Partial entanglement network and geometry reconstruction in AdS/CFT

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中国科学技术大学

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Based on my recent work with

Lin Jiong (中科大) **Zhong Yiwei（中科大）** **Lu Yizhou (南方科大，SIMIS)**

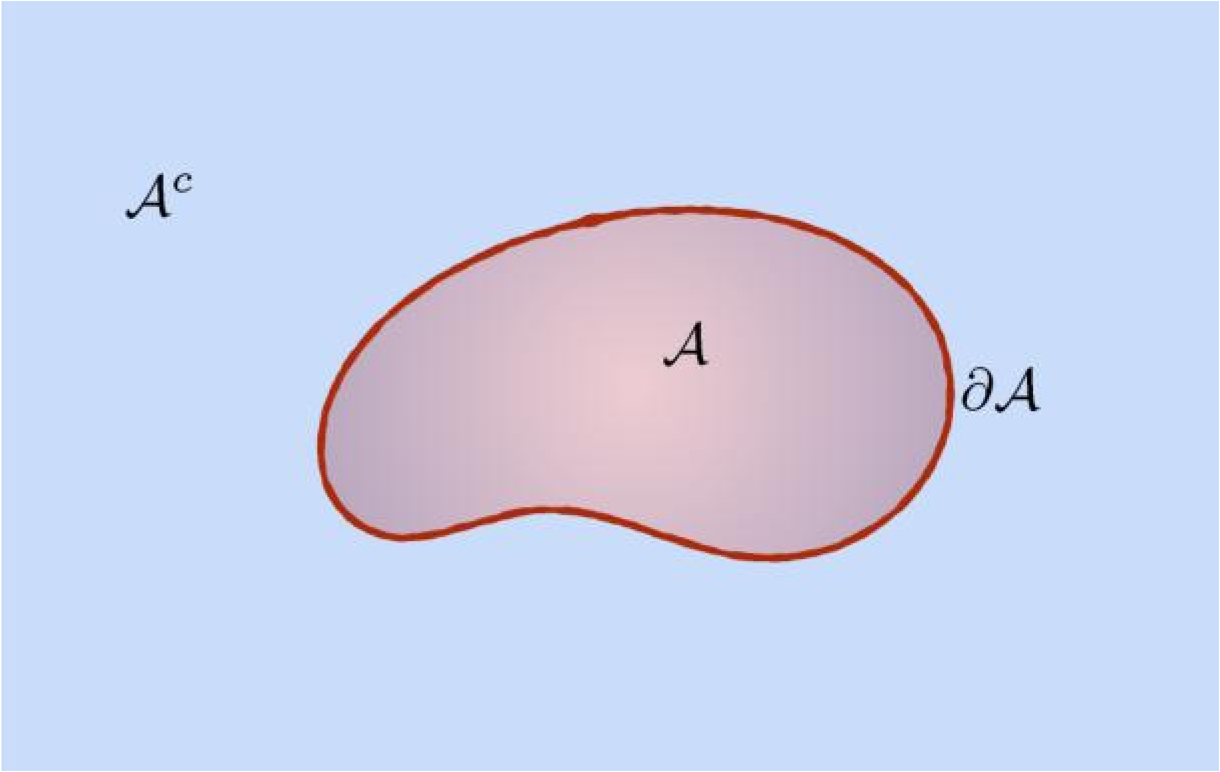
- Lin, Lu and Wen, Geometrizing the Partial Entanglement Entropy: from PEE Threads to Bit Threads, 2311.02301 JHEP
- Lin, Lu and Wen, Partial entanglement network and geometry reconstruction in AdS/CFT, 2401.07471
- Lu, Wen and Zhong **in progress**

Outline

- Holographic entanglement entropy
- Partial entanglement entropy and PEE treads
- Single intervals and spherical regions
- Multi-intervals and weight of the PEE threads
- Entropy interpretation of the Crofton formula

Holographic entanglement entropy

Entanglement Entropy



- **Reduced density matrix**

$$\rho_{\mathcal{A}} = \mathrm{Tr}_{\mathcal{A}^c}\left(|\Psi\rangle\langle\Psi|\right)$$

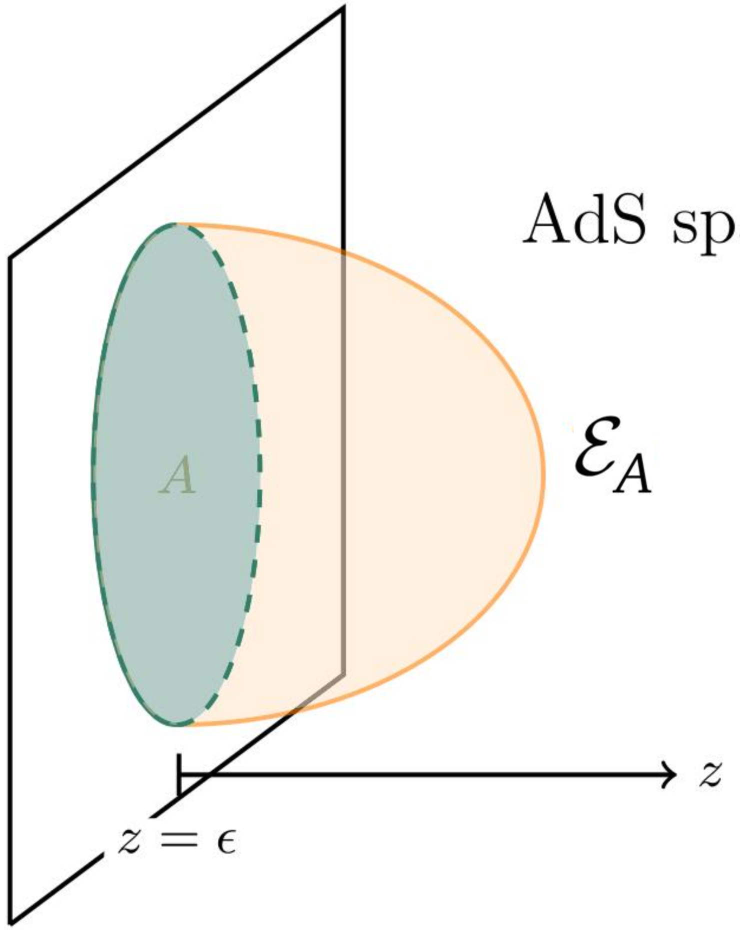
- **Entanglement entropy**

$$S_{\mathcal{A}} = -\mathrm{Tr}_{\mathcal{A}}\left(\rho_{\mathcal{A}}\log\rho_{\mathcal{A}}\right)$$

Holographic entanglement entropy in AdS /CFT

Ryu-Takayanagi formula 06'

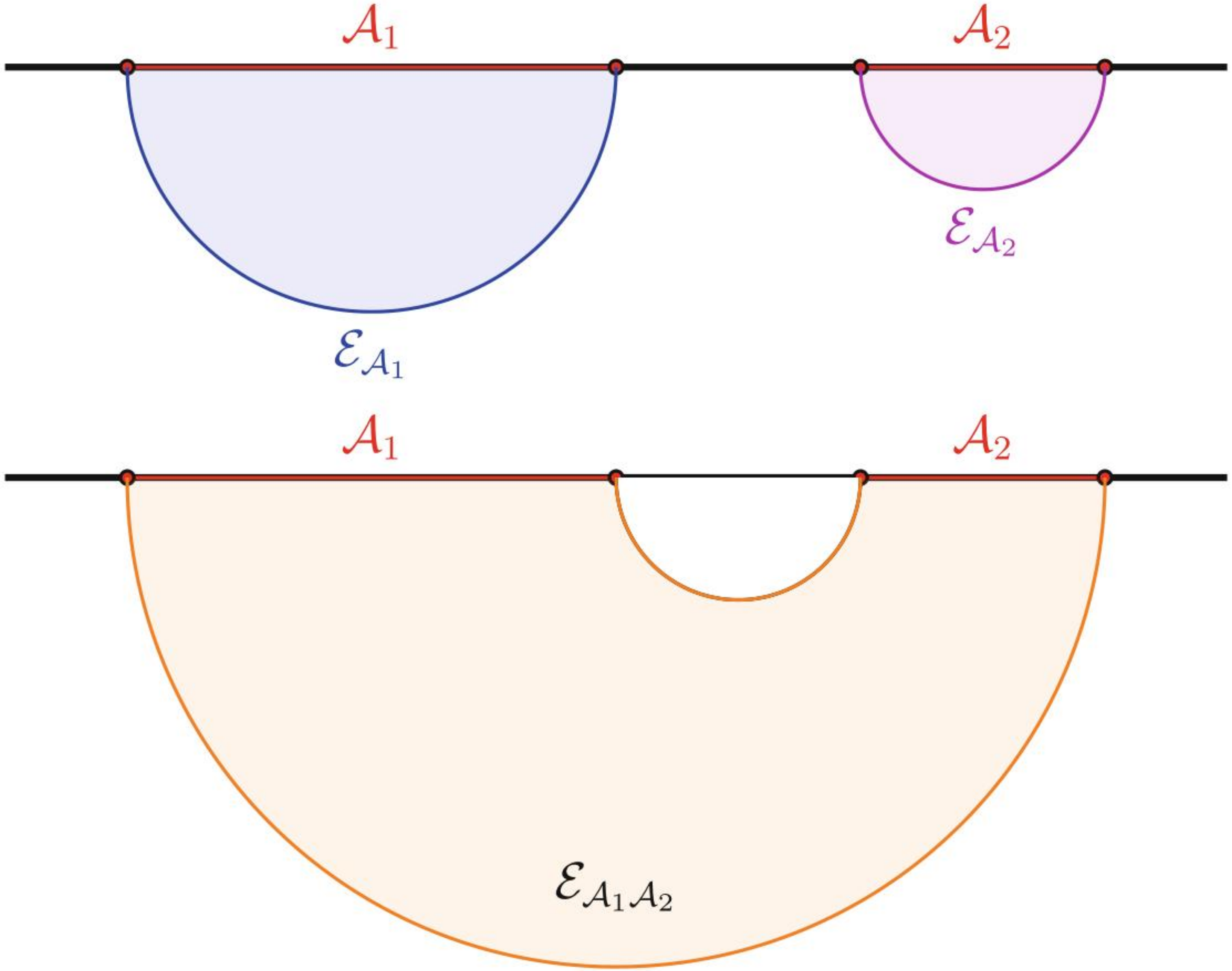
$$S_A = \frac{\text{Area}(\mathcal{E}_A)}{4G_N}$$



Quantum entanglement ⟷ Bulk geometry

Consistent with the field theory calculation at large c limit

Hartman 2011’



Homology surface

Entanglement wedge

Partial entanglement entropy (PEE) and PEE threads!

Partial entanglement entropy

Chen, Vidal, JSTAT, 2014
Wen, PRR, 2020

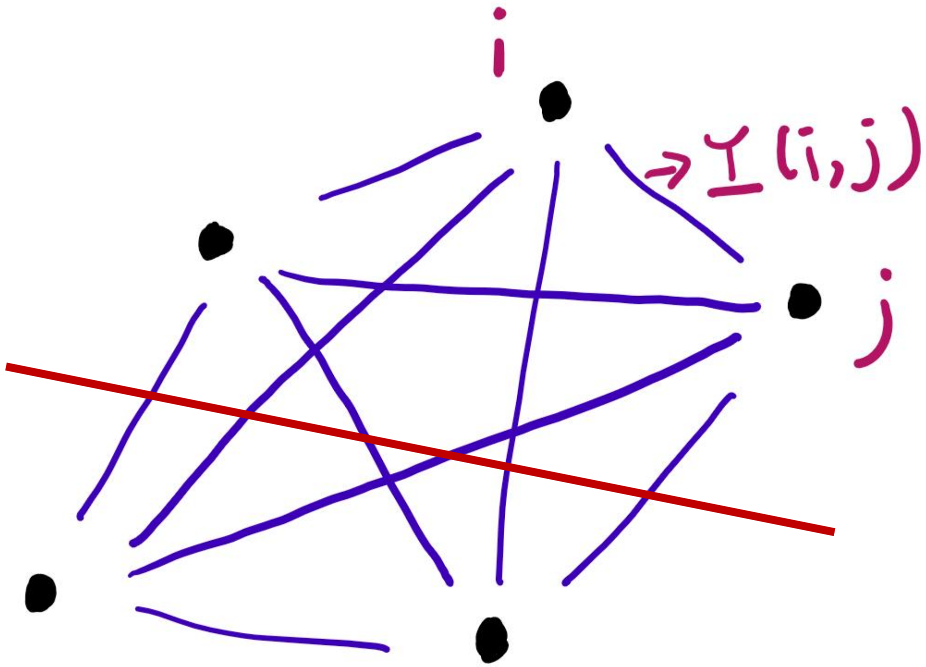
1. *Additivity:* $\mathcal{I}(A,B\cup C)=\mathcal{I}(A,B)+\mathcal{I}(A,C)$;
2. *Permutation symmetry:* $\mathcal{I}(A,B)=\mathcal{I}(B,A)$;
3. *Normalization:* $\mathcal{I}(A,\bar{A})=S_A$; For spherical regions
4. *Positivity:* $\mathcal{I}(A,B)>0$;
5. *Upper bounded:* $\mathcal{I}(A,B)\leq \min\{S_A,S_B\}$;
6. $\mathcal{I}(A,B)$ *should be Invariant under local unitary transformations inside* A *or* B;
7. *Symmetry:* For any symmetry transformation $\mathcal{T}$ under which $\mathcal{T}A=A'$ and $\mathcal{T}B=B'$, we have $\mathcal{I}(A,B)=\mathcal{I}(A',B')$.

Unique solution of Poincare invariant theories.
Explicit formula for CFTs

Wen, PRD, 2018
Wen, PRR, 2020
Casini, Huerta, JHEP, 2010

Partial Entanglement Entropy

Chen, Vidal, JSTAT, 2014
Wen, PRR, 2020



$$\mathcal{I}(A,B) = \sum_{i\in A, j\in B} \mathcal{I}(i,j)$$

$$\mathcal{I}(A,B) = \int_A \mathrm{d}^{d-1}\mathbf{x} \int_B \mathrm{d}^{d-1}\mathbf{y}\, \mathcal{I}(\mathbf{x},\mathbf{y}) \,.$$

$$S_A = \int_A \mathrm{d}^{d-1}\mathbf{x} \int_{\bar{A}-\epsilon} \mathrm{d}^{d-1}\mathbf{y}\, \mathcal{I}(\mathbf{x},\mathbf{y}),$$

For spherical regions

Wen, PRD, 2018
Wen, PRR, 2020
Casini, Huerta, JHEP, 2010

- **Unique solutions exist at least in:**
- 1, Generic two-dimensional theories where all the degrees of freedom settled with a natural order
- 2, Highly symmetric higher dimensional theories where the entanglement structure only vary along one spatial direction configurations
- 3, Higher dimensional theories with Poincaré symmetry

Two point PEE for (Vacuum) CFTs

Basu, Lin, Lu, Wen, Scipost Phys., 2023
Casini, Huerta, JHEP, 2010

Additivity and permutation symmetry give:

$$\mathcal{I}(A,B) = \int_A \mathrm{d}\sigma_{\mathbf{x}} \int_B \mathrm{d}\sigma_{\mathbf{y}}\, \mathcal{I}(\mathbf{x},\mathbf{y}).$$

In CFT2

$$\mathcal{I}(x,y) = \frac{c}{6}\frac{1}{(x-y)^2}$$

In d-dimentional CFT,

$$\mathcal{I}(\mathbf{x}_1,\mathbf{x}_2) = \frac{c}{6}\frac{2^{d-1}(d-1)}{\Omega_{d-2}|\mathbf{x}_2-\mathbf{x}_1|^{2(d-1)}}.$$

Geometrizing the PEE 1:
Representing the two-point PEE via the bulk geodesics

PEE threads!

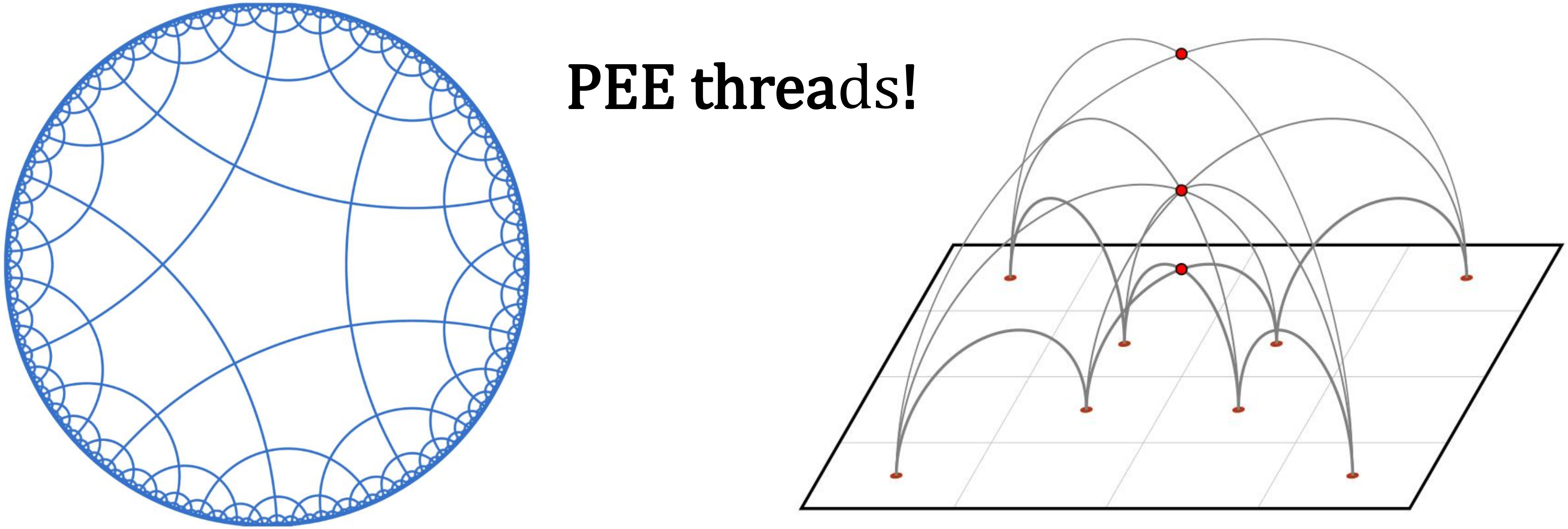
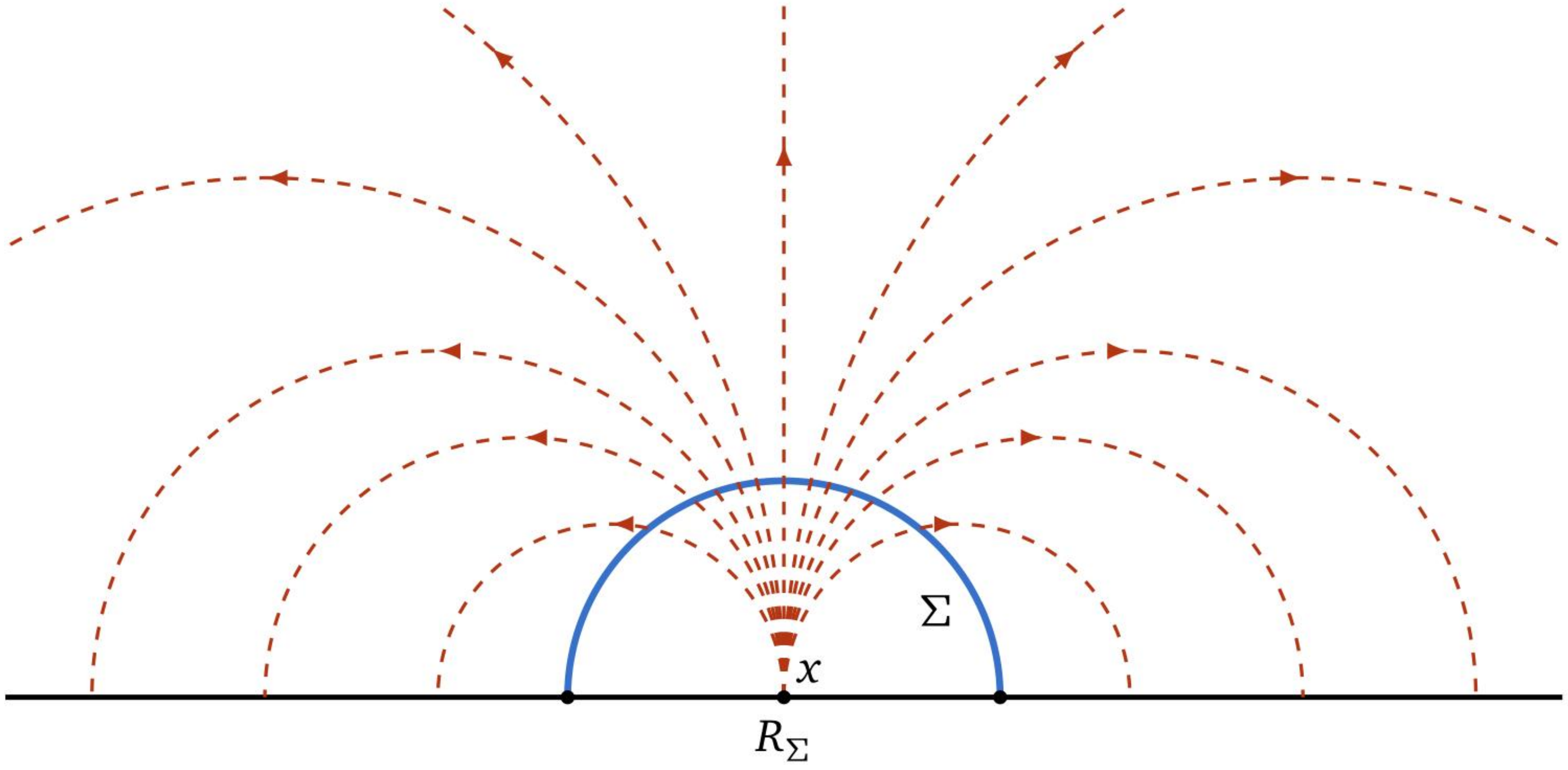


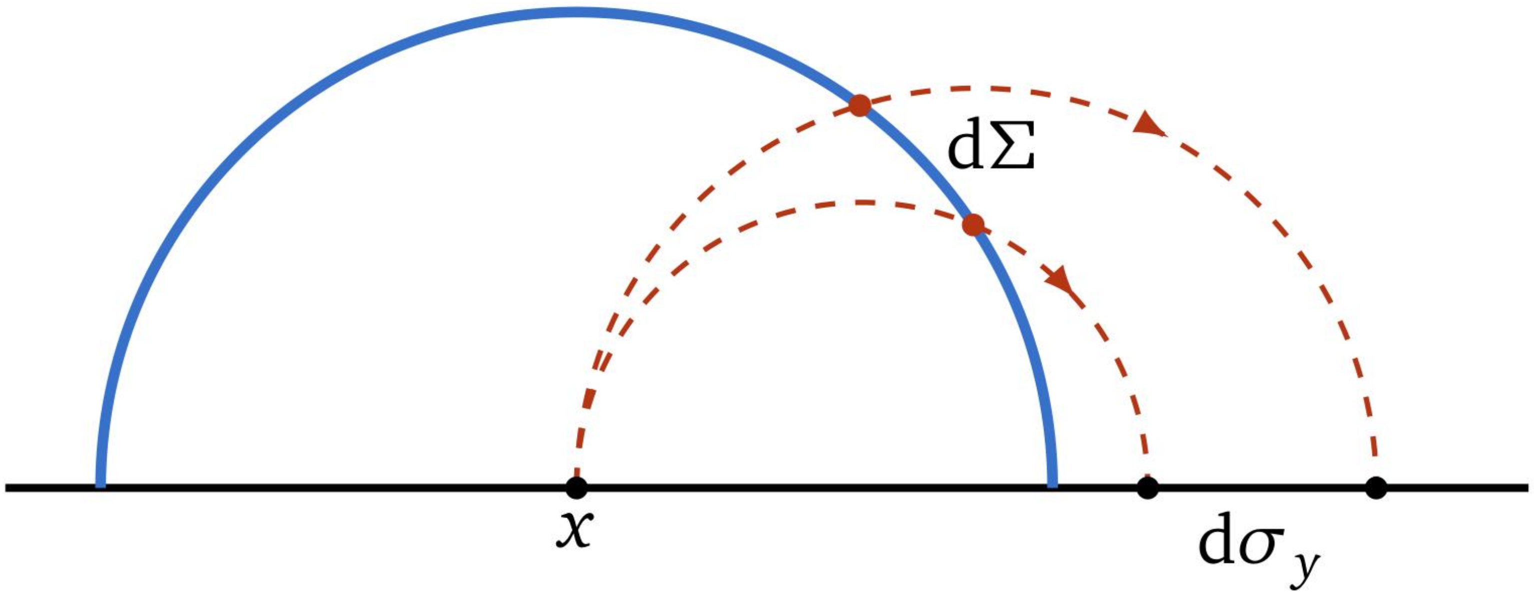
Figure 1: PEE threads on a static time slice of global AdS_3/CFT_2.



PEE flow from x: $$V_{\mathbf{x}}^{\mu} = |V_{\mathbf{x}}|\tau_{\mathbf{x}}^{\mu}$$

Geometrizing the PEE 2:

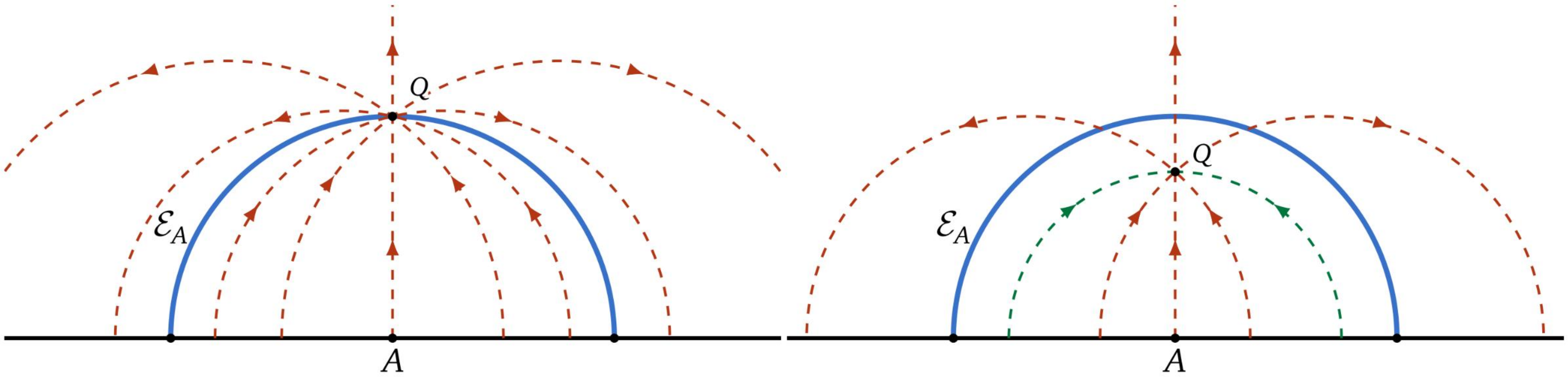
Flux description of the PEE structure



$$\mathrm{d}\sigma_{\mathbf{y}}\,\mathcal{I}(\mathbf{x},\mathbf{y}) = \mathrm{d}\Sigma\,\sqrt{h}V_{\mathbf{x}}^{\mu}n_{\mu}$$

$\mathcal{I}(A,\bar{A}) = S_A$ **:The flux of PEE threads from A to barA** the *PEE thread flow* V_O^μ at the point $Q=(\bar r,\bar z)$

$$V_O^\mu(Q)=\frac{1}{4G_N}\frac{2\bar z^2\bar r}{(\bar r^2+\bar z^2)^2}\left(\bar z,\frac{\bar z^2-\bar r^2}{2\bar r}\right).$$



$$V_A^\mu(Q)=\int_{-R}^{R}\mathrm{d}r_0V_{r_0}^\mu=\frac{\bar z^2}{4G_N}\frac{2R}{((R-\bar r)^2+\bar z^2)((R+\bar r)^2+\bar z^2)}\left(2\bar z\bar r,R^2-\bar r^2+\bar z^2\right)$$

PEE threads and bit threads in AdS_{d+1}

$$V_O^{\mu}(Q) = \frac{2^d \bar{z}^d}{4G_N}\frac{(d-1)}{\Omega_{d-2}}\frac{\bar{r}}{(\bar{r}^2+\bar{z}^2)^d}\left(\bar{z}, \frac{\bar{z}^2-\bar{r}^2}{2\bar{r}}\right)$$

Static spherical region A

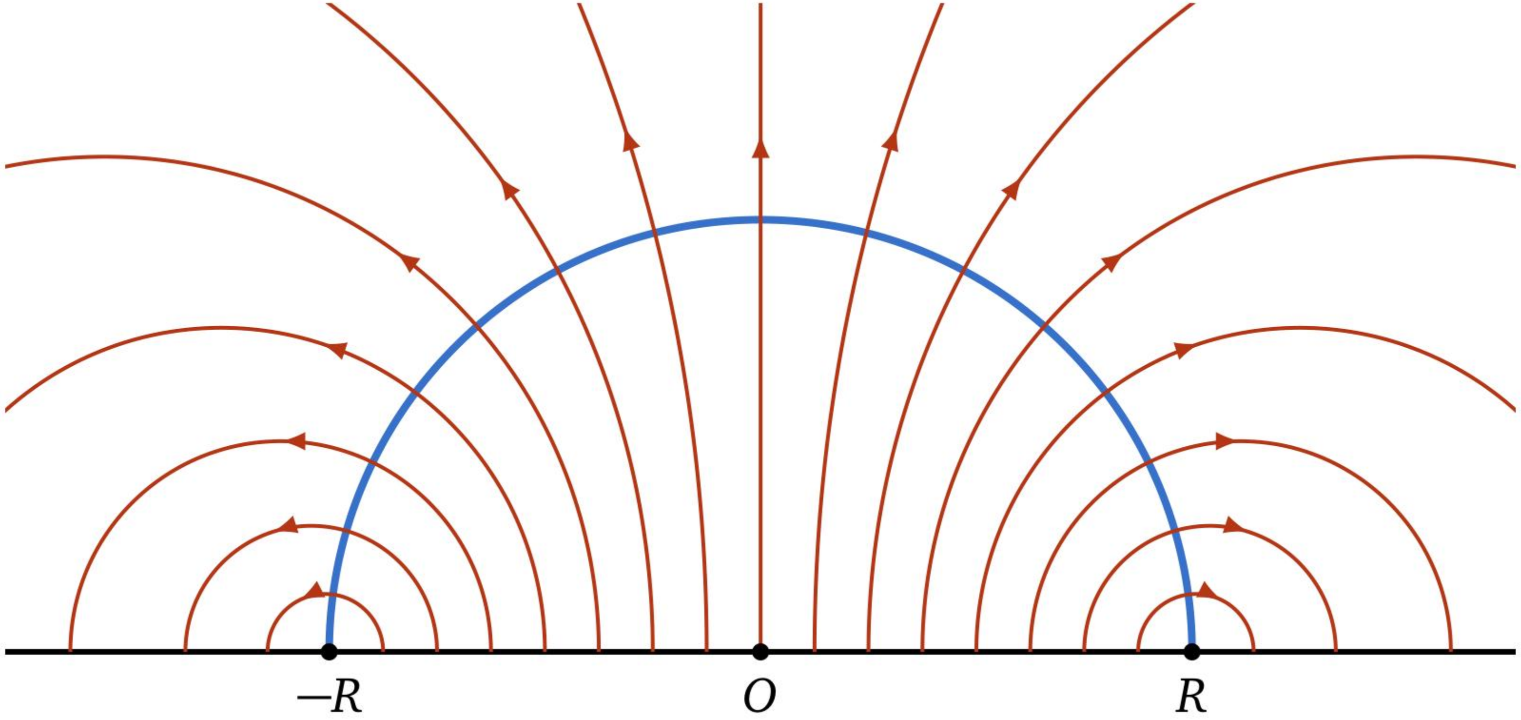
Agón, De Boer, Pedraza, JHEP 05, 075

$$V_A^r(Q) = \int_{\bar{r}-R}^{\bar{r}+R} \mathrm{d}r_0 \, V^r_{\text{partial}\,\circ_{r_0}} = \frac{1}{4G_N}\frac{\bar{r}\bar{z}}{R}\left(\frac{2R\bar{z}}{\sqrt{(R^2+\bar{r}^2+\bar{z}^2)^2-4R^2\bar{r}^2}}\right)^d,$$

$$V_A^z(Q) = \int_{\bar{r}-R}^{\bar{r}+R} \mathrm{d}r_0 \, V^z_{\text{partial}\,\circ_{r_0}} = \frac{1}{4G_N}\frac{R^2-\bar{r}^2+\bar{z}^2}{2R}\left(\frac{2R\bar{z}}{\sqrt{(R^2+\bar{r}^2+\bar{z}^2)^2-4R^2\bar{r}^2}}\right)^d$$

Other A? No!

$V_A^{\mu}(Q)$



Agón, De Boer, Pedraza, JHEP 05, 075

$$\int_A V_{\mathbf{x}}^{\mu}(Q)\mathrm{d}^{d-1}\mathbf{x} = \frac{1}{4G} n^{\mu}(Q),$$

point Q on the RT surface $\mathcal{E}_A$

Multi-intervals and Weighting the PEE threads!

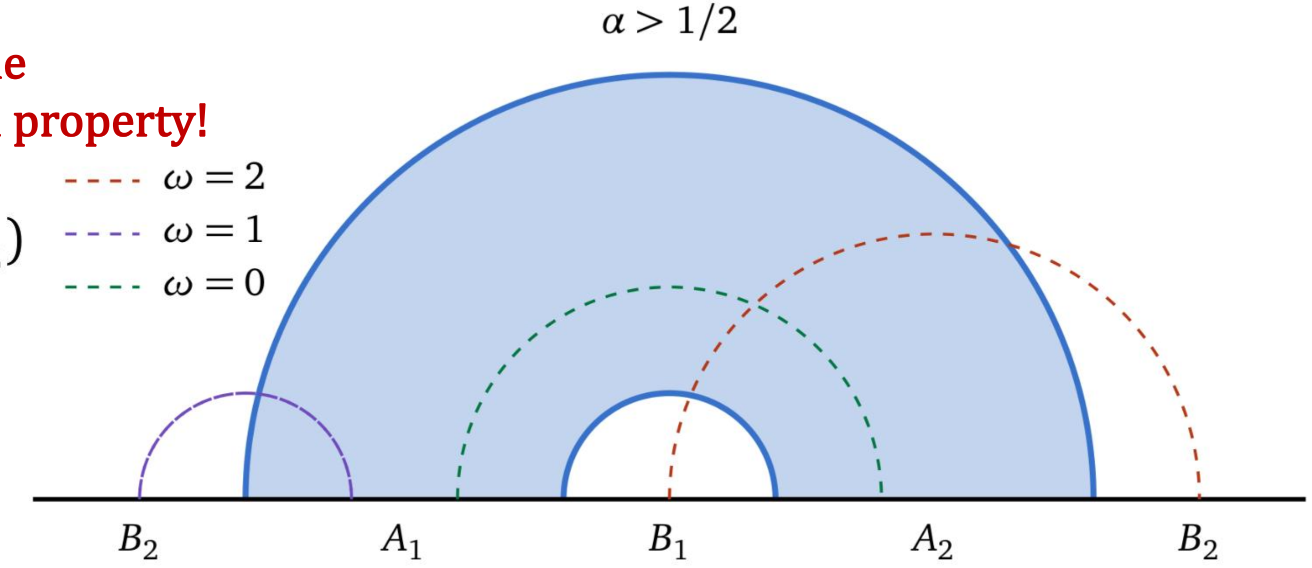
Provided RT formula is right!

PEE threads for multi-intervals

$\mathcal{I}(A,\bar{A})$

Violation of the normalization property!

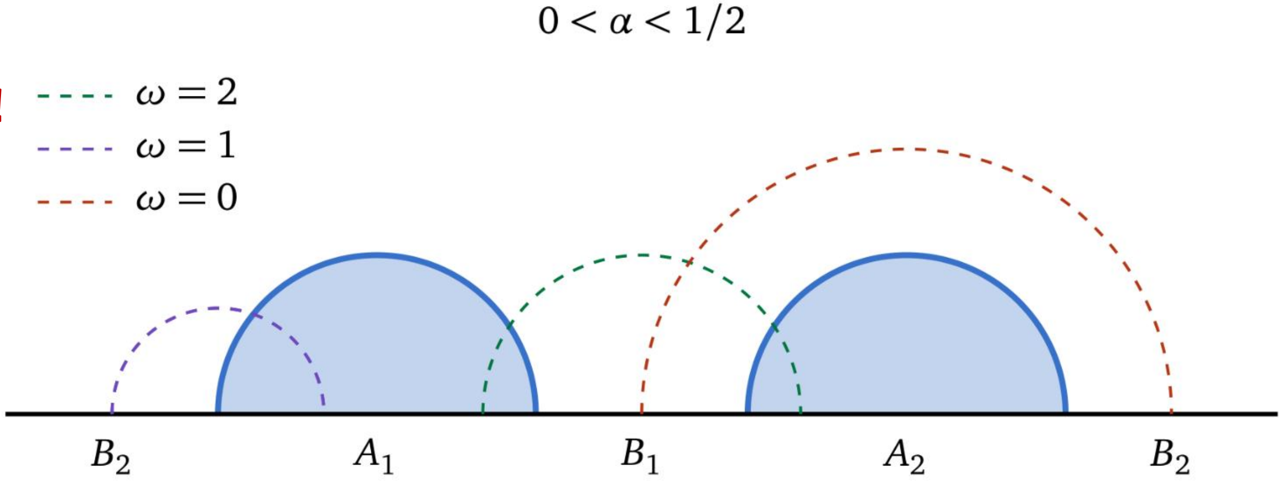
$$S_A = \mathcal{I}(A_1, B_1 \cup B_2) + \mathcal{I}(A_2, B_1 \cup B_2) + 2\mathcal{I}(B_1, B_2)$$
$$= \mathcal{I}(A_1 \cup A_2 \cup B_2, B_1) + \mathcal{I}(A_1 \cup A_2 \cup B_1, B_2)$$
$$= S_{B_1} + S_{B_2},$$



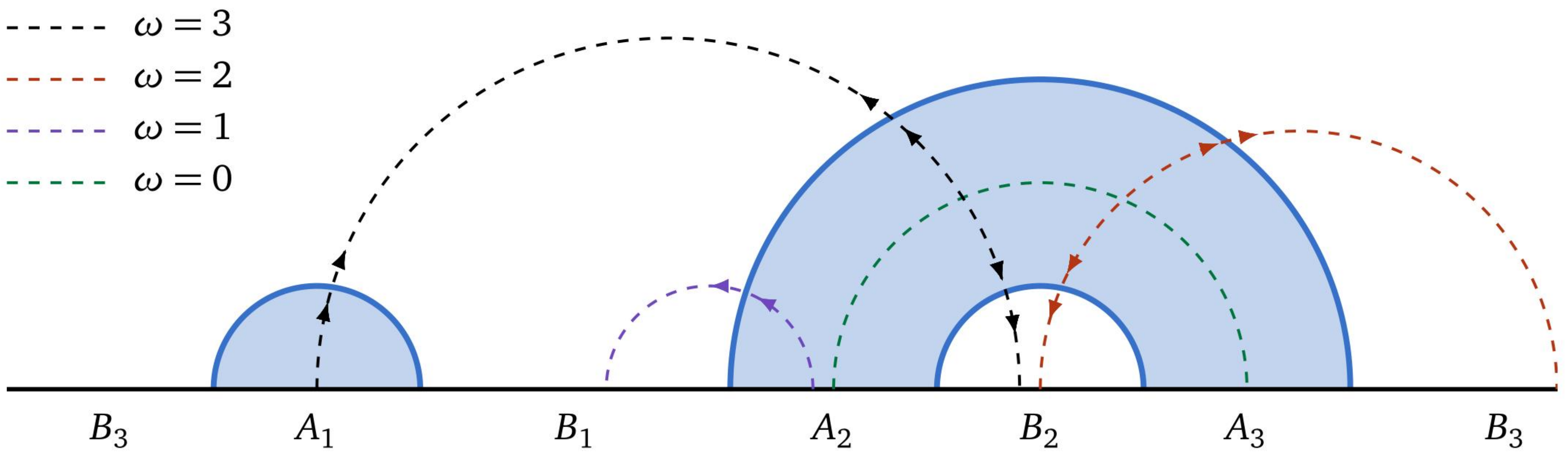
$\mathcal{I}(A,\bar{A})$

Violation of the normalization property!

$$S_A = \mathcal{I}(A_1, B) + \mathcal{I}(A_2, B) + 2\mathcal{I}(A_1, A_2)$$
$$= S_{A_1} + S_{A_2}$$

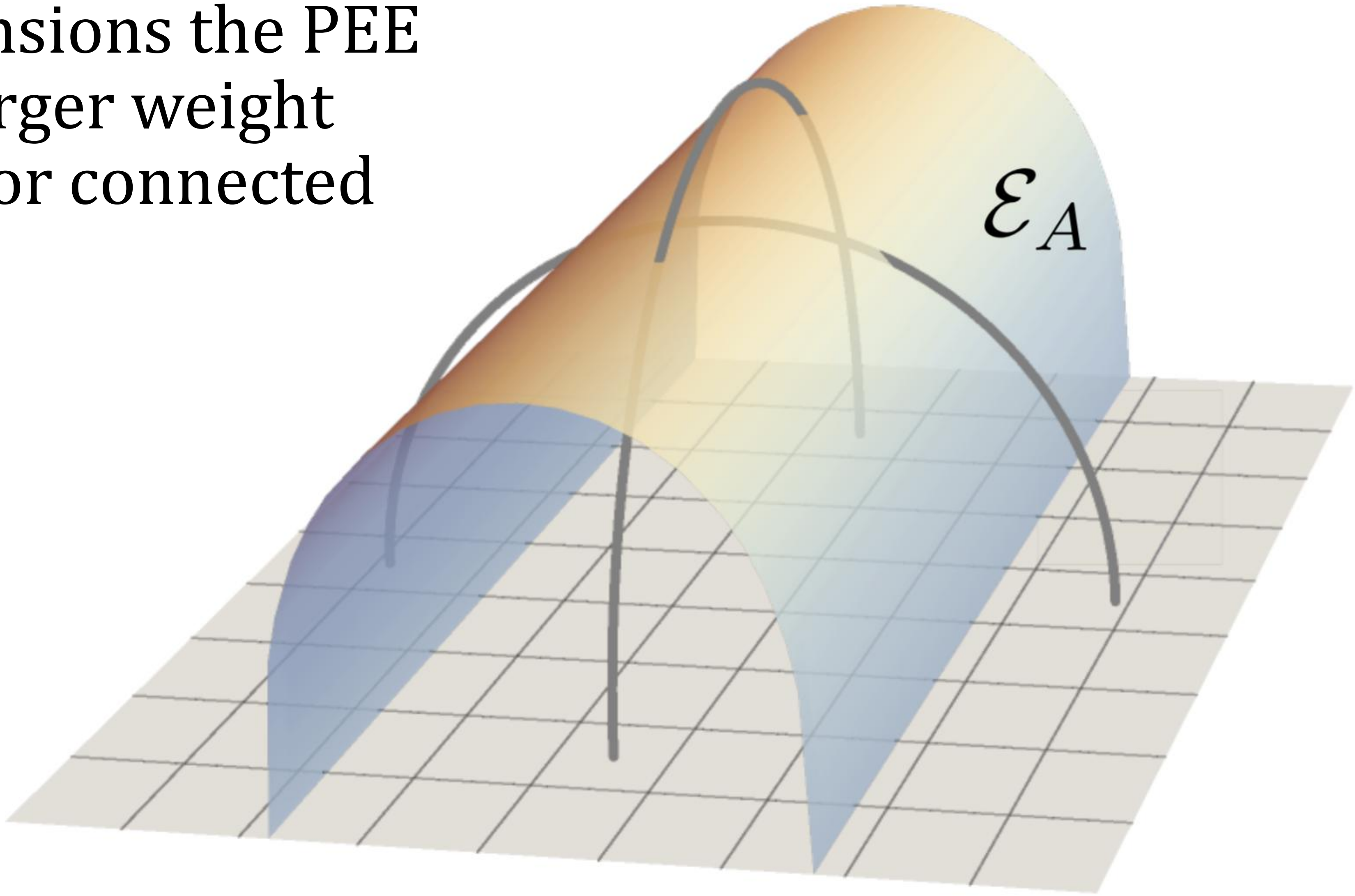


Flux from A to $\overline{\text{A}}$ ⇒ Flux from $\mathcal{W}_A$ to $\mathcal{W}_B$



$$
\begin{aligned}
S_A =& \mathcal{I}(A_1, B_1B_3) + 2\mathcal{I}(A_1, A_2A_3) + 3\mathcal{I}(A_1, B_2) + \mathcal{I}(A_2A_3, B_1B_2B_3) + 2\mathcal{I}(B_1B_3, B_2) \\
=& \mathcal{I}(A_1, A_2A_3B_1B_2B_3) + \mathcal{I}(B_2, A_1A_2A_3B_1B_3) + \mathcal{I}(A_2B_2A_3, A_1B_1B_3) \\
=& S_{A_1} + S_{B_2} + S_{A_2 \cup B_2 \cup A_3},
\end{aligned}
$$

- In higher dimensions the PEE threads with larger weight happens even for connected regions



A reformulation of the RT formula
and
A reconstruction for the bulk geometric quantities

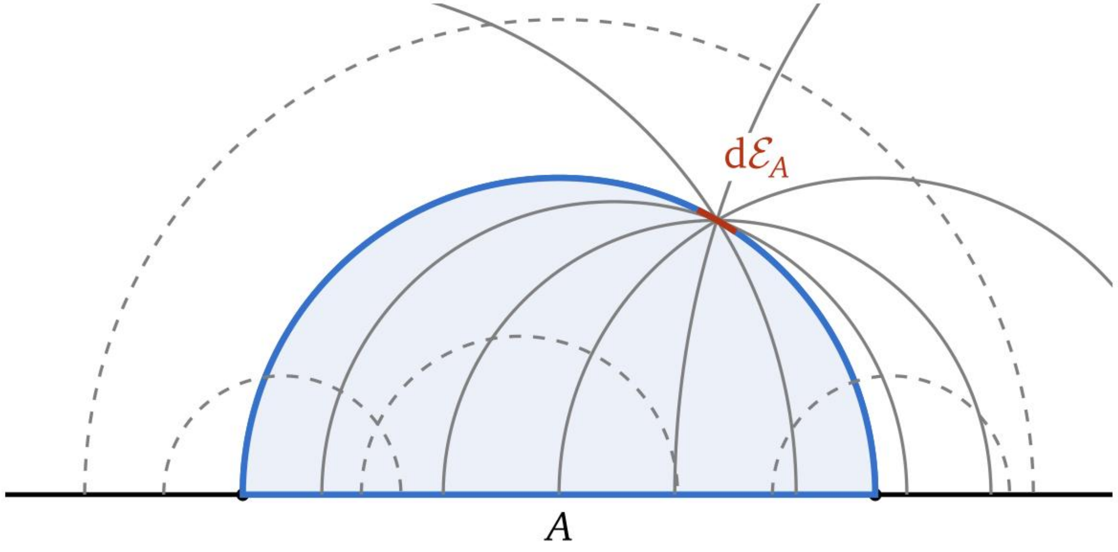
Conjecture: A reformulation of RT formula based on the configurations of the PEE threads (or network)

$$S_A = \min_{\Sigma_A} \frac{1}{2} \int_{\partial\mathcal{M}} \mathrm{d}^{d-1}\mathbf{x} \int_{\partial\mathcal{M}} \mathrm{d}^{d-1}\mathbf{y}\ \omega_{\Sigma_A}(\mathbf{x},\mathbf{y})\mathcal{I}(\mathbf{x},\mathbf{y}),$$

Or

$$S_A = \min_{\Sigma_A} \frac{1}{2} \int_{\Sigma_A} \mathrm{d}\Sigma_A \sqrt{h} \int_{\partial\mathcal{M}} \mathrm{d}^{d-1}\mathbf{x} |V^{\mu}_{\mathbf{x}} n_{\mu}|.$$

Proof:



Consider an area element on the RT surface for any spherical region.

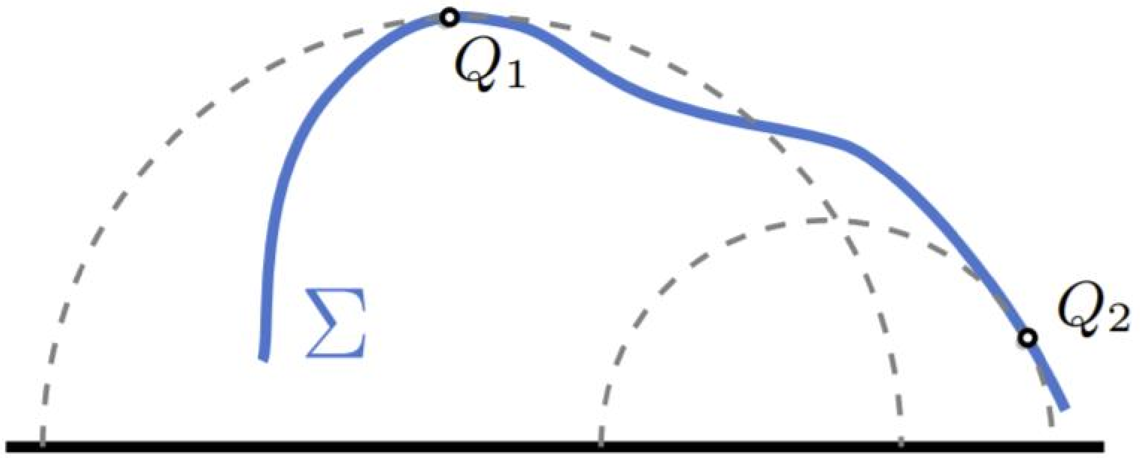
Only the third class of PEE threads will pass the area element

Consider a spherical region A. Its RT surface $\mathcal{E}_A$ is just a hemisphere, and the PEE threads can be classified into three classes,

1. $\omega_{\mathcal{E}_A}(\mathbf{x}, \mathbf{y}) = 0$ for $\mathbf{x}, \mathbf{y} \in A$;
2. $\omega_{\mathcal{E}_A}(\mathbf{x}, \mathbf{y}) = 0$ for $\mathbf{x}, \mathbf{y} \in \bar{A}$;
3. $\omega_{\mathcal{E}_A}(\mathbf{x}, \mathbf{y}) = 1$ for $\mathbf{x} \in A, \mathbf{y} \in \bar{A}$ or $\mathbf{x} \in \bar{A}, \mathbf{y} \in A$.

The flux strength is 1/4G!

$$\int_A d^{d-1}\mathbf{x} |V^\mu_\mathbf{x}(Q) n_\mu(Q)| = \frac{1}{2}\int_{\partial\mathcal{M}} d^{d-1}\mathbf{x} |V^\mu_\mathbf{x}(Q) n_\mu(Q)| = \frac{1}{4G}$$



Consider an arbitrary surface in the bulk

Any element area can be embedded in the RT surface for a spherical region!

Again the flux strength across this area element is 1/4G!

Conclusion: the strength of the PEE flux at any point, along any direction, is always 1/4G!

In other words, if we set 1/4G as the upper limit for the strength of the PEE threads, **then AdS space is full of PEE threads!**

For an arbitrary homologous surface, the PEE flux from one side of it to the other side is proportional to its area.

$$S_A = \frac{1}{4G} \min_{\Sigma_A} \int_{\Sigma_A} \mathrm{d}\Sigma_A \sqrt{h}$$
$$= \min_{\Sigma_A} \frac{\mathrm{Area}[\Sigma_A]}{4G} = \frac{\mathrm{Area}[\mathcal{E}_A]}{4G},$$

This is exactly the RT formula!

Reconstruction for more generic geometric quantities

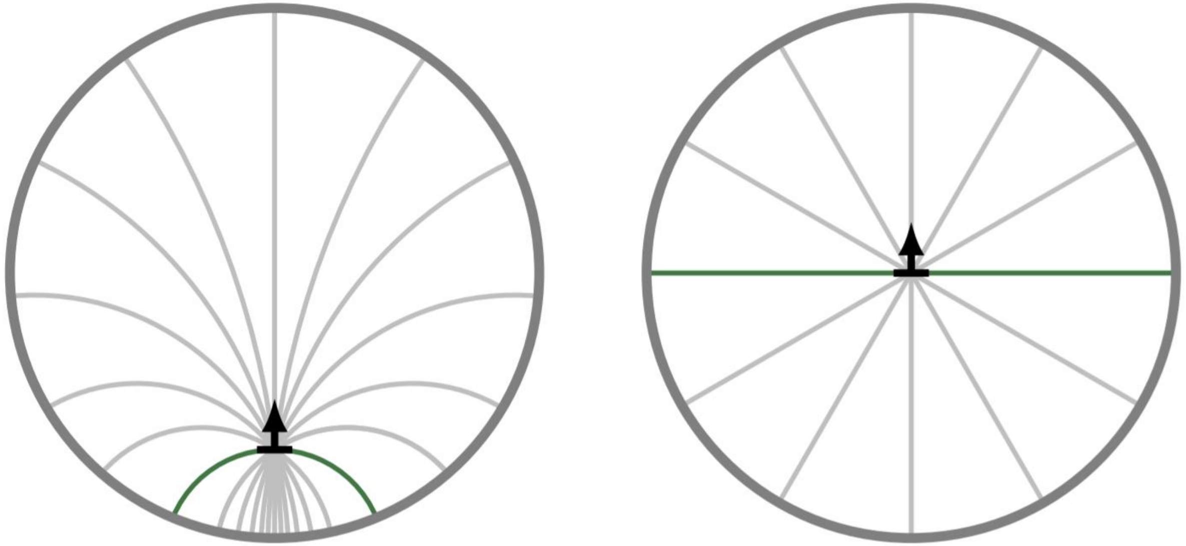
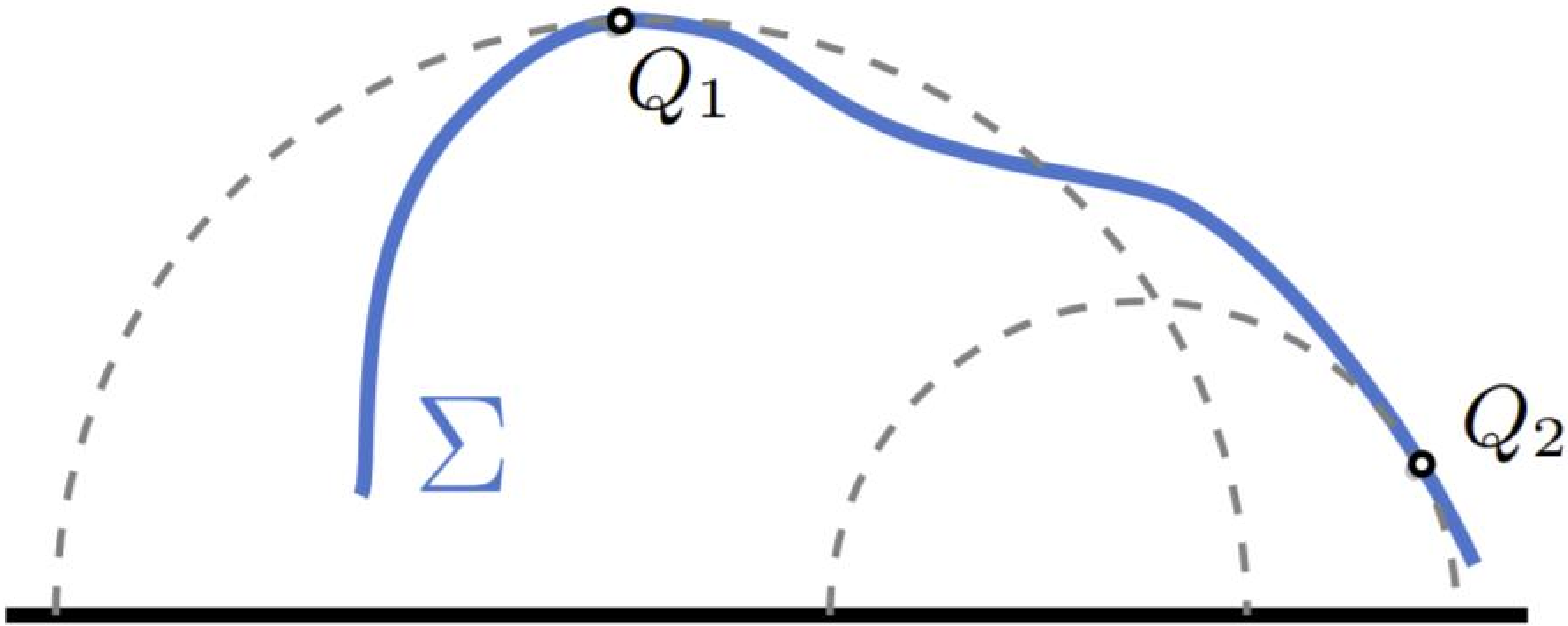


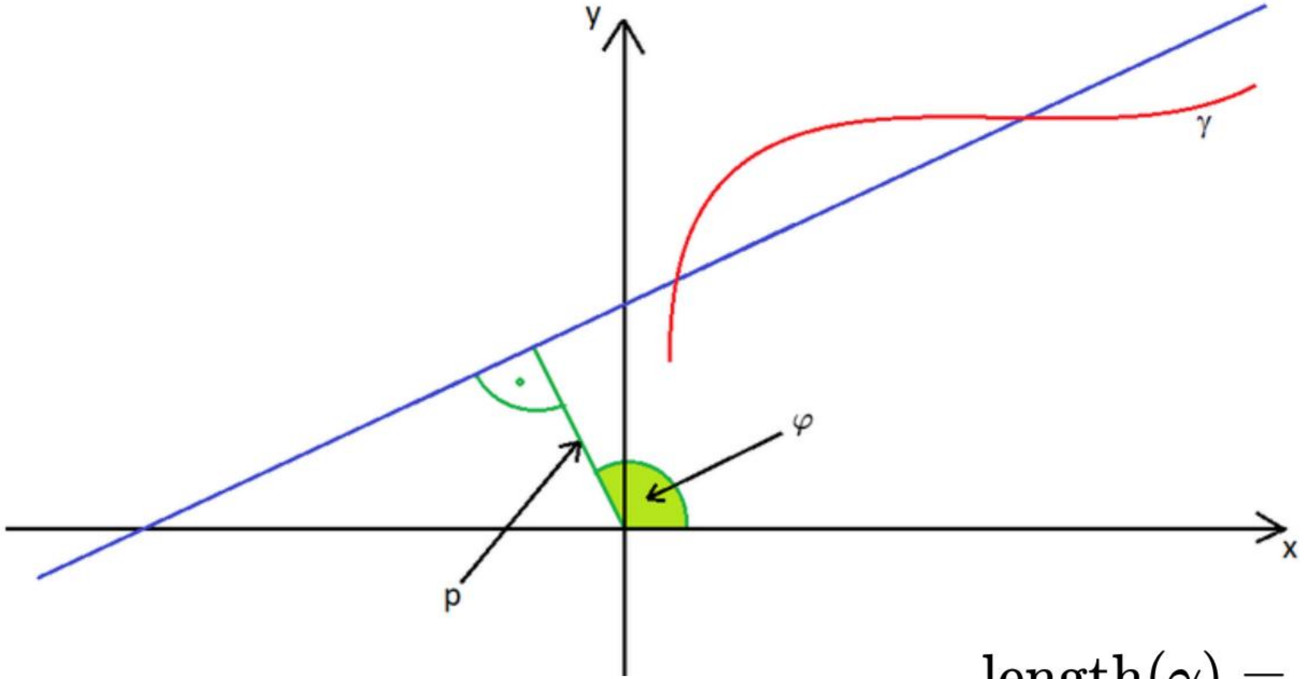
FIG. 6. The area element $d\Sigma$ and its direction is represented by the black arrow. The gray curves represent all the PEE threads that involve in the reconstruction of $d\Sigma$, and the green curve represents the PEE thread that saturates the lower bound.



$$\frac{\mathrm{Area}(\mathrm{d}\Sigma)}{4G} = \frac{1}{2}\int_{\partial\mathcal{M}} \mathrm{d}^{d-1}\mathbf{x} \int_{\partial\mathcal{M}} \mathrm{d}^{d-1}\mathbf{y}\ \omega_{\mathrm{d}\Sigma}(\mathbf{x},\mathbf{y})\mathcal{I}(\mathbf{x},\mathbf{y})\,,$$

Entropy interpretation of the Crofton formula

See also
Bartek Czech eta.al JHEP 2015, 2016
Xing Huang and Fengli Lin, JHEP 2015



$$\text{length}(\gamma) = \frac{1}{4} \iint n_\gamma(\varphi, p)\, d\varphi\, dp.$$

The differential form

$$d\varphi \wedge dp$$

is invariant under rigid motions of $\mathbb{R}^2$, so it is a natural integration measure for speaking of an "average" number of intersections. It is usually called the **kinematic measure**.

Crofton formula in more generic mainfold

$$\boxed{\frac{1}{2}\frac{d-1}{\Omega_{d-2}}\int_{\Gamma\cap\Sigma\neq\varnothing} n(\Sigma\cap\Gamma)\mathrm{d}\Gamma = \mathrm{Area}(\Sigma).}$$

$$\begin{aligned}\mathrm{d}\Gamma &= (\mathrm{d}p_\mu \wedge \mathrm{d}x^\mu)^{d-1}\\ &= \sum_{i=1}^{d} \mathrm{d}p_1 \wedge \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}p_{i-1} \wedge \mathrm{d}x^{i-1} \wedge \mathrm{d}p_{i+1} \wedge \mathrm{d}x^{i+1} \wedge \cdots \mathrm{d}p_d \wedge \mathrm{d}x^d,\end{aligned}$$

Kinematic measure: An invariant measure of $(2d-2)$-dimensional set of geodesics

For a time slice of d+1 AdS spacetime

$$p_\mu|_{x_2} = \partial\ell(x_1, x_2)/\partial x_2^\mu$$

$$\mathrm{d}\Gamma = \det\left(\frac{\partial^2\ell(\mathbf{x}_1, \mathbf{x}_2)}{\partial\mathbf{x}_1\partial\mathbf{x}_2}\right)\mathrm{d}\mathbf{x}_1 \wedge \mathrm{d}\mathbf{x}_2,$$

$$\mathrm{d}\mathbf{x}_{1,2} = \mathrm{d}x_{1,2}^1 \wedge \mathrm{d}x_{1,2}^2 \wedge \cdots \wedge \mathrm{d}x_{1,2}^{d-1}$$

$$\ell(\mathbf{x}_1, \mathbf{x}_2) = 2\log\frac{|\mathbf{x}_1 - \mathbf{x}_2|}{\epsilon}$$

Kinematic measure:

$$\det\left(\frac{\partial^2\ell(\mathbf{x}_1, \mathbf{x}_2)}{\partial\mathbf{x}_1\partial\mathbf{x}_2}\right) = \det\left(2\frac{-\delta_{ij}(\mathbf{x}_1 - \mathbf{x}_2)^2 + 2\left(x_1^i - x_2^i\right)\left(x_1^j - x_2^j\right)}{(\mathbf{x}_1 - \mathbf{x}_2)^4}\right) = \frac{2^{d-1}}{(\mathbf{x}_1 - \mathbf{x}_2)^{2d-2}}.$$

$$\mathcal{I}(\mathbf{x}_1, \mathbf{x}_2) = \frac{c}{6} \frac{2^{d-1}(d-1)}{\Omega_{d-2}|\mathbf{x}_1 - \mathbf{x}_2|^{2(d-1)}} \Rightarrow \mathcal{I}(\mathbf{x}_1, \mathbf{x}_2) = \frac{c}{6} \frac{d-1}{\Omega_{d-2}} \det\left(\frac{\partial^2 \ell(\mathbf{x}_1, \mathbf{x}_2)}{\partial \mathbf{x}_1 \partial \mathbf{x}_2}\right)$$

$$\frac{\text{Area}(\Sigma)}{4G} = \frac{1}{2} \int_{\partial\mathcal{M}} \mathrm{d}^{d-1}\mathbf{x}_1 \int_{\partial\mathcal{M}} \mathrm{d}^{d-1}\mathbf{x}_2 \; \omega_\Sigma(\mathbf{x}_1, \mathbf{x}_2) \mathcal{I}(\mathbf{x}_1, \mathbf{x}_2)$$

$$\Updownarrow$$

$$\boxed{\frac{1}{2} \frac{d-1}{\Omega_{d-2}} \int_{\Gamma\cap\Sigma\neq\varnothing} n(\Sigma \cap \Gamma) \mathrm{d}\Gamma = \text{Area}(\Sigma).}$$

Summary

- The static AdS space is full of PEE threads with no empty space, hence we can interpret the area of any geometric quantity in terms of the number of PEE threads passing through it.

- Mathematically this context if a special application of the Crofton formula, but in the context of AdS/CFT with entropy interpretation for all the bulk geodesics.

- The contribution from “inner threads” to entanglement entropy gives us new understanding about the contribution distribution of entanglement entropy.

Future directions

- How to go beyond Poincare AdS? Black hole? With matter fields? Non-AdS spacetime?
- Go beyond static scenarios?
- Toy models of quantum gravity based on the tensor network of PEE threads?

Thanks!