Trans-series for Hofstadter Butterfly

顾杰

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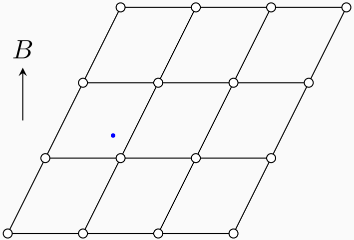
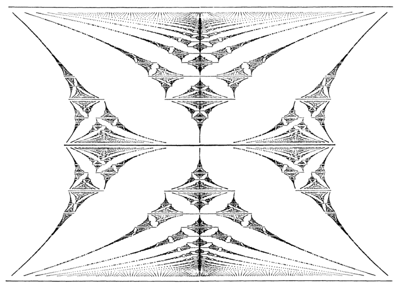
Hofstadter butterfly

- In 1979, D. Hofstadter (侯世达) considered an interesting 2d electron model in a magnetic field. [Hofstadter'79]





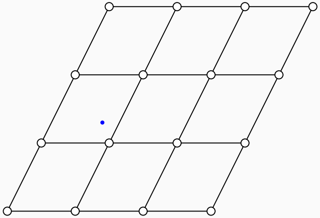
- How to explain? Relation with supersymmetric field theory! [Hatsuda,Katsura,Tachikawa'16]

Hofstadter butterfly

2d electron in lattice with magnetic field

- 2d electron in a square lattice with spaceing a: by tight binding approximation

$$H = 2\cos\frac{p_x a}{\hbar} + 2\cos\frac{p_y a}{\hbar}$$



2d electron in lattice with magnetic field

- 2d electron in a square lattice with spaceing a: by tight binding approximation

$$\mathrm{H} = 2\cos\frac{\mathrm{p}_x a}{\hbar} + 2\cos\frac{\mathrm{p}_y a}{\hbar}$$

- Adding a uniform and perpendicular magnetic field B

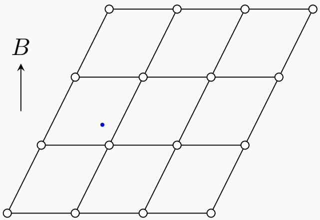
$$\vec{\mathrm{p}} \to \vec{\Pi} = \vec{\mathrm{p}} + e\vec{A}$$

which satisfy commutation relation

$$[\vec{\Pi}_x, \vec{\Pi}_y] = -\mathrm{i}\hbar e(\partial_x A_y - \partial_y A_x) = -\mathrm{i}\hbar e B$$

Hamiltonian of electron becomes

$$\mathrm{H} = \mathrm{e}^{\frac{\mathrm{i}a}{\hbar}\Pi_x} + \mathrm{e}^{-\frac{\mathrm{i}a}{\hbar}\Pi_x} + \mathrm{e}^{\frac{\mathrm{i}a}{\hbar}\Pi_y} + \mathrm{e}^{-\frac{\mathrm{i}a}{\hbar}\Pi_y}$$



Harper's equation

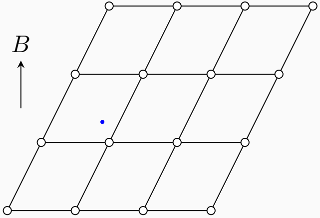
- Replacing $(a/\hbar)\Pi_{x,y}$ by operators x, y

$$\mathrm{H} = \mathrm{e}^{\mathrm{i}\mathrm{x}} + \mathrm{e}^{-\mathrm{i}\mathrm{x}} + \mathrm{e}^{\mathrm{i}\mathrm{y}} + \mathrm{e}^{-\mathrm{i}\mathrm{y}}$$

with the commutation relation

$$[\mathrm{x}, \mathrm{y}] = \frac{\mathrm{i}a^2 eB}{\hbar} =: \mathrm{i}\phi \quad \text{magnetic flux through a plaquette}$$

Equivalent to a **1d relativistic QM model** where ϕ is $\hbar$.



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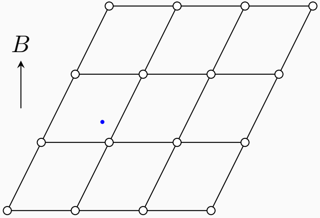
- Harper's equationn

$$\psi(x+\phi) + \psi(x-\phi) + 2\cos(x)\psi(x) = E\psi(x)$$

  Introducing $x = n\phi + \delta$ and $\psi_n(\delta) = \psi(n\phi + \delta)$

$$\psi_{n+1} + \psi_{n-1} + 2\cos(n\phi+\delta)\psi_n = E\psi_n$$



Energy spectrum at rational flux

- The model simplifies when flux is **rational** [Hofstadter'79]

$$\phi = 2\pi\alpha = 2\pi\frac{P}{Q}, \quad P, Q \in \mathbb{N}, (P, Q) = 1$$

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$$\phi = 2\pi\alpha = 2\pi\frac{P}{Q}, \quad P, Q \in \mathbb{N}, (P, Q) = 1$$

- Harper's equation is periodic $n \to n + Q$. With Bloch wavefunciton

$$\psi_n(\delta) = e^{ikn} u_n(\delta, k) \quad \text{w/ } u_{n+Q}(\delta, k) = u_n(\delta, k)$$

energy spectrum is computed by the **polynomial characteristic (secular) equation**

$$F_{P/Q}(E, \delta, k) := \det(H_Q - E\mathbf{1}_Q) = 0$$

with

$$H_Q(\delta, k) = \begin{pmatrix} 2\cos\delta & e^{ik} & & & e^{-ik} \\ e^{-ik} & 2\cos(\delta + 2\pi\frac{P}{Q}) & e^{ik} & & \\ & e^{-ik} & 2\cos(\delta + 4\pi\frac{P}{Q}) & e^{ik} & \\ & & \ddots & \ddots & \ddots \\ e^{ik} & & & e^{-ik} & 2\cos(\delta + 2\pi(Q-1)\frac{P}{Q}) \end{pmatrix}$$

Energy spectrum at rational flux, two Bloch angles

- It can be shown [Hasegawa,Hatsugai,Kohmoto,Montambaux'90]

$$F_{P/Q}(E,0,0) = 2(\cos Qk + \cos Q\delta) =: 2(\cos\theta_x + \cos\theta_y)$$

  where θ_x, θ_y are on equal footing: symmetric, both periodic by $\theta_{x,y} \to \theta_{x,y} + 2\pi$.
- When $\phi = 2\pi P/Q$, $\mathrm{H} = \mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{-\mathrm{i}x} + \mathrm{e}^{\mathrm{i}y} + \mathrm{e}^{-\mathrm{i}y}$ allows two Bloch angles in both directions, identified with θ_x, θ_y. [Duan,**JG**,Hatsuda,Sulejmanpasic'18]

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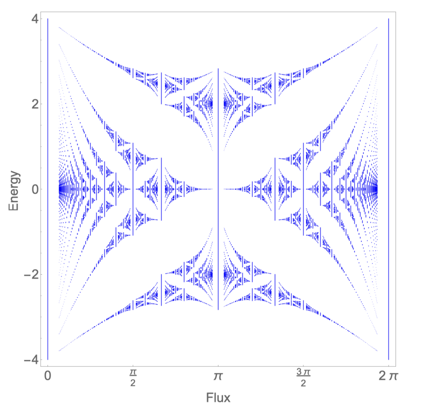
- Varying $\cos\theta_x + \cos\theta_y \in [-2, 2]$, degree Q polynomial $F_{P/Q}(E,0,0)$ yields Q energy bands.

Hofstadter butterfly



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  - Rational vs irrational magnetic fluxes
  - Fractal structure

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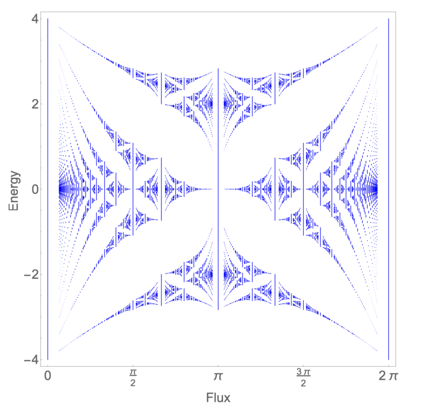

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- Problems of the energy spectrum
  - How to understand this picture? What is E as a function of ϕ? **Highly non-perturbative**!

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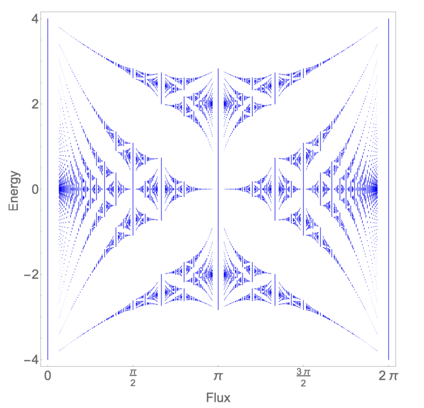

- Features of the energy spectrum
  - Rational vs irrational magnetic fluxes
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- Problems of the energy spectrum
  - How to understand this picture? What is E as a function of ϕ? **Highly non-perturbative**!
- Results:
  - Energy trans-series for $\phi = 2\pi/Q$ that includes **full non-pert. corrections**.
  - Reply heavily on **relation with supersymmetric field theory**!

Energy trans-series of Hofstadter butterfly

Semi-classical analsys of energy series

- Hamiltonian for the Harper-Hofstadter model

$$\mathrm{H} = \mathrm{e}^{\mathrm{i}\mathrm{x}} + \mathrm{e}^{-\mathrm{i}\mathrm{x}} + \mathrm{e}^{\mathrm{i}\mathrm{y}} + \mathrm{e}^{-\mathrm{i}\mathrm{y}}, \quad [\mathrm{x}, \mathrm{y}] = \mathrm{i}\phi.$$

- The perturbative energy series can be efficiently calculated by `BenderWu` package with Landau level $N = 0, 1, 2, \ldots$ [Bender,Wu'73; Sulejmanpasic,Unsal'16; **JG**,Sulejmanpasic'17]

$$E(N, \phi) = 4 - (1 + 2N)\phi + \frac{1}{8}(1 + 2N + 2N^2)\phi^2 + -\frac{1}{192}(1 + 2N)(1 + N + N^2)\phi^3 + \ldots$$

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- By path integral analysis of twisted thermal partition function, one finds that for $\phi = 2\pi/Q$, there are instanton and anti-instanton in both x- and y-directions

[Duan,**JG**,Hatsuda,Sulejmanpasic'18]

$$E^{(1)}_{(\theta_x,\theta_y)}(N=0,\phi) = 8(\cos\theta_x + \cos\theta_y)\left(\frac{\phi}{2\pi}\right)^{1/2} \mathrm{e}^{-S_c/\phi}(1+\ldots), \quad S_c = 8C.$$

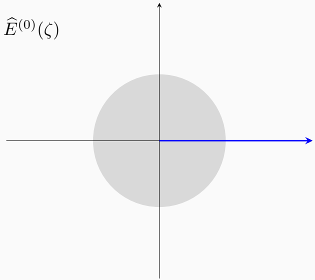
Borel resummation and Stokes ambiguity

- Perturbative energy series is divergent

$$E^{(0)}(\phi) = \sum a_k \phi^k, \quad a_k \sim k!$$

- Method of (naive) Borel resummation

$$\mathscr{S}(E^{(0)})(\phi) = \phi^{-1} \int_0^\infty \widehat{E}^{(0)}(\zeta) e^{-\zeta/\phi} d\zeta, \quad \widehat{E}^{(0)}(\zeta) = \sum \frac{a_k}{k!} \zeta^k$$



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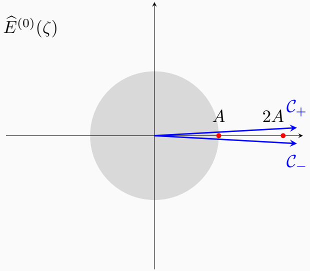
- As there exist singular points at $A, 2A, \ldots$ along $\mathbb{R}^+$, have to use lateral Borel resummations which are **ambiguous**

$$\mathscr{S}^{(\pm)}(E^{(0)})(\phi) = \phi^{-1} \int_0^{e^{\pm i0}\infty} \widehat{E}^{(0)}(\zeta) e^{-\zeta/\phi} d\zeta$$

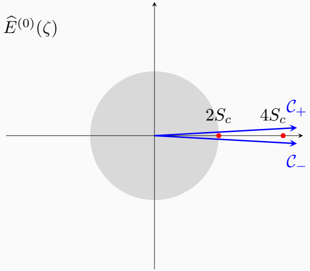
- Both $\mathscr{S}^{\pm}(E^{(0)})(\phi)$ have small **imaginary** parts.



Borel resummation of energy series

- As $A = 2S_c$, the singular points correspond to $2n$-instanton corrections,

$$\mathscr{S}^{(+)}E^{(0)} - \mathscr{S}^{(-)}E^{(0)} = 2\mathscr{S}^{(-)}E^{(2)} + \dots$$



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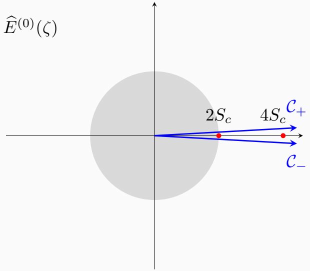
- After including $2n$-instanton corrections, Borel resummed energy trans-series is **both real and unique**

$$\mathscr{S}^{(\pm)}(E^{(0)} \mp E^{(2)} + \ldots) > 0 \quad \text{are the same}$$



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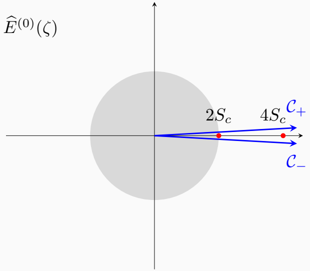
- The exact energy is Borel resummation of **full trans-series**

$$E_{\text{ex}} = \mathscr{S}^{(\pm)}(E^{(0)} + E^{(1)} \mp E^{(2)} + \dots)$$

with Stokes ambiguity $\epsilon = \pm$.



Exact WKB method

- WKB ansatz for the 1d non-rel. QM

$$\mathrm{H}(\mathrm{x},\mathrm{y})\psi(x) = E\psi(x), \quad \psi(x) = \exp\left(\frac{\mathrm{i}}{\phi}\int_{*}^{x} P(x',\phi)\mathrm{d}x'\right)$$

where

$$P(x,\phi) = \sum_{n\geq 0} P_n(x)\phi^n, \quad P_0(x) \text{ momentum}$$

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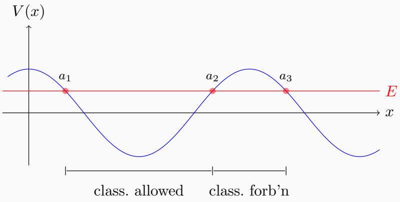
- Quantum periods [Voros'83]
  - Perturbative quantum period

$$t = \frac{1}{\pi}\int_{a_1}^{a_2} P(x,\phi)\mathrm{d}x$$

  - Non-perturbative quantum period

$$t_D = -2\mathrm{i}\int_{a_2}^{a_3} P(x,\phi)\mathrm{d}x$$



Exact quantisation conditions

- Exact Quantisation Conditions (EQCs) usually take the form [Delabaere'92; Zinn-Justin,Jentschura'04; ...]

$$1 + \mathcal{V}_A = f(\mathcal{V}_A^{1/2}, \mathcal{V}_B^{1/2}) \xrightarrow{\phi \to 0} 0$$

  with Voros symbols

$$\mathcal{V}_A = \mathrm{e}^{2\pi\mathrm{i}\frac{t(E,\phi)}{\phi}}, \quad \mathcal{V}_B = \mathrm{e}^{-\frac{t_D(E,\phi)}{\phi}}.$$

- E.g. for cosine model

$$1 + \mathcal{V}_A^{\pm 1}(1 + \mathcal{V}_B) - 2\sqrt{\mathcal{V}_A^{\pm 1}\mathcal{V}_B}\cos\theta = 0$$

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$$1 + \mathcal{V}_A^{\pm 1}(1 + \mathcal{V}_B) - 2\sqrt{\mathcal{V}_A^{\pm 1}\mathcal{V}_B}\cos\theta = 0$$

- This implies the **universal structure** of the full trans-series [van Spaendonck,Vonk'23]

$$E(N,\phi) = E^{(0)}(N,\phi) + \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} u_{n,m}(\theta_{x,y},\epsilon) E^{(n,m)}(N,\phi)$$

with *n*-**instanton corrections**

$$E^{(n,m)} = \left(\frac{\partial}{\partial N}\right)^m \left(\frac{\partial E^{(0)}(N,\phi)}{\partial N}\mathrm{e}^{-\frac{n\, t_D(N,\phi)}{\phi}}\right), \quad t_D(N,\phi) = S_c + \mathcal{O}(\phi)$$

where coefficient $u_{n,m}$ depends on Bloch angles θ_x, θ_y and Stokes ambiguity $\epsilon = \pm 1$.

5d SYM

- 5d $\mathcal{N} = 1$ SYM with $G = SU(2)$ on S^1 (radius 1) is described by Seiberg-Witten curve

$$e^x + e^{-x} + e^y + e^{-y} - u = 0, \quad \lambda = y dx.$$

- In NS limit of Omega background, the curve is promoted to quantum operator

$$\mathrm{H}_{\mathrm{SYM}} = e^{\mathrm{x}} + e^{-\mathrm{x}} + e^{\mathrm{y}} + e^{-\mathrm{y}} - u, \quad [\mathrm{x}, \mathrm{y}] = \mathrm{i}\hbar.$$

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- Interesting physical observables: [See Wang Xin's talk on Monday]
  - 1/2-BPS Wilson loop along S^1

$$W_{\Box}(t,\hbar) = \sum_{n\geq 0} W_n(t)\hbar^{2n}.$$

  - NS free energy

$$F_{\mathrm{NS}}(t,\hbar) = \sum_{n\geq 0} F_n(t)\hbar^{2n}.$$

5d SYM and butterfly

- **5d SYM is closely related to Harper-Hofstadter**: $\mathsf{H}_{\mathsf{SYM}}$ is identified with Harper equation by

$$x, y, \hbar \quad \rightarrow \quad \mathrm{i}x, \mathrm{i}y, -\phi$$

while [Hatsuda,Katsura,Tachikawa'16; Duan,**JG**,Hatsuda,Sulejmanpasic'18; See also Chen Jin's talk on Monday]

$$\begin{aligned} E^{(0)}(N,\phi) =& W_{\Box}(t=-\phi\nu, \hbar=-\phi) \\ t_D(N,\phi) =& \frac{\partial}{\partial t} F_{\mathrm{NS}}(t=-\phi\nu, \hbar=-\phi). \end{aligned}$$

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$$t_D(N,\phi) = \frac{\partial}{\partial t} F_{\mathrm{NS}}(t=-\phi\nu, \hbar=-\phi).$$

- W, F_{NS} can be computed efficiently via holomorphic anomaly equations, which allows efficient calculation of $E^{(0)}(N,\phi), t_D(N,\phi)$ as well. [BCOV'93; Huang,Klemm'10; Krefl,Walcher'10; Huang,Lee,Wang'22; Wang'23]

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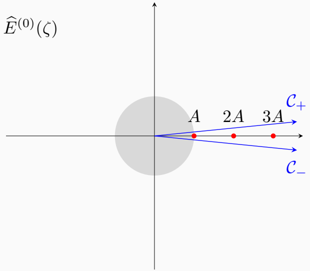
- Both $W_{\Box}(t,\hbar)$ and $F_{\mathrm{NS}}(t,\hbar)$ are divergent series, and we have a good understanding of their Borel singularities as well as Stokes discontinuities, which are controlled by **BPS invariants of 5d SYM**. [**JG**,Marino'23; **JG**,Guo'24]
- This knowledge can be exported to Harper-Hofstadter!

Resurgent strcuture of Harper-Hofstadter

- The **Borel singularity** A corresponds to **BPS state** of D2 brane wrapping either $\mathbb{P}^1$ of local $\mathbb{P}^1 \times \mathbb{P}^1$ underlying 5d SYM



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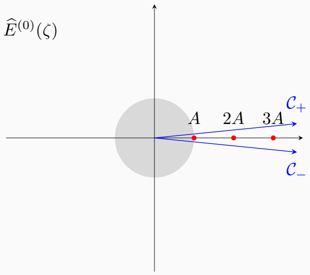
- **Stokes discontinuity** in terms of contributions from individual singularities encoded by alien derivatives [JG,Xu'24]

$$(\mathscr{S}^{(+)} - \mathscr{S}^{(-)})E^{(0)} = \Big(\exp\Big(\sum_{\ell} \dot{\Delta}_{\ell A}\Big) - 1\Big)E^{(0)}$$

$$\dot{\Delta}_{\ell A} E^{(0)} = \frac{S_A}{2\pi \mathrm{i}} \frac{(-1)^{\ell}}{\ell} E^{(2\ell,0)}$$

$$\dot{\Delta}_{\ell A} E^{(n,m)} = \frac{S_A}{2\pi \mathrm{i}} \frac{(-1)^{\ell}}{\ell} E^{(n+2\ell,m+1)}$$



the **Stokes constant** S_A is the **BPS multiplicity**

$$S_A = 2\chi_{1/2}(1) = 4.$$

Minimal trans-series

- Define minimal trans-series

$$E^{(0)}_{\text{min}}(\sigma) = E^{(0)} + \sum_{n'=1}^{\infty} \sum_{m'=0}^{n'-1} \sigma^{m'+1} v_{n',m'} E^{(2n',m')}$$

where

$$v_{n,m} = \frac{1}{n!} B_{n,m+1}(1!s_1, 2!s_2, \ldots),\ s_j = \frac{(-1)^{j-1}}{j \cdot 2\pi \mathrm{i}}.$$

m	0	1	2
$v_{1,m}$	$-\frac{\mathrm{i}}{2\pi}$		
$v_{2,m}$	$\frac{\mathrm{i}}{4\pi}$	$-\frac{1}{8\pi^2}$	
$v_{3,m}$	$-\frac{\mathrm{i}}{6\pi}$	$\frac{1}{8\pi^2}$	$-\frac{\mathrm{i}}{48\pi^3}$

Minimal trans-series

- Define minimal trans-series

$$E^{(0)}_{\text{min}}(\sigma) = E^{(0)} + \sum_{n'=1}^{\infty} \sum_{m'=0}^{n'-1} \sigma^{m'+1} v_{n',m'} E^{(2n',m')}$$

where

$$v_{n,m} = \frac{1}{n!} B_{n,m+1}(1!s_1, 2!s_2, \ldots),\ s_j = \frac{(-1)^{j-1}}{j \cdot 2\pi \mathrm{i}}.$$

- **Real** and **ambiguity-free**

$$\mathscr{S}^{(+)} E^{(0)}_{\text{min}}(-2) = \mathscr{S}^{(-)} E^{(0)}_{\text{min}}(+2).$$

m	0	1	2
$v_{1,m}$	$-\frac{\mathrm{i}}{2\pi}$		
$v_{2,m}$	$\frac{\mathrm{i}}{4\pi}$	$-\frac{1}{8\pi^2}$	
$v_{3,m}$	$-\frac{\mathrm{i}}{6\pi}$	$\frac{1}{8\pi^2}$	$-\frac{\mathrm{i}}{48\pi^3}$

$$N = 0, \phi = 2\pi/13$$

n	$\mathscr{S}^{(+)} E^{(0)}_{\text{min}}(N, \phi, -2)$	$\mathscr{S}^{(-)} E^{(0)}_{\text{min}}(N, \phi, +2)$
0	$3.545 + 3.794 \times 10^{-12}\mathrm{i}$	$3.545 - 3.794 \times 10^{-12}\mathrm{i}$
2	$3.545 - 2.485 \times 10^{-23}\mathrm{i}$	$3.545 + 2.485 \times 10^{-23}\mathrm{i}$
4	$3.545 - 6.074 \times 10^{-33}\mathrm{i}$	$3.545 + 6.074 \times 10^{-33}\mathrm{i}$
6	$3.545 - 2.074 \times 10^{-38}\mathrm{i}$	$3.545 + 2.074 \times 10^{-38}\mathrm{i}$

Full trans-series

- Full trans-series in terms of minimal trans-series

$$E_{\theta_{x,y},\sigma}(N,\phi) = E^{(0)}_{\rm min}(\sigma) + \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} w_{n,m}(\theta_{x,y}) E^{(n,m)}_{\rm min}(\sigma)$$

such that

$$E^{\rm ext}_{\theta_{x,y}}(N,\phi) = \mathscr{S}^{(\pm)} E_{\theta_{x,y},\mp 2}(N,\phi).$$

- By comparing with numerical spectrum, we computed $w_{n,m}(\theta_{x,y})$ up to 6-instanton order,

For $\phi = 2\pi/Q$

m	0	1	2
$w_{1,m}$	$\frac{\Theta}{\pi}$		
$w_{2,m}$	0	$\frac{\Theta^2}{2\pi^2}$	
$w_{3,m}$	$-\frac{\Theta}{\pi} + \frac{\Theta^3}{6\pi}$	0	$\frac{\Theta^3}{6\pi^3}$

$$\Theta = (-1)^{N+1}(\cos\theta_x + \cos\theta_y)$$

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- and found a conjectural formula for **all coefficients**.

[**JG**,Xu'24]

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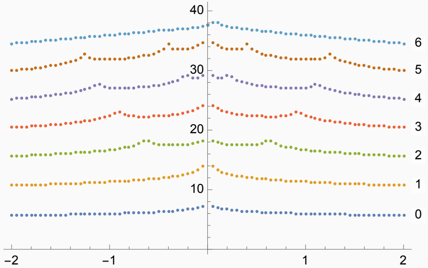
$$\Theta = (-1)^{N+1}(\cos\theta_x + \cos\theta_y)$$

$$w_{n,m} = \frac{1}{n!} B_{n,m+1}(1! r_1, \ldots)$$

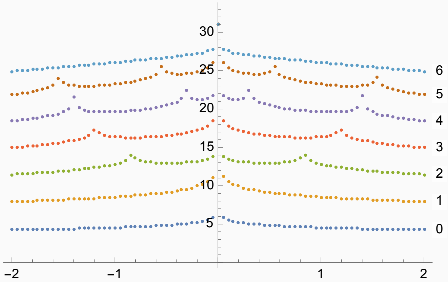
$$\sum_{j\geq 1} r_j \lambda^j = \frac{1}{\pi} \arcsin \frac{\Theta}{\lambda + \lambda^{-1}}.$$

Full trans-series

- Number of matching digits between $E^{\text{ext}}_{\theta_{x,y}}(N,\phi)$ and $\mathscr{S}^{(\pm)}E_{\theta_{x,y},\mp 2}(N,\phi)$ as a function of Θ with increasing instanton orders.



$\phi = 2\pi/13, N = 0$



$\phi = 2\pi/13, N = 1$

Conclusion and discussion

- We have found the **full energy trans-series** for Harper-Hofstadter model when $\phi = 2\pi/Q$.

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- The implied EQC [**JG**,Xu'24]

$$D^{\pm}_{\theta_{x,y}} = 1 + \mathcal{V}_A^{\pm 1}(1+\mathcal{V}_B)^2 - 2\sqrt{\mathcal{V}_A^{\pm 1}\mathcal{V}_B}(\cos\theta_x + \cos\theta_y) = 0.$$

is a "**double cover**" of the EQC for non-relativistic cosine model

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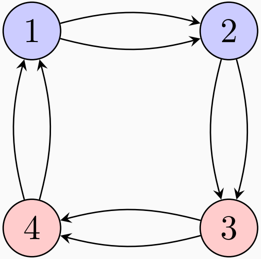
which is related to that the BPS quiver of 5d $SU(2)$ SYM is a **double copy** of that of 4d $SU(2)$ SYM.



Conclusion and discussion

- We have found the **full energy trans-series** for Harper-Hofstadter model when $\phi = 2\pi/Q$.
- The implied EQC [**JG**,Xu'24]

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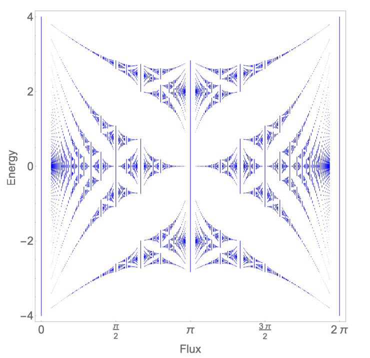
- Future: What about $\phi = 2\pi P/Q$ and irrational ϕ? Perturbative expansion at $\phi = 2\pi P/Q$?



Thank you for your attention!