Aspects of Carrollian holography

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合肥

Introduction

1. Gravitational physics
   - Holographic principle (various examples in AdS/CFT)
   - Most gravitational processes occur in asymptotically flat spacetime(AFS)
2. Flat holography?
   - Celestial holography — Strominger et al. (2017)
     - ⋆ 4d/2d correspondence
     - ⋆ Four dimensional field theory in AFS spacetime is dual to a two dimensional putative CFT living on S^2
   - Carrollian holography
     - ⋆ 4d/3d correspondence
     - ⋆ Four dimensional field theory in AFS spacetime is dual to a three dimensional putative "CFT" living on future/past null infinty $\mathcal{I}^{\pm}$

   Donnay, Fiorucci, Herfray, Ruzziconi, Bagchi, Banerjee, Basu, Dutta et al. (2022)

Carrollian holography

Is there any systematic method to implement the Carrollian holography?

W.-B.Liu and JL (2022)

1. Bulk/boundary geometry
   - Poincaré invariance
   - Asymptotic symmetry (BMS)
   - Diffeomorphism
2. Bulk/boundary field
   - Transformation law
   - Reconstruction
3. Amplitude
   - Carrollian amplitude (the most natural object in Carrollian holography)
4. Applications

Preliminaries

1. Intrinsic point of view: Carrollian manifold $\mathcal{I}^+ = \mathbb{R} \times S$ with a degenerate metric and a null tangent vector

$$\gamma = \gamma_{AB} d\theta^A d\theta^B, \quad \chi = \partial_u, \quad \gamma(\bullet, \chi) = 0.$$

Lévy-Leblon(1965), Gupta(1966)

2. Embedding into higher dimensions: null hypersurfaces
   - Future/past null infinity
     - ★ Gravitational waves (observation)
     - ★ Asymptotic symmetries (flat holography)
   - Black hole horizon
     - ★ Area law
     - ★ Hawking radiation
   - Rindler horizon
     - ★ Unruh effect (accelerated observer)
   - Causal diamond
     - ★ Causal structure of spacetime
     - ★ Entanglement entropy
   - . . .

Poincaré group

1. Poincaré group

$$\mathrm{ISO}(1,3) = \mathrm{SO}(1,3) \ltimes \text{translation}$$

2. Spacetime translation
   - bulk: Minkowski spacetime

$$x^\mu \to x'^\mu = x^\mu + a^\mu$$

   - boundary: future/past null infinity in retarded/advanced coordinates

$$u' = u - a\cdot n, \quad \Omega' = \Omega \quad \Leftrightarrow \quad \boldsymbol{\xi} = -a\cdot n\partial_u$$

★ Null vector $n^\mu = (1, \boldsymbol{n})$
★ Unit normal vector of sphere $\boldsymbol{n} = \frac{1}{r}(x, y, z)$
★ Spherical coordinates $\Omega = (\theta, \phi)$.

Poincaré group

1. Lorentz transformation
   - bulk:

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$

   - boundary: time dilation and Möbius transformation $\mathrm{SL}(2,\mathbb{C})$

$$u' = \Gamma^{-1} u, \quad z' = \frac{az+b}{cz+d}.$$

★ Stereographic coordinates

$$z = \cot\frac{\theta}{2} e^{i\phi}, \quad \bar{z} = \cot\frac{\theta}{2} e^{-i\phi} \quad \Rightarrow \quad ds^2 = \frac{4}{(1+z\bar{z})^2} dz d\bar{z}.$$

★ Redshift factor

$$\Gamma = \left|\frac{\partial\Omega'}{\partial\Omega}\right|^{1/2} \left(\frac{\det\gamma(\Omega)}{\det\gamma(\Omega')}\right)^{-1/4} = \frac{|az+b|^2 + |cz+d|^2}{1+z\bar{z}}.$$

★ Globally conformal transformation on S^2

$$\xi = \frac{1}{2} u \nabla_A Y^A \partial_u + Y^A \partial_A.$$

with $\nabla_A Y_B + \nabla_B Y_A - \gamma_{AB} \nabla_C Y^C = 0$.

BMS group

1. Bulk: large diffeomorphism which preserve Bondi fall-off conditions

$$\begin{aligned}\xi_f &= f(\Omega)\partial_u + \frac{1}{2}\nabla_A\nabla^A f(\Omega)\partial_r - \frac{\nabla^A f(\Omega)}{r}\partial_A + \cdots,\\ \xi_Y &= \frac{1}{2}u\nabla_A Y^A\partial_u - \frac{1}{2}(u+r)\nabla_A Y^A\partial_r + (Y^A - \frac{u}{2r}\nabla^A\nabla_B Y^B)\partial_A + \cdots\end{aligned}$$

Bondi, van der Burg, Metzner and Sachs (1962)

$$\text{BMS}_4 = \text{SO}(1,3) \ltimes \text{supertranslation}$$

2. Boundary: conformal Carroll group of level 2

$$\text{CCarr}_k(\mathcal{I}^+, \gamma, \chi)$$

which is generated by the vector ξ such that

$$\mathcal{L}_\xi\gamma = \lambda\gamma, \quad \mathcal{L}_\xi\chi = \mu\chi, \quad \lambda + k\mu = 0,$$

where λ and μ are conformal factors and k is called the level

$$\xi = Y^A(\Omega)\partial_A + (f(\Omega) + \frac{u}{k}\nabla_A Y^A(\Omega))\partial_u.$$

Carrollian diffeomorphism

1. Boundary geometric transformation
2. Carrollian diffeomorphism is the transformation that preserves the null structure

$$\mathcal{L}_\xi \chi \propto \chi \quad \Rightarrow \quad \xi = f(u,\Omega)\partial_u + Y^A(\Omega)\partial_A$$

   which forms an infinite dimensional Lie algebra

$$[\xi_{f_1,Y_1}, \xi_{f_2,Y_2}] = \xi_{f_1\dot{f}_2 - f_2\dot{f}_1 + Y_1^A\partial_A f_2 - Y_2^A\partial_A f_1, [Y_1,Y_2]}$$

3. Carrollian diffeomorphism can be extended into bulk
   - BMS group is a subgroup
   - In general, Carrollian diffeomorphism is not the usual asymptotic symmetry group since they break standard fall-off conditions
4. We will use the following terminology

$$f(u,\Omega) \leftrightarrow \text{supertranslation}, \quad Y^A(\Omega) \leftrightarrow \text{superrotation}.$$

Fundamental fields (classical aspects)

1. Living on the boundary $\Sigma(u,\Omega)$
2. Asymptotic expansion

$$\Phi(x) = \frac{\Sigma(u,\Omega)}{r} + \mathcal{O}(r^{-2})$$

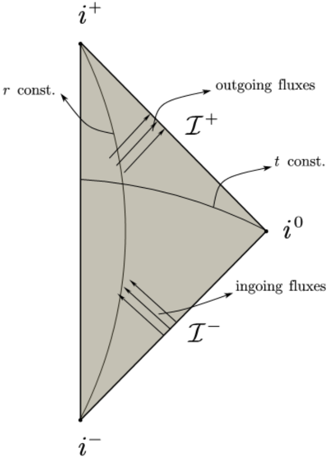
3. Characterize the radiative information
4. Transformation law

$$\begin{aligned} \not\delta_f \Sigma &= f(u,\Omega)\dot{\Sigma}, \\ \not\delta_Y \Sigma &= Y^A \nabla_A \Sigma + \frac{1}{2} \nabla_A Y^A \Sigma. \end{aligned}$$

5. Construct local operators

$$\begin{aligned} T(u,\Omega) &= \dot{\Sigma}(u,\Omega)^2, \\ M_A(u,\Omega) &= \frac{1}{2}(\dot{\Sigma}\nabla_A \Sigma - \Sigma \nabla_A \dot{\Sigma}). \end{aligned}$$

energy and angular momentum flux densities



Fundamental fields (quantum aspects)

1. Canonical quantization of the bulk field (plane wave expansion)

$$\Phi(t,\boldsymbol{x}) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}(e^{-i\omega t+i\boldsymbol{p}\cdot\boldsymbol{x}}b_{\boldsymbol{p}}+e^{i\omega t-i\boldsymbol{p}\cdot\boldsymbol{x}}b^\dagger_{\boldsymbol{p}})$$

2. Bulk reduction: canonical quantization of the fundamental field (spherical wave)

$$\Sigma(u,\Omega) = \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}}\sum_{\ell m}[a_{\omega,\ell,m}e^{-i\omega u}Y_{\ell,m}(\Omega)+a^\dagger_{\omega,\ell,m}e^{i\omega u}Y^*_{\ell,m}(\Omega)]$$

3. A state located at (u,Ω)

$$|\Sigma(u,\Omega)\rangle = \Sigma(u,\Omega)|0\rangle = \frac{i}{8\pi^2}\int_0^\infty d\omega e^{i\omega u}|\boldsymbol{p}\rangle \quad \Leftarrow \quad \text{outgoing state}$$

Fourier transform of the momentum state

4. A state located at (v,Ω)

$$|\Xi(v,\Omega)\rangle \equiv \Xi(v,\Omega)|0\rangle = -\frac{i}{8\pi^2}\int_0^\infty d\omega e^{i\omega v}|\boldsymbol{p}^P\rangle, \quad \Leftarrow \quad \text{incoming state}$$

with

$$\boldsymbol{p}^{\mathrm{P}} = (p,\Omega^{\mathrm{P}}) = (p,\pi-\theta,\pi+\phi)$$

Fock space

1. n-particle outgoing states

$$|\boldsymbol{p}_1\boldsymbol{p}_2\cdots\boldsymbol{p}_n\rangle = :\prod_{j=1}^{n}(-4\pi i)\int du_j e^{-i\omega_j u_j}\Sigma(u_j,\Omega_j):|0\rangle$$
$$= \int d\mu_{1,2,\cdots,n}|\prod_{k=1}^{n}\Sigma(u_k,\Omega_k)\rangle.$$

where the integration measure is defined as

$$d\mu_{1,2,\cdots,n} = \prod_{j=1}^{n}(-4\pi i)du_j e^{-i\omega_j u_j}.$$

2. m-particle incoming states

$$|\boldsymbol{p}_1\boldsymbol{p}_2\cdots\boldsymbol{p}_m\rangle = :\prod_{j=1}^{m}(4\pi i)\int dv_j e^{-i\omega_j v_j}\Xi(v_j,\Omega_j^P):|0\rangle$$
$$= \int d\nu_{1,2,\cdots,m}|\prod_{j=1}^{m}\Xi(v_j,\Omega_j^P)\rangle$$

with

$$d\nu_{1,2,\cdots,m} = \prod_{j=1}^{m}(4\pi i)dv_j e^{-i\omega_j v_j}.$$

Carrollian amplitude

1. $m \to n$ scattering process in momentum representation

$$
{}_{\text{out}}\langle \boldsymbol{p}_{m+1}\boldsymbol{p}_{m+2}\cdots\boldsymbol{p}_{m+n}|\boldsymbol{p}_1\boldsymbol{p}_2\cdots\boldsymbol{p}_m\rangle_{\text{in}} = \langle \boldsymbol{p}_{m+1}\boldsymbol{p}_{m+2}\cdots\boldsymbol{p}_{m+n}|S|\boldsymbol{p}_1\boldsymbol{p}_2\cdots\boldsymbol{p}_m\rangle.
$$

2. In Carrollian space

$$
\begin{aligned}
&\langle \boldsymbol{p}_{m+1}\boldsymbol{p}_{m+2}\cdots\boldsymbol{p}_{m+n}|S|\boldsymbol{p}_1\boldsymbol{p}_2\cdots\boldsymbol{p}_m\rangle \\
= &\int d\mu^*_{m+1,\cdots,m+n} d\nu_{1,\cdots,m}\langle \prod_{k=m+1}^{m+n}\Sigma(u_k,\Omega_k)|S|\prod_{k=1}^{m}\Xi(v_k,\Omega_k^P)\rangle \\
= &\int d\mu^*_{m+1,\cdots,m+n} d\nu_{1,\cdots,m}\ {}_{\text{out}}\langle \prod_{k=m+1}^{m+n}\Sigma(u_k,\Omega_k)|\prod_{k=1}^{m}\Xi(v_k,\Omega_k^P)\rangle_{\text{in}}
\end{aligned}
$$

where we have defined

$$
{}_{\text{out}}\langle \prod_{k=m+1}^{m+n}\Sigma(u_k,\Omega_k)|\prod_{k=1}^{m}\Xi(v_k,\Omega_k^P)\rangle_{\text{in}} = \langle \prod_{k=m+1}^{m+n}\Sigma(u_k,\Omega_k)|S|\prod_{k=1}^{m}\Xi(v_k,\Omega_k^P)\rangle
$$

Carrollian amplitude

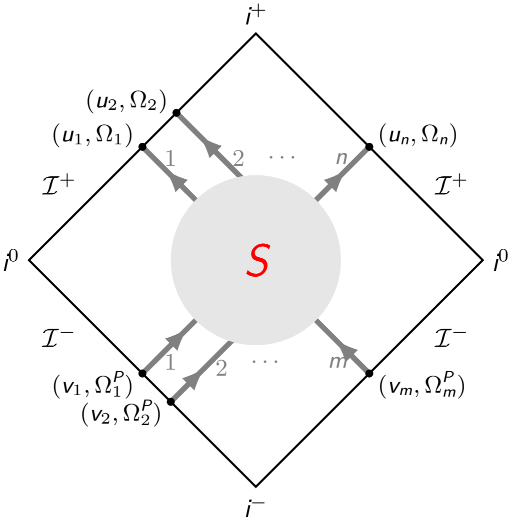


图: Carrollian amplitude in Penrose diagram.

Antipodal map

1. Map the Carrollian amplitude to the putative "CFT" at null infinity

$$v_j \to u_j, \quad \Omega_j \to \Omega_j^{\mathrm{P}}, \quad \Xi_j \to \Sigma_j$$

2. Correlator = Carrollian amplitude ⇔ Scattering amplitude

$$\begin{aligned}
&{}_{\text{out}}\langle \prod_{k=m+1}^{m+n} \Sigma(u_k, \Omega_k) | \prod_{k=1}^{m} \Sigma(u_k, \Omega_k) \rangle_{\text{in}} \\
= \quad & \left(\frac{1}{8\pi^2 i}\right)^{m+n} \prod_{j=1}^{m+n} \int d\omega_j e^{-i\sigma_j\omega_j u_j} \langle \boldsymbol{p}_{m+1}\boldsymbol{p}_{m+2}\cdots\boldsymbol{p}_{m+n}|S|\boldsymbol{p}_1\boldsymbol{p}_2\cdots\boldsymbol{p}_m\rangle
\end{aligned}$$

The symbol $\sigma_j, j = 1, 2, \cdots, m+n$ is designed to distinguish the outgoing and incoming states through the relation

$$\sigma_j = \begin{cases} +1 & \text{outgoing state,} \\ -1 & \text{incoming state.} \end{cases}$$

Carrollian amplitude

1. Connected part $\Rightarrow \mathcal{M}$ matrix

$$S = 1 + iT$$

$$\Rightarrow \quad \langle \boldsymbol{p}_{m+1}\boldsymbol{p}_{m+2}\cdots\boldsymbol{p}_{m+n}|iT|\boldsymbol{p}_1\boldsymbol{p}_2\cdots\boldsymbol{p}_m\rangle = (2\pi)^4\delta^{(4)}(\sum_{j=1}^{m+n} p_j)i\mathcal{M}(p_1,p_2,\cdots,p_{m+n}).$$

2. Carrollian amplitude (essential part)

$$i\mathcal{C}(u_1,\Omega_1,\sigma_1;\cdots;u_{m+n},\Omega_{m+n},\sigma_{m+n}) = \langle\prod_{j=1}^{m+n}\Sigma_j(u_j,\Omega_j;\sigma_j)\rangle$$

$$= \quad (\frac{1}{8\pi^2 i})^{m+n}\prod_{j=1}^{m+n}\int d\omega_j e^{-i\sigma_j\omega_j u_j}(2\pi)^4\delta^{(4)}(\sum_{j=1}^{m+n} p_j)i\mathcal{M}(p_1,p_2,\cdots,p_{m+n}).$$

Poincaré invariance of Carrollian amplitude

1. Spacetime translation
   - Field transformation

$$\Sigma'(u',\Omega') = \Sigma(u,\Omega), \quad u' = u - a\cdot n, \quad \Omega' = \Omega$$

   - Invariance of Carrollian amplitude

$$\langle \prod_{j=1}^{n} \Sigma_j(u_j',\Omega_j)\rangle = \langle \prod_{j=1}^{n} \Sigma_j(u_j,\Omega_j)\rangle$$

2. Lorentz transformation
   - Field transformation

$$\Sigma'(u',\Omega') = \Gamma\Sigma(u,\Omega), \quad u' = \Gamma^{-1}u, \quad z' = \frac{az+b}{cz+d}$$

   - Invariance of Carrollian amplitude

$$\langle \prod_{j=1}^{n} \Sigma_j(u_j',\Omega_j')\rangle = \left(\prod_{j=1}^{n}\Gamma_j\right) \langle \prod_{j=1}^{n} \Sigma_j(u_j,\Omega_j)\rangle.$$

Completeness relation and Unitarity

1. Inverse transform

$$|\boldsymbol{p}\rangle = -4\pi i \int_{-\infty}^{\infty} du e^{-i\omega u} |\Sigma(u,\Omega)\rangle.$$

2. Completeness relation for one particle states

$$1 = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3 2\omega_{\boldsymbol{p}}} |\boldsymbol{p}\rangle\langle\boldsymbol{p}|$$

is transformed to

$$1 = i \int du d\Omega \left(|\Sigma(u,\Omega)\rangle\langle\dot{\Sigma}(u,\Omega)| - |\dot{\Sigma}(u,\Omega)\rangle\langle\Sigma(u,\Omega)|\right)$$

3. Unitarity

$$S^\dagger S = 1 \quad \Rightarrow \quad -i(T - T^\dagger) = T^\dagger T.$$

4. In Carrollian space

$$C(m \to n) - C^*(n \to m) = \sum_k \left(\prod_{j=1}^k 2i \int du_j d\Omega_j\right) C^*(n \to k) \left(\prod_{j=1}^k \frac{\partial}{\partial u_j}\right) C(m \to k).$$

Triality

1. Celestial amplitudes are obtained from momentum space scattering amplitudes by performing Mellin transforms

$$\mathcal{A}(\Delta_1,\Omega_1,\sigma_1;\cdots;\Delta_n,\Omega_n,\sigma_n)=\prod_{j=1}^{n}\int_0^{\infty}d\omega_j\omega_j^{\Delta_j-1}\mathcal{M}(\boldsymbol{p}_1,\boldsymbol{p}_2,\cdots,\boldsymbol{p}_n)(2\pi)^4\delta^{(4)}(\sum_j p_j)$$

2. Carrollian amplitudes are obtained from momentum space scattering amplitudes by performing "Fourier transforms"

$$\mathcal{C}(u_1,\Omega_1,\sigma_1;\cdots;u_n,\Omega_n,\sigma_n)=\prod_{j=1}^{n}\int_0^{\infty}d\omega_j e^{-i\sigma_j\omega_j u_j}\mathcal{M}(\boldsymbol{p}_1,\boldsymbol{p}_2,\cdots,\boldsymbol{p}_n)(2\pi)^4\delta^{(4)}(\sum_j p_j)$$

3. $\mathcal{B}$ transform (inverse Mellin transform and Fourier transform): from Celestial amplitudes to Carrollian amplitudes

Amplitude triangle

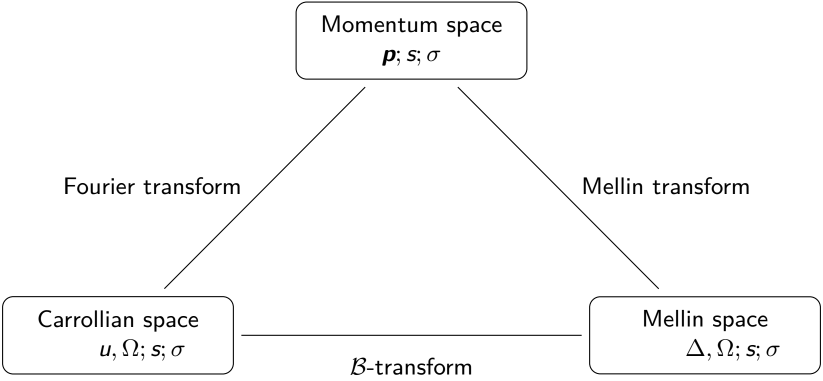


图: Interplay among the three bases of scattering amplitudes

Donnay, Fiorucci, Herfray, Ruzziconi (2022)

Two point Carrollian amplitude

1. Spacetime translation

$$i\,\mathcal{C}(u_1,\Omega_1;u_2,\Omega_2)=\langle\Sigma(u_2,\Omega_2)\Sigma(u_1,\Omega_1)\rangle=i\,\mathcal{C}(u_1-u_2-a\cdot(n_1-n_2),\Omega_1;0,\Omega_2)$$

2. Independent of a

$$i\,\mathcal{C}(u_1,\Omega_1;u_2,\Omega_2)=g(u_1-u_2)\delta(\Omega_1-\Omega_2).$$

3. Lorentz transformation

$$i\,\mathcal{C}(\Gamma_1^{-1}u_1,\Omega_1';\Gamma_2^{-1}u_2,\Omega_2')=(\Gamma_1\Gamma_2)\,i\,\mathcal{C}(u_1,\Omega_1;u_2,\Omega_2)$$

4. Invariance of the metric γ_{AB}

$$\gamma'_{AB}(\Omega)=\gamma_{AB}(\Omega)\quad\Rightarrow\quad\delta(\Omega_1'-\Omega_2')=\Gamma_1^2\delta(\Omega_1-\Omega_2).$$

5. Constraint for the function $g(u_1-u_2)$

$$g(u_1'-u_2')=g(u_1-u_2)\quad\Rightarrow\quad g(u)=g(\Gamma u).$$

6. Trivial solution

$$g(u)=\text{const}.$$

Two point Carrollian amplitude

1. Inconsistent with canonical quantization

$$i\,\mathcal{C}(u_1,\Omega_1;u_2,\Omega_2)=\beta(u_2-u_1)\delta(\Omega_1-\Omega_2)$$

with

$$\beta(u_2-u_1)=\frac{1}{4\pi}\int_0^\infty\frac{d\omega}{\omega}e^{-i\omega(u_2-u_1)}$$

2. $i\epsilon$ prescription
3. Regularization: IR cutoff

$$\beta(u_2-u_1)=\frac{1}{4\pi}\int_{\omega_0}^\infty\frac{d\omega}{\omega}e^{-i\omega(u_2-u_1-i\epsilon)}=\frac{1}{4\pi}\Gamma[0,i\omega_0(u_2-u_1-i\epsilon)]$$

4. Incomplete Gamma function

$$\Gamma(q,x)=\int_x^\infty dt\ t^{q-1}e^{-t}.$$

5. IR cutoff $\omega_0\to 0^+$ W.-B.Liu and JL (2022)

$$\boxed{i\,\mathcal{C}(u_1,\Omega_1;u_2,\Omega_2)=\frac{1}{4\pi}I_0(\omega_0(u_2-u_1-i\epsilon))\delta(\Omega_1-\Omega_2)}.$$

with

$$I_0(\omega_0(u-i\epsilon))=\gamma_E+\log i\omega_0(u-i\epsilon)$$

Lessons from two point Carrollian amplitude

1. Modified incoming/outgoing state for Carrollian amplitude

$$|\Sigma(u,\Omega;\omega_0)\rangle = \frac{i}{8\pi^2}\int_{\omega_0}^{\infty} d\omega e^{-i\omega u}|\boldsymbol{p}\rangle.$$

2. Loop hole in the symmetry arguments: the Carrollian amplitude may depend on the IR cutoff

$$i\,\mathcal{C}(u_1,\Omega_1;u_2,\Omega_2) = g(u_1-u_2;\omega_0)\delta(\Omega_1-\Omega_2).$$

with

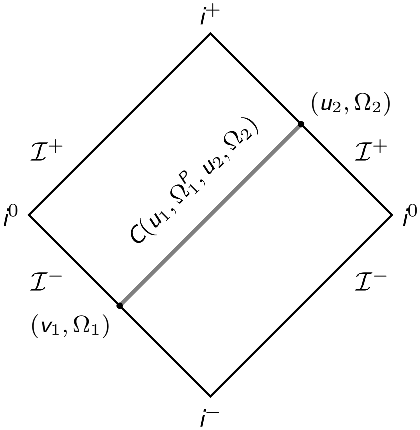
$$g(u;\omega_0) = g(\Gamma^{-1}u;\Gamma\omega_0) = g(\omega_0 u)$$

3. Metric γ_{AB} of the Carrollian manifold is invariant under Lorentz transformation, which is not equivalent to the coordinate transformation of S^2

$$\gamma'_{AB}(\Omega') \neq \frac{\partial\theta^C}{\partial\theta'^A}\frac{\partial\theta^D}{\partial\theta'^B}\gamma_{CD}(\Omega).$$

Feynman rules

1. Physical meaning of the integral transform of the scattering amplitude is unclear
2. Q: Geometric/graphic interpretation of the Carrollian amplitudes?
3. In AdS/CFT, Witten diagram is used to compute boundary CFT correlators
4. Two point Carrollian amplitude as a boundary-to-boundary propagator



Bulk-to-boundary propagator

1. A bulk state inserted at x

$$|\Phi(x)\rangle = \Phi(x)|0\rangle.$$

2. Insersion of one-particle state completeness relation

$$\begin{aligned} |\Phi(x)\rangle &= -2i\int du d\Omega |\dot{\Sigma}(u,\Omega)\rangle\langle\Sigma(u,\Omega)|\Phi(x)\rangle \\ &= -2i\int du d\Omega |\dot{\Sigma}(u,\Omega)\rangle D_+(u,\Omega;x) \\ &= 2i\int du d\Omega K_+(u,\Omega;x)|\Sigma(u,\Omega)\rangle \end{aligned}$$

3. Bulk-to-boundary propagator

$$\boxed{K_+(u,\Omega;x) = 2i\partial_u D_+(u,\Omega;x) = \frac{i}{4\pi^2(u+n\cdot x-i\epsilon)^2}}.$$

4. External line

$$\boxed{D_+(u,\Omega;x) = \langle 0|\Sigma(u,\Omega)\Phi(x)|0\rangle = -\frac{1}{8\pi^2(u+n\cdot x-i\epsilon)}}.$$

W.-B.Liu, JL and X.Q.Ye (2024)

Bulk-to-bulk propagator

1. Feynman propagator

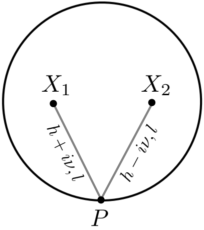
$$G_F(x,y) = \langle \mathrm{T}\Phi(x)\Phi(y)\rangle = \frac{1}{4\pi^2\left((x-y)^2 + i\epsilon\right)}.$$

2. Split representation

$$G_F(x,y) = \int du d\Omega \left(\theta(x^0 - y^0) D^*(u,\Omega;x) K(u,\Omega;y) + \theta(y^0 - x^0) D^*(u,\Omega;y) K(u,\Omega;x)\right)$$

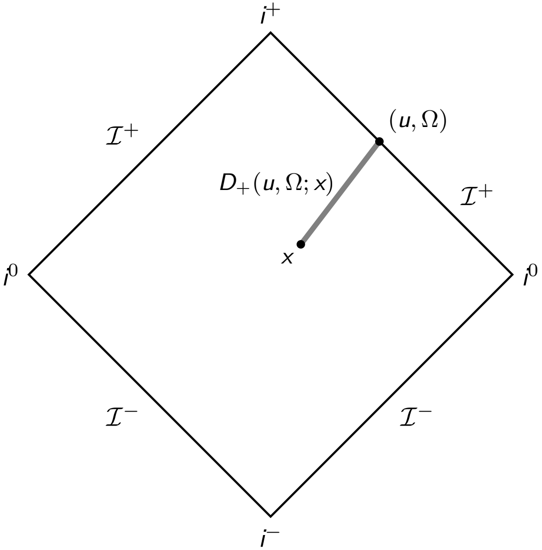
3. Analogy in AdS

$$\sum_{l=0}^{J} \int d\nu \, a_l(\nu)$$

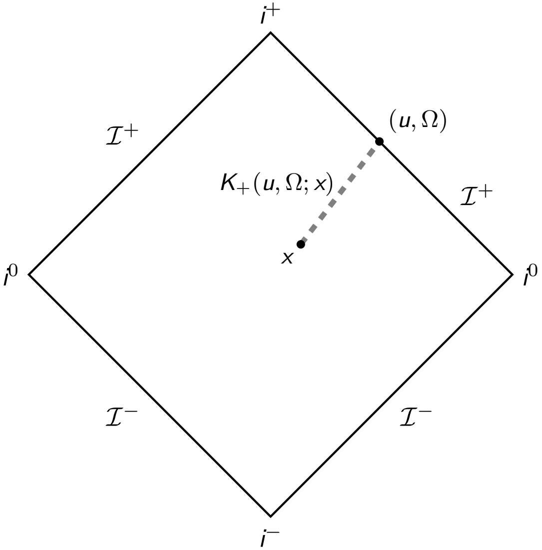


M.S. Costa, V.Goncalves, J.Penedones (2014)

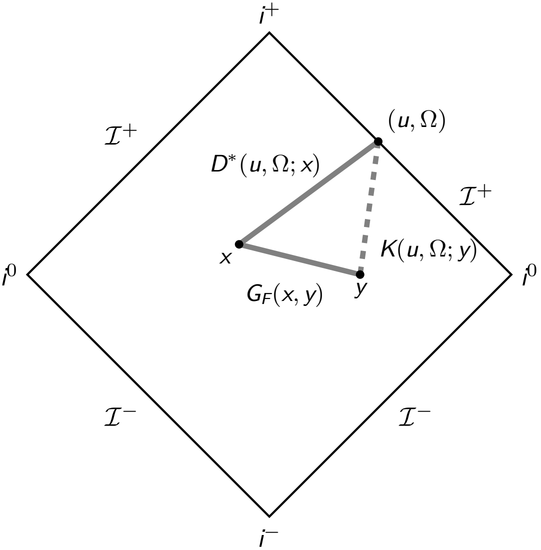
External line in Penrose diagram



Bulk-to-boundary propagator in Penrose diagram



Feynman propagator in split representation

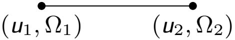


Feynman rules

1. For each state, we define a signature σ

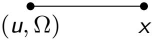
$$\sigma = \begin{cases} 1 & \text{outgoing state} \\ -1 & \text{incoming state} \end{cases}$$

2. For each boundary-to-boundary propagator,



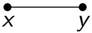
$$= C(u_1, \Omega_1; u_2, \Omega_2) = -\frac{1}{4\pi} I_0(\omega_0(u_2 - u_1 - i\epsilon))\delta(\Omega_1 - \Omega_2)$$

3. For each external line,



$$= D(u, \Omega; x) = -\frac{1}{8\pi^2(u + n\cdot x - i\sigma\epsilon)} = \frac{\sigma}{8\pi^2 i}\int_0^\infty d\omega e^{-i\sigma\omega(u+n\cdot x - i\sigma\epsilon)}$$

4. For each bulk-to-bulk propagator (Feynman propagator),



$$= G_F(x, y) = \int \frac{d^4 p}{(2\pi)^4} G_F(p) e^{ip\cdot(x-y)} = \frac{1}{4\pi^2\left((x-y)^2 + i\epsilon\right)}$$

5. Vertex
6. Symmetry factor

W.-B.Liu, JL & X.Q.Ye (2024)

Example: Φ^4 theory

1. Four point Carrollian amplitude

$$i\,\mathcal{C}(u_1,\Omega_1,\sigma_1;u_2,\Omega_2,\sigma_2;u_3,\Omega_3,\sigma_3;u_4,\Omega_4,\sigma_4)$$

2. 12 independent variables − 10 constraints from Poincaré symmetry
3. Möbius transformation: cross ratio is an invariant under Poincaré transformation

$$(z_1,z_2,z_3,z_4)\to(0,z,1,\infty)\quad \text{with} \quad z=\frac{z_{12}z_{34}}{z_{13}z_{24}}$$

4. Spacetime translation invariant χ

$$\chi=u_4-\frac{1}{z}u_1-\frac{1+z^2}{z(z-1)}u_2+\frac{2}{z-1}u_3.$$

   - Lorentz tranformation

$$\chi\to\Gamma_4^{-1}\chi.$$

5. Invariants under Poincaré transformation

$$z,\quad \omega_0\chi,\quad \frac{\omega_0}{M}.$$

Example: Φ^4 theory

1. Four point Carrollian amplitude (after fixing $z_1 = 0, z_2 = z, z_3 = 1, z_4 = \infty$)

$$\mathcal{C} = \mathcal{C}(\chi, z) = \mathcal{C}(\omega_0\chi, z, \frac{\omega_0}{M}).$$

2. Loop corrections (polynomial of two point Carrollian amplitude)

$$\boxed{C_{(n)}^{n\text{-loop}} = F(z)\sum_{j=0}^{n+1} a_j(z;\lambda)[I_0(\omega_0\chi)]^j}$$

with

$$F(z) = \frac{1}{(4\pi)^4}\frac{1+z^2}{2}\delta(\bar{z}-z)\Theta(z-1).$$

and

$$a_j(z;\lambda) = \sum_{k=j+1}^{\infty} a_j^{(k)}(z)\lambda^k.$$

Example: Φ^4 theory

1. Tree level

$$C^{\text{tree}} = \lambda F(z) I_0(\omega_0 \chi).$$

2. One-loop level

$$C^{\text{1-loop}} = F(z)[\frac{\pi^2}{12} a_1 + (-a_0 - a_1 \log\frac{\omega_0}{M}) I_0 + \frac{a_1}{2} I_0^2],$$

with

$$a_0 = -\lambda + \frac{\lambda^2}{32\pi^2}[\log z(z-1) + \log z + \log(1-z)], \quad a_1 = -\frac{3\lambda^2}{16\pi^2}.$$

3. Two-loop level ✓

Renormalization flow

1. Carrollian amplitude and Green's function (definition and *LSZ* reduction formula)

$$\langle\prod_{j=1}^{n}\Sigma_j(u_j,\Omega_j)\rangle=\prod_{j=1}^{n}\int d^4x_j\prod_{j=1}^{n}D(u_j,\Omega_j;x_j)\,G_{\text{connected and amputated}}(x_1,x_2\cdots,x_n)$$

2. Bare and renormalized Carrollian amplitude

$$\mathcal{C}_{(n)}=Z^{n/2}\mathcal{C}^0_{(n)}.$$

3. Callan-Symanzik equation for Carrollian amplitude

$$M\frac{\partial}{\partial M}\mathcal{C}_{(n)}+\beta\frac{\partial}{\partial\lambda}\mathcal{C}_{(n)}-n\gamma\mathcal{C}_{(n)}=0,$$

where we have defined the β and γ function as

$$\beta=M\frac{\partial\lambda}{\partial M},\qquad\gamma=\frac{1}{2}M\frac{\partial\log Z}{\partial M}.$$

4. Four point Carrollian amplitude for Φ^4 theory at two loop level ✓

Extension to spinning theory

1. Photon or gluon state inserted at (u, Ω)

$$|A_A(u,\Omega)\rangle = A_A(u,\Omega)|0\rangle = -\frac{i}{8\pi^2}\int_0^\infty d\omega e^{i\omega u} Y_A^i \sum_a \epsilon_i^a(\boldsymbol{p})|\boldsymbol{p}; a\rangle$$

where $a = +, -$ to denote the polarization and $Y_A^i = -\nabla_A n^i$.

2. Vielbein field on S^2

$$e_A^a(\Omega) = -Y_A^i(\Omega)\epsilon_i^a(\boldsymbol{p}), \quad \boldsymbol{p} = (\omega, \Omega).$$

- Orthogonality and completeness relation

$$e_A^a e_B^b \gamma^{AB} = \gamma^{ab}, \quad e_A^a e_B^b \gamma_{ab} = \gamma^{AB}.$$

3. State in Cartesian frame

$$|A_a(u,\Omega)\rangle = e_a^A(\Omega) A_A(u,\Omega)|0\rangle = \frac{i}{8\pi^2}\int_0^\infty d\omega e^{i\omega u}|\boldsymbol{p}; a\rangle.$$

Carrollian amplitude for vector theory

1. Definition

$$i\,\mathcal{C}_{a_1\cdots a_n}(u_1,\Omega_1,\sigma_1;\cdots;u_n,\Omega_n,\sigma_n) = \langle \prod_{j=1}^{n} A_{a_j}(u_j,\Omega_j)\rangle$$

$$= \left(\frac{i}{8\pi^2}\right)^n \left(\prod_{i=1}^{n}\int d\omega_i e^{-i\sigma_i\omega_i u_i}\right)(2\pi)^4\delta(\sum_{j=1}^{n} p_j)\, i\mathcal{M}_{a_1\cdots a_n}(\boldsymbol{p}_1,\cdots,\boldsymbol{p}_n)$$

with

$$\mathcal{M}_{a_1\cdots a_n} = \mathcal{M}_{\mu_1\cdots\mu_n}\epsilon^{\mu_1}_{a_1}(\sigma_1)\cdots\epsilon^{\mu_n}_{a_n}(\sigma_n).$$

2. Spacetime translation invariance

$$\boxed{\langle \prod_{j=1}^{n} A_{a_j}(u'_j,\Omega_j)\rangle = \langle \prod_{j=1}^{n} A_{a_j}(u_j,\Omega_j)\rangle}$$

with

$$u' = u - a\cdot n, \quad A'_a(u',\Omega) = A_a(u,\Omega).$$

Lorentz transformation

1. Möbius transformation is a conformal transformation of S^2
2. Fix the metric

$$\gamma'_{AB}(\Omega') = \Gamma^{-2}\frac{\partial\theta^C}{\partial\theta'^A}\frac{\partial\theta^D}{\partial\theta'^B}\gamma_{CD}(\Omega), \quad \gamma'^{AB}(\Omega') = \Gamma^2\frac{\partial\theta'^A}{\partial\theta^C}\frac{\partial\theta'^B}{\partial\theta^D}\gamma^{CD}(\Omega).$$

3. Vector field (naive computation)

$$A'_A(u',\Omega') = \frac{\partial\theta^B}{\partial\theta'^A}A_B(u,\Omega), \quad A'^A(u',\Omega') = \Gamma^2\frac{\partial\theta'^A}{\partial\theta^B}A^B(u,\Omega).$$

4. Vielbein

$$e'^a_A(\Omega') = \Gamma^{-1}\frac{\partial\theta^C}{\partial\theta'^A}S^a{}_b e^b_C(\Omega), \quad e'^A_a(\Omega') = \Gamma\frac{\partial\theta'^A}{\partial\theta^C}S_a{}^b e^C_b(\Omega)$$

where $S^a{}_b$ is a local $SO(2)$ rotation in the Cartesian frame

$$S^a{}_b S^c{}_d\gamma_{ac} = \gamma_{bd}$$

Therefore, $S_a{}^b$ is a local $SO(2)$ rotation in the Cartesian frame.

5. The ambiguity for the choice of the vielbein field
6. Short little group $SO(2)$

Lorentz transformation

1. In Cartesian frame

$$A'_a(u',\Omega') = \Gamma S_a{}^b A_b(u,\Omega), \quad A'^a(u',\Omega') = \Gamma S^a{}_b A^b(u,\Omega).$$

2. Transformation law of Carrollian amplitude

$$\langle \prod_{j=1}^n A_{a_j}(u'_j,\Omega'_j)\rangle = \left(\prod_{j=1}^n \Gamma_j\right)\left(\prod_{j=1}^n S_{a_j}{}^{b_j}(\sigma_j)\right)\langle \prod_{j=1}^n A_{a_j}(u_j,\Omega_j)\rangle.$$

- The form of $S_a{}^b$ depends on the convention
- Diagonal matrix $\Leftarrow$ convenient

$$S_a{}^b = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad S^a{}_b = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}$$

with t a local phase factor ($SO(2)$ rotation)

$$t^{-1} = t^*.$$

Extension to gravity

1. Radiative modes are encoded in the shear tensor

$$g_{AB} = r^2 \gamma_{AB} + r\, C_{AB} + \cdots .$$

2. State in the Cartesian frame

$$|C_{ab}\rangle = e_a^A e_b^B C_{AB}|0\rangle .$$

3. Lorentz transformation of gravitational Carrollian amplitude

$$\boxed{\langle \prod_{j=1}^{n} C_{a_j b_j}(u_j', \Omega_j')\rangle = \left(\prod_{j=1}^{n} \Gamma_j\right) \left(\prod_{j=1}^{n} S_{a_j}^{\ c_j}(\sigma_j) S_{b_j}^{\ d_j}(\sigma_j)\right) \langle \prod_{j=1}^{n} C_{c_j d_j}(u_j, \Omega_j)\rangle}.$$

W.I.P

Convention

1. Momentum

$$p^{\mu} = \sigma\omega n^{\mu} = \sigma\omega(1, \frac{z+\bar{z}}{1+z\bar{z}}, -i\frac{z-\bar{z}}{1+z\bar{z}}, -\frac{1-z\bar{z}}{1+z\bar{z}}).$$

2. Polarization vector

$$\epsilon_{\mu}^{(+)} = \frac{\sqrt{2}}{1+z\bar{z}}\left(0, \frac{1-z^2}{2}, -\frac{1+z^2}{2i}, z\right), \quad \epsilon_{\mu}^{(-)} = \left(\epsilon_{\mu}^{(+)}\right)^{*}.$$

3. Vielbein

$$e_A^a = \frac{\sqrt{2}}{1+z\bar{z}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = z, \bar{z} \quad a = +, -.$$

4. Flat metric

$$\gamma^{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

5. Phase factor

$$t = \left(\frac{\partial z'}{\partial z}\right)^{-1/2}\left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{1/2} = \frac{cz+d}{\bar{c}\bar{z}+\bar{d}}.$$

Example: MHV amplitude

1. Four gluons MHV amplitude Mason, Ruzziconi, and Srikan (2023)

$$\mathcal{M}^{-,-,+,+}[1,2,3,4] = \mathcal{M}[1^{-1},2^{-1},3^{+1},4^{+1}] = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

2. Fix $z_1 = 0, z_2 = 1, z_3 = \infty, z_4 = z$ and using the momentum conservation and on-shell condition

$$\mathcal{M}^{-,-,+,+}[1,2,3,4] = \frac{1}{z}$$

3. Carrollian amplitude (tree level)

$$\begin{aligned}\mathcal{C}^{-,-,+,+}[1,2,3,4] &= \left(\frac{i}{8\pi^2}\right)^4 \int_0^\infty \prod_{j=1}^4 d\omega_j e^{-i\sigma_j\omega_j u_j}(2\pi)^4\delta(\sum_{j=1}^4 p_j)\mathcal{M}^{-,-,+,+}[1,2,3\\ &= -F(z)I_0(\omega_0\chi)\frac{1}{z}.\end{aligned}$$

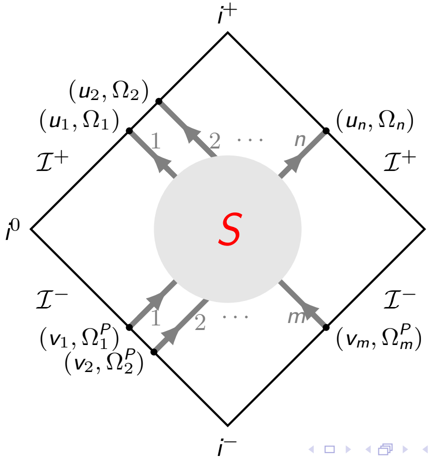
with $F(z) = \frac{1}{(4\pi)^4}\Theta(z)\Theta(1-z)\delta(\bar{z}-z)\frac{1+z^2}{2}$

4. Lorentz transformation

$$\mathcal{C}^{-,-,+,+}[1',2',3',4'] = \Gamma_1\Gamma_2\Gamma_3\Gamma_4 t_1 t_2 t_3^{-1} t_4^{-1}\mathcal{C}^{-,-,+,+}[1,2,3,4].$$

Summary

1. We provide a unified framework to derive Carrollian amplitude
2. Carrollian amplitude is equivalent to scattering amplitude
3. Carrollian amplitude is the natural object to study flat holography
4. Physical meaning: probability amplitude in Carrollian space



Application I: superduality transformation

1. There is a helicity flux operator that implements the local rotation

$$\mathcal{O}_g = \int du d\Omega g(\Omega) : \dot{A}^C A^B \epsilon_{BC} :$$

2. Superduality transformation W.-B.Liu & JL (2023)

$$[\mathcal{O}_g, A_A(u,\Omega)] = ig(\Omega)\epsilon_{AB}A^B(u,\Omega).$$

3. Finite superduality transformation of Carrollian amplitude

$$\langle \prod_{j=1}^{n} A'_{a_j}(u_j,\Omega_j)\rangle = \left(\prod_{j=1}^{n} S_{a_j}{}^{b_j}\right)\langle \prod_{j=1}^{n} A_{a_j}(u_j,\Omega_j)\rangle.$$

- Intertwined with Carrollian diffeomorphism: for any massless gauge theory with non-vanishing spin, there is always an associated superduality transformation

W.-B.Liu, JL & X.H.Zhou (2023)

Intertwined Carrollian diffeomorphism

1. Quantum flux operators which generate Carrollian diffeomorphism

$$
\begin{aligned}
[\mathcal{T}_{f_1}, \mathcal{T}_{f_2}] &= C_T(f_1, f_2) + i\mathcal{T}_{f_1\dot{f}_2 - f_2\dot{f}_1}, \\
[\mathcal{T}_f, \mathcal{M}_Y] &= -i\mathcal{T}_{Y^A\nabla_A f}, \\
[\mathcal{T}_f, \mathcal{O}_g] &= 0, \\
[\mathcal{M}_Y, \mathcal{M}_Z] &= i\mathcal{M}_{[Y,Z]} + is\mathcal{O}_{o(Y,Z)}, \\
[\mathcal{M}_Y, \mathcal{O}_g] &= i\mathcal{O}_{Y^A\nabla_A g}, \\
[\mathcal{O}_{g_1}, \mathcal{O}_{g_2}] &= 0.
\end{aligned}
$$

W.-B.Liu and JL (2023)

2. Energy and angular momentum flux operators are physical observables
3. Why helicity flux?

$$\mathcal{O}_{g=1} = \#\ \text{left hand helicity} - \#\ \text{right hand helicity}.$$

4. Spin 1: Optical helicity

R.P. Cameron, S. M. Barnett and A.M. Yao (2012)

Gravitational helicity flux density

1. Gyroscope's spin precession caused by the burst of GWs

   A.Seraj and B.Oblak (2022)

2. Differential formula

$$\frac{dH}{du d\Omega} = \mathcal{O}(u,\Omega) \quad \text{with} \quad \mathcal{O}(u,\Omega) = \epsilon_{AB}\dot{C}^{BC}C_C^{\ A}.$$

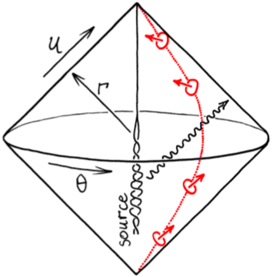
3. Quadrupole formula

   J.Dong, JL & R.-Z.Yu(2024)

$$\frac{dE}{du d\Omega} = -\frac{G}{8\pi}\dddot{M}_{ij}\dddot{M}_{kl}E^{ijkl}, \quad \frac{dH}{du d\Omega} = \frac{G}{8\pi}\dddot{M}_{ij}\ddot{M}_{kl}Q^{ijkl}.$$

with

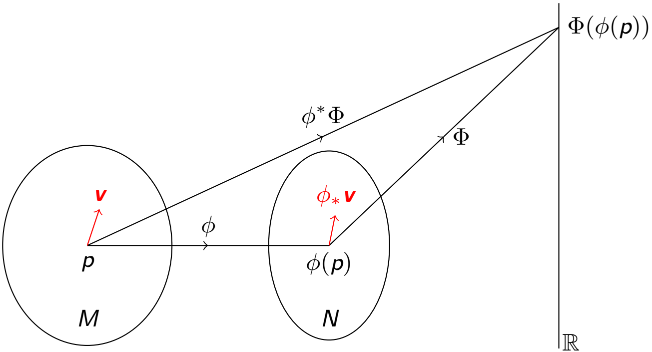
$$E^{ijkl} = \delta^{ik}\delta^{jl} - 2\delta^{ik}n^j n^l + \frac{1}{2}n^i n^j n^k n^l,$$
$$Q^{ijkl} = -(\delta^{jk} - n^j n^k)\epsilon^{ilm}n_m.$$



Application II: Carrollian diffeomorphism

1. Bulk diffeomorphism & fields
2. Scalar field

$$\phi^*\Phi = \Phi \quad \Leftrightarrow \quad \Phi'(x') = \Phi(x).$$



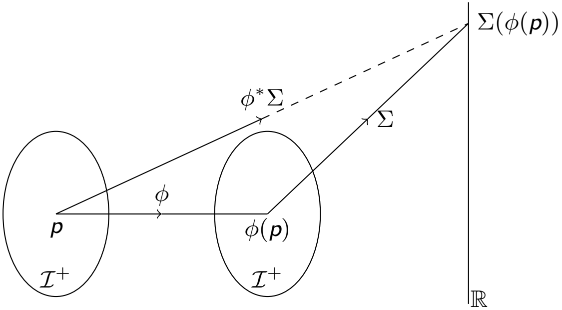
Carrollian diffeomorphism

1. Boundary diffeomorphism & fields
2. "Scalar field"

$$\phi^*\Sigma = W\Sigma \quad \text{for superrotation}$$

3. "Vector field"

$$\phi^* A_a = W S_a{}^b A_b$$



A geometric point of view on Carrollian amplitude

1. n point Carrollian amplitude is defined on the space

$$\mathcal{C}_{(n)} : \underbrace{\mathcal{I}^+ \otimes \mathcal{I}^+ \otimes \cdots \otimes \mathcal{I}^+}_{n} \to \mathbb{C}.$$

2. Carrollian diffeomorphism $\phi : \mathcal{I}^+ \to \mathcal{I}^+$

$$(u, \Omega) \to (u', \Omega')$$

3. Pull back of the Carrollian amplitude

$$\phi^* \mathcal{C}_{(n)}$$

4. Q: physical consequence of Carrollian diffeomorphism?

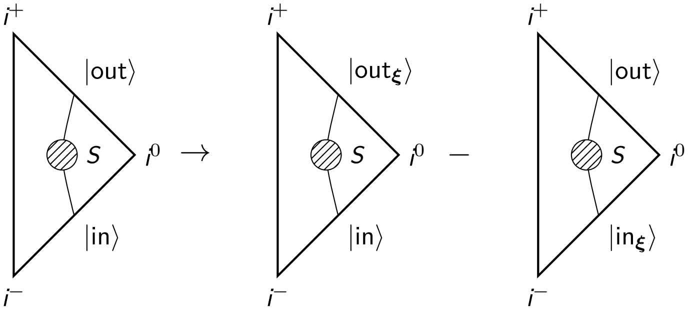
A geometric point of view on Carrollian amplitude

1. Variation of S matrix under infinitesimal Carrollian diffeomorphism along ξ

$$\delta_\xi \mathcal{C}_{(n)} = \lim_{\epsilon \to 0} \frac{\phi^* \mathcal{C}_{(n)} - \mathcal{C}_{(n)}}{\epsilon}$$

2. Carrollian diffeomorphism is generated by quantum flux operators

$$\delta_\xi \mathcal{C}(m \to n) = \langle \prod_{k=m+1}^{m+n} \Sigma(u_k, \Omega_k) | Q_\xi^{(+)} S - S Q_\xi^{(-)} | \prod_{k=1}^{m} \Xi(v_k, \Omega_k^P) \rangle.$$



A geometric point of view on Carrollian amplitude

1. Physical meaning of quantum flux operators and Stokes' theorem

$$Q_\xi^{(+)} - Q_\xi^{(-)} = \frac{1}{2}\int d^4x T^{\mu\nu}\delta_\xi g_{\mu\nu}$$

A.Li, W.-B.Liu, JL & R.Z.Yu (2023)

2. Einstein equation

$$\text{RHS} = \frac{1}{16\pi G}\int d^4x G^{\mu\nu}\delta_\xi g_{\mu\nu} = \frac{1}{8\pi G}\int (d^3x)_\mu G^{\mu\nu}\xi_\nu = -\tilde{Q}_\xi^{(+)} + \tilde{Q}_\xi^{(-)}.$$

3. Inserting into Carrollian amplitude

$$\boxed{\delta_\xi \mathcal{C}(m \to n) = \text{Carrollian amplitude with graviton insertion}}$$

W.I.P

Comments on flat holography

1. Comparison between Carrollian holography and AdS/CFT

AdS/CFT	Carrollian holography
$d \to d-1$	$d \to d-1$
Fefferman Graham expansion	Asymptotic expansion
AdS scattering	Flat space scattering
Witten diagram	Feynman diagram

2. Concrete example on the boundary side $\times$
3. Define boundary correlators through Carrollian amplitude ✓