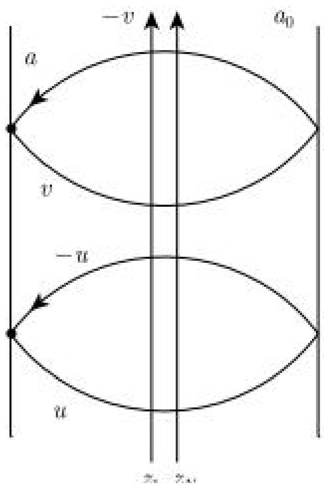
(7) $$t^{(p)}(\theta_j)\,\tilde{t}^{(p)}(\theta_j-\frac{9}{2}) = \prod_{i=1}^{N}(\theta_j-\theta_i+6)\,t^{(p)}(\theta_j-2)t^{(p)}(\theta_j-5),$$

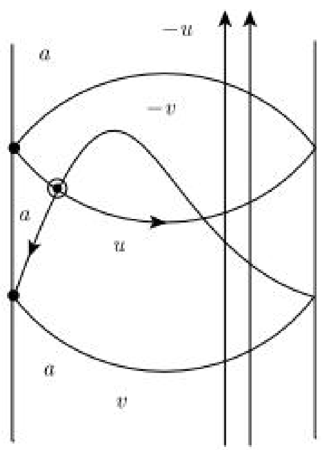
(8) $$t^{(p)}(\theta_j)\,\tilde{t}^{(p)}(\theta_j-\frac{13}{2}) = \prod_{i=1}^{N}(\theta_j-\theta_i-4)a(\theta_j-\theta_i)\,t^{(p)}(\theta_j-7),$$

(9) $$t^{(p)}(\theta_j)\,\tilde{t}^{(p)}(\theta_j-\frac{3}{2}) = \prod_{i=1}^{N}(\theta_j-\theta_i-1)(\theta_j-\theta_i-6)\,t^{(p)}(\theta_j-2)\bar{t}^{(p)}(\theta_j-\frac{1}{2}),$$



# 开边聚合公式




$$
\begin{aligned}
t_a(u)t_b(u+\delta) &= tr_a\{K_a^+(u)T_a(u)K_a^-(u)\hat{T}_a(u)\} \\
&\quad \times tr_b\{K_b^+(u+\delta)T_b(u+\delta)K_b^-(u+\delta)\hat{T}_b(u+\delta)\}^{t_b} \\
&= tr_{ab}\{K_a^+(u)T_a(u)K_a^-(u)\hat{T}_a(u)[T_b(u+\delta)K_b^-(u+\delta)\hat{T}_b(u+\delta)]^{t_b}[K_b^+(u+\delta)]^{t_b}\} \\
&= [\tilde{\rho}_{ab}(2u+\delta)]^{-1}tr_{ab}\{K_a^+(u)T_a(u)K_a^-(u)\hat{T}_a(u)[T_b(u+\delta)K_b^-(u+\delta) \\
&\quad \times\hat{T}_b(u+\delta)]^{t_b}R_{ba}^{t_b}(2u+\delta)R_{ab}^{t_b}(-2u-2\kappa-\delta)[K_b^+(u+\delta)]^{t_b}\} \\
&= [\tilde{\rho}_{ab}(2u+\delta)]^{-1}tr_{ab}\{[K_b^+(u+\delta)R_{ab}(-2u-2\kappa-\delta)K_a^+(u)T_a(u) \\
&\quad \times K_a^-(u)\hat{T}_a(u)]^{t_b}[R_{ba}(2u+\delta)T_b(u+\delta)K_b^-(u+\delta)\hat{T}_b(u+\delta)]^{t_b}\} \\
&= [\tilde{\rho}_{ab}(2u+\delta)]^{-1}tr_{ab}\{K_b^+(u+\delta)R_{ab}(-2u-2\kappa-\delta)K_a^+(u)T_a(u) \\
&\quad \times K_a^-(u)\hat{T}_a(u)R_{ba}(2u+\delta)T_b(u+\delta)K_b^-(u+\delta)\hat{T}_b(u+\delta)\} \\
&= [\tilde{\rho}_{ab}(2u+\delta)]^{-1}tr_{ab}\{\underline{K_b^+(u+\delta)R_{ab}(-2u-2\kappa-\delta)K_a^+(u)}\underline{T_a(u)T_b(u+\delta)} \\
&\quad \underline{\times K_a^-(u)R_{ba}(2u+\delta)K_b^-(u+\delta)}\underline{\hat{T}_a(u)\hat{T}_b(u+\delta)}\}.
\end{aligned}
\tag{2.28}
$$

# 算子恒等式

(1) $$t(\pm\theta_j)\,t(\pm\theta_j-6)\sim \mathrm{id},$$

(2) $$t(\pm\theta_j)\,t(\pm\theta_j-4)\sim t(\pm\theta_j-2),$$

(3) $$t(\pm\theta_j)\,t(\pm\theta_j-1)\sim \bar{t}^{(p)}(\pm\theta_j-\frac{1}{2}),$$

(4) $$t(\pm\theta_j)\,\bar{t}(\pm\theta_j-\frac{11}{2})\sim t(\pm\theta_j-5),$$

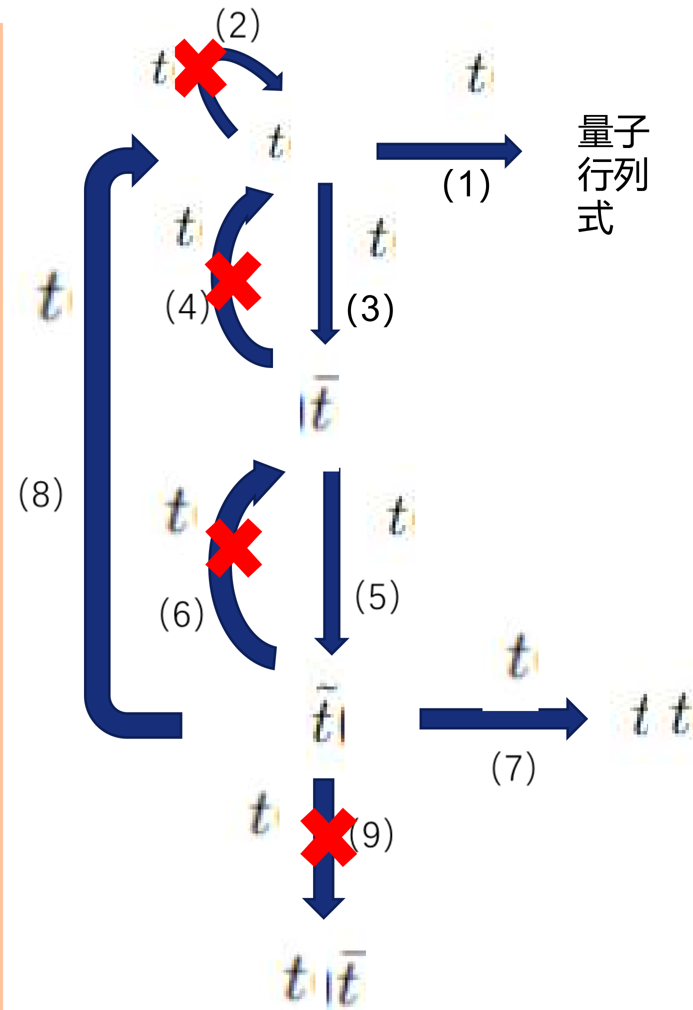
(5) $$t(\pm\theta_j)\,\bar{t}(\pm\theta_j-\frac{7}{2}\sim \tilde{t}(\pm\theta_j-\frac{5}{2}),$$

(6) $$t(\pm\theta_j)\,\tilde{t}(\pm\theta_j-\frac{7}{2})\sim \bar{t}(\pm\theta_j-\frac{5}{2}),$$

(7) $$t(\pm\theta_j)\,\tilde{t}(\pm\theta_j-\frac{9}{2})\sim t(\pm\theta_j-2)t(\pm\theta_j-5),$$

(8) $$t(\pm\theta_j)\,\tilde{t}(\pm\theta_j-\frac{13}{2})\sim t(\pm\theta_j-7),$$

(9) $$t(\pm\theta_j)\,\tilde{t}(\pm\theta_j-\frac{3}{2})\sim t(\pm\theta_j-2)\bar{t}(\pm\theta_j-\frac{1}{2}),$$



# 交叉幺正性

$$R_{12}(u) = -V_1 R_{21}^{t_1}(-u-6) V_1^{-1} = -V_2^{t_2} R_{21}^{t_2}(-u-6)[V_2^{t_2}]^{-1}$$

$$T_0^{t_0}(-u-6) = (-1)^N [V_0^{t_0}]^{-1} \hat{T}_0(u) V_0^{t_0} \qquad \hat{T}_0^{t_0}(-u-6) = (-1)^N V_0^{-1} T_0(u) V_0.$$

$$tr_1\{R_{12}(0)R_{12}(2u)V_1[K_1^-(-u-6)]^{t_1}[V_1^{t_1}]^{-1}\} = f(u)K_2^-(u),$$

$$tr_2\{R_{12}(0)R_{12}(2u)K_2^+(u)\} = f(u)V_1^{t_1}K_1^+(-u-6)^{t_1}V_1^{-1},$$

$$\begin{aligned}
t(-u-6) &= tr_0\{K_0^+(-u-6)T_0(-u-6)\}^{t_0}\{K_0^-(-u-6)\hat{T}_0(-u-6)\}^{t_0} \\
&= tr_0\hat{T}_0(u)V_0^{t_0}\{K_0^+(-u-6)\}^{t_0}V_0^{-1}T_0(u)V_0\{K_0^-(-u-6)\}^{t_0}[V_0^{t_0}]^{-1} \\
&= tr_0\hat{T}_0(u)tr_1R_{01}(0)R_{01}(2u)K_1^+(u)T_0(u)V_0\{K_0^-(-u-6)\}^{t_0}[V_0^{t_0}]^{-1}/f(u) \\
&= tr_1tr_0R_{10}(0)\hat{T}_1(u)R_{01}(2u)T_0(u)V_0\{K_0^-(-u-6)\}^{t_0}[V_0^{t_0}]^{-1}K_1^+(u)/f(u) \\
&= tr_1tr_0R_{10}(0)T_0(u)R_{01}(2u)\hat{T}_1(u)V_0\{K_0^-(-u-6)\}^{t_0}[V_0^{t_0}]^{-1}K_1^+(u)/f(u) \\
&= tr_1T_1(u)tr_0R_{01}R_{01}(2u)V_0\{K_0^-(-u-6)\}^{t_0}[V_0^{t_0}]^{-1}\hat{T}_1(u)K_1^+(u)/f(u) \\
&= tr_1K_1^+(u)T_1(u)K_1^-(u)\hat{T}_1(u) = t(u).
\end{aligned}$$

## T-Q关系

$$t(-u-6) = t(u).$$

$$\bar{t}(u) = \bar{t}(-u-6).$$

$$\tilde{t}(u) = \tilde{t}(-u-6).$$

$$\Lambda(\theta_j)\Lambda(-\theta_j) = 4^2 \frac{(\theta_j-1)(-\theta_j)(\theta_j+1)(\theta_j+6)}{(\theta_j-2)(\theta_j-3)(\theta_j+2)(\theta_j+3)}$$

$$\times(\theta_j-\frac{1}{2})(\theta_j-\frac{5}{2})(\theta_j+\frac{1}{2})(\theta_j+\frac{5}{2})\prod_{i=1}^{N}\rho_{12}(\theta_j-\theta_i)\rho_{12}(\theta_j+\theta_i),$$

$$\Lambda(\pm\theta_j)\Lambda(\pm\theta_j-1) = -\frac{(\pm\theta_j-1)(\pm\theta_j+6)(\pm\theta_j+\frac{5}{2})^2}{(\pm\theta_j+2)(\pm\theta_j+3)}$$

$$\times\prod_{i=1}^{N}(\pm\theta_j-\theta_i-1)(\pm\theta_j+\theta_i-1)a(\pm\theta_j-\theta_i)a(\pm\theta_j+\theta_i)\bar{\Lambda}(\pm\theta_j-\frac{1}{2}),$$

# T-Q关系

$$\Lambda(\pm\theta_j)\bar{\Lambda}(\pm\theta_j-\frac{7}{2})=-\frac{(\pm\theta_j+1)(\pm\theta_j+6)}{(\pm\theta_j-\frac{1}{2})(\pm\theta_j-\frac{3}{2})(\pm\theta_j+3)(\pm\theta_j+4)}$$
$$\times\prod_{i=1}^{N}(\pm\theta_j-\theta_i+6)(\pm\theta_j+\theta_i+6)\tilde{\Lambda}(\pm\theta_j-\frac{5}{2}),$$

$$\Lambda(\pm\theta_j)\tilde{\Lambda}(\pm\theta_j-\frac{9}{2})=2^4\frac{(\pm\theta_j+1)(\pm\theta_j+6)}{(\pm\theta_j+3)(\pm\theta_j+4)}(\pm\theta_j-\frac{1}{2})(\pm\theta_j-\frac{7}{2})(\pm\theta_j+\frac{1}{2})(\pm\theta_j+\frac{5}{2})$$
$$\times\prod_{i=1}^{N}(\pm\theta_j-\theta_i+6)(\pm\theta_j+\theta_i+6)\Lambda(\pm\theta_j-2)\Lambda(\pm\theta_j-5). \quad (3.4)$$

$$\Lambda(\pm\theta_j)\tilde{\Lambda}(\pm\theta_j-\frac{13}{2})=-2^6\frac{(\pm\theta_j-4)(\pm\theta_j+1)(\pm\theta_j+6)}{(\pm\theta_j-2)(\pm\theta_j+2)(\pm\theta_j+3)}$$
$$\times(\pm\theta_j-\frac{11}{2})(\pm\theta_j-\frac{5}{2})(\pm\theta_j-\frac{3}{2})(\pm\theta_j-\frac{1}{2})(\pm\theta_j+\frac{1}{2})(\pm\theta_j+\frac{5}{2})$$
$$\times\prod_{i=1}^{N}(\pm\theta_j-\theta_i+4)(\pm\theta_j+\theta_i+4)a(\pm\theta_j-\theta_i)a(\pm\theta_j+\theta_i)\Lambda(\pm\theta_j-7).$$

$$\Lambda(u)|_{u\to\pm\infty} = Au^{6N+2} + \cdots, \quad A = -2[-2 + 2(c_1\tilde{c}_1 + c_2\tilde{c}_2 + c_3\tilde{c}_3) + \frac{c_1c_2\tilde{c}_1\tilde{c}_2}{c_3\tilde{c}_3} + \frac{c_1c_3\tilde{c}_1\tilde{c}_3}{c_2\tilde{c}_2} + \frac{c_3c_2\tilde{c}_3\tilde{c}_2}{c_1\tilde{c}_1}].$$

$$\bar{\Lambda}(u)|_{u\to\pm\infty} = 16[-(\frac{A}{8})^2 + \frac{A}{8} + \frac{3}{4}]u^{4N+2} + \cdots,$$

$$\tilde{\Lambda}(u)|_{u\to\pm\infty} = -128[\frac{3}{2}(\frac{A}{8})^2 + \frac{A}{16} + \frac{3}{8}]u^{8N+6} + \cdots,$$

3N+2N+4N+1+1+
1+1+3+1=9N+8

$$\Lambda(0) = -5\prod_{l=1}^{N}\rho_{12}(\theta_l),$$

$$\Lambda(-1) = -\frac{5}{4}\prod_{l=1}^{N}(\theta_l - 1)(-\theta_l - 1)\bar{\Lambda}(-\frac{1}{2}),$$

$$\tilde{\Lambda}(-\frac{5}{2}) = -\frac{15}{2}\prod_{l=1}^{N}(\theta_l + 1)(-\theta_l + 1)(\theta_l + 4)(-\theta_l + 4)\bar{\Lambda}(-\frac{7}{2}),$$

$$\tilde{\Lambda}(-\frac{13}{2}) = 330\prod_{l=1}^{N}(\theta_l - 4)(-\theta_l - 4)\Lambda(-7),$$

$$\tilde{\Lambda}(-1) = 0.$$

$$P_{\bar{1}2}^{(7)\perp} = 1 - P_{\bar{1}2}^{(7)},$$

$$t(u)\tilde{t}(u - \frac{13}{2}) = -2^6\frac{(u-4)(u+1)(u+6)}{(u-2)(u+2)(u+3)}$$
$$\times(u - \frac{11}{2})(u - \frac{5}{2})(u - \frac{3}{2})(u - \frac{1}{2})(u + \frac{1}{2})(u + \frac{5}{2})$$
$$\times\prod_{i=1}^{N}(u - \theta_i + 4)(u + \theta_i + 4)a(u - \theta_i)a(u + \theta_i)t(u - 7)$$
$$+\frac{u(u - \frac{11}{2})(u - \frac{3}{2})}{2^8(u+3)(u+2)(u+\frac{3}{2})(u+\frac{1}{2})(u-1)(u-2)(u-\frac{5}{2})(u-\frac{7}{2})}$$
$$\times\prod_{i=1}^{N}(u - \theta_i)(u + \theta_i)\tilde{t}^{\perp}(u - 7).$$

$$\Lambda(u) = Z_1(u) + Z_2(u) + Z_3(u) + Z_4(u) + Z_5(u) + Z_6(u) + Z_7(u) + f_1(u) + f_2(u),$$

$$Z_1(u) = -4\frac{(u+1)(u+6)}{(u+2)(u+3)}(u+\frac{1}{2})(u+\frac{5}{2})\prod_{j=1}^{N} a(u-\theta_j)a(u+\theta_j)\frac{Q^{(1)}(u-1)}{Q^{(1)}(u)}$$

$$Z_2(u) = -4\frac{u(u+6)}{(u+2)(u+3)}(u+\frac{1}{2})(u+\frac{5}{2})\prod_{j=1}^{N} b(u-\theta_j)b(u+\theta_j)\frac{Q^{(1)}(u+1)Q^{(2)}(u-3)}{Q^{(1)}(u)Q^{(2)}(u)}$$

$$Z_3(u) = -4\frac{u(u+6)}{(u+2)(u+3)}(u+\frac{7}{2})(u+\frac{5}{2})\prod_{j=1}^{N} b(u-\theta_j)b(u+\theta_j)\frac{Q^{(1)}(u+1)Q^{(2)}(u+3)}{Q^{(1)}(u+3)Q^{(2)}(u)}$$

$$Z_4(u) = -4\frac{u(u+6)}{(u+2)(u+4)}(u+\frac{7}{2})(u+\frac{5}{2})\prod_{j=1}^{N} c(u-\theta_j)c(u+\theta_j)\frac{Q^{(1)}(u+1)Q^{(1)}(u+4)}{Q^{(1)}(u+2)Q^{(1)}(u+3)}$$

$$Z_5(u) = -4\frac{u(u+6)}{(u+3)(u+4)}(u+\frac{7}{2})(u+\frac{5}{2})\prod_{j=1}^{N} d(u-\theta_j)d(u+\theta_j)\frac{Q^{(1)}(u+4)Q^{(2)}(u-1)}{Q^{(1)}(u+2)Q^{(2)}(u+2)}$$

$$Z_6(u) = -4\frac{u(u+6)}{(u+3)(u+4)}(u+\frac{7}{2})(u+\frac{11}{2})\prod_{j=1}^{N} d(u-\theta_j)d(u+\theta_j)\frac{Q^{(1)}(u+4)Q^{(2)}(u+5)}{Q^{(1)}(u+5)Q^{(2)}(u+2)}$$

$$Z_7(u) = -4\frac{u(u+5)}{(u+3)(u+4)}(u+\frac{7}{2})(u+\frac{11}{2})\prod_{j=1}^{N} e(u-\theta_j)e(u+\theta_j)\frac{Q^{(1)}(u+6)}{Q^{(1)}(u+5)}$$

$$Q^{(1)}(u)=\prod_{k=1}^{L_1}(iu+\mu_k^{(1)}+i\frac{1}{2})(iu-\mu_k^{(1)}+i\frac{1}{2}),$$

$$Q^{(2)}(u)=\prod_{k=1}^{L_2}(iu+\mu_k^{(2)}+2i)(iu-\mu_k^{(2)}+2i).$$

$$f_1(u)=-4\frac{u(u+6)}{(u+3)}(u+\frac{1}{2})(u+\frac{5}{2})(u+\frac{7}{2})$$
$$\times\prod_{j=1}^{N}b(u-\theta_j)b(u+\theta_j)\frac{Q^{(1)}(u+1)}{Q^{(2)}(u)}x,$$

$$f_2(u)=-4\frac{u(u+6)}{(u+3)}(u+\frac{5}{2})(u+\frac{7}{2})(u+\frac{11}{2})$$
$$\times\prod_{j=1}^{N}d(u-\theta_j)d(u+\theta_j)\frac{Q^{(1)}(u+4)}{Q^{(2)}(u+2)}x,$$

$$\frac{Q^{(1)}(i\mu_k^{(1)}+\frac{1}{2})Q^{(2)}(i\mu_k^{(1)}-\frac{7}{2})}{Q^{(1)}(i\mu_k^{(1)}-\frac{3}{2})Q^{(2)}(i\mu_k^{(1)}-\frac{1}{2})}=-\frac{(i\mu_k^{(1)}+\frac{1}{2})\prod_{j=1}^{N}(i\mu_k^{(1)}-\theta_j+\frac{1}{2})(i\mu_k^{(1)}+\theta_j+\frac{1}{2})}{(i\mu_k^{(1)}-\frac{1}{2})\prod_{j=1}^{N}(i\mu_k^{(1)}-\theta_j-\frac{1}{2})(i\mu_k^{(1)}+\theta_j-\frac{1}{2})},$$
$$k=1,2,\cdots,L_1, \quad (3.$$

$$\frac{(i\mu_l^{(2)}-\frac{3}{2})}{(i\mu_l^{(2)})}\frac{Q^{(2)}(i\mu_l^{(2)}-5)}{Q^{(1)}(i\mu_l^{(2)}-2)}+\frac{(i\mu_l^{(2)}+\frac{3}{2})}{(i\mu_l^{(2)})}\frac{Q^{(2)}(i\mu_l^{(2)}+1)}{Q^{(1)}(i\mu_l^{(2)}+1)}$$
$$=-x(i\mu_l^{(2)}-\frac{3}{2})(i\mu_l^{(2)}+\frac{3}{2}),\quad l=1,2,\cdots,L_2. \quad (3.$$

$$L_2=L_1,$$

$$x=\frac{1}{4}[-16+2(c_1\tilde{c}_1+c_2\tilde{c}_2+c_3\tilde{c}_3)+\frac{c_1c_2\tilde{c}_1\tilde{c}_2}{c_3\tilde{c}_3}+\frac{c_1c_3\tilde{c}_1\tilde{c}_3}{c_2\tilde{c}_2}+\frac{c_3c_2\tilde{c}_3\tilde{c}_2}{c_1\tilde{c}_1}].$$

$$\bar{\Lambda}(u-\frac{1}{2}) = -\frac{(u+2)(u+3)}{(u-1)(u+6)(u+\frac{5}{2})(u+\frac{5}{2})}$$

$$\times[\prod_{i=1}^{N}(u+\theta_i-1)(u-\theta_i-1)a(u-\theta_i)a(u+\theta_i)]^{-1}$$

$$\times\{Z_1(u)[\sum_{i=2}^{7} Z_i(u-1)+f_1(u-1)+f_2(u-1)]$$

$$+[\sum_{i=2}^{6} Z_i(u)+f_1(u)+f_2(u)]Z_7(u-1)$$

$$+(Z_2(u)+f_1(u)+Z_3(u))(Z_5(u-1)+f_2(u-1)+Z_6(u-1))\},$$

$$\tilde{\Lambda}(u-\frac{5}{2}) = \frac{u(u-1)(u+4)(u-\frac{1}{2})(u-\frac{3}{2})}{(u+1)(u+6)(u-4)(u-\frac{1}{2})^2}$$

$$\times[\prod_{i=1}^{N}(u+\theta_i-4)(u-\theta_i-4)(u+\theta_i+6)(u-\theta_i+6)a(u+\theta_i-3)a(u-\theta_i-3)]^{-1}$$

$$\times\{(\sum_{i=1}^{4} Z_i(u)+f_1(u))(\sum_{k=1}^{6} Z_k(u-3)+f_1(u-3)+f_2(u-3))Z_7(u-4)$$

$$+Z_1(u)Z_1(u-3)(\sum_{k=4}^{6} Z_k(u-4)+f_2(u-4))+Z_5(u)Z_6(u-3)Z_7(u-4)$$

$$+Z_1(u)(Z_2(u-3)+f_1(u-3)+Z_3(u-3))(Z_5(u-4)+f_2(u-4)+Z_6(u-4))$$

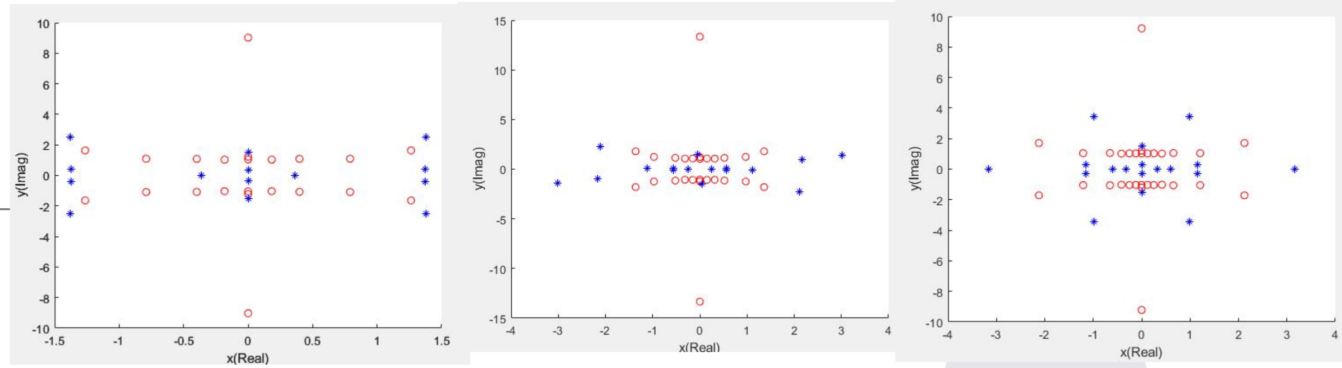
$$+Z_1(u)Z_3(u-3)(Z_5(u-4)+f_2(u-4)+Z_6(u-4))\}.$$

# 04

# 问题与展望

04

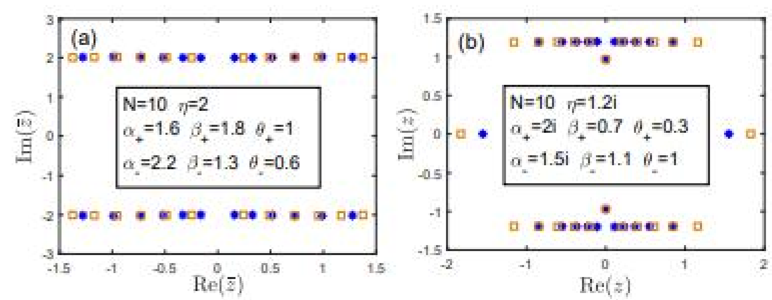
# 问题与展望




**存在的问题：**

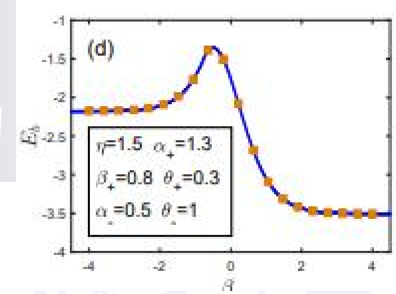
1. $B_n$，$D_n$模型的封闭关系个数随秩的增高会加倍增加，需要的计算量越大；
2. 例外李代数 $G_2^{(1)}, D_4^{(3)}, E_6^{(1)}, E_6^{(2)}$
3. 超对称模型。

$$\Lambda(u) = \Lambda_0 \prod_{j=1}^{N+2} \sinh(u - z_j + \frac{\eta}{2}) \sinh(u + z_j + \frac{\eta}{2}).$$




**未来工作方向：**

1. 寻找Bethe根的规律，零根分布性质；
2. 计算系统对应的本征态，计算关联函数和热力学性质；
3. 模型的应用，例如黑洞、强关联系统等；




Y. Qiao, J. Cao, W.-L. Yang, K. Shi and Y. Wang, Phys. Rev. B 103(2021), L220401

# 谢谢大家！